Exercises to „Algebra II“, 1

All rings are supposed to be commutative and contain a 1.

Exercise 1 (2 points). Let $I \subseteq R$ be an ideal. Show that

$$\sqrt{I} = \left\{ f \in R \;\middle|\; f^a \in I \text{ for some } a \in \mathbb{N} \right\}$$

is an ideal.

Recall that, for any commutative ring R and any ideal $I \subseteq R$, we define

$$V(I) = \left\{ \mathfrak{p} \in \mathrm{Spec} R \;\middle|\; \mathfrak{p} \supseteq I \right\}$$

Exercise 2 (5 points). Show that

- $V\left(\sum_{\lambda \in \Lambda} I_\lambda\right) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$
- $V(I \cdot J) = V(I \cap J) = V(I) \cup V(J)$
- $V(\sqrt{I}) = V(I)$.

where I, J and the I_λ are ideals in R and the index set Λ may be infinite. Give a counterexample to the second equality of the second point for infinite intersections of ideals.

Exercise 3 (5 points). For a topological space X, show the equivalence of the following conditions:

- Any open subset of X is quasi-compact.
- There is no strictly descending chain $Z_0 \supsetneq Z_1 \supsetneq \dots$ of closed subsets of X.
- Any non-empty set of closed subsets of X contains a $\subseteq$-minimal element.

Remark 1. Recall that such topological spaces are called *Noetherian*. The equivalence depends on the axiom of choice.

Exercise 4 (5 points). For a topological space X, show the equivalence of the following conditions:

- If $X = A \cup B$ where A and B are closed subsets, then $A = X$ or $B = X$. Moreover, we have $X \neq \emptyset$.
- Any non-empty open subset of X is dense in X. Moreover, we have $X \neq \emptyset$.
- The intersection of two non-empty open subsets of X is non-empty. Moreover, we have $X \neq \emptyset$.
- If $n \in \mathbb{N}$ and if $X = \bigcup_{i=1}^{n} Z_i$ is a covering of X by n closed subsets Z_i, then there is a natural number i such that $1 \leq i \leq n$ and $Z_i = X$.

Remark 2. Recall that such topological spaces are called *irreducible*

Exercise 5 (3 points). Let X be a Noetherian topological space and $\mathfrak{A}$ the set of closed subsets of X containing an interior point (i. e., containing a non-empty open subset). Show that the $\subseteq$-minimal elements of $\mathfrak{A}$ are precisely the elements of $\mathfrak{A}$ which are irreducible (when equipped with the induced topology). Also, show that the $\subseteq$-minimal elements for $\mathfrak{A}$ cover X.

Solutions should be submitted Monday, October 23, in the lecture.