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Étude d'espaces de modules et dynamique des catégories dérivées

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Résumé

Cette thèse se divise en deux parties. D'une part, elle a pour but l'étude d'espaces de modules de faisceaux cohérents sur une 3-variété de Fano X complexe et une surface K3 $S \subset X$. Plus précisément, on considère une 3-variété de Fano primitive d'indice 1 et de genre 9. On montre que la restriction des faisceaux sur X à une surface K3 $S \subset X$ de genre 9 obtenue comme section hyperplane donne un morphisme

$$res : \mathcal{M}_X(2,1,7) \to \mathcal{M}_S(2,1,7)$$

entre les espaces de modules de faisceaux sur X et S, qui identifie l'image de $\mathcal{M}_X(2,1,7)$ à une sous-variété lagrangienne de $\mathcal{M}_S(2,1,7)$ singulière en un nombre fini de point. De plus, lorsque l'on fait varier X dans le système linéaire $\mathcal{W} \subset \mathbb{P}^3$ des Fano lisses qui contiennent S, les morphismes de restriction se recollent et on obtient une fibration lagrangienne rationnelle

$$\mathcal{M}_S(2,1,7) \dashrightarrow \mathbb{P}^3$$

définie par l'assignation d'un faisceaux $F \in \mathcal{M}_S(2,1,7)$ globalement engendré provenant d'une Fano X à sa classe $[X] \in \mathcal{W}$. En outre, il existe un modèle birationnel $\mathcal{M} \dashrightarrow \mathcal{M}_S(2,1,7)$ qui étend la fibration rationnelle en une fibration lagrangienne

$$\mathcal{M} \to \mathbb{P}^3.$$

Dans un second temps, cette thèse se penche sur l'étude de système dynamique dans un contexte catégorique. Dans ce manuscrit, on se concentre sur le cas d'une surface projective lisse complexe S et de sa catégorie dérivée $\mathrm{D}^b(S)$. Étant donné une autoéquivalence $\varphi : \mathrm{D}^b(S) \to \mathrm{D}^b(S)$, on s'intéresse à deux quantités :

- l'*entropie catégorique* $h_{cat}(\varphi)$, qui calcule la complexité du système dynamique $(\mathrm{D}^b(S), \varphi)$, et
- l'*entropie topologique généralisée* $\log \rho(\varphi^H)$, qui mesure l'action de φ sur la cohomologie $H^*(S, \mathbb{C})$ de S.

D'une part, on explicite un nouvel exemple où entropie catégorique et topologique ne coïncident pas : on considère la composé $\varphi = T_{\mathcal{O}_C} \circ (- \otimes \mathcal{L})$ du twist sphérique le long d'une (-2)-courbe rationnelle lisse $C \subset S$ et d'un fibré en droites bien choisi, et on montre que $h_{cat}(\varphi) > 0 = \log \rho(\varphi^H)$. D'autre part, on étudie les valeurs que peut prendre l'entropie topologique généralisée $\log \rho(\varphi^H)$ lorsque φ parcourt $\mathrm{Aut}(\mathrm{D}^b(S))$, et on montre que sous certaines conditions ces valeurs sont uniquement déterminées par $\mathrm{Aut}(S)$. Couplé au célèbre résultat de Cantat sur la dynamique des automorphismes des surfaces, on en déduit un premier résultat de classification des surfaces admettant une autoéquivalence d'entropie topologique généralisée strictement positive.

Abstract

This thesis splits in two parts. First, it aims to study moduli spaces of coherent sheaves on a complex Fano threefold X and a K3 surface $S \subset X$. More precisely, we consider a primitive Fano threefold of index 1 and genus 9. We show that the restriction of sheaves on X to a K3 surface $S \subset X$ of genus 9 obtained as a hyperplane section of X gives a morphism

$$res : \mathcal{M}_X(2,1,7) \to \mathcal{M}_S(2,1,7)$$

between the moduli spaces of sheaves on S and X, which identifies the image of $\mathcal{M}_X(2,1,7)$ with a Lagrangian subvariety of $\mathcal{M}_S(2,1,7)$ singular along finitely many points. Moreover, when we vary X in the linear system $\mathcal{W} \subset \mathbb{P}^3$ of smooth Fanos containing S, the restriction morphisms glue and we obtain a rational Lagrangian fibration

$$\mathcal{M}_S(2,1,7) \dashrightarrow \mathbb{P}^3$$

defined by mapping a sheaf $F \in \mathcal{M}_S(2,1,7)$ globally generated coming from a Fano X to its class $[X] \in \mathcal{W}$. Furthermore, there exists a birational model $\mathcal{M} \dashrightarrow \mathcal{M}_S(2,1,7)$ extending the rational fibration to a Lagrangian fibration

$$\mathcal{M} \to \mathbb{P}^3.$$

In a second part, this thesis adresses the study of dynamical systems in a categorical point of view. In this text, we focus on the case of a smooth projective surface S and its derived category $\mathrm{D}^b(S)$. Given an autoequivalence $\varphi : \mathrm{D}^b(S) \to \mathrm{D}^b(S)$, we take a look at two quantities:

- the *categorical entropy* $h_{cat}(\varphi)$, which computes the complexity of the dynamical system $(\mathrm{D}^b(S), \varphi)$, and
- the *generalized topological entropy* $\log \rho(\varphi^H)$, which measures the action of φ on the cohomology $H^*(S, \mathbb{C})$ of S.

One one hand, we exhibit a new example for which categorical and topological entropy do not coincide: we consider the composition $\varphi = T_{\mathcal{O}_C} \circ (- \otimes \mathcal{L})$ of the spherical twist along a smooth rational (-2)-curve $C \subset S$ and a specific line bundle, and we show that $h_{cat}(\varphi) > 0 = \log \rho(\varphi^H)$. On the other hand, we study the values that can take the generalized topological entropy $\log \rho(\varphi^H)$ when φ ranges in $\mathrm{Aut}(\mathrm{D}^b(S))$, and we show that under mild assumptions these values are uniquely determined by $\mathrm{Aut}(S)$. Combined with the celebrated Cantat result concerning dynamics of surfaces automorphisms, we deduce a first result of classification of surfaces admitting an autoequivalence with positive generalized topological entropy.

Contents

Introduction

Le coeur de cette thèse se découpe en deux parties. Le deuxième chapitre concerne l'étude d'espaces de modules de faisceaux sur des surfaces K3 et des variétés de Fano de dimension 3 complexes. Le troisième chapitre concerne l'étude de systèmes dynamiques d'un point de vue catégorique. Bien que les études menées dans ces deux parties soient différentes, elles utilisent de manière centrale les *catégories dérivées*.

Catégories dérivées

Considérons une variété lisse et projective X sur un corps K. Afin d'étudier la géométrie de X, il est naturel de s'intéresser sa catégorie abélienne de faisceaux cohérents $\mathbf{Coh}(X)$. Cependant, les foncteurs les plus fréquemment utilisés en géométrie algébrique, tels que le produit tensoriel de faisceaux ou le tiré en arrière par un morphisme, ne préservent pas la structure abélienne. L'exemple fondateur est le foncteur Γ des sections globales. Si

$$0 \to E \to F \to G \to 0 \tag{1}$$

est une suite exacte de faisceaux cohérents, on obtient une suite exacte longue

$$0 \to \Gamma(E) \to \Gamma(F) \to \Gamma(G) \to H^1(X,E) \to H^1(X,F) \to H^1(X,G) \to H^2(X,F) \to \cdots$$

d'espaces vectoriels. Les espaces $H^i(X,F)$ contiennent de nombreuses informations sur F. Un moyen de les construire est de *remplacer* F par une résolution $F \to I^\bullet$ de faisceaux injectifs, et de définir $H^i(X,F)$ comme le i-ème espace de cohomologie du complexe $\Gamma(I^\bullet)$.

Dans la catégorie $\mathrm{D}^b(X) := \mathrm{D}^b(\mathbf{Coh}(X))$, dont les objets sont les complexes de faisceaux cohérents, le faisceau F et le complexe $I^\bullet$ sont *isomorphes*, et ainsi on peut définir un foncteur *dérivé* $R\Gamma(F) := \Gamma(I^\bullet)$. L'avantage de cette construction est, d'une part, que le complexe $R\Gamma(F)$ contient toute l'information cohomologique de F, et d'autre part que le foncteur $R\Gamma$ préserve la structure *triangulée* de $\mathrm{D}^b(X)$. En particulier, la suite exacte (1) induit un triangle

$$R\Gamma(E) \to R\Gamma(F) \to R\Gamma(G)$$

dans la catégorie dérivées des espaces vectoriels $\mathrm{D}^b(K)$. Ainsi, les catégories dérivées forment le bon environnement pour l'étude cohomologique des faisceaux cohérents. Elles donnent un cadre plus général et naturel pour de nombreux résultats (dualité de Serre, formule de projection, adjonction tiré en arrière - poussé en avant...).

Inventées dans les années soixante par Grothendieck, puis développées en grande partie par Verdier [Del77], [Ver96], le succès des catégories dérivées en géométrie complexe est d'abord apparu dans les années quatre-vingt suite aux travaux de Mukai [Muk81] dans lesquels il étudie des équivalences dérivées de variétés abéliennes. Depuis lors, l'utilisation des catégories dérivées s'est répandu dans de nombreux domaines. On peut citer par exemple les travaux de Kontsevich [Kon95] qui utilisent les catégories dérivées comme cadre mathématique pour la description de phénomènes de symétrie miroir en physique (voir [KO04] pour une exposition sur le sujet).

La catégorie dérivée $\mathrm{D}^b(X)$ d'une variété X n'est pas un invariant complet de X en général. Cependant, lorsque le fibré (anti)-canonique $\pm\omega_X$ de X est ample, Bondal et Orlov [BO01] ont montré que la catégorie $\mathrm{D}^b(X)$ caractérise complètement X. Ce résultat est faux lorsque $\omega_X \simeq \mathcal{O}_X$, par exemple lorsque X est une surface K3 ou un variété abélienne ([Orl03]).

Plus récement se sont développé des travaux en direction des *variétés noncommutatives*, c'est à dire des catégories triangulées qui partagent de nombreuses propriétés similaires aux catégories dérivées de variétés algébriques. Un exemple typique d'apparition de variétés noncommutatives sont les composantes de décompositions semiorthogonales (section 1.2.1). Dans de nombreux cas, ces composantes détiennent des informations géométriques sur la variété de base (voir par exemple [LNSZ20], [BLMS17]).

Espaces de modules de faisceaux

Les *espaces de modules* (dont le nom est principalement due à Riemann [Rie57]) sont par définition des espaces qui classifient des objets mathématiques. Dans les années soixante, Mumford [Mum63] étudia les fibrés vectoriels à première classe de Chern prescrite sur les courbes, ce qui aboutit à la première notion de *stabilité* de fibré. Par la suite, cette notion a été généralisée pour les faisceaux cohérents et en dimensions supérieures par de nombreux auteurs (Bogomolov, Gieseker, Maruyama, Simpson).

Le cas d'une surface K3 S a été, dans un premier temps, étudié par Mukai [Muk84], [Muk87]. Les espaces de modules de faisceaux $\mathcal{M}_S[v]$ de vecteur de Mukai v fixé fournissent des exemples de variétés à la géométrie très riche : des variétés hyperkähler (HK), voir exemples 1.1.19. Si l'on choisit v primitif de telle sorte que $\langle v, v\rangle = 0$, alors l'espace de module $\mathcal{M}_S[v]$ est lui-même une surface K3, pas forcément isomorphe à S mais dont la catégorie dérivée $\mathrm{D}^b(\mathcal{M}_S[v])$ est (quitte à considérer des faisceaux tordus par une classe de Brauer) équivalente à celle de S. En fait, une surface K3 S' qui satisfait $\mathrm{D}^b(S') \simeq \mathrm{D}^b(S)$ est toujours de la forme $S' \simeq \mathcal{M}_S[v]$ pour un bon choix de v.

Les espaces de modules sur les variétés de Fanos sont plus complexes à étudier. La section 1.1.4 à pour but de citer certains des résultats à ce sujet. D'une autre part, les variétés de Fano sont fortement liées aux variétés HK. Par exemple, la variété des droites [BD85] ou la variété construite à partir de courbes rationnelles de degré 3 [LLSvS17] [AL17] dans une cubique lisse de $\mathbb{P}^5$ est une variété HK de type $\mathrm{K3}^{[n]}$.

Le lien entre les variété de Fano de dimension 3 et les variétés HK reste mystérieux. D'une part, Laza, Saccà and Voisin [LSV17] ont construit une fibration lagrangienne sur une variété HK de dimension 10 (de type OG10) comme compactification d'une fibration sur un ouvert de $\mathbb{P}^5$ dont les fibres sont des les jacobiennes intermédiaires de cubiques de $\mathbb{P}^4$ (obtenues comme sections hyperplanes d'une cubique de $\mathbb{P}^5$ fixée). D'autre part, en se basant sur une remarque de Tyurin, Beauville [Bea19] montre que si X est une 3-variété de Fano et que $S \subset X$ est une surface anticanonique (K3), alors la restriction $res : \mathcal{M}_X \to \mathcal{M}_S$ de faisceaux stables sur X à S, sous certaines hypothèses, est une immersion et l'image de $\mathcal{M}_X$ dans $\mathcal{M}_S$ est une sous-variété lagrangienne. Il est alors naturel de se demander :

Question. *L'image de $\mathcal{M}_X$ dans $\mathcal{M}_S$ est-elle la fibre d'une fibration Lagrangienne $p : \mathcal{M}_S \to B$?*

Le but du chapitre 2 de cette thèse est d'expliciter un exemple de réponse positive à cette question. Plus précisément, nous considérons une famille $\mathfrak{X}$ sur un ouvert $\mathcal{W} \subset \mathbb{P}^3$ de 3-variétés de Fano d'indice 1 et de genre 9 qui contiennent toutes une même surface K3 S comme section hyperplane. Les restrictions fibre à fibre de faisceaux stables des Fanos sur la surface S se recollent en une restriction $\mathcal{M}_{\mathfrak{X}/\mathcal{W}} \to \mathcal{M}_S$, qui induit un isomorphisme de l'ouvert $\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}}$ des faisceaux globalement engendré vers un ouvert $\mathcal{M}^o_S$ de $\mathcal{M}_S$.

Théorème (voir Corollaire 2.3.8 et Théorème 2.3.9)**.** *Le morphisme* $\mathcal{M}_S^o \to \mathcal{W}$ *qui envoie un faisceau de la forme* $res(F) \in \mathcal{M}_S$*, avec* $X \subset \mathfrak{X}$ *une Fano et* $F \in \mathcal{M}_X$ *globalement engendré, sur* $[X] \in \mathcal{W}$ *donne une fibration lagrangienne rationnelle*

$$\mathcal{M}_S \dashrightarrow \mathbb{P}^3,$$

où la fibre au dessus d'un point $[X] \in \mathcal{W}$ *est l'ouvert* $\mathcal{M}_X^o \subset \mathcal{M}_X$ *des faisceaux globalement engendrés.*

De plus, il existe une surface K3 S' *et une application birationnelle entre un espace de modules de faisceaux tordus* $\mathcal{M}_{S'}$ *sur* S' *et* $\mathcal{M}_S$ *au dessus de* $\mathcal{W}$*, qui étend la fibration rationnelle en une fibration lagrangienne*

$$\mathcal{M}_{S'} \to \mathbb{P}^3.$$

L'application birationnelle $\mathcal{M}_{S'} \dashrightarrow \mathcal{M}_S$ *est un flop le long d'un fibré en* $\mathbb{P}^2$ *sur* S*.*

Pour prouver ce théorème, on utilise les constructions suivantes (section 2.1). On associe à la surface S (resp. à une Fano $X \in \mathcal{W}$) une surface K3 duale S' (resp. une courbe quartique plane $\Gamma \subset S'$). De plus, les catégories dérivées de S et S' (resp. X et Γ) sont reliés par *dualité projective homologique*, et on obtient un diagramme commutatif

$$\begin{array}{ccc} \mathrm{D}^b(\Gamma) & \xrightarrow{\phi_{11}} & \mathrm{D}^b(X) \\ {\scriptstyle (i_{\Gamma S'})_*}\downarrow & & \downarrow{\scriptstyle i_{SX}^*} \\ \mathrm{D}^b(S', \alpha) & \xrightarrow{\phi_{10}} & \mathrm{D}^b(S), \end{array}$$

où $i_{SX} : S \hookrightarrow X$ et $i_{\Gamma S'} : \Gamma \hookrightarrow S'$ sont des immersions fermées. Brambilla et Faenzi [BF13] ont montré que l'adjoint à droite $\phi_{11}^! : \mathrm{D}^b(X) \to \mathrm{D}^b(\Gamma)$ de ϕ_{11} se restreint en un éclatement $\mathcal{M}_X \to \mathrm{Pic}^2(\Gamma)$.

Enfin, le foncteur ϕ_{10} permet de définir une application birationnelle (au dessus de $\mathcal{W}$) entre un espace de modules de faisceaux tordus $\mathcal{M}_{S'}$ sur S' et $\mathcal{M}_S$, qui étend la fibration rationnelle sur $\mathcal{M}_S$ en une fibration lagrangienne

$$\mathcal{M}_{S'} \to \mathbb{P}^3$$

dont la fibre au dessus d'un point $w \in \mathcal{W}$ représentant une courbe Γ est isomorphe à $\mathrm{Pic}^2(\Gamma)$. En étudiant section 2.4 les conditions de stabilités de Bridgeland sur S et en utilisant les résultats de Bayer et Macrì [BM14a], on montre que $\mathcal{M}_{S'}$ est lié à $\mathcal{M}_S$ par un flop le long d'un fibré en $\mathbb{P}^2$ au dessus de S.

On peut alors se demander si toute 3-variété de Fano (sous certaines hypothèses) est reliée à une variété abélienne qui apparait comme fibre d'une fibration lagrangienne sur une variété HK, comme dans les exemples cités ci-dessus. Les pistes pour construire ces liens sont multiples, on peut penser à la Jacobienne intermédiare de la Fano (comme dans [LSV17]), la construction de Serre (comme dans [Bea19]) ou encore la dualité projective homologique (section 1.2.2) comme utilisé dans le Chapitre 2.

Dynamique des autoéquivalences de catégories triangulées

L'étude des systèmes dynamiques est omniprésente en mathématique. La notion d'*entropie topologique* $h_{top}(f)$ a été developpé pour quantifier la complexité d'un système dynamique (X, f), où X est un espace topologique et $f : X \to X$ est une application continue. Lorsque l'espace X est une variété complexe et que f est en endomorphisme, de nombreux résultats ont mis en exergue le lien entre l'entropie topologique et la géométrie de X. On peut par exemple citer les travaux de Gromov [Gro87] [Gro03] et Yomdin [Yom87] qui montrent que l'entropie $h_{top}(f)$ est égale au logarithme du rayon spectral de l'action de f sur la cohomologie $H^*(X,\mathbb{C})$ de X. Des liens entre l'entropie et les degrés dynamiques pour le cas où f est seulement rationnelle ont été établi par Dinh et Sibony [DS05a], [DS05b].

D'une autre part, Cantat a étudié l'entropie dans le cas d'une surface S. Son célèbre résultat [Can99] assure qu'une surface admettant un automorphisme d'entropie strictement positive est birationnellement équivalente à $\mathbb{P}^2$, une surface K3, une surface d'Enriques ou un tore de dimension 2. Nous renvoyons au papier d'Oguiso [Ogu14] qui expose de nombreux exemples de phénomènes apparaissant sur les variétés de dimension supérieures.

Remplaçons désormais le système dynamique classique par son analogue catégorique : on considère une catégorie $\mathcal{T}$ et un foncteur $F : \mathcal{T} \to \mathcal{T}$. Puisque l'on veut étudier des variétés algébriques, il est naturel d'imposer une structure triangulée sur $\mathcal{T}$, de supposer qu'elle admet un générateur G et de supposer que F préserve la structure triangulée. Dans ce contexte, Dimitrov, Haiden, Katzarkov et Kontsevich [DHKK14] proposent une définition de *complexité* du système $(\mathcal{T}, F)$: cette fonction calcule combien d'étapes sont nécessaires pour construire $F^{\circ n}(G)$ (à une somme directe près) à partir de G à l'aide d'extensions dans $\mathcal{T}$. L'entropie catégorique $h_{cat}(F)$ de F mesure la croissance exponentielle de cette complexité en fonction de n.

Lorsque $\mathcal{T} = \mathrm{D}^b(X)$ pour une variété projective X, on peut comparer l'entropie catégorique $h_{cat}(F)$ avec l'action F^H de F sur la cohomologie $H^*(X,\mathbb{C})$ de X. Un résultat similaire au théorème de Gromov-Yomdin (voir Conjecture 3.1.10) est vérifié pour certains types de variétés (abélienne, Fano...) ou de foncteur (tirés en arrières, twists sphériques...) mais n'est pas vrai en général. Dans le chapitre 3, nous constuisons un nouveau contre-exemple pour n'importe quelle surface S contenant une courbe rationnelle lisse $C \subset S$ d'autointersection $C^2 = -2$.

Théorème (voir Théorème 3.4.1)**.** *Soit S une surface projective lisse et $C \subset S$ une (-2)-courbe. Soit $\mathcal{L} \in \mathrm{Pic}(S)$ un fibré en droite satisfaisant $\deg_C(\mathcal{L}|_C) < 0$ et considérons l'autoéquivalence $\varphi = T_{\mathcal{O}_C} \circ (- \otimes \mathcal{L})$ de $\mathrm{D}^b(S)$. On a*

$$h_0(\varphi) > 0 = \log \rho(\varphi^H).$$

Puisqu'une telle (-2)-courbe peut être produite en éclatant deux points lisses infiniment proches dans S, ce contre-exemple montre que l'existence d'un foncteur d'entropie catégorique positive n'est pas un invariant birationnel.

On définit alors l'*entropie topologique généralisée* d'un endofoncteur $F : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ comme le rayon spectral de l'action en cohomologie $F^H : H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C})$ de F. Puisque tout automorphisme de X induit une autoéquivalence de $\mathrm{D}^b(X)$, cette notion étend celle d'entropie topologique classique. Il serait intéressant de comprendre quelles propriétés birationnelles de X, où catégoriques de $\mathrm{D}^b(X)$, peuvent être déduites de cette notion d'entropie. Un premier résultat similaire au théorème de Cantat dans le cadre catégorique est développé en section 3.2.

Théorème (voir Corollaire 3.2.7)**.** *Soit S une surface lisse projective avec $K_S \not\equiv_{num} 0$ qui n'admet pas de fibration elliptique minimale. Supposons que l'ensemble des (-2)-courbes de S forment une union disjointe de configuration de type Dynkin A. Alors, s'il existe une autoéquivalence $\varphi \in \mathrm{Aut}(\mathrm{D}^b(S))$ avec $\rho(\varphi^H) > 1$, S est rationnelle.*

Ce résultats se base sur une classification des surface par rapport à leur groupe d'autoéquivalences proposée par Uehara [Ueh19]. Dans le cas traité par ce théorème, le groupe des autoéquivalences de $\mathrm{D}^b(S)$ est engendré par les équivalences standarts $\mathbb{Z}[1] \times (\mathrm{Pic}(S) \rtimes \mathrm{Aut}(S))$ et les twists sphériques $T_{\mathcal{O}_C(a)}$ le long des (-2)-courbes $C \subset S$. L'étude de l'action en cohomologie de ces foncteurs (section 1.3.3) permet de réduire l'action de l'itération $(\varphi^H)^{\circ n}$ à l'action $(f^*)^{\circ n}$ d'un automorphisme $f \in \mathrm{Aut}(S)$. On est alors ramené aux cas explicités par le théorème de Cantat.

Introduction

The core of this thesis splits in two parts. The second chapter studies moduli spaces of sheaves on complex K3 surfaces and Fano threefolds. The third chapter studies dynamical systems with a categorical point of view. While these two parts are different, they both relies on *derived categories*.

Derived categories

Consider a smooth projective variety X over a field K. In order to study the geometry of X, it is natural to look at the abelian category $\mathbf{Coh}(X)$ of coherent sheaves on X. However, the most common functors used in algebraic geometry, as the pullback along a morphism or the tensor product of sheaves, do not preserve the abelian structure. The most foundational example is the global section functor Γ. If

$$0 \to E \to F \to G \to 0 \tag{2}$$

is a short exact sequence of coherent sheaves, we obtain a long exact sequence

$$0 \to \Gamma(E) \to \Gamma(F) \to \Gamma(G) \to H^1(X, E) \to H^1(X, F) \to H^1(X, G) \to H^2(X, E) \to \cdots$$

of vector spaces. The spaces $H^i(X, F)$ contain a lot of informations on F. A way to construct them is to *replace* F with an injective resolution $F \to I^\bullet$ of injective sheaves, and to define $H^i(X, F)$ as the i-th cohomology space of the complex $\Gamma(I^\bullet)$.

In the category $\mathrm{D}^b(X) := \mathrm{D}^b(\mathbf{Coh}(X))$, whose objects are complexes of coherent sheaves, the sheaf F and the complex $I^\bullet$ are isomorphic, therefore we can define a *derived* functor $R\Gamma(F) := \Gamma(I^\bullet)$. The advantage of this construction, on one hand, is that the complex $R\Gamma(F)$ contains all the cohomological information of F, and on the other hand the functor $R\Gamma$ preserves the *triangulated* structure of $\mathrm{D}^b(X)$. In particular, the exact sequence (2) induces a triangle

$$R\Gamma(E) \to R\Gamma(F) \to R\Gamma(G)$$

in the derived category $\mathrm{D}^b(K)$ of K-vector spaces. As a consequence, derived categories form the correct environment for the cohomological study of coherent sheaves. They give a most general and natural framework for many results (Serre duality, projection formula, adjunction pullback-pushforward...).

Invented by Grothendieck during the sixties, then mostly developed by Verdier [Del77], [Ver96], the succes of derived categories in complex geometry first appeared in the eighties after the works of Mukai [Muk81] in which he studies derived equivalences of abelian varieties. Since then, application of derived categories have spread in numerous fields. For instance, we can cite the works of Kontsevich [Kon95] in which he uses derived categories as a mathematical framework for the description of mirror symmetry phenomenons in physics (see [KO04] for a survey on the subject).

The derived category $\mathrm{D}^b(X)$ of a variety X is not a complete invariant of X in general. Though, when the (anti)-canonical bundle $\pm\omega_X$ of X is ample, Bondal and Orlov [BO01] proved that $\mathrm{D}^b(X)$ completely caracterizes X. This result is false when $\omega_X \simeq \mathcal{O}_X$, for instance when X is a K3 surface or an abelian variety ([Orl03]).

Most recently have been developed works in direction of *noncommutative varieties*, that is triangulated categories sharing a lot of properties with the derived categories of varieties. A typical example of apparition of noncommutative varieties are components of semiorthogonal decompositions (section 1.2.1). In many cases, a lot of geometric information about the ground variety lie in these components (see for instance [LNSZ20], [BLMS17]).

Moduli spaces of sheaves

Moduli spaces of sheaves (whose name is mostly due to Riemann [Rie57]) are by definition spaces classifying mathematical objects. During the sixties, Mumford [Mum63] studied vector bundles with prescribed first Chern class on curves, which led to the first notion of *stability* of bundles. Afterwards, this notion have been generalized for coherent sheaves and higher dimensions by many authors (Bogomolov, Gieseker, Maruyama, Simpson).

The case of a K3 surfaces S have been first studied by Mukai [Muk84], [Muk87]. The moduli spaces of sheaves $\mathcal{M}_S[v]$ with fixed Mukai vector v provides examples of varieties rich in geometry: hyperKähler varieties (HK), see examples 1.1.19. If v is primitive and chosen such that $\langle v, v\rangle = 0$, then the moduli space $\mathcal{M}_S[v]$ is itself a K3 surface, not necessarily isomorphic to S, but whose derived category $\mathrm{D}^b(\mathcal{M}_S[v])$ is (up to consider sheaves twisted by a Brauer class) equivalent to the one of S. In fact, any K3 surface S' satisfying $\mathrm{D}^b(S') \simeq \mathrm{D}^b(S)$ is of the form $S' \simeq \mathcal{M}_S[v]$ for a good choice of v.

Moduli spaces on Fano varieties are more difficult to study. Section 1.1.4 aims to gather some results on this subject. Moreover, Fano varieties are closely related to HK varieties. For instance, the variety of lines [BD85] or the variety constructed from rational curves of degree 3 [LLSvS17], [AL17] in a smooth cubic of $\mathbb{P}^5$ is a HK variety of $\mathrm{K3}^{[n]}$-type.

The link between Fano threefolds and HK varieties remains mysterious. On one hand, Laza, Saccà and Voisin [LSV17] have constructed a Lagrangian fibration on a HK variety of dimension 10 (of type OG10) as a compactification of a fibration over an open subset of $\mathbb{P}^5$ whose fibres are the intermediate Jacobians of cubics of $\mathbb{P}^4$ (obtained as hyperplane sections of a fixed cubic of $\mathbb{P}^5$). On the other hand, following a remark of Tyurin, Beauville [Bea19] shows that if X is a Fano threefold and $S \subset X$ is a anticanonical K3 surface, then the restriction map $res : \mathcal{M}_X \to \mathcal{M}_S$ of stable sheaves on X to S, under mild assumptions, is an immersion and the image of $\mathcal{M}_X$ in $\mathcal{M}_S$ is a Lagrangian subvariety. Therefore it is natural to ask:

Question. *Is the image of $\mathcal{M}_X$ inside $\mathcal{M}_S$ the fibre of a Lagrangian fibration $p : \mathcal{M}_S \to B$?*

The goal of chapter 2 of this thesis is to provide an example of positive answer to this question. More precisely, we consider a family $\mathfrak{X}$ over an open $\mathcal{W} \subset \mathbb{P}^3$ of Fano threefolds of index 1 and genus 9 all containing a fixed K3 surface S as hyperplane section. The fibrewise restrictions of stable sheaves from the Fanos to S glue into a restriction map $\mathcal{M}_{\mathfrak{X}/\mathcal{W}} \to \mathcal{M}_S$, which induces an isomorphism from the open $\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}}$ of globally generated sheaves to an open $\mathcal{M}^o_S \subset \mathcal{M}_S$.

Theorem (see Corollary 2.3.8 and Theorem 2.3.9)**.** *The morphism $\mathcal{M}^o_S \to \mathcal{W}$ sending a sheaf of the form $res(F) \in \mathcal{M}_S$, with $X \subset \mathfrak{X}$ a Fano and $F \in \mathcal{M}_X$ globally generated, to $[X] \in \mathcal{W}$ gives a rational Lagrangian fibration*

$$\mathcal{M}_S \dashrightarrow \mathbb{P}^3,$$

where the fibre over a point $[X] \in \mathcal{W}$ is the open $\mathcal{M}^o_X \subset \mathcal{M}_X$ of globally generated sheaves.

Furthermore, there exists a K3 surface S' and a birational map between a moduli space of twisted sheaves $\mathcal{M}_{S'}$ over S' and $\mathcal{M}_S$ over $\mathcal{W}$, which extends the rational fibration to an actual

Lagrangian fibration

$$\mathcal{M}_{S'} \to \mathbb{P}^3.$$

The birational map $\mathcal{M}_{S'} \to \mathcal{M}_S$ is a flop along a $\mathbb{P}^2$-bundle over S.

To prove this theorem, we use the following constructions (see 2.1). We associate to the surface S (resp. to a Fano $X \in \mathcal{W}$) a dual K3 surface S' (resp. a quartic plane curve $\Gamma \subset S'$). Moreover, the derived categories of S and S' (resp. X and Γ) are related by *homological projective duality*, and we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{D}^b(\Gamma) & \xrightarrow{\phi_{11}} & \mathrm{D}^b(X) \\ {\scriptstyle (i_{\Gamma S'})_*}\downarrow & & \downarrow{\scriptstyle i^*_{SX}} \\ \mathrm{D}^b(S', \alpha) & \xrightarrow{\phi_{10}} & \mathrm{D}^b(S), \end{array}$$

where $i_{SX} : S \hookrightarrow X$ and $i_{\Gamma S'} : \Gamma \hookrightarrow S'$ are closed immersion. Brambilla and Faenzi [BF13] have shown that the right adjoint $\phi_{11}^! : \mathrm{D}^b(X) \to \mathrm{D}^b(\Gamma)$ of ϕ_{11} restricts to a blowup $\mathcal{M}_X \to \mathrm{Pic}^2(\Gamma)$.

Finally, the functor ϕ_{10} permits to define a birational map (over $\mathcal{W}$) between a moduli space of twisted sheaves $\mathcal{M}_{S'}$ over S' to $\mathcal{M}_S$, which extends the rational fibration over $\mathcal{M}_S$ to an actual Lagrangian fibration

$$\mathcal{M}_{S'} \to \mathbb{P}^3$$

whose fibre over a point $w \in \mathcal{W}$ representing a curve Γ is isomorphic to $\mathrm{Pic}^2(\Gamma)$. Studying, section 2.4, Bridgeland stability condition on S and using results of Bayer and Macrí [BM14a], we show that $\mathcal{M}_{S'}$ and $\mathcal{M}_S$ are related by a flop along a $\mathbb{P}^2$-bundle along S.

Therefore, we can ask if any Fano threefold (under mild hypotheses) is related to an abelian variety which appears as the fibre of a Lagrangian fibration over a HK variety, as in the examples above. Clues to construct such links are multiple, we can think to the intermediate Jacobian of the Fano (as in [LSV17]), Serre construction (as in [Bea19]), or even to homological projective duality (section 1.2.2) as used in Chapter 2.

Dynamics of autoequivalences of triangulated categories

The study of dynamical systems is ubiquitous in mathematics. The notion of *topological entropy* h_{top} have been developed to quantify the complexity of a dynamical system (X, f), where X is a topological space and $f : X \to X$ is a continuous map. When X is a complex manifold and f is an endomorphism, a lot of results highlighted relations between the topological entropy and the geometry of X. For instance, we can cite the works of Gromov [Gro87] [Gro03] and Yomdin [Yom87] in which they show that the topological entropy $h_{top}(f)$ is equal to the logarithm of the spectral radius of the action of f on the cohomology $H^*(X, \mathbb{C})$ of X. Links between entropy and dynamical degrees when f is only rational have been etablished by Dinh and Sibony [DS05b] [DS05a].

Furthermore, Cantat studied entropy when the manifold is a surface S. His celebrated result [Can99] assures that a surface admitting an automorphism with positive entropy is birationally equivalent to $\mathbb{P}^2$, a K3 surface, an Enriques surface or a 2-dimensional torus. We refer to Oguiso's paper [Ogu14] in which he presents many phenomenons appearing on higher dimensional varieties.

Now, we replace the classical dynamical system by its categorical analogue: we consider a category $\mathcal{T}$ and a functor $\mathcal{T} \to \mathcal{T}$. As we want to study varieties, it is natural to ask for a triangulated structure on $\mathcal{T}$, to assume that it admits a generator G and that F preserves the triangulated structure. In this settings, Dimitrov, Haiden, Katzarkov and Kontsevich [DHKK14] propose a definition of *complexity* of the system $(\mathcal{T}, F)$: this function computes how many steps are necessary to construct $F^{\circ n}(G)$ (up to take a direct summand) from G by extensions. The categorical entropy $h_{cat}(F)$ of F measures the exponential growth of this complexity function with respect to n.

When $\mathcal{T} = \mathrm{D}^b(X)$ for a projective variety X, we can compare categorical entropy $h_{cat}(F)$ with the action F^H of F on the cohomology $H^*(X, \mathbb{C})$ of X. A Gromov-Yomdin type theorem (see Conjecture 3.1.10) hold for certain kind of varieties (abelian, Fano...) or functors (pullbacks, spherical twists...), but do not hold in general. In Chapter 3, we construct a new counter-example for any surface S containing a smooth rational curve $C \subset S$ of auto-intersection $C^2 = -2$.

Theorem (see Theorem 3.4.1)**.** *Let S be a smooth projective surface and $C \subset S$ a (-2)-curve. Let $\mathcal{L} \in \mathrm{Pic}(S)$ be a line bundle satisfying $\deg_C(\mathcal{L}|_C) < 0$ and consider the autoequivalence $\varphi = T_{\mathcal{O}_C} \circ (- \otimes \mathcal{L})$ of $\mathrm{D}^b(S)$. We have*

$$h_0(\varphi) > 0 = \log \rho(\varphi^H).$$

Such a (-2)-curve can always be produced by blowing up two infinitely near smooth points in S, thus this counter-example shows that the existence of a functor with positive categorical entropy is not a birational invariant.

Therefore we define the *generalized topological entropy* of an endofunctor $F : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ as the spectral radius of the action $F^H : H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$ of F. As any automorphism of X induces an autoequivalence of $\mathrm{D}^b(X)$, this notion extends the one of classical topological entropy. It would be interesting to understand which birational (resp. categorical) properties of X (resp. $\mathrm{D}^b(X)$) can be deduced from this notion of entropy. A first result similar to Cantat theorem is developed section 3.2.

Theorem (see Corollary 3.2.7)**.** *Let S be a smooth projective surface with $K_S \not\equiv_{num} 0$ which admits no minimal elliptic fibration. Assume that the union of all (-2)-curves on S form a disjoint union of configuration of Dynkin type A. Then, if there exists an equivalence $\varphi \in \mathrm{Aut}(\mathrm{D}^b(S))$ with $\rho(\varphi^H) > 1$, then S is rational.*

This result is based on a classification of surfaces with respect to their group of auto-equivalences proposed by Uehara [Ueh19]. In the case treated by this theorem, the group of autoequivalences of $\mathrm{D}^b(X)$ is generated by standard equivalences $\mathbb{Z}[1] \times (\mathrm{Pic}(S) \rtimes \mathrm{Aut}(S))$ and spherical twists $T_{\mathcal{O}_{C(a)}}$ along (-2)-curves $C \subset S$. The study of the action on cohomology of these functors (section 1.3.3) permits to restrict the action of the iteration $(\varphi^H)^{\circ n}$ to the action $(f^*)^{\circ n}$ of an automorphism $f \in \mathrm{Aut}(S)$. Hence we are reduce to the cases decribed by Cantat's theorem.

Notations and terminology

Throughout this thesis, a *scheme* will mean a Noetherian separated scheme of finite type over a field, and a *variety* means a integral scheme. A *sheaf* on a scheme X will always mean a coherent sheaf of $\mathcal{O}_X$-modules. We denote $i_{XY} : X \hookrightarrow Y$ for a *closed immersion* between two schemes X, Y.

The equality between divisor on a variety X always mean equality up to *linear* equivalence. The notation $\equiv$ is used for *numerical* equivalence.

For a scheme X over a field K, we denote $\mathbf{Coh}(X)$ the abelian category of coherent sheaves on X, and $\mathrm{D}^b(X) := \mathrm{D}^b(\mathbf{Coh}(X))$ its bounded derived category. Given a complex $F^\bullet \in \mathrm{D}^b(X)$, we denote $\mathcal{H}^i(F^\bullet)$ for the i-th cohomological sheaf of F. Moreover, for sheaves F, G we denote $h^i(F) = \dim_K H^i(X, F)$ and $\mathrm{ext}^i(F, G) = \dim_K \mathrm{Ext}^i(F, G)$.

In sections 1.2, 1.3 and chapter 3, derived functors are written with their underived notations (e.g. $\otimes$ instead of $\otimes^L$), with only exception the functor $R\operatorname{Hom}(-,-)$ to distinguish with morphisms in the category. Recall that closed immersions have no higher direct image, flat pullback is exact, and that tensor products and the pullbacks do not need to be derived when applied to locally free sheaves.

The classical dual of a sheaf F is denoted $F^* := \operatorname{Hom}(F, \mathcal{O}_X)$ and the derived dual is denoted $F^\vee$. Note that $F^* = F^\vee$ for locally free sheaves (in particular, for vector spaces), and we will often use the derived notation in this case.

Chapter 1

Surfaces, threefolds and their derived categories of coherent sheaves

This first chapter aims to recall the materials that will be used in the next chapters.

Section 1.1 is mainly devoted to the classical theory of moduli spaces of sheaves. We first recall some general definitions and results concerning the theory of moduli spaces of sheaves (1.1.1), and we inspect the cases of K3 surfaces (1.1.3) and prime Fano threefolds (1.1.4) with more details. We also recall the celebrated Enriques-Kodaira classification of algebraic surfaces (1.1.2) and study the existence of smooth rational curves of self-intersection (-2) with respect to this classification. In a last part (1.1.5), we recall the definitions of *hyperKähler* varieties, central objects when it comes to classify varieties with trivial first Chern class. Examples of such varieties are often related to moduli spaces of sheaves on K3 and abelian surfaces. We finish by inspecting the existence of Lagrangian fibrations on hyperKähler varieties.

The next sections focus on constructions related to the derived category $\mathrm{D}^b(X)$ of a smooth projective variety X. In section 1.2, we recall the notion of *homological projective duality* and its application to computations of semiorthogonal decompositions. In section 1.3, we expose some examples of autoequivalences of $\mathrm{D}^b(X)$ and describe their action on the cohomology of X. We finish by recalling, section 1.4, the definition of *Bridgeland stability conditions* on a triangulated category, with a particular attention to the case of the derived category of a K3 surface.

1.1 Algebraic varieties and moduli spaces of sheaves

Throughout this section, we assume that all schemes are defined over $\mathbb{C}$, even if most of the general results about moduli spaces of sheaves hold for any algebraically closed field of characteristic 0.

1.1.1 Moduli spaces of sheaves

Let X be a smooth projective variety. We fix H_X an ample divisor of X.

We define the *Hilbert polynomial* $P(E,m)$ of a sheaf E as

$$P(E,m) := \chi(E(mH_X)) = \sum_{i=0}^{d} \alpha_i(E)\frac{m^i}{i!}$$

where $d := \dim E := \dim \mathrm{Supp}(E)$ and $\alpha_0(E),\dots,\alpha_d(E) \in \mathbb{Q}$.

Similarly, we define the *reduced Hilbert polynomial* of E to be

$$p(E,m) := \frac{P(E,m)}{\alpha_d(E)}.$$

We will say that a sheaf E is *pure* if any non-trivial subsheaf $F \subset E$ satisfies $\dim E = \dim F$. In particular, if $\dim E = \dim X$, the sheaf E is pure if and only if it is torsion-free.

Definition 1.1.1. A sheaf E on X is called (Gieseker)-*semistable* if it is pure and $p(F, m) \leq p(E, m)$, $m \gg 0$, for all proper non-trivial subsheaves $F \subset E$. It is called (Gieseker)-*stable* if the inequality is strict for all F.

Another definition of stability is also useful when E is torsion-free. We define the *slope* of a torsion-free sheaf E as

$$\mu(E) = \frac{c_1(E) \cdot H_X^{\dim(X)-1}}{\operatorname{rk} E},$$

and we call *degree* of E the number $c_1(E) \cdot H_X^{\dim(X)-1}$.

In the particular case $\operatorname{Pic}(X) = \mathbb{Z} \cdot H_X$, we get $c_1(E) = \alpha H$, $\alpha \in \mathbb{Z}$ for all E. For this reason we sometimes rescale the definition of μ by dividing it by $H^{\dim X}$ as our most interest consists in comparing slopes.

Definition 1.1.2. A torsion-free sheaf E is called *slope-semistable* or *μ-semistable* if for all $F \subset E$ with $0 < \operatorname{rk} F < \operatorname{rk} E$ one has

$$\mu(F) \leq \mu(E).$$

The sheaf E is called *slope-stable* or *μ-stable* if the inequality is strict for all such F.

We can state a very useful criterion for stability in the case of vector bundles.

Theorem 1.1.3 (Hoppe's criterion, [Hop84])**.** *Let X be smooth projective variety over an algebraically closed field K of characteristic 0 with Picard group generated by an ample line bundle $\mathcal{O}_X(1)$. For any vector bundle E of rank r on X, we have*

- *If $H^0(X, (\Lambda^q E)_{\text{norm}}(-1)) = 0$ for $0 < q < r$, then E is μ-semistable.*
- *If $H^0(X, (\Lambda^q E)_{\text{norm}}) = 0$ for $0 < q < r$, then E is μ-stable, and the converse is true when $r = 2$.*

Here $E_{\text{norm}} := E(-k_E)$ with $k_E \in \mathbb{Z}$ unique so that $-r + 1 \leq c_1(E_{\text{norm}}) \leq 0$.

There exists generalization of this theorem in positive characteristic, see for instance [KK12].

Any semistable sheaf E admits a *Jordan-Hölder* filtration

$$0 \subset E_0 \subset \cdots \subset E_n = E$$

of subsheaves such that all quotients E_{i+1}/E_i are stable with reduced Hilbert polynomial $p(E, m)$. The isomorphism type of the graded object $JH(E) = \bigoplus_i E_{i+1}/E_i$ is independant of the filtration.

We say that two semistable sheaves E and F are *S-equivalent* if $JH(E) \simeq JH(F)$ as graded objects.

Theorem 1.1.4. *For a fixed Hilbert polynomial P there exist a projective coarse moduli space $\mathcal{M}_X(P)$ for the moduli problem*

$$\begin{array}{rcl} \mathbf{M}_X(P) : (Sch/\mathbb{C})^o & \to & Sets \\ S & \mapsto & \{E \in \mathbf{Coh}(S \times X) \mid E \text{ is } S\text{-flat}, P(E_s) = P, E_s \text{ is semistable}\}/\sim \end{array}$$

where the equivalence relation is the S-equivalence. Moreover, there is an open subscheme $\mathcal{M}_X^s(P) \subset \mathcal{M}_X(P)$ which parametrizes stable sheaves.

Recall that a *coarse moduli space* for $\mathbf{M}_X(P)$ is a scheme $\mathcal{M}_X(P)$ over $\mathbb{C}$ with a natural transformation $\mathbf{M}_X(P) \to h(\mathcal{M}_X(P))$ (functor of points, namely $h(\mathcal{M}_X(P))(X) = \mathrm{Hom}(X, \mathcal{M}_X(P))$) which induces a bijection $\mathbf{M}_X(P)(\mathbb{C}) \xrightarrow{1:1} \mathcal{M}_X(P)(\mathbb{C})$ and which is *universal* among such transformation: any other transformation $\mathbf{M}_X(P) \to h(N)$ for N a scheme over $\mathbb{C}$ factorizes over a uniquely determined map $M \to N$, i.e.

$$\begin{array}{ccc} \mathbf{M}_X(P) & \longrightarrow & h(M) \\ & \searrow & \Big\downarrow \exists! \\ & & h(N). \end{array}$$

In particular, the set of S-equivalent class of semistable sheaves on X is in one-to-one correspondance with the closed points of $\mathcal{M}_X(P)$.

Proof of Theorem 1.1.4. A complete proof can be found in [HL10]. We briefly sketch the construction.

Let $m \in \mathbb{Z}$ be a positive integer such that any semistable sheaf F with $P(F) = P$ is globally generated and $P(m) = h^0(F(m))$ (such an integer always exists). Define $V := \mathbb{C}^{\oplus P(m)}$ and $\mathcal{H} := V \otimes_X \mathcal{O}_X(-m)$. For any semistable sheaf F there exists a surjective map $\rho : \mathcal{H} \to F$, which gives a point of the Quot scheme $\mathrm{Quot}(\mathcal{H}, P)$. This point $[\rho]$ lies in the subscheme $\mathcal{R} \subset \mathrm{Quot}(\mathcal{H}, P)$ of surjections $[\mathcal{H} \to F]$ for which F is semistable and the induced map

$$H^0(X, \mathcal{H}) \simeq V \to H^0(X, F)$$

is an isomorphism. But for the same semistable sheaf F, the choice of a basis of $H^0(X, F)$ gives different point of $\mathcal{R}$. To obtain the moduli space $\mathcal{M}_X(P)$ one needs to use *Geometric Invariant Theory* to take the quotient with respect to the action of $\mathrm{GL}(V)$. It turns out that the action factorizes through $\mathrm{SL}(V)$, and we get

$$\mathcal{M}_S := \mathcal{R} /\!\!/ \mathrm{SL}(V).$$

The subscheme $\mathcal{M}^s_X(P)$ of stable sheaves is obtain in a similar fashion by considering the subscheme $\mathcal{R}^s \subset \mathcal{R}$ parametrizing stable sheaves. □

The moduli space $\mathcal{M}_X(P)$ is called *fine* if we have an isomorphism of functors $\mathbf{M}_X(P) \xrightarrow{\sim} h(\mathcal{M}_X(P))$. In this case, we obtain a universal family of sheaves $\mathcal{E}$ on $\mathcal{M}_X(P) \times X$. This family $\mathcal{E}$ is the image of $\mathrm{Id}_{\mathcal{M}_X(P)}$ via the isomorphism

$$\mathrm{Hom}(\mathcal{M}_X(P), \mathcal{M}_X(P)) = h(\mathcal{M}_X(P))(\mathcal{M}_X(P)) \simeq \mathbf{M}_X(P)(\mathcal{M}_X(P))$$

and it is universal in the sense that any family $\mathcal{F}$ of semistable sheaves with Hilbert polynomial P over a base T (that is, $\mathcal{F} \in \mathbf{M}_X(P)(T)$) is the pullback for $\mathcal{E}$ along a unique map $T \to \mathcal{M}_X(P)$ up to a twist by a line bundle on T.

Any semistable sheaf F on X corresponding to the point $[F] = m \in \mathcal{M}_X(P)$ is obtained as $F \simeq \mathcal{E}_m$, where $\mathcal{E}_m := p^*(\mathcal{E})$ for $p : X \simeq X \times \{m\} \hookrightarrow X \times \mathcal{M}_X(P)$ is the fibre of $\mathcal{E}$.

Often, instead of fixing a Hilbert polynomial, it is more natural to fix a rank $r \in \mathbb{Z}_{\geq 0}$ and Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$. Let P the Hilbert polynomial associated to r, c_i, $i = 0, \dots, \dim X$ via the Hirzebruch-Riemann-Roch theorem. We introduce the moduli spaces

$$\mathcal{M}_X(r, c_1, \dots, c_{\dim X}) \subset \mathcal{M}_P$$

of semistable sheaves on X with rank r and Chern classes c_i. These moduli spaces are open and closed in $\mathcal{M}_X(P)$, hence projective. Once again, we denote $\mathcal{M}^s_X(r, c_1, \dots, c_{\dim X}) \subset \mathcal{M}_X(r, c_1, \dots, c_{\dim X})$ the open subscheme parametrizing the stable sheaves.

We state now a theorem describing local properties of $\mathcal{M}_X(P)$, which is very useful when it comes to compute dimension and smoothness of moduli spaces.

Theorem 1.1.5. *Let F be a semistable sheaf on X with Hilbert polynomial P. There is a natural isomorphism $T_{[F]}\mathcal{M}_X(P) \simeq \operatorname{Ext}^1(F,F)$. Moreover, if $\operatorname{Ext}^2(F,F) = 0$, then $\mathcal{M}_X(P)$ is smooth at $[F]$.*

When $\mathcal{M}_X(P)$ is fine, the isomorphism of functors $h(\mathcal{M}_X(P)) \simeq \mathbf{M}_X(P)$ applied to the space of dual numbers $D = \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$ gives an isomorphism $\mathbf{t} : \operatorname{Hom}(D, \mathcal{M}_X(P)) \xrightarrow{\sim} \mathbf{M}_X(P)(D)$. If we restrict the former space to the morphisms $f : D \to \mathcal{M}_X(P)$ with $f_{|\operatorname{Spec}\mathbb{C}}(\operatorname{Spec}\mathbb{C}) = [F]$, we obtain the tangent space $T_{[F]}\mathcal{M}_X(P)$, and the image of these morphisms by $\mathbf{t}$ describes exactly the space of *first order deformations* of $[F]$, which is naturally isomorphic to $\operatorname{Ext}^1(F,F)$.

Finally, sometimes one needs to study families of moduli spaces (see for instance section 2.3.1). The following theorem is a relative version of Theorem 1.1.4.

Theorem 1.1.6. *Let S be a $\mathbb{C}$-scheme (not necessarily smooth nor proper), let $f : X \to S$ be a projective morphism with connected fibres. Let $\mathcal{O}_X(1)$ be a line bundle on X very ample relative to S. Then for a fixed Hilbert polynomial P there exist a coarse moduli space $\mathcal{M}_{X/S}(P)$ and a projective morphism $\mathcal{M}_{X/S}(P) \to S$ for the moduli problem*

$$\begin{array}{rcl} \mathbf{M}_{X/S}(P) : (Sch/S)^o & \to & Sets \\ T & \mapsto & \{E \in \mathbf{Coh}(T \times_S X) \mid E_t \text{ is } T\text{-flat}, P(E_t) = P, E_t \text{ is semistable}\}/\sim, \end{array}$$

where E_t denotes the fibre of E over the map $T \times_S X \to T$.

In particular, for any closed point $s \in S$ we have $\mathcal{M}_{X/S}(P)_s \simeq \mathcal{M}_{X_s}(P)$. Moreover, there is an open subscheme $\mathcal{M}^s_{X/S}(P) \subset \mathcal{M}_{X/S}(P)$ which parametrizes stable sheaves.

The construction is really close to the absolute case, we refer to [HL10] Theorem 4.3.7 for a proof.

1.1.2 Algebraic surfaces and (-2)-curves

In this short section, let us first recall the so-called *Enriques-Kodaira* classification of surfaces (see [BPV84] for more details).

Theorem 1.1.7. *Let S_0 be a smooth projective surface. Then S_0 admits a minimal model S which lies in exactly one of the following case.*

- *If $\kappa(S) = -\infty$:*
  - ***Rational surface:*** *S is birational to $\mathbb{P}^2$. In fact, S is either $\mathbb{P}^2$ or a Hirzebruch surface Σ_r, $r \geq 0, r \neq 1$.*
  - ***Ruled surface of genus*** $\mathbf{g} \geq \mathbf{1}$: *S is birational to $C \times \mathbb{P}^1$ for some smooth curve C with $g(C) = g \geq 1$.*
- *If $\kappa(S) = 0$:*
  - ***K3 surface:*** *S satisfies $K_S = 0$ and $H^1(S, \mathcal{O}_S) = 0$.*
  - ***Enriques surface:*** *S satisfies $K_S \neq 0$, $2K_S = 0$ and $H^1(S, \mathcal{O}_S) = 0$. In fact, S is the quotient of a K3 surface by a fixed-point-free involution.*
  - ***Bielliptic:*** *$S \simeq (E \times F)/G$, where E, F are elliptic curves and G is a finite group of translations of E acting on F such that $F/G \simeq \mathbb{P}^1$.*
  - ***Abelian surface:*** *S is an abelian variety (connected projective algebraic group). The complex points $S(\mathbb{C})$ form a 2-dimensional complex torus.*

- *If $\kappa(S) = 1$: **Properly elliptic surface.** S is elliptic, that is there exists a morphism $f : S \to C$ with C a smooth curve such that the general fibre S_c, $c \in C$, is a smooth curve of genus 1, and S has Kodaira dimension 1. Beware that there exists elliptic surface with Kodaira dimension 0 and $-\infty$.*

- *If $\kappa(S) = 2$: **General type.***

In view of sections 3.2, 3.3, the categorical dynamics of a surface S is closely related (at least of $\kappa(S) \neq 0$) to the existence and properties of spherical twists (see section 1.3.2) along (-2)-curves on S. By (-2)-curves, we mean *smooth rational* curves $\mathbb{P}^1 \simeq C \subset S$ such that $C^2 = -2$.

The case $\kappa(S) = 0$ is wild. Abelian varieties contain no rational curves at all. On K3 surfaces, rational curves do no come in family. Though, if S is a K3 surface with infinite automorphism group, and if there exist a (-2)-curve on S, then there exist infinitely many of them (see [Huy16], Corollary 8.4.7).

From adjunction formula, it is easy to see that there exist no (-2)-curve on $\mathbb{P}^2$. For other minimal surfaces with $\kappa(S) = -\infty$, we have the following result.

Proposition 1.1.8. *Let $\pi : S \simeq \mathbb{P}(\mathcal{E}) \to C$ be a minimal ruled surface, with C a smooth surface of genus g, $\mathcal{E}$ a vector bundle of rank 2, with invariant $e = -\deg(\mathcal{E})$. Then there exists at most one (-2)-curve on S.*

Proof. We know that $\mathrm{Num}(S) = \mathbb{Z}\sigma + \mathbb{Z}f$ for σ the image of a section of π and f a fibre of π, with $\sigma^2 = -e$, $f^2 = 0$ and $f \cdot \sigma = 1$. Consider D an irreducible curve in S, $D \neq \sigma, f$, and let $D = a\sigma + bf$ be its class in $\mathrm{Num}(S)$. We get $D^2 = 2ab - a^2 e$.

First assume $e \geq 0$. From [Har77] Proposition 2.20 we must have $a > 0$ and $b > ae$. The latter implies $D^2 = 2ab - a^2 e > a^2 e > 0$ so we cannot have $D^2 = -2$.

If $e < 0$, either $a = 1$ and $b \geq 0$ or $a \geq 2$ and $b > \frac{1}{2}ae$. In the former case we get $D^2 = 2b - e$ which is positive, and in the latter case we get $D^2 = 2ab - a^2 e > 0$. In both cases we cannot have $D^2 = -2$.

Hence, the only possible (-2)-curve in S is σ with $\sigma^2 = -2$ (that is $e = 2$). We obtain $C \simeq \mathbb{P}^1$ and $\mathcal{E} \simeq \mathcal{O}_C \oplus \mathcal{O}_C(-2)$. We compute

$$H^0(S, \mathcal{O}_S(\sigma)) = H^0(S, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq H^0(C, \pi_* \mathcal{O}_S(1)) = H^0(C, \mathcal{E}) \simeq \mathbb{C}$$

and we conclude that the only integral curve in $|\sigma|$ is σ itself. □

For minimal general type surfaces, we can bound the number of (-2)-curves.

Proposition 1.1.9 ([BPV84] VII. Proposition 2.5)**.** *If S be a smooth minimal surface of general type. Then the number of (-2)-curves on X is at most $\rho(X) - 1$.*

Proof. Let $C_1, \ldots, C_N$ be distinct (-2)-curves on S for some $N \geq 0$. It is enough to prove that their images in $\mathrm{NS}(S)_{\mathbb{Q}}$, denote again $C_1, \ldots, C_N$, are independant over $\mathbb{Q}$: since an ample divisor H satisfies $H \cdot K_S > 0$, H cannot be cohomologous to the (-2)-curves and we obtain the result. Consider a trivial linear combinations of the C_i's, $i = 1, \ldots, N$. Reordering the indices we can assume that we have

$$\sum_{i=1}^{k} \lambda_i C_i = \sum_{j=k+1}^{N} \lambda_j C_j$$

with $\lambda_i, \lambda_j > 0$ for all i, j. Note that $K_S^2 > 0$ and $K_S \cdot C = 0$ for any (-2)-curve C, we can apply the algebraic index theorem [BPV84], IV Corollary 2.16, to conclude that

$$(\sum_{i=1}^{k} \lambda_i C_i)^2 \leq 0.$$

But in the other hand, since the intersection number of distinct curves is positive, we have

$$(\sum_{i=1}^{k}\lambda_i C_i)^2 = (\sum_{i=1}^{k}\lambda_i C_i)(\sum_{j=k+1}^{N}\lambda_j C_j) \geq 0.$$

Hence from the algebraic index theorem again we obtain that the C_i, C_j are independant over $\mathbb{Q}$. □

Typical examples of general type surfaces with (-2)-curves are minimal resolution of du Val singularities.

Finally, consider an elliptic surface $f : S \to B$, where B is a smooth curve, f is surjective, such that the general fibre is a smooth curve of genus 1.

Proposition 1.1.10. *If $f : S \to B$ is an minimal elliptic surface and $\kappa(S) = 1$, then the only (-2)-curves lying in S are component of singular fibres. In particular, they are in finite number.*

Proof. Assume $C \subset S$ is a (-2)-curve which does not lie in a fibre of f. Then the restricted morphism $f : C \to B$ is surjective, and by Hurwitz formula we obtain that $B \simeq \mathbb{P}^1$. By [Fri98], Chapter 7 exercice 7, $K_X \equiv rF$, for F the numerical class of a fibre and $r \in \mathbb{Q}$. In particular, by adjunction formula $C^2 = -2$ gives $C \cdot K_X = rC \cdot F = 0$. Since C is effective and is not contained in a fibre, $C \cdot F \geq 0$. Hence we obtain $r = 0$, that is K_X is numerically trivial. But this contradicts $\kappa(S) = 1$.

Finally, there is finitely many singular fibres by genericity of smoothness of f. □

1.1.3 Moduli space of sheaves on K3 surfaces and Mukai lattice

In this text a K3 surface S is a smooth projective surface with $\omega_S \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. The alternating pairing

$$\Omega^1_S \times \Omega^1_S \to \omega_S \simeq \mathcal{O}_S$$

gives S a structure of hyperKähler variety (see section 1.1.5). In the following, we will consider *polarized* K3 surfaces (S, H), that is S is a K3 surface and H is an ample class in $\mathrm{Pic}(S)$.

For classification of polarized K3 surfaces, we have the following theorem

Theorem 1.1.11. *For any integer $d > 0$, there exists a 19-dimensional quasi-projective coarse moduli space M_d parametrizing polarized K3 surfaces (S, H) of degree $H^2 = 2d$.*

Proof. We only sketch the construction. Given (S, H) a polarized K3 surface with $H^2 = 2d$, it can be shown that $3H$ is very ample, in particular we obtain an embedding

$$S \hookrightarrow \mathbb{P}H^0(S, \mathcal{O}(3H))^\vee \simeq \mathbb{P}^{9d+1}.$$

Let P be the Hilbert polynomial of S in $\mathbb{P}^{9d+1}$. Consider the Hilbert scheme $\mathrm{Hilb}^P_{\mathbb{P}^{9d+1}}$. It is projective. The K3 surfaces obtained as above forms an open subscheme $\mathcal{H} \subset \mathrm{Hilb}^P_{\mathbb{P}^{9d+1}}$. finally, we obtain M_d as the GIT quotient

$$M_d := \mathcal{H} /\!\!/ \mathrm{PGL}_{9d+2}(\mathbb{C}).$$

□

Another way to classify K3 surfaces is to study the Hodge structure on their second cohomology group $H^2(S, \mathbb{Z})$. In fact, K3 surfaces are completely determined by the Hodge structure. For a proof of the next theorem, we refer to [Huy16], Theorem 5.3.

Theorem 1.1.12 (Global Torelli Theorem)**.** *Two K3 surface S and S' are isomorphic if and only if there exists a Hodge isometry $H^2(S, \mathbb{Z}) \simeq H^2(S', \mathbb{Z})$.*

Moduli spaces of sheaves on K3 surfaces

Let S be a projective K3 surface. We fix an ample class H. We endow $H^*(X,\mathbb{Z})$ with a weight 2 Hodge structure, which we denote $\widetilde{H}(S,\mathbb{Z})$ and call *Mukai lattice*, which is induced by the usual Hodge structure on $H^2(S,\mathbb{Z})$ and by setting

$$\widetilde{H}^{1,1}(S) := (H^0 \oplus H^{1,1} \oplus H^4)(S) \text{ and } \widetilde{H}^{2,0}(S) := H^{2,0}(S).$$

We define a *Mukai pairing* on $\widetilde{H}(S,\mathbb{Z})$ as follows. For $v = (v_0, v_1, v_2)$ and $w = (w_0, w_1, w_2)$ in $\widetilde{H}(S,\mathbb{Z})$ we set

$$\langle v, w\rangle := v_1 w_1 - v_0 w_2 - v_2 w_0 \tag{1.1}$$

Denote $\mathrm{NS}(S)$ the *Néron-Severi* group of S. It is well known that $\mathrm{NS}(S) = \mathrm{Pic}(S) = H^{1,1}(S) \cap H^2(S,\mathbb{Z})$. We consider the sublattice

$$\Lambda := (H^0 \oplus \mathrm{NS} \oplus H^4)(S) \subset \widetilde{H}(S,\mathbb{Z}).$$

Given a sheaf $F \in \mathbf{Coh}(S)$ we consider its *Mukai vector* $v(F) = (\mathrm{rk}(F), c_1(F), \mathrm{ch}_2(F) + \mathrm{rk}(F)) \in \Lambda$. In view of Theorem 1.1.4, we consider the projective moduli spaces $\mathcal{M}_S[v]$ of stable sheaves F with Mukai vector $v(F) = v$.

Theorem 1.1.13. *Let $v \in \Lambda$ be a primitive vector. Then, with respect to a generic choice of H, any semistable sheaf F with $v(F) = v$ is stable and $\mathcal{M}_S[v]$ is a smooth projective hyperKähler manifold of dimension $\langle v, v\rangle + 2$ if not empty. Moreover, if $\langle v, v\rangle \geq -2$ with $v_0 > 0$ or $v_0 = 0$ and v_1 ample, then $\mathcal{M}_S[v]$ is not empty.*

Proof. For all the statement of this proof, we refer to [Huy16], Chapter 10 section 2.

Note that for any sheaf $E \in \mathbf{Coh}(S)$, $P(E,m) = -\langle v(F), v(\mathcal{O}(mH))\rangle$. Hence for any subsheaf $E \subset F$ with $p(F,m) \equiv p(E,m)$, it is easy to see that we obtain $v(F) = rv(E)$ for some integer $r > 1$. This contradicts the fact that v is primitive.

By Serre duality, if F is stable then $\mathrm{Hom}(F,F) = \mathrm{Ext}^2(F,F) \simeq \mathbb{C}$. Since $\langle v(F), v(F)\rangle = \chi(F,F)$ we obtain $\mathrm{ext}^1(F,F) = \langle v,v\rangle + 2$. Hence, if $\mathcal{M}_S[v]$ is smooth at $[F]$, by 1.1.5 we get $\dim T_{[F]}\mathcal{M}_S[v] = \mathrm{ext}^1(F,F) = \langle v,v\rangle + 2$. In fact, the trace map $\mathrm{Ext}^2(F,F) \to H^2(S,\mathcal{O}_S)$ is an isomorphism, and by mean of local deformation one can prove that $\mathcal{M}_S[v]$ is smooth.

Consider the composition

$$\mathrm{Ext}^1(F,F) \times \mathrm{Ext}^1(F,F) \to \mathrm{Ext}^2(F,F) \xrightarrow{tr} H^2(S,\mathcal{O}_S) \simeq \mathbb{C}.$$

It gives a natural non-degenerate pairing, and we obtain a everywhere non-degenerate regular 2-form $\sigma \in H^0(\mathcal{M}_S[v], \Omega^2_{\mathcal{M}_S[v]})$.

The most difficult part is the non-emptyness of $\mathcal{M}_S[v]$ whenever $\langle v,v\rangle \geq -2$, and we omit the proof in this text. □

Finally, we introduce the *twisted* Mukai lattice associated to a K3 surface S. The idea is that the obstruction of the existence of a universal family over some moduli space of sheaves $\mathcal{M}_S[v]$ on S is contained in a Brauer class $\alpha \in \mathrm{Br}(S)$, and there exists an α-twisted universal family.

Hence, let S be a K3 surface, and define $\mathrm{Br}(S) := H^2(S,\mathcal{O}_S)^\vee_{tors}$. We will call a rational cohomology class $B \in H^2(S,\mathbb{Q})$ a *B-field* lift of α if its $(0,2)$-part $B^{0,2}$ is sent to α via the exceptional sequence. A B-field lift of α is unique up to $\mathrm{NS}(S)_\mathbb{Q}$ and $H^2(S,\mathbb{Z})$.

Definition 1.1.14. Given B a B-field lift of a Brauer class $\alpha \in \mathrm{Br}(S)$, we define a Hodge structure of weight 2, denoted $\widetilde{H}(S,\alpha,\mathbb{Z})$, on $H^*(S,\mathbb{Z})$ as the Mukai lattice $\widetilde{H}(S,\mathbb{Z})$ with (p,q)-part given by

$$\widetilde{H}^{p,q}(S,\alpha) := \exp(B)\cdot \widetilde{H}^{p,q}(S).$$

We see that $\widetilde{H}^{2,0}(S,\alpha)$ is spanned by $\sigma + B\wedge\sigma \in H^{2,0}(S)\oplus H^4(S,\mathbb{Z})$ for $\sigma \neq 0 \in H^{2,0}(S,\mathbb{Z})$. Moreover, if $B \in H^2(S,\mathbb{Z})$ then multiplication by $\exp(B)$ gives an isometry of Hodge structure $\widetilde{H}(S,\mathbb{Z}) \simeq \widetilde{H}(S,\alpha,\mathbb{Z})$.

1.1.4 Moduli spaces of sheaves on prime Fano threefolds

A smooth projective threefold X with $-K_X$ ample is called *Fano*, and if $\mathrm{Pic}(X) = \mathbb{Z}\cdot K_X$ it is called *prime*. In particular, such a Fano X has Picard rank $\rho(X) = 1$ and index $i_X = 1$. These varieties have been extensively studied by Iskovskikh and Prokhorov [IP99]. They are classified in 10 classes of deformations, each class is caracterized by the *genus*

$$g(X) := \frac{1}{2}(-K_X^3 + 2).$$

The value of the genus $g := g(X)$ ranges in $\{2,3,\dots,9,10,12\}$. If $-K_X$ is *very* ample, we say that X is *non-hyperelliptic*. In this case we have $g \geq 3$.

The cohomology groups $H^{k,k}(X)$ of a prime Fano threefold X are generated by

- the fundamental class $[X]$ for $k = 0$,
- the class $H_X = -K_X$ of a hyperplane section of X for $k = 1$,
- the class L_X of a line contained in X for $k = 2$.
- the class P_X of a point contained in X for $k = 3$.

For this reason, we will denote

$$\mathcal{M}_X(r, c_1, c_2, c_3),$$

$r, c_1, c_2, c_3 \in \mathbb{Z}$, the moduli space of sheaves F on X with rank $\mathrm{rk}(F) = r$ and chern classes $c_1(F) = c_1 H_X$, $c_2(F) = c_2 L_X$, $c_3(F) = c_3 P_X$. When $c_3 = 0$, we simply write $\mathcal{M}_X(r, c_1, c_2)$.

The moduli spaces $\mathcal{M}_X(2, 1, c_2, c_3)$ have been studied by many authors: we can cite Arrondo, Brambilla, Faenzi, Iliev, Manivel, Markushevich and Ranestad in [IM00a] for $g = 3$, [IM04],[IM07b], [BF14] for $g = 7$, [IM00b], [IM07a] for $g = 8$, [IR05], [BF13] for $g = 9$, [AF06] for $g = 12$.

A first observation is a criterion for non-emptyness.

Proposition 1.1.15. *Let X be a smooth non-hyperelliptic Fano threefold of genus g. Let F be a rank 2 stable sheaf on X with $c_1(F) = 1$. Then*

$$c_2(F) \geq \frac{g+2}{2}$$

Proof. A very general member of $|H_X|$ is a K3 surface S with Picard group $\mathrm{Pic}(S) = \mathbb{Z}\cdot H_S$, where H_S is the restriction of H_X to S. By generality of S and by Maruyama's theorem [Mar80], the restricted sheaf F_S is semistable with Mukai vector $v(F_S) = (2, 1, g - c_2(F) + 1)$. Computing the dimension of the latter space (Theorem 1.1.13), we find that it is not empty for $c_2(F) \geq \frac{g}{2} + 1$. □

We denote $m_g := \left\lceil \frac{g+2}{2} \right\rceil$ this lower bound. The moduli space $\mathcal{M}_X(2, 1, m_g)$ is never empty (explicit examples have been constructed for each value of $g(X)$). Brambilla and Faenzi give in [BF11] a description of $\mathcal{M}_X(2, 1, m_g)$.

Theorem 1.1.16 ([BF11], Theorem 3.2)**.** *Let X be a smooth non-hyperelliptic prime Fano threefold of genus g. Then $\mathcal{M}_X(2, 1, m_g)$ can be described as follows:*

($g = 3$) the curve $\mathcal{H}_1^0(X)$ of lines contained in X,

($g = 4$) a scheme of length two, smooth if and only if X is contained in a smooth quadric,

($g = 5$) the double cover of the discriminant septic curve of the net of 6-dimensional quadrics defining X,

($g = 6, 8, 10, 12$) a single smooth point,

($g = 7$) a smooth non-tetragonal curve of genus 7,

($g = 9$) a smooth plane quartic curve.

Moreover, the authors describe recursively in [BF14] a "good" component of $\mathcal{M}_X(2, 1, c_2)$ for $c_2 \geq m_g$.

Theorem 1.1.17 ([BF14], Theorem 3.7)**.** *Let X be a smooth non-hyperelliptic prime Fano threefold of genus g. If $g = 4$, we assume that X is contained in a smooth quadric in $\mathbb{P}^5$. Then for any $c_2 \geq m_g$, there exists a non-empty generically smooth irreducible component*

$$M(c_2) \subset \mathcal{M}_X(2, 1, c_2)$$

of dimension $2d - g - 2$. Its very general element F is locally free and satisfies

$$\operatorname{Ext}^2(F, F) = 0,$$
$$H^1(X, F(-1)) = 0.$$

The case $g = 9$ will be described with more details in section 2.1.

1.1.5 HyperKähler manifolds and Lagrangian fibrations

In this thesis, a *hyperKähler* manifold (*HK manifold* for short), also called *irreducible symplectic manifold*, is defined as a compact Kähler simply-connected manifold X such that $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate holomorphic two-form σ. A projective HK manifold is called *HK variety*. The study of HK manifolds is motivated by the following theorem.

Theorem 1.1.18 (Beauville-Bogomolov decomposition, [Bea83])**.** *Let Y be a compact Kähler manifold with $c_1(Y) = 0$. There exists an étale finite covering*

$$\prod_{i=1}^{d} M_i \to Y$$

where each of the factors M_i is either a compact complex torus, a Calabi-Yau variety or a HK manifold.

Hence HK varieties are building blocks for smooth projective varieties with trivial first Chern class. However, not some many families of HK varieties are known. We list them here. In Example 1.1.19, the families 2 and 3 have been constructed by Beauville [Bea83] and the two sporadic examples 4 and 5 have been discovered by O'Grady [O'G99], [O'G03].

Example 1.1.19. 1. The only 2-dimensional HK varieties are K3 surfaces.

2. Let S be a K3 surface, and let $X = S^{[n]}$ be the Hilbert scheme of zero-dimensional subscheme of S of length $n \geq 2$. Then X is a HK variety of dimension $2n$ with $b_2(X) = 23$. HK varieties deformation equivalent to $S^{[n]}$ for some K3 surface S are called *HK varieties of* K3$^{[n]}$*-type*.

3. Let A be an abelian surface. The Hilbert scheme $A^{[n]}$ carries a symplectic form but is not HK. Though, there exists a morphism

$$\begin{array}{rccc} s: & A^{[n+1]} & \longrightarrow & A \\ & Z & \longmapsto & \sum_{p\in A} l(\mathcal{O}_{Z,p})p \end{array}$$

and the variety

$$X = K^{[n]}(A) := s^{-1}(0)$$

is a HK variety of dimension $2n$ with $b_2(X) = 7$. HK varieties deformation equivalent to $K^{[n]}(A)$ for some abelian variety A and some integer n are called *HK varieties of Kummer type.*

4. Let S be a K3 surface. Consider a Mukai vector $v = (v_0, v_1, v_2) \in \widetilde{H}^{1,1}(S)_{\mathbb{Z}}$ primitive, with either $v_0 \neq 0$ or v_1 effective, and $v^2 = 2$. Then the moduli space $\mathcal{M}_S[2v]$ of semistable sheaves with Mukai vector v is non-empty, irreducible of dimension $2 + v^2$. Moreover, there exists a symplectic desingularization

$$f : X = \widetilde{\mathcal{M}_S}[2v] \to \mathcal{M}_S[2v]$$

so that X is a 10-dimensional HK variety with $b_2(X) = 24$. HK varieties deformation equivalent to X for some K3 surface S are called *HK varieties of OG*10*-type.*

5. We can perform the same construction as the previous example 4 replacing the K3 surface S by an abelian surface T. In this case, one can consider the subvariety

$$Y = f^{-1}(\mathcal{M}_T[2v]^0) \subset \widetilde{\mathcal{M}}_T[2v]$$

where $\mathcal{M}_T[2v]^0 \subset \mathcal{M}_T[2v]$ is defined as the subspace of sheaves F with fixed $c := c_1^{CH}(F)$ (where $c_i^{CH}(-)$ is the i-th Chern class with values in the Chow group of T) and with $\sum c_2^{CH}(F) = 0$ (where the sum is taken on T via the map $\mathrm{CH}^2(T) \to T$). Then Y is a 6-dimensional HK variety with $b_2(Y) = 8$. HK varieties deformation equivalent to Y for some abelian surface T are called *HK varieties of OG*6*-type.*

Remark 1.1.20. Unfortunately, Kaledin, Lehn and Sorger [KLS06] proved that O'Grady's examples 1.1.19, 4 and 5 cannot be generalized for $m \geq 2, v_0^2 > 2$ or $m > 2$: the moduli spaces $\mathcal{M}_S(mv)$, resp. $\mathcal{M}_T(mv)$, do not admit symplectic resolutions.

Consider X a HK variety of dimension $2n$. A subvariety $L \subset X$ of dimension n is called *Lagrangian* if the holomorphic symplectic form $\sigma \in H^0(X, \Omega_X^2)$ restricts to the trivial 2-form on the smooth locus of L.

Definition 1.1.21. Let X be a HK variety. A *Lagrangian fibration* on X is a morphism $f : X \to B$ with connected fibre to a normal variety B such that every irreducible component of the reduction of every fibre of f is a Lagrangian subvariety of X.

From the work of Matsushita [Mat99], [Mat00], it turns out that any morphism $f : X \to B$ with connected fibre onto a normal variety B with $0 < \dim B < \dim X$ is a Lagrangian fibration. In particular, f is equidimensional and $\dim B = n$. By Liouville's theorem, the generic fibre is a complex torus.

It is conjecture that the base B of any Lagrangian fibration $f : X \to B$ is the projective space $\mathbb{P}^n$. Hwang [Hwa08] proved that this is true when B is smooth, and Huybrechts and Xu [HX19] proved that the conjecture is true for any normal variety B when $\dim X = 4$. Let us introduce a weakened notion of Lagrangian fibration.

Definition 1.1.22. Let X be a HK variety. A *rational Lagrangian fibration* on X is a rational map $f : X \to B$ such that there exists another HK variety Y and a birational map $g : Y \dashrightarrow X$ such that $f \circ g$ is a Lagrangian fibration.

Example 1.1.23. 1. A K3 surface which admits a Lagrangian fibration is called *elliptic*. There exists a lot of example of elliptic K3 surfaces. For instance, if you consider the Kummer surface X associated to the abelian surface $A = E_1 \times E_2$, where E_1, E_2 are elliptic curves, then the projections $X \to E_i/_{\pm} \simeq \mathbb{P}^1$, $i = 1, 2$, give elliptic fibrations.

2. Consider $X = S^{[n]}$ for $f : S \to \mathbb{P}^1$ an elliptic surface. Consider the *Hilbert-Chow* morphism

$$\begin{array}{cccc} \gamma: & X & \longrightarrow & S^{(n)} \\ & Z & \longmapsto & \sum_{p\in S} l(\mathcal{O}_{Z,p})p, \end{array}.$$

Then the composition

$$S^{[n]} \to S^{(n)} \xrightarrow{f\times\cdots\times f} (\mathbb{P}^1)^{(n)} \simeq \mathbb{P}^n$$

is a Lagrangian fibration, where $S^{(n)}$ and $(\mathbb{P}^1)^{(n)}$ denote the *symmetric powers*.

3. If $X \simeq K^{[n]}(A)$ for some principally polarized abelian surface A of Picard number 1, Gulbrandsen [Gul07] proved that X admits a rational Lagrangian fibration over $\mathbb{P}^n$ if and only if n is a perfect square.

4. Let $Y \subset \mathbb{P}^5$ be a general cubic fourfold. There is a universal family $\mathcal{Y} \to B := (\mathbb{P}^5)^\vee$ of cubic threefolds obtained as hyperplane sections of Y. Denoting $\mathcal{U} \subset B$ the locus of smooth hyperplane sections, taking the intermediate Jacobian of the cubic threefolds leads to a relative intermediate Jacobian fibration

$$\mathcal{J} \to \mathcal{U}.$$

In [LSV17], Laza, Saccà and Voisin construct a compactification $p : \overline{\mathcal{J}} \to B$ of $\mathcal{J} \to \mathcal{U}$ such that $\overline{\mathcal{J}}$ is a HK variety of $OG10$-type, and for which p is a Lagrangian fibration.

The goal of Chapter 2 of this thesis is to construct a rational Lagrangian fibration over some moduli spaces of sheaves on a K3 surface of Picard rank 1 and genus 9 (see Corollary 2.3.8 and Theorem 2.3.9).

1.2 Decompositions of the derived category of a smooth projective variety

We start by gathering some useful construction that will be used in Chapter 2. For a general background about triangulated categories and derived categories of coherent sheaves, we refer to [Huy06].

1.2.1 Semi-orthogonal decompositions

Let $\mathcal{T}$ be a triangulated category linear over a field K. Recall that a thick triangulated subcategory $\mathcal{A} \subset \mathcal{T}$ is called *admissible* if the embedding functor $i : \mathcal{A} \hookrightarrow \mathcal{T}$ admits left and right adjoints $i^*, i^!$.

Definition 1.2.1. A *semi-orthogonal decomposition* (*SOD* for short) for $\mathcal{T}$ is a family of admissible triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{T}$ such that

1. For any $A_j \in \mathcal{A}_j, A_i \in \mathcal{A}_i$ with $j > i$ we have $\mathrm{Hom}_{\mathcal{T}}(A_j, A_i) = 0$.

2. The smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{A}_1, \dots, \mathcal{A}_n$ is $\mathcal{T}$ itself.

We denote $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$. A way to construct semiorthogonal decomposition is to find a collection of *exceptional objects* $E_1, \dots, E_m$, that is for all $i = 1, \dots, n$ we have $\mathrm{Hom}(E, E) = K$ and $\mathrm{Hom}(E, E[t]) = 0$ for $t \neq 0$. Denoting the subcategory generated by E_i as E_i again, we obtain

$$\mathcal{T} = \langle \mathcal{A}, E_1, \dots, E_n \rangle$$

where $\mathcal{A} = \langle E_1, \dots, E_n \rangle^{\perp} := \{ T \in \mathcal{T} \mid \mathrm{Hom}(E_i[t], \mathcal{A}) = 0 \ \forall t \in \mathbb{Z}, i = 1, \dots, n \}$.

Example 1.2.2. 1. *Beilinson decomposition* [Bei79], [Bei84]:

$$\mathrm{D}^b(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle. \tag{1.2}$$

It easy to see that $\{\mathcal{O}_{\mathbb{P}^n}, \dots, \mathcal{O}_{\mathbb{P}^n}(n)\}$ is exceptional as $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(j), \mathcal{O}_{\mathbb{P}^n}(i)[t]) = H^t(\mathbb{P}^n, \mathcal{O}(j - i)$. The most difficult fact is to show that $\mathrm{D}^b(\mathbb{P}^n)$ is indeed generated by the exceptional objects.

2. *Orlov projective bundle formula* [Orl92]: Let S be a K-scheme and E be a vector bundle of rank $r+1$ on it. Let $f : \mathbb{P}_S(E) \to S$ be the projectivization of E over S and let $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}_S(E)/S}(1)$ be the Grothendieck line bundle on $\mathbb{P}_S(E)$. Then we have

$$\mathrm{D}^b(\mathbb{P}_S(E)) = \langle f^*(\mathrm{D}^b(S)) \otimes \mathcal{O}, f^*(\mathrm{D}^b(S)) \otimes \mathcal{O}(1), \dots, f^*(\mathrm{D}^b(S)) \otimes \mathcal{O}(r) \rangle \tag{1.3}$$

3. Let X be a Fano variety of index i, and fix an ample divisor $\mathcal{O}_X(1)$ such that that $-K_X = \mathcal{O}_X(i)$. By Kodaira vanishing and using Serre duality, we see that the family $\{\mathcal{O}_X, \dots, \mathcal{O}_X(i-1)\}$ is exceptional in $\mathrm{D}^b(X)$, and we get a SOD

$$\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(i-1) \rangle.$$

If $X \simeq \mathbb{P}^n$, we get $\mathcal{A}_X = 0$ (example 1). We will see in section 2.1.1 the example of a prime Fano threefold X of index 1 and degree 16, for which $\mathcal{A}_X \simeq \langle \mathrm{D}^b(\Gamma), \mathcal{U} \rangle$ for Γ a plane quartic curve. See [Kuz16] for Fano threefolds and [Kuz10] for cubic fourfolds.

1.2.2 Homological projective duality

Let X be a smooth projective variety over an algebraically closed field K of characteristic 0 with a morphism

$$f : X \to \mathbb{P}V$$

for V a finite dimensional K-vector space. Set $\mathcal{O}_X(1) := f^*\mathcal{O}_{\mathbb{P}V}(1)$.

Assume X is endowed with a right Lefschetz decomposition

$$\mathrm{D}^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_m(m) \rangle, \tag{1.4}$$

that is a SOD such that $\mathcal{A}_m \subset \mathcal{A}_{m-1} \subset \cdots \subset \mathcal{A}_1 \subset \mathcal{A}_0$ (note that 1.2.2 example 1, with $\mathcal{A}_0 = \cdots = \mathcal{A}_m = \langle \mathcal{O}_{\mathbb{P}^n} \rangle$, is an example of such).

From [Kuz11], the product $X \times \mathbb{P}V^\vee$ inherits a semiorthogonal decomposition

$$\mathrm{D}^b(X \times \mathbb{P}V^\vee) = \langle \mathcal{A}_0 \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee), \mathcal{A}_1(1) \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee), \dots, \mathcal{A}_m(m) \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee) \rangle. \tag{1.5}$$

Now define $Q \subset \mathbb{P}V \times \mathbb{P}V^\vee$ the *incidence quadric* of point $(Kv, H) \in \mathbb{P}V \times \mathbb{P}V^\vee$ with $v \in H$. We consider the *universal hyperplane section* of X as the fibre product

$$\mathcal{X}_1 := X \times_{\mathbb{P}V} Q.$$

There is a natural embedding $\mathcal{X}_1 \hookrightarrow X \times \mathbb{P}V^\vee$ and hence a map $p : \mathcal{X}_1 \to \mathbb{P}V^\vee$. The fibre of p over $H \in \mathbb{P}V^\vee$ is the hyperplane section $X \cap H$. Thus, $\mathcal{X}_1$ can be thought as the family of hyperplane sections of X over $\mathbb{P}V^\vee$.

Proposition 1.2.3 ([Kuz07] Lemma 5.3)**.** *The pullback by the closed immersion $\mathcal{X}_1 \subset X \times \mathbb{P}V^\vee$ is fully faithful on each component $\mathcal{A}_i(i) \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee)$ of (1.5) and induces a semiorthogonal decomposition*

$$\mathrm{D}^b(\mathcal{X}_1) = \langle \mathcal{C}, \mathcal{A}_0 \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee), \mathcal{A}_1(1) \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee), \dots, \mathcal{A}_m(m) \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee) \rangle \tag{1.6}$$

The orthogonal complement $\mathcal{C}$ is called the *HP dual category*. When it is the derived category of a variety, we call the latter the *HP dual variety* of X.

Definition 1.2.4. A variety Y equipped with a morphism $g : Y \to \mathbb{P}V^\vee$ is called *homologically projective dual* to $f : X \to \mathbb{P}V$ with respect to a Lefschetz decomposition (1.4) if there is an object $\mathcal{E} \in \mathrm{D}^b(Q(X,Y))$ such that the Fourier-Mukai functor

$$\phi_{\mathcal{E}} : \mathrm{D}^b(Y) \to \mathrm{D}^b(\mathcal{X}_1)$$

is an equivalence onto the subcategory $\mathcal{C} \subset \mathrm{D}^b(\mathcal{X}_1)$ of (1.6).

Here, $Q(X,Y) := \mathcal{X}_1 \times_{\mathbb{P}V^\vee} Y = (X \times Y) \times_{\mathbb{P}V \times \mathbb{P}V^\vee} Q$.

Theorem 1.2.5 ([Kuz07])**.** *Let $g : Y \to \mathbb{P}V^\vee$ be an HP dual variety for $f : X \to \mathbb{P}V$ with respect to 1.4.*

1. *Y is smooth and there is a subcategory $\mathcal{B}_0 \simeq \mathcal{A}_0$ and a Lefschetz decomposition*

$$\mathrm{D}^b(Y) = \langle \mathcal{B}_n(-n), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle, \ \mathcal{B}_n \subset \dots \subset \mathcal{B}_1 \subset \mathcal{B}_0 \tag{1.7}$$

2. *(X, f) is HP dual to (Y, g) with respect to (1.7).*

3. *The set of critical values of g is the classical projective dual of X.*

4. *For any linear subspace $L \subset V^\vee$, if $X_L = X \times_{\mathbb{P}V} \mathbb{P}L^\perp$ and $Y_L = Y \times_{\mathbb{P}V^\vee} L$ have expected dimensions, then there are semiorthogonal decompositions*

$$\mathrm{D}^b(X_L) = \langle \mathcal{C}_L, \mathcal{A}_{\dim L}(\dim L), \dots, \mathcal{A}_m(m) \rangle \tag{1.8}$$
$$\mathrm{D}^b(Y_L) = \langle \mathcal{B}_n(-n), \dots, \mathcal{B}_{\dim V - \dim L}(\dim L - \dim V), \mathcal{C}_L \rangle. \tag{1.9}$$

Example 1.2.6 ([Kuz07])**.** Let V be a K-vector space of dimension N. Let $E \subset V$ be a linear subspace of dimension $i+1$ and let $X := \mathbb{P}E \hookrightarrow \mathbb{P}V$ be the natural embedding. We will prove that $g : Y := \mathbb{P}E^\perp \hookrightarrow \mathbb{P}V^\vee$ is the HP dual variety for X.

Note that $Q(X,Y) = Y \times_{\mathbb{P}V^\vee} \mathcal{X}_1 = Y \times X$. Consider the diagram induced by the natural projections from $Y \times_{\mathbb{P}V^\vee} \mathcal{X}_1 \simeq Y \times X$:

$$\begin{array}{ccc} Y \times X & \xrightarrow{\ p\ } & \mathcal{X}_1 \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\ g\ } & \mathbb{P}V^\vee. \end{array}$$

Using the projection formula, it is easy to see that $\phi_{\mathcal{O}_{(Y \times X)}} \simeq p_*\phi^*$. Let $F \in \mathrm{D}^b(\mathbb{P}V^\vee)$ and $E \in \mathrm{D}^b(Y)$. We have

$$\begin{aligned}
\mathrm{Hom}(f^*F \otimes \mathcal{O}_X(k), p_*\phi^*G) &= \mathrm{Hom}(f^*F, p_*\phi^*G \otimes \mathcal{O}_X(-k)) \\
&= \mathrm{Hom}(f^*F, p_*(\phi^*G \otimes \mathcal{O}_{X \times Y}(-k))) \\
&= \mathrm{Hom}(F, f_*p_*(\phi^*G \otimes \mathcal{O}_{X \times Y}(-k))) \\
&= \mathrm{Hom}(F, g_*\phi_*(\phi^*G \otimes \mathcal{O}_{X \times Y}(-k))) \\
&= \mathrm{Hom}(F, g_*(G \otimes \phi_*\mathcal{O}_{X \times Y}(-k)))
\end{aligned}$$

and $\phi_*\mathcal{O}_{X\times Y}(-k) \simeq \mathcal{O}_Y(-k) = 0$ for all $k > 0$. Hence we have

$$\mathrm{D}^b(\mathcal{X}_1) \supset \langle \phi_{\mathcal{O}_{X\times Y}}(\mathrm{D}^b(Y)), \mathcal{O}_X \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee), \dots, \mathcal{O}_X(i) \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee)\rangle. \tag{1.10}$$

To conclude, it suffices to prove that the latter inclusion is an equality. Pick G in the *left* orthogonal to the RHS of (1.10). For any $F \in \mathrm{D}^b(Y)$ we have

$$\mathrm{Hom}(G, p_*\phi^*F \otimes \mathcal{O}_X(k)) = \mathrm{Hom}(p^*G, \phi^*F \otimes \mathcal{O}_{X\times Y}(k)) = 0$$

and hence $p^*G = 0$ since the subcategories $\phi^*\,\mathrm{D}^b(Y) \otimes \mathcal{O}_{X\times Y}(k)$ generates $\mathrm{D}^b(X \times Y)$ (1.2.2 Example 2). Hence we obtain that G is supported on $\mathcal{X}_1 \smallsetminus X \times Y$ (note that, by definition of Y, the projection p is an embedding). But the fibre of $\mathcal{X}_1$ over a point $H \in \mathbb{P}V^\vee \backslash Y$ is a hyperplane of X, so that $\mathcal{X}_1 \smallsetminus X \times Y$ is a $\mathbb{P}^{i-1}$-bundle over $\mathbb{P}V^\vee \smallsetminus Y$. Once again using 1.2.2 Example 2, we conclude by pulling back the orthogonality of G with $\langle \mathcal{O}_X \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee), \dots, \mathcal{O}_X(r) \boxtimes \mathrm{D}^b(\mathbb{P}V^\vee)\rangle$ to $\mathcal{X}_1 \smallsetminus X \times Y$ that $G = 0$.

In general, the HP dual category $\mathcal{C}$ in (1.6) is not equivalent to the derived category of a variety. Let us point out two possible phenomenons:

- Sometimes there is a variety Y with a map $g : Y \to \mathbb{P}V^\vee$, a sheaf of finite $\mathcal{O}_Y$-algebras $\mathcal{A}$ on Y, and an object $\mathcal{E} \in \mathrm{D}^b(Q(X, Y), \mathcal{A})$ which induces an equivalence

  $$\phi_\mathcal{E} : \mathrm{D}^b(Y, \mathcal{A}) \xrightarrow{\sim} \mathcal{C},$$

  where $\mathrm{D}^b(Y, \mathcal{A})$ denotes the bounded derived category of of coherent $\mathcal{A}$-modules. We call such a variety Y as a *noncommutative* HP dual of X.

- Moreover, sometimes the (noncommutative) variety Y is a HP dual of X only over an open dense subset $\mathcal{U} \subset \mathbb{P}V^\vee$. Denote $\mathcal{C}_\mathcal{U}$ the base change of $\mathcal{C}$ with respect to the open immersion $U \hookrightarrow \mathbb{P}V^\vee$. When there exists an object $\mathcal{E} \in \mathrm{D}^b(Q(X, Y)_\mathcal{U}, \mathcal{A})$ which gives an equivalence

  $$\phi_\mathcal{E} : \mathrm{D}^b(Y_\mathcal{U}, \mathcal{A}) \xrightarrow{\sim} \mathcal{C}_\mathcal{U},$$

  we say that Y_U is an *incomplete* (noncommutative) HP dual of X over $\mathcal{U}$.

Proposition 1.2.7 ([Kuz14], Proposition 4.6)**.** *In both cases described above, the semiorthogonal decompositions (1.8) and (1.9) hold true for any subspace $L \subset V^\vee$ such that $\mathbb{P}L \subset \mathcal{U}$ and X_L, Y_L have expected dimensions.*

Of course, one have to replace $\mathrm{D}^b(Y_L)$ by $\mathrm{D}^b(Y_L, \mathcal{A})$ and possibly apply base change along $\mathcal{U} \hookrightarrow \mathbb{P}V^\vee$. En explicit example of such an incomplete HP duality will be stated in section 2.1.1.

Remark 1.2.8. In fact, there exists a stronger statement that Theorem 1.2.5. One can generalize the construction of $\mathcal{X}_1$ to higher codimensional linear sections: for each $r \in \{1, \dots, \dim V\}$ there is a *universal family of linear sections*

$$\mathcal{X}_r \subset X \times G(r, V^*)$$

such that the fibre over a linear subspace $L \in G(r, V^*)$ is $X_L := X \cap L^\perp$. It turns out that $\mathcal{X}_r \to G(r, V^*)$ is projective and $\mathcal{X}_r \to X$ is smooth, in particular $\mathcal{X}_r$ is smooth.

In [Kuz07], the author proves that the decompositions (1.8) and (1.9) hold true in the universal context. Denote $\mathcal{Y}_r$ the universal families of linear sections of Y. The object $\mathcal{E} \in \mathrm{D}^b(Q(X, Y))$ pullbacks to $\mathcal{E}_r \in \mathrm{D}^b(\mathcal{X}_r \times_{G(r,V^\vee)} \mathcal{Y}_r)$, which as a kernel provides a functor $\phi_{\mathcal{E}_r}$:

$D^b(\mathcal{Y}_r) \to D^b(\mathcal{X}_r)$ (and left and right adjoint $\phi^*_{\mathcal{E}_r}, \phi^!_{\mathcal{E}_r} : D^b(\mathcal{X}_r) \to D^b(\mathcal{Y}_r)$. Kuznetsov proves that these functors give the decompositions

$$D^b(\mathcal{X}_r) = \langle \mathcal{C}_r, \mathcal{A}_{\dim L}(\dim L) \boxtimes D^b(G(r,V^\vee)), \dots, \mathcal{A}_m(m) \boxtimes D^b(G(r,V^\vee)) \rangle \quad (1.11)$$
$$D^b(\mathcal{Y}_r) = \langle \mathcal{B}_n(-n) \boxtimes D^b(G(r,V^\vee)), \dots, \mathcal{B}_{\dim V - \dim L}(\dim L - \dim V) \boxtimes D^b(G(r,V^\vee)), \mathcal{C}_r \rangle. \quad (1.12)$$

The decompositions (1.8) and (1.9) are obtained from (1.11) and (1.12) by base change $\operatorname{Spec}\mathbb{C} \to G(r,V^\vee), * \mapsto L$.

Finally, the same statement hold true when Y is *noncommutative* or for *incomplete* HPD, replacing $D^b(\mathcal{Y}_r)$ by $D^b(\mathcal{Y}_r, \mathcal{A})$ and pulling back to an open of $G(r,V^\vee)$ when needed.

1.3 Autoequivalences of derived categories

We consider in this section a smooth projective variety X over an arbitrary field K.

1.3.1 Standard autoequivalences and the Bondal-Orlov theorem

To understand $D^b(X)$, one wants to compute the group of autoequivalences $\operatorname{Aut}(D^b(X))$. Here, by autoequivalence we mean exact K-linear equivalence $D^b(X) \to D^b(X)$, and $\operatorname{Aut}(D^b(X))$ is defined as the group of isomorphism classes of autoequivalences. The easiest examples of autoequivalences are the following ones:

1. The shift functor $[1] : D^b(X) \to D^b(X)$.
2. Let $f : X \to X$ be an automorphism. Then the derived pushforward $Rf_* : D^b(X) \to D^b(X)$ is an autoequivalence, whose inverse is given as the derived pullback Lf^*.
3. Let $L \in \operatorname{Pic}(X)$ be a line bundle. The derived tensor product $(- \otimes^L L) : D^b(X) \to D^b(X)$ is another example of autoequivalence, whose inverse is $(- \otimes^L L^\vee)$

Remark 1.3.1. One can check that two automorphisms f, g induce the same autoequivalence if and only if $f = g$. Similarly, the autoequivalence $(- \otimes^L L)$ is isomorphic to the identity if and only if L is trivial. In particular, examples (2) and (3) give subgroups of autoequivalences

$$\operatorname{Aut}(X) \hookrightarrow \operatorname{Aut}(D^b(X)), \quad \operatorname{Pic}(X) \hookrightarrow \operatorname{Aut}(D^b(X)).$$

Note that the shift functor commutes with any autoequivalence. Moreover, given $F \in D^b(X)$, $L \in \operatorname{Pic}(X)$ and $f \in \operatorname{Aut}(X)$, we have $Lf^*(F \otimes^L L) \simeq Lf^*F \otimes^L Lf^*L$, so that $\operatorname{Pic}(X) \rtimes \operatorname{Aut}(X)$ is a normal subgroup of $\operatorname{Aut}(D^b(X))$.

Definition 1.3.2. The equivalences lying in the subgroup

$$\mathbb{Z} \cdot [1] \times (\operatorname{Pic}(X) \rtimes \operatorname{Aut}(X)) \subset \operatorname{Aut}(D^b(X)) \quad (1.13)$$

are called *standard* autoequivalences.

Proposition 1.3.3 ([BO01], Theorem 3.1)**.** *Assume ω_X or ω_X^* is ample. Then*

$$\operatorname{Aut}(D^b(X)) \simeq \mathbb{Z} \cdot [1] \times (\operatorname{Pic}(X) \rtimes \operatorname{Aut}(X)).$$

A particularly interesting example of autoequivalence of $\mathrm{D}^b(X)$ is the *Serre functor*

$$\begin{array}{rrcl} S: & \mathrm{D}^b(X) & \longrightarrow & \mathrm{D}^b(X) \\ & F & \longmapsto & F \otimes \omega_X[\dim X]. \end{array} \tag{1.14}$$

The name come from the categorical version of Serre duality: for any $F, G \in \mathrm{D}^b(X)$ there exists a functorial isomorphism

$$\mathrm{Hom}(F, G) \simeq \mathrm{Hom}(G, S(F))^\vee. \tag{1.15}$$

On a triangulated category $\mathcal{T}$ over K, an autoequivalence S inducing a functorial isomorphism of the form (1.15) for any $F, G \in \mathcal{T}$ is called a Serre functor.

1.3.2 Spherical twists

Spherical twists, introduced by Seidel and Thomas [ST01], are a really important example of autoequivalence of $\mathrm{D}^b(X)$.

Definition 1.3.4. An object $\mathcal{E} \in \mathrm{D}^b(X)$ is called *spherical* if

1. $\mathcal{E} \otimes \omega_X \simeq \mathcal{E}$
2. $R\,\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \simeq K \oplus K[-\dim X]$.

The name *spherical* comes from the remark that the second condition is equivalent to ask $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \simeq H^*(S^{\dim X}, K)$, where $S^{\dim X}$ is the real sphere of dimension $\dim X$.

Remark 1.3.5. If $\mathcal{E}$ is spherical, then for any autoequivalence $\phi \in \mathrm{Aut}(\mathrm{D}^b(X))$ the object $\phi(\mathcal{E})$ is also spherical. This is a consequence of the general fact that the Serre functors of triangulated categories commute with equivalences ([Huy06], Lemma 1.30).

Example 1.3.6.

- On a K3 surface (over any field), every line bundle is spherical.
- Assume $K = \mathbb{C}$ and let $X = S$ be a surface, and $C \overset{i}{\hookrightarrow} S$ a (-2)-curve, that is, a curve $C \simeq \mathbb{P}^1$ with self-intersection -2. Then any line bundle $\mathcal{O}_C(a)$, $a \in \mathbb{Z}$, is a spherical object. Indeed, $K_S \cdot C = 0$ by adjunction formula and thus $i_*(\mathcal{O}_C(a)) \otimes \omega_S \simeq i_*(\mathcal{O}_C(a) \otimes i^*\omega_S) \simeq i_*\mathcal{O}_C(a)$ by projection formula. Now, using [Huy06], section 11:

$$\begin{aligned} R\,\mathrm{Hom}_S(i_*\mathcal{O}_C(a), i_*\mathcal{O}_C(a)) &= R\,\mathrm{Hom}_C(i^*i_*\mathcal{O}_C(a), \mathcal{O}_C(a)) \\ &= R\,\mathrm{Hom}_C(\mathcal{O}_C(a) \oplus \mathcal{O}_C(a+2)[1], \mathcal{O}_C(a)) \\ &= R\,\mathrm{Hom}_C(\mathcal{O}_C, \mathcal{O}_C \oplus \mathcal{O}_C(-2)[1]), \end{aligned}$$

  and the result comes from direct computation of these Ext-groups since $C \simeq \mathbb{P}^1$.

Given any object $\mathcal{E} \in \mathrm{D}^b(X)$, consider the object $\mathcal{P}_\mathcal{E} \in \mathrm{D}^b(X \times X)$ constructed as follow. We denote $p, q : X \times X \to X$ the first, resp. second, projection. Set $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$ the diagonal embedding. Consider the composition of the restriction

$$q^*\mathcal{E}^\vee \otimes p^*\mathcal{E} \to \iota_*\iota^*((q^*\mathcal{E}^\vee \otimes p^*\mathcal{E})) \simeq \iota_*(\mathcal{E}^\vee \otimes \mathcal{E})$$

and the pushforward of the trace map $\mathcal{E}^\vee \otimes \mathcal{E} \to \mathcal{O}_X$ through ι. We obtain a map

$$q^*\mathcal{E}^\vee \otimes p^*\mathcal{E} \to \mathcal{O}_\Delta.$$

Now we set

$$\mathcal{P}_\mathcal{E} := C(q^*\mathcal{E}^\vee \otimes p^*\mathcal{E} \to \mathcal{O}_\Delta).$$

Definition 1.3.7. The *spherical twist* $T_\mathcal{E}$ with respect to a spherical object $\mathcal{E} \in \mathrm{D}^b(X)$ is the Fourier-Mukai transform with kernel $\mathcal{P}_\mathcal{E}$.

On the level of objects it is given by

$$A \mapsto T_\mathcal{E}(A) := C(R\operatorname{Hom}(\mathcal{E}, A) \otimes \mathcal{E} \xrightarrow{\mathrm{ev}} A),$$

i.e. $T_\mathcal{E}(A)$ is given by the cone of the natural evaluation map.

Remark 1.3.8. Quick computations give $T_\mathcal{E}(\mathcal{E}) = \mathcal{E}[1 - \dim X]$ and $\mathcal{T}_\mathcal{E}(F) \simeq F$ for all $F \in \mathrm{D}^b(X)$ with $\operatorname{Hom}(\mathcal{E}, F[i]) = 0\ \forall i \in \mathbb{Z}$ (i.e. $F \in \mathcal{E}^\perp$).

Proposition 1.3.9 ([ST01])**.** *The spherical twist $T_\mathcal{E}$ is an equivalence.*

Proof. We only sketch the proof. First, one can check that $\Omega = \{\mathcal{E}\} \cup \mathcal{E}^\perp$ is a *spanning class* of $\mathrm{D}^b(X)$. In view of [Huy06], Corollary 1.56, to prove that $T_\mathcal{E}$ is fully faithful it is enough to prove that

$$T_\mathcal{E} : \operatorname{Hom}(F, G[i]) \to \operatorname{Hom}(T_\mathcal{E}(F), T_\mathcal{E}(G)[i])$$

is an isomorphism for all $i \in \mathbb{Z}$ and $F, G \in \Omega$. The ingredients to prove the latter are the descriptions $T_\mathcal{E}(\mathcal{E}) \simeq \mathcal{E}[1-\dim X]$ and $T_\mathcal{E}(F) \simeq F$ for all $F \in \mathcal{E}^\perp$ (Remark 1.3.8), Serre duality and the properties of spherical objects.

To conclude, since $T_\mathcal{E}$ admits left and right adjoints (as it is a FM transform) it suffices to prove that for any $G \in \Omega$, we have $T_\mathcal{E}(G \otimes \omega_X[n]) \simeq T_\mathcal{E}(G) \otimes \omega_X[n]$. This is once again a consequence of the fact that $\mathcal{E} \otimes \omega_X \simeq \mathcal{E}$ and Remark 1.3.8.

□

Spherical twists form a normal subgroup of the group of autoequivalences $\operatorname{Aut}(\mathrm{D}^b(X))$. We have the following formula, really useful when it comes to computations.

Proposition 1.3.10. *Let $\phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(X)$ be an autoequivalence. Let $\mathcal{E} \in \mathrm{D}^b(X)$ be a spherical object. Then we have*

$$\phi \circ T_\mathcal{E} \simeq T_{\phi(\mathcal{E})} \circ \phi.$$

Proof. This is easy to see on the level of objects. Indeed, by definition of spherical twists we have

$$T_{\phi(\mathcal{E})}(\phi(F)) = C\left(R\operatorname{Hom}(\phi(\mathcal{E}), \phi(F)) \otimes \phi(\mathcal{E}) \to \phi(F)\right)$$

and on the other hand

$$\phi(T_\mathcal{E}(F)) = \phi\left(C\left(R\operatorname{Hom}(\mathcal{E}, F) \otimes \mathcal{E} \to F\right)\right).$$

Note that $R\operatorname{Hom}(\phi(\mathcal{E}), \phi(F)) \simeq R\operatorname{Hom}(\mathcal{E}, F)$ and that the cone of the image of ϕ is the image of the cone. The identification of both maps $R\operatorname{Hom}(\mathcal{E}, F) \otimes \phi(\mathcal{E}) \to \phi(F)$ permits to obtain an isomorphism $T_{\phi(\mathcal{E})}(\phi(F)) \simeq \phi(T_\mathcal{E}(F))$. The functoriality of this isomorphism is more difficult to prove, we refer to [Huy06], Lemma 8.21. □

Later on, we will study a subgroup of $\operatorname{Aut}(\mathrm{D}^b(X))$, for X a surface, which is generated by spherical twists along the structural sheaves of curves in X. We give now a more general statement.

Assume you are given a finite family $V = \{\mathcal{E}_1, \dots, \mathcal{E}_m\}$, $m \geq 1$, of spherical objects in $\mathrm{D}^b(X)$. Assume that for any $i, j \in \{1, \dots, m\}$ with $i \neq j$, we have

$$\mathcal{E}_i \cdot \mathcal{E}_j := \sum_k \dim \operatorname{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0 \text{ or } 1.$$

Consider the graph Γ_V which set of vertices is $V = \{\mathcal{E}_1, \dots, \mathcal{E}_m\}$ and two vertices $\mathcal{E}_i, \mathcal{E}_j$, $i \neq j$, are linked by an edge if and only if $\mathcal{E}_i \cdot \mathcal{E}_j = 1$. We say that V is an A_m (resp. D_m, E_m, $\bar{A}_m$, $\bar{D}_m$, $\bar{E}_m$) *configuration of spherical objects* if the graph Γ_V is the Dynkin diagram of the corresponding type.

Proposition 1.3.11. *Assume $V = \{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ is a configuration of spherical objects of one of the ADE (or $\tilde{A}\tilde{D}\tilde{E}$) type. Denote $T_i := T_{\mathcal{E}_i}$. Then we have*

$$T_i \circ T_j = T_j \circ T_i \quad \text{if} \quad \mathcal{E}_i \cdot \mathcal{E}_j = 0, \tag{1.16}$$
$$T_i \circ T_j \circ T_i = T_j \circ T_i \circ T_j \quad \text{if} \quad \mathcal{E}_i \cdot \mathcal{E}_j = 1. \tag{1.17}$$

Proof. The proof of [Huy06], Proposition 8.22 works for this more general case. □

We will see in the next section 1.3.3 that the relation $T_i^2 = 1$ for all $i \in \{1, \ldots, m\}$ can be added when we descend to cohomology, so that we end up with a Coxeter system $\{T_1, \ldots, T_m\}$. We will use this fact sections 3.2 and 3.3.

1.3.3 Action on cohomology

For this section we assume $K = \mathbb{C}$. In this part we will work with the cohomology of X with coefficients in $\mathbb{C}$ for convenience, but most of the theory can be made with rational cohomology.

Definition 1.3.12. We define the *Grothendieck group* of X, denoted $K_0(X)$, as the free group generated by coherent sheaves $F \in \mathbf{Coh}(X)$ subject to the relation

$$[F] = [E] + [G]$$

whenever there exists an exact sequence of coherent sheaves

$$0 \to E \to F \to G \to 0.$$

In particular, we have $[E] + [G] = [E \oplus G]$ for any coherent sheaves E, G. Since we assumed X smooth, every coherent sheaf admits a finite locally free resolution, so that any element $E \in K_0(X)$ can be written as $E = \sum_k a_k [E_k]$ with E_k a locally free sheaf.

Moreover, $K_0(X)$ admits a ring structure by setting

$$[E] \cdot [F] = [E \otimes F],$$

with neutral element $[\mathcal{O}_X]$.

We consider the map

$$\begin{array}{rccl} [-]: & \mathrm{D}^b(X) & \longrightarrow & \mathrm{K}(X) \\ & F^\bullet & \longmapsto & \sum(-1)^i [F^i] = \sum(-1)^i [\mathcal{H}^i(F^\bullet)]. \end{array} \tag{1.18}$$

Note that $[F^\bullet[k]] = (-1)^k [F^\bullet]$, and $F^\bullet \simeq G^\bullet$ in $\mathrm{D}^b(X)$ implies that $[F^\bullet] = [G^\bullet]$.

Some notations:

We can extend the characteristic classes of coherent sheaves to objects in $\mathrm{D}^b(X)$ using 1.18. For instance, for $F^\bullet \in \mathrm{D}^b(X)$ we can define $\mathrm{ch}(F^\bullet) := \mathrm{ch}([F^\bullet])$. As such, we will omit the bracket $[-]$ to simplify the notations.

In order to obtain a well-behaved map from the derived category to the Betti cohomology of X, we need to twist its Chern character by the Todd class, in order to take Grothendieck-Riemann-Roch formula in account.

Definition 1.3.13. We define the *Mukai vector* $v(F)$ of an object $F \in \mathrm{D}^b(X)$ as the cohomology class

$$v(F) := \mathrm{ch}(F) \cdot \sqrt{\mathrm{td}(X)}.$$

Now let Y be another smooth projective variety over $\mathbb{C}$. Let $\mathcal{P} \in \mathrm{D}^b(X \times Y)$ be an object and consider $\phi_{\mathcal{P}}$ the associated FM transform.

Definition 1.3.14. The *cohomological Fourier-Mukai transform* $\phi_{\mathcal{P}}^H$ associated to $\phi_{\mathcal{P}}$ is the $\mathbb{C}$-linear morphism

$$\begin{array}{rccc} \phi_{\mathcal{P}}^H : & H^*(X,\mathbb{C}) & \longrightarrow & H^*(Y,\mathbb{C}) \\ & \alpha & \longmapsto & p_*(q^*(\alpha) \cdot v(\mathcal{P})) \end{array}$$

Any equivalence $\phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is a FM transform. In particular, it induces a cohomological morphism which we denote ϕ^H. The cohomological FM transform need not respect the grading nor the ring structure of the cohomology spaces, though it is well behaved for many reasons that we gather here.

1. The diagram

$$\begin{array}{ccc} \mathrm{D}^b(X) & \xrightarrow{\phi_{\mathcal{P}}} & \mathrm{D}^b(Y) \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v} \\ H^*(X,\mathbb{C}) & \xrightarrow{\phi_{\mathcal{P}}^H} & H^*(Y,\mathbb{C}) \end{array}$$

   commutes.

2. The morphism $\phi_{\mathcal{P}}^H$ respect the parity of classes, that is $\phi_{\mathcal{P}}^H(H^{even}(X,\mathbb{C})) \subset H^{even}(Y,\mathbb{C})$ and $\phi_{\mathcal{P}}^H(H^{odd}(X,\mathbb{C})) \subset H^{odd}(Y,\mathbb{C})$.

3. If Z is a third smooth projective variety and $\phi_{\mathcal{Q}} : \mathrm{D}^b(Y) \to \mathrm{D}^b(Z)$ is a FM transform, then

$$(\phi_{\mathcal{Q}} \circ \phi_{\mathcal{P}})^H \simeq \phi_{\mathcal{Q}}^H \circ \phi_{\mathcal{P}}^H. \tag{1.19}$$

4. If $\phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ is an equivalence, then $\phi^H : H^*(X,\mathbb{C}) \xrightarrow{\sim} H^*(Y,\mathbb{C})$ is a $\mathbb{C}$-linear isomorphism.

5. If $\phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ is an equivalence, then ϕ^H induces an isomorphism

$$\bigoplus_{p-q=k} H^{p,q}(X) \simeq \bigoplus_{p-q=k} H^{p,q}(Y) \tag{1.20}$$

   for all $k \in \{-\dim X, \cdots, \dim X\}$.

Remark 1.3.15. From 1.19 we deduce that $(\mathrm{Id}_{\mathrm{D}^b(X)})^H = \mathrm{Id}_{H^*(X,\mathbb{C})}$, and for an equivalence $\phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ we have $(\phi^H)^{-1} = (\phi^{-1})^H$. Thus we obtain a group morphism

$$\mathrm{Aut}(\mathrm{D}^b(X)) \to \mathrm{GL}(H^*(X,\mathbb{C}))$$

This morphism is never injective (see Example 1.3.19, 1) and need not be surjective (see for instance Remark 1.3.22).

Finally, we would like to define a pairing on $H^*(X,\mathbb{C})$ and $H^*(Y,\mathbb{C})$ such that any equivalence $\phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ would lead to an *isometry* $\phi^H : H^*(X,\mathbb{C}) \xrightarrow{\sim} H^*(Y,\mathbb{C})$. To do so, we introduce the following notation. For $v \in H^*(X,\mathbb{C})$ with homogeneous components $v = \sum_j v_j$, we set

$$v^\vee := \sum_j \sqrt{-1}^j v_j.$$

Definition 1.3.16. Let $v, w \in H^*(X, \mathbb{C})$. We define the *Mukai pairing* on $H^*(X, \mathbb{C})$ as the quadratic form

$$\langle v, w \rangle := \int_X \exp(c_1(X)/2) \cdot v^\vee \cdot w. \tag{1.21}$$

For $F, G \in \mathrm{D}^b(X)$, we obtain

$$\chi(F, G) = \langle v(F), v(G) \rangle.$$

Proposition 1.3.17. *Let $\phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ be an equivalence. Then the induced cohomological morphism $\phi^H : H^*(X, \mathbb{C}) \xrightarrow{\sim} H^*(Y, \mathbb{C})$ is an isometry with respect to the Mukai pairing on X and Y, that is*

$$\langle v, w \rangle_X = \langle \phi^H(v), \phi^H(w) \rangle_Y.$$

Remark 1.3.18. Note that this Mukai pairing is the *opposite* of the Mukai pairing used on K3 surface section 1.1.3.

Some examples

We can illustrates the previous section by computing the action on cohomology induced by the standard autoequivalences and spherical twists defined before.

Example 1.3.19. 1. The shift functor $[1] : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ acts by multiplication by (-1) on $H^*(X, \mathbb{C})$ (in fact, this fact is already a consequence of the passage through the Grothendieck group $\mathrm{K}(X)$ between the derived category and the cohomology space). From (1.19) we obtain that $[k]$, $k \in \mathbb{Z}$, acts by multiplication by $(-1)^k$ on $H^*(X, \mathbb{C})$. In particular, $[2k]$ is an example of nontrivial autoequivalence (for $k \neq 0$) which acts trivially in cohomology.

2. If $L \in \mathrm{Pic}(X)$ is a line bundle, then $(- \otimes L)$ acts on $H^*(X, \mathbb{C})$ by

$$v \mapsto v \cdot \exp(c_1(L)).$$

3. If $f : X \to Y$ is a morphism, then $(f_*)^H$ coincide with the cohomological pushforward $f_* : H^*(X, \mathbb{C}) \to H^*(Y, \mathbb{C})$, and similarly for f^*.

4. Let $\mathcal{E} \in \mathrm{D}^b(X)$ be a spherical object and consider the spherical twist $T_{\mathcal{E}}$. Then the cohomological action on $H^*(X, \mathbb{C})$ is given by

$$v \mapsto v - \langle v(\mathcal{E}), v \rangle v(\mathcal{E}).$$

In particular, $T^H_{\mathcal{E}}(v(\mathcal{E})) = \begin{cases} -v(\mathcal{E}) & \text{if } \dim X \equiv_2 0 \\ v(\mathcal{E}) & \text{if } \dim X \equiv_2 1 \end{cases}$. Moreover, when $\dim X$ is even $(T^H_{\mathcal{E}})^2 = \mathrm{Id}_{H^*(X,\mathbb{C})}$, hence $T^2_{\mathcal{E}}$ gives another example of nontrivial autoequivalence which acts trivially in cohomology.

Remark 1.3.20. For computations, it is useful to have a description of the matrices of the cohomological FM morphisms with respect to some fixed basis.

Endow $H^*(X, \mathbb{C})$ with a basis B_X which is the concatenation of basis of each $H^k(X, \mathbb{C})$ for $k = 0, \ldots, 2 \dim X$ (and similarly for $H^*(Y, \mathbb{C})$). In other words, the elements of the basis are homogeneous with increasing degrees.

- The matrix of $(- \otimes L)^H$ with respect to B_X is lower triangular. The restricted map $H^k(X, \mathbb{C}) \to H^k(X, \mathbb{C})$ is the multiplication by $\mathrm{rk}(L) = 1$, hence the corresponding block is the identity matrix:

$$\begin{pmatrix} \mathrm{Id} & & 0 \\ & \ddots & \\ * & & \mathrm{Id} \end{pmatrix}$$

- The matrix of f^* with respect to B_X and B_Y is diagonal by block, each block corresponding to the restricted map $H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$:

$$\begin{pmatrix} f_0^* & & 0 \\ & \ddots & \\ 0 & & f_{2\dim X}^* \end{pmatrix}$$

- For a spherical object $\mathcal{E} \in \mathrm{D}^b(X)$, it is conveniant to change the basis: fill the set $\{v(\mathcal{E})\}$ into a basis $\{e_1, \ldots, e_m, v(\mathcal{E})\}$ of $H^*(X, \mathbb{C})$. Then the matrix of $T_{\mathcal{E}}^H$ with respect to this basis is the identity matrix except for its last line:

$$\left(\begin{array}{ccc|c} & \mathrm{Id} & & 0 \\ \hline -\langle v(\mathcal{E}), e_1\rangle, & \ldots, & -\langle v(\mathcal{E}), e_m\rangle & 1 - \langle v(\mathcal{E}), v(\mathcal{E})\rangle \end{array}\right)$$

For K3 surfaces

For K3 surfaces, the Todd class is integral, which allows to work over $\mathbb{Z}$. Let us explain here how it works in this case.

Let S be a projective K3 surface. Recall the definition of the Hodge structure on the Mukai lattice $\widetilde{H}(S, \mathbb{Z})$ in section 1.1.3.

Proposition 1.3.21 (Mukai)**.** *If $\phi : \mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S')$ is an equivalence between the derived categories of two K3 surfaces, then the cohomological map induces a Hodge isometry*

$$\phi^H : \widetilde{H}(S, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(S', \mathbb{Z}).$$

Remark 1.3.22. There exists, under some assumptions, converses of this statement.

1. First, $\mathrm{D}^b(S)$ and $\mathrm{D}^b(S')$ are equivalent if and only if there exists a Hodge isometry $\widetilde{H}(S, \mathbb{Z}) \simeq \widetilde{H}(S', \mathbb{Z})$. In fact, two cases can appear in this situation. Given an isometry

$$\varphi : \widetilde{H}(S, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(S', \mathbb{Z}),$$

either φ induces an isometry $H^2(S, \mathbb{Z}) \simeq H^2(S', \mathbb{Z})$ and in this case $S \simeq S'$ by Global Torelli Theorem 1.1.12, or one can prove that S is isomorphic to a fine moduli space of sheaves M on S' (in fact, it is a moduli space of *μ-stable vector bundles* on S'). In the latter case, the universal family $\mathcal{E}$ on $S \times S'$ (identifying M and S), as a kernel, induces a derived equivalence

$$\mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S').$$

2. Moreover, any Hodge isometry $\varphi : \widetilde{H}(S, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(S', \mathbb{Z})$ *which preserves the natural orientation of the positive directions* is induced by an equivalence, that is there is an equivalence $\phi : \mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S')$ such that $\phi^H = \varphi$. The natural orientation of the positive directions is given by the arbitrary choice of the orientation for the two-dimensional subspace of $\widetilde{H}^{1,1}(S, \mathbb{Z})$ defined by the basis $\{(1, 0, -H^2/2), (0, H, 0)\}$, for $H \in \mathrm{NS}(S)$ an ample class. See [Huy06], Corollary 10.3.

Remark 1.3.23 ([HS05]). Proposition 1.3.21 also holds in the twisted case. Namely, given two K3 surfaces S, S' and $\alpha \in \mathrm{Br}(S)$, $\beta \in \mathrm{Br}(S')$, any equivalence

$$\phi : \mathrm{D}^b(S, \alpha) \xrightarrow{\sim} \mathrm{D}^b(S', \beta)$$

between the derived categories of *twisted* sheaves on S and S' induces an isometry

$$\phi^H : \widetilde{H}(S, \alpha, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(S', \beta, \mathbb{Z})$$

with respect to their twisted Mukai lattice, see Definition 1.1.14.

1.4 Bridgeland stability conditions

This last section aims to recall the general definition of stability conditions and to gather results for the particular case of K3 surface. The theory is originally due to Bridgeland [Bri07] [Bri08], and we refer to [MS17] for a survey on the topic.

1.4.1 Stability conditions on a triangulated category

General definitions and properties

We first start with the general definition of stability conditions. Let $\mathcal{D}$ be a triangulated category, and denote $[1] : \mathcal{D} \to \mathcal{D}$ its shift functor. Let $K(\mathcal{D})$ be the Grothendieck group of $\mathcal{D}$, i.e. the group generated by all objects of $\mathcal{D}$ subject to the relation

$$[F] = [E] + [G]$$

whenever a distinguished triangle $E \to F \to G \to E[1]$ exists. When $\mathcal{D} = \mathrm{D}^b(X)$ for X a smooth projective variety over K, we have $K(\mathrm{D}^b(X)) = K_0(X)$ (see section 1.3.3). In particular, $[F] = \sum_i (-1)^i \mathcal{H}^i(F)$. We will omit the bracket $[-]$ to simplify the notations.

Definition 1.4.1. A *Bridgeland stability condition* on $\mathcal{D}$ is a pair $\sigma = (\mathcal{P}, Z)$ where:

- $\mathcal{P}$ is a slicing of $\mathcal{D}$, that is $\mathcal{P}$ is a collection of subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for all $\phi \in \mathbb{R}$ such that
  1. $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$,
  2. for $\phi_1 > \phi_2$ and $A_1 \in \mathcal{P}(\phi_1), A_2 \in \mathcal{P}(\phi_2)$ we have $\mathrm{Hom}(A_1, A_2) = 0$,
  3. for all $E \in \mathcal{D}$ there exist real numbers $\phi_1 > \cdots > \phi_m$, objects $E_i \in \mathcal{D}$ for $i = 1, \ldots, m$ and a collection of triangles

     $$\begin{array}{ccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_{m-1} & \longrightarrow & E_m = E \\ & \nwarrow & \downarrow & & & \nwarrow & \downarrow & \nwarrow & \downarrow \\ & & A_1 & & & & A_{m-1} & & A_m \end{array}$$

     with $A_i \in \mathcal{P}(\phi_i)$.
- $Z : \Lambda \to \mathbb{C}$ is an additive morphism, called *central charge*, such that for all nonzero object $E \in \mathcal{D}$, we have

  $$Z(E) \in \mathbb{R}_{>0} e^{i\pi\phi},$$

The collection of triangles in condition 3 is called the *Harder-Narasimhan filtration* (HN filtration for short) of E. The objects $A_i, i = 1, \dots, m$ are called the *HN factors* of E.

We call objects in $\mathcal{P}(\phi)$ *semistable* of phase ϕ. The simple objects in $\mathcal{P}(\phi)$ are called *stable*. Often we will write $\mathcal{P}([\phi_a, \phi_b])$ for the extension closure in $\mathcal{D}$ of the categories $\mathcal{P}(\phi)$ for $\phi \in [\phi_a, \phi_b]$.

When $\mathcal{D}$ is linear over a field K, we can consider the *numerical* Grothendieck group $\mathcal{N}(\mathcal{D})$ defined as

$$\mathcal{N}(\mathcal{D}) = K(\mathcal{D})/K(\mathcal{D})^{\perp}$$

where the orthogonal is taken with respect to the bilinear form χ on $K(\mathcal{D})$ given by

$$\chi : (E, F) \mapsto \sum_i (-1)^i \dim_K \operatorname{Hom}(E, F[i]).$$

We say that $\mathcal{D}$ is *numerically finite* if $\mathcal{N}(\mathcal{D})$ has finite rank. A stability condition $\sigma = (\mathcal{P}, Z)$ for which the central charge Z factors through $\mathcal{N}(\mathcal{D})$ is called *numerical*.

Definition 1.4.2. A numerical stability condition $\sigma = (\mathcal{P}, Z)$ is said to be *full* if it satisfies the support property:

- For any given norm $\|\cdot\|$ on $\mathcal{N}(\mathcal{D})_{\mathbb{R}}$, there exists a constant $C > 0$ such that for all semistable object $F \in \mathcal{D}$,
$$C|Z(F)| \geq \|F\|.$$

The set of full numerical stability conditions is denoted $\operatorname{Stab}(\mathcal{D})$.

For now one, we will always assume that the stability conditions we consider are numerical and full.

The set $\operatorname{Stab}(\mathcal{D})$ can be endowed with a natural topology. It is the coarsest topology such, that for any $F \in \mathcal{D}$, the maps $(\mathcal{P}, Z) \mapsto Z \in \operatorname{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$, $(\mathcal{P}, Z) \mapsto \phi_m$, $(\mathcal{P}, Z) \mapsto \phi_1$ are continuous (with the linear topology on $\operatorname{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$), where we denote ϕ_1 and ϕ_m the extremal phases of the HN factors of F (see Definition 1.4.1).

More precisely, the topology is induced by the generalized metric

$$d(\mathcal{P}, \mathcal{Q}) = \inf\{\epsilon \in \mathbb{Z}_{\geq 0} \parallel \mathcal{Q}(\phi) \subset \mathcal{P}([\phi - \epsilon, \phi + \epsilon]) \text{ for all } \phi \in \mathbb{R}\},$$

for $\mathcal{P}, \mathcal{Q}$ slicings of $\mathcal{D}$, and we consider the product topology on $\{\text{Slicings of } \mathcal{D}\} \times \operatorname{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$.

Theorem 1.4.3 ([Bri07]). *Let $\mathcal{D}$ be a numerically finite triangulated category. The map*

$$(\mathcal{P}, Z) \mapsto Z$$

gives a local isomorphism between $\operatorname{Stab}(\mathcal{D})$ *and* $\operatorname{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$. *In particular,* $\operatorname{Stab}(\mathcal{D})$ *is a complex manifold of finite dimension* $\operatorname{rk} \mathcal{N}(\mathcal{D})$.

Nice proofs of this result can also be found in [Bay11] and[Bay19].

Wall and chamber decomposition

A natural question to ask is how vary the set of semistable objects when one changes continuously the stability condition in $\operatorname{Stab}(\mathcal{D})$. The following theorem gives a precise answer, see for instance [BM11], Proposition 3.3.

Theorem 1.4.4. *Let $v \in \mathcal{N}(\mathcal{D})$ be a fixed primitive vector and consider an arbitrary set $S \subset \mathcal{D}$ of objects of class v. Consider* $\operatorname{Stab}^* \subset \operatorname{Stab}(\mathcal{D})$ *a connected component of stability conditions. Then* Stab^* *admits a wall and chamber structure, that is there exists a locally finite family W_w, $w \in \mathcal{N}(\mathcal{D})$, of real codimension* 1 *submanifolds with boundaries , called walls, with the following properties.*

1. *For every stability condition $(\mathcal{P}, Z) \in W_w$ there exists a phase ϕ and a non trivial inclusion $E_w \hookrightarrow F_v$ with $[E_w] = w$ and $[F_v] = v$ in $\mathcal{P}(\phi)$, for some $F_v \in S$.*

2. *If $C \subset \mathrm{Stab}^*$ is a connected component of the complement of $\bigcup_{w \in \mathcal{N}(\mathcal{D})} W_w$ and $\sigma, \tau \in C$ are two stability conditions, then an object $F_v \in S$ is σ-stable if and only if it is τ-stable.*

A component $C \subset \mathrm{Stab}^*$ as in property 2 is called a chamber.

Define V_w as the subset of stability conditions for which it exists an inclusion as in property 1. Then the walls W_v are defined as the codimension one components of V_w.

We call a stability condition v-*generic* if it does not lie on a wall with respect to v.

Stability functions and geometric stability conditions

Let $\mathcal{D}$ be a numerically finite triangulated category over a field K. The next definitions and results give tools to construct explicit stability conditions, see section 1.4.2.

Consider $\mathcal{A} \subset \mathcal{D}$ the *heart of a bounded t-structure*. In otherword, $\mathcal{A}$ is an full abelian subcategory of $\mathcal{D}$ such that

1. for all integers $i < j$ and $A, B \in \mathcal{A}$, we have $\mathrm{Hom}(A[j], B[i]) = 0$,

2. for any $E \in \mathcal{D}$ there exist integers $k_1 > \cdots > k_m$ and objects $E_i \in \mathcal{D}$, $A_i \in \mathcal{A}$ for $i = 1, \dots, m$ and a collection of triangles

$$\begin{array}{ccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow \cdots \longrightarrow & E_{m-1} & \longrightarrow & E_m = E \\ & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow \\ & & A_1[k_1] & & A_{m-1}[k_{m-1}] & & A_m[k_m]. \end{array}$$

In view of the condition 2 we can identify $K(\mathcal{D}) = K(\mathcal{A})$.

Definition 1.4.5. A group morphism $Z : K(\mathcal{A}) \to \mathbb{C}$ is called *stability function* if for any non-zero object $A \in \mathcal{A}$ we have

$$\Im Z(A) \geq 0, \text{ and } \Im Z(A) = 0 \Rightarrow \Re Z(A) < 0. \tag{1.22}$$

In particular, for any object $A \in \mathcal{A}$, there is a well defined notion of *phase*

$$\phi(A) = (1/\pi) \arg Z(A) \in (0, 1].$$

We see that $\Im Z(-)$ and $\Re Z(-)$ are additive on exact sequence in $\mathcal{A}$, and condition 1.22 tells us that $\Im Z(-)$ and $-\Re Z(-)$ act exactly as the *rank* and *degree* of sheaves.

We will say that an object $F \in \mathcal{A}$ is *(semi)stable* if for all proper non-zero subobject $E \subset F$ the inequality

$$\mu_Z(E) := \frac{-\Re Z(E)}{\Im Z(E)} < (\leq) \frac{-\Re Z(F)}{\Im Z(F)} = \mu_Z(F)$$

holds. This is equivalent to $\phi(E) < (\leq) \phi(F)$. The value $\mu_Z(F)$ is called the (Z)-*slope* of F, and we set $\mu_Z(F) = +\infty$ whenever $\Im Z(F) = 0$.

Proposition 1.4.6. *The data of a stability condition $\sigma = (\mathcal{P}, Z)$ is equivalent to the data $(\mathcal{A}, Z)$ of the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$ and a stability function $Z : K(\mathcal{A}) \to \mathbb{C}$ on $K(\mathcal{A})$ such that any object $F \in \mathcal{A}$ admits a Harder-Narasimhan filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = F$$

whose factors $A_i = F_i/F_{i-1}$ are semistable for $i = 1, \dots, m$ and $\mu_Z(A_m) < \cdots < \mu_Z(A_1)$.

Via this identification, μ_Z-(semi)stable objects are exactly the ones which lie in $\mathcal{P}(\phi)$, with $\phi \in (0,1]$. Hence can extend the notion of μ_Z-stability for any object $F \in \mathcal{D}$: F is μ_Z-(semi)stable if there is an integer $k \in \mathbb{Z}$ such that $F[k] \in \mathcal{A}$ and $F[k]$ is μ_Z-(semi)stable.

Proof of Proposition 1.4.6. If $(\mathcal{P}, Z)$ is a stability condition, then define $\mathcal{A} := \mathcal{P}(0,1]$. One can check that $\mathcal{A}$ is the heart of a bounded t-structure, and the central charge Z gives a stability function.

On the other hand, if $(\mathcal{A}, Z)$ is as in the proposition, for each $\phi \in (0,1]$ define $\mathcal{P}(\phi)$ as the full additive subcategory of $\mathcal{D}$ consisting of μ_Z-semistable objects of phase ϕ together with the zero object of $\mathcal{D}$. Then set $\mathcal{P}(\phi+1) := \mathcal{P}(\phi)[1]$ to obtain a slicing of $\mathcal{D}$. □

Example 1.4.7. Let C be a smooth projective curve over $\mathbb{C}$ and set $\mathcal{A} = \mathbf{Coh}(C)$. Then the stability function

$$Z : K_0(C) \to \mathbb{C}, Z(F) = -\deg(F) + \sqrt{-1}\,\mathrm{rk}(F)$$

permits to define a stability condition σ on $\mathrm{D}^b(C)$. The σ-(semi)stable objects are exactly the shifts of slope-(semi)stable sheaves on C.

1.4.2 Stability conditions on K3 surfaces

Geometric stability conditions

Let S be a K3 surface. In this case, we can identify the numerical Grothendieck group $\mathcal{N}(S) = \mathcal{N}(\mathrm{D}^b(S))$ with the extended Néron-Severi lattice $\Lambda := \widetilde{NS}(S) = (H^0 \oplus \mathrm{NS} \oplus H^4)(S)$ via the Mukai vector

$$v : F \mapsto \mathrm{ch}(F)\sqrt{\mathrm{td}(S)}$$

(see section 1.1.3). We denote $\langle -,- \rangle$ the Mukai pairing on Λ and $\Lambda_{\mathbb{C}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. We fix an ample divisor H on S.

Fix $\alpha, \beta \in \mathbb{R}$ two real numbers, assume $\alpha \geq 0$. For an object $F \in \mathrm{D}^b(S)$, consider the group morphism

$$\begin{array}{cccc} Z_{\alpha,\beta} : & \Lambda & \longrightarrow & \mathbb{C} \\ & v & \longmapsto & \langle \exp(\sqrt{-1}\alpha H + \beta H, v\rangle \end{array}.$$

Explicitely, we have

$$Z_{\alpha,\beta}(v_0, v_1, v_2) = \sqrt{-1}\alpha H \cdot (v_1 - H\beta v_0) - v_2 + \beta H \cdot v_1 + \frac{H^2}{2}(\alpha^2 - \beta^2)v_0. \tag{1.23}$$

Let us now construct a heart of a bounded t-structure on $\mathrm{D}^b(S)$ as follow. For any sheaf $F \in \mathbf{Coh}(S)$, consider its β-slope as

$$\mu_\beta(F) := \frac{H \cdot c_1 F}{H^2\,\mathrm{rk}\,F} - \beta,$$

with $\mu_\beta(F) = \infty$ if $\mathrm{rk}(F) = 0$. Consider the abelian full subcategory

$$\mathbf{Coh}^\beta(S) := \langle \mathbf{T}^\beta, \mathbf{F}^\beta[1] \rangle \subset \mathbf{Coh}(S),$$

where

$$\begin{array}{lll} \mathbf{T}^\beta & := \{F \in \mathbf{Coh}(S) & : \text{ all Harder-Narasimhan factors of } F \text{ satisfy } \mu_\beta(\_) > 0\} \\ \mathbf{F}^\beta & := \{F \in \mathbf{Coh}(S) & : \text{ all Harder-Narasimhan factors of } F \text{ satisfy } \mu_\beta(\_) \leq 0\}. \end{array}$$

Hence, any element $F \in \mathbf{Coh}^\beta(S)$ is such that $\mathcal{H}^i(F) = 0$ for $i \neq 0, -1$, $\mathcal{H}^0(F) \in \mathbf{T}^\beta$ and $\mathcal{H}^{-1}(F) \in \mathbf{F}^\beta$.

One can check that $\Im Z(-)$ and $\Re Z(-)$ are additive on exact sequence in $\mathbf{Coh}^\beta(S)$.

Proposition 1.4.8. *The subcategory $\mathbf{Coh}^\beta(S)$ is the heart of a bounded t-structure. Moreover, the data $\sigma_{\alpha,\beta} = (\mathbf{Coh}^\beta(S), Z_{\alpha,\beta})$ defines a stability condition on $\mathrm{D}^b(S)$ if $\Re Z_{\alpha,\beta}(\delta) > 0$ for all $\delta \in \Lambda$ with $\delta^2 = -2$, $\mathrm{rk}(\delta) > 0$ and $\mu_\beta(\delta) = 0$. In particular, the conditions hold for $\alpha^2 H^2 \geq 2$.*

Proof. In view of Proposition 1.4.6, we first need to prove that $Z := Z_{\alpha,\beta}$ is a stability function. Let $E \in \mathbf{Coh}^\beta(S)$. It is clear by definition of Z and $\mathbf{Coh}^\beta(S)$ that $\Im Z(v(E)) \geq 0$. Assume $\Im Z(v(E)) = 0$. In particular we get $\Im Z(v(\mathcal{H}^{-1}E)) = 0 = \Im Z(v(\mathcal{H}^0 E)))$. Since $v(E) = v(\mathcal{H}^0 E) - v(\mathcal{H}^{-1})$, it is enough to prove

1. $\Re Z(v(\mathcal{H}^0 E))) < 0$ whenever $\mathcal{H}^0 E \neq 0$,
2. $\Re Z(v(\mathcal{H}^{-1} E))) > 0$.

Note that if $\mathcal{H}^0 E \neq 0$, then $\Im Z(v(\mathcal{H}^0 E)) = 0$ and $\mathcal{H}^0 E \in \mathbf{T}^\beta$ implies $\mathrm{rk}(\mathcal{H}^0 E) = c_1(\mathcal{H}^0 E) = 0$, so that $\mathcal{H}^0 E$ is a sheaf with zero-dimensional support. Hence $\Re Z(v(\mathcal{H}^0 E))) = -\mathrm{ch}_2(\mathcal{H}^0 E) < 0$.

On the other hand, set $v = (v_0, v_1, v_2) := v(\mathcal{H}^{-1}E)$. By definition of $\mathbf{T}^\beta$, the condition $\Im Z(v(\mathcal{H}^{-1}E)) = 0$ implies that $\mathcal{H}^{-1}E$ is a torsionfree semistable sheaf with $H \cdot v_1 = H^2 \beta v_0$. In particular, we get $v^2 \geq -2$. If $v^2 = -2$ then the hypotheses of the proposition permit to conclude. Assume otherwise $v^2 \geq 0$. We obtain

$$\begin{aligned} & v^2 \geq 0 \\ \iff\quad & v_1^2 - 2v_0 v_2 \geq 0 \\ \iff\quad & H^2 \beta^2 v_0^2 \geq 2 v_0 v_2 \\ \iff\quad & \frac{H^2}{2} \beta^2 v_0 \geq v_2 \end{aligned}$$

This gives $\Re Z(v(\mathcal{H}^{-1}E))) > 0$.

To conclude, it remains to show that any object in $\mathbf{Coh}^\beta(S)$ admits a Harder-Narashiman filtration with semistable factors. We refer to [Bri08], sections 7 and 11, for a proof of this fact. □

It can be proved that $\sigma_{\alpha,\beta}$ depends *continuously* on $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$. In fact, it is possible to extend the previous constructions on all surfaces and to replace $\alpha H, \beta H$ by divisors class $\omega, B \in \mathrm{NS}(X)_\mathbb{R}$ with ω ample. In this case, the map

$$\mathrm{Amp}(S) \times \mathrm{NS}(S) \to \mathrm{Stab}(S), (\omega, B) \mapsto (\mathbf{Coh}^{\omega,B}(S), Z_{\omega,B})$$

is a continuous embedding (see [MS17], Theorem 6.10).

There are two groups acting naturally on the space of stability conditions:

- The universal cover $G := \widetilde{GL}_2^+(\mathbb{R})$ of $\mathrm{GL}_2^+(\mathbb{R})$ acts on the right of $\mathrm{Stab}(S)$. Any element of G can be represented by a couple (M, f) where $M \in \mathrm{GL}_2^+(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is an increasing function satisfying $f(\phi + 1) = f(\phi) + 1$ and such that $f_{|\mathbb{R}/2\mathbb{Z}} = M_{|(\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0}}$. The action is given by $(M, f) \cdot (\mathcal{P}, Z) = (\mathcal{P}_f, T^{-1} \circ Z)$ whith $\mathcal{P}_f(\phi) = \mathcal{P}(f(\phi))$. In particular, given $g \in G$ and $\sigma \in \mathrm{Stab}(S)$, the sets of semistable objects of with respect to σ and $\sigma \cdot g$ are the same but the phases have been relabelled
- The set $\mathrm{Aut}(\mathrm{D}^b(S))$ acts on the left by isometries on $\mathrm{Stab}(S)$ by defining, for $\varphi \in \mathrm{Aut}(\mathrm{D}^b(S))$ and $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(S)$, the stability condition $\varphi(\sigma) = (\varphi(\mathcal{P}), Z \circ \varphi^{-1})$ where $\varphi(\mathcal{P})(\phi) = \varphi(\mathcal{P}(\phi))$.

Definition 1.4.9. A stability condition $\sigma \in \mathrm{Stab}(S)$ is called *geometric* if, up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, it is of the form $(\mathbf{Coh}^\beta(S), Z_{\alpha,\beta})$ (in the notation of Proposition 1.4.8).
The subset of geometric stability condition is denoted $U(S) \subset \mathrm{Stab}(S)$.

The subset $U(S)$ turns out to be *open* and *connected*. Moreover, the boundary $\partial U(S) = \overline{U}(S) \setminus U(S)$ is a union of walls and can be described thanks to the following theorem.

Theorem 1.4.10 ([Bri08], Theorem 12.1)**.** *Let $\sigma = (\mathcal{P}, Z) \in \partial U(S)$ be a stability condition lying on the boundary of $U(S)$. Then exactly one of the following possibilities hold:*

(A^+) *There is a spherical vector bundle A such that the Jordan-Hölder filtration of any sheaf $\mathcal{O}_x$, $x \in S$, is of the form*

$$0 \to A^{\oplus \operatorname{rk} A} \to \mathcal{O}_x \to T_A(\mathcal{O}_x) \to 0,$$

where T_A is the spherical twist associated to A.

(A^-) *There is a spherical vector bundle A (see Definition 1.3.4) such that the Jordan-Hölder filtration of any sheaf $\mathcal{O}_x$, $x \in S$, is of the form*

$$0 \to T_A^{-1}(\mathcal{O}_x) \to \mathcal{O}_x \to A^{\oplus \operatorname{rk} A}[2] \to 0,$$

where T_A is the spherical twist associated to A.

(C_k) *There is a non-singular rational curve $C \subset S$ and an integer k such that $\mathcal{O}_x$ is stable with respect to σ for $x \notin C$ and such that the Jordan-Hölder filtration of $\mathcal{O}_x$ for $x \in C$ is*

$$0 \to \mathcal{O}_C(k+1) \to \mathcal{O}_x \to \mathcal{O}_C(k)[1] \to 0.$$

Recall that for any stability condition $\sigma \in \mathrm{Stab}(S)$, any semistable object $F \in \mathrm{D}^b(S)$ admits a *Jordan-Hölder filtration*, that is a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{l-1} \subset F_l = F$$

whose factors $A_i = F_i/F_{i-1}$, $i = 1, \dots, l$, are *stable* with same phase $\phi(A_i) = \phi(F)$. This is a consequence of the support property (see Definition 1.4.2). While the Jordan-Hölder filtration is non-unique, they all have the same lenght and the factors A_i, $i = 1, \dots, l$ are unique up to reordering (Jordan-Hölder theorem).
In the following we denote

$$\mathrm{Stab}^+(S) \subset \mathrm{Stab}(S)$$

the connected component of $\mathrm{Stab}(S)$ containing $U(S)$.

Proposition 1.4.11. *Let $\sigma \in \mathrm{Stab}^+(S)$ be a stability condition. Then, there exist $\varphi \in \mathrm{Aut}(\mathrm{D}^b(S))$ such that $\varphi(\sigma)$ lies in $\overline{U(S)}$. In particular, if σ is generic, then there exists $M \in \widetilde{GL}_2^+(\mathbb{R})$ such that $\varphi(\sigma) \cdot M$ is of the form $\sigma_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$.*

Proof. Let $\sigma \in \mathrm{Stab}^+(S)$ be a stability condition, pick $\sigma_0 \in \overline{U(S)}$ and consider a path $\gamma : [0,1] \to \mathrm{Stab}^+(S)$ such that $\gamma(0) = \sigma_0$ and $\gamma(1) = \sigma$. We can assume that γ hits only a finite number of walls, so that there is numbers $0 \le t_1 < \cdots < t_n \le 1$ such that $\gamma(t_i)$ lies on a wall and $\gamma((t_i, t_{i+1}))$ lies in a chamber for all i. We can assume $\gamma(t_1) \in \partial U(S)$.
Now we follow the proof of [Bri08] Proposition 13.2: there is an autoequivalence $\Phi \in \mathrm{Aut}(\mathrm{D}^b(S))$ such that $\Phi(\gamma(t_1))$ still lies in $\partial U(S)$, but the orientation of $\partial U(S)$ is reversed: for any $t > t_1$ close to t_1, $\Phi(\gamma(t)) \in U(S)$. Depending on the situations listed in Theorem 1.4.10, the autoequivalence Φ is either of the form T_A^2 for A a spherical vector bundle in case $(A^\pm)$, or of the form T_C for C a non-singular rational curve in case (C_k).
By induction, we obtain an autoequivalence φ such that $\varphi(\sigma_0) \in \overline{U(S)}$. Moreover the autoequivalences send walls to walls, hence if σ is generic so is $\varphi(\sigma)$. □

Moduli spaces of semistable objects

The question of existence of moduli *spaces* (rather than stacks) of semistable objects with fixed numerical class with respect to a stability condition on a variety X is very complicated. Though, for K3 surfaces, we have the following results, see [BM14a], Theorem 2.15.

Theorem 1.4.12 (Bayer, Macrì, Toda, Yoshioka)**.** *Let $v \in \Lambda$ be a primitive Mukai vector, and $\sigma \in \mathrm{Stab}^+(S)$ be a generic stability condition on S. Then there exists a coarse moduli space $\mathcal{M}_\sigma[v]$ parametrizing S-equivalent classes of σ-semistable objects of class v. It is a smooth projective hyperKähler variety.*

Moreover, either $\mathcal{M}_\sigma[v]$ is empty or $v^2 \geq -2$ and $\dim \mathcal{M}_\sigma[v] = v^2 + 2$.

Remark 1.4.13. In fact by definition of walls (see [BM11] Proposition 3.3) and by formula 1.23, one can prove, for v generic, that a stability condition $\sigma \in \mathrm{Stab}^+(S)$ lies on a wall if and only if there exist a strictly σ-semistable object. In particular the moduli spaces $\mathcal{M}_\sigma[v]$ with σ generic parametrizes the classes of σ-*stable* objects.

We know from Theorem 1.4.4 that the space of stable objects is invariant when one changes the stability condition within a chamber. The behaviour of wallcrossing is first described by the following theorem. Further studies will be recalled section 2.4.1.

Theorem 1.4.14 ([BM14a], Theorem 1.1)**.** *Let $v \in \Lambda$ be a primitive Mukai vector, and $\sigma, \tau \in \mathrm{Stab}^+(S)$ be generic stability conditions on S. Then $\mathcal{M}_\sigma[v]$ and $\mathcal{M}_\tau[v]$ are birational to each other, and coincide (up to an autoequivalence of $\mathrm{D}^b(S)$) on an open subset of codimension at least 2.*

A corollary of Proposition 1.4.11 is that if we work on the distinguished component $\mathrm{Stab}^+(S)$ it is enough to compute moduli spaces with respect to stability conditions of the form $\sigma_{\alpha,\beta}$, with $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$.

Corollary 1.4.15. *Let $v \in \Lambda$ be a primitive Mukai vector. Let $\sigma \in \mathrm{Stab}^+(S)$ be a generic stability condition. Then there exists $\varphi \in \mathrm{Aut}(\mathrm{D}^b(S))$ such that*

$$\mathcal{M}_\sigma[v] \simeq \mathcal{M}_{\sigma_{\alpha,\beta}}[\varphi^H(v)].$$

We finish with the important fact that there always exists a Gieseker chamber.

Theorem 1.4.16. *Let $v = (v_0, v_1, v_2) \in \Lambda$ be a primite Mukai vector. Assume $v_0 > 0$ or $v_0 = 0$ and v_1 is effective. Then there exists $\alpha_0 > 0$ such that for any $\alpha \geq \alpha_0$ and all $\beta < \frac{Hv_1}{H^2 v_0}$ (or β arbitrary in case $v_0 = 0$), the moduli space $\mathcal{M}_{\sigma_{\alpha,\beta}}[v]$ is isomorphic to the moduli space $\mathcal{M}_S[v]$ of Gieseker-stable sheaves on S of class v. More precisely, an object $F \in \mathrm{D}^b(S)$ with $v(F) = v$ is $\sigma_{\alpha,\beta}$-stable if and only if it is the shift of a Gieseker-stable sheaf.*

For the following proof, note that from Remark 1.4.13 and as a consequence of Proposition 1.4.18, for $\alpha \gg 0$ the stability condition $\sigma_{\alpha,\beta}$ is generic with respect to v. Hence the notions of $\sigma_{\alpha,\beta}$-stability and $\sigma_{\alpha,\beta}$-semistability coincide.

Proof of Theorem 2.4.14. Fix $\beta < \frac{H \cdot v_1}{H^2 v_0}$ (or beta arbitrary in case $v_0 = 0$). We will write $Z := Z_{\alpha,\beta}$ and $\sigma = \sigma_{\alpha,\beta}$ for simplicity. Given a class $w = (w_0, w_1, w_2)$ with $\Im Z(w) \neq 0$. We see from equation (1.23) that

$$\mu_Z(w) \xrightarrow[\alpha \to +\infty]{} \begin{cases} \dfrac{-\alpha}{2\mu_\beta(w)} & \text{if } w_0 \neq 0 \\ 0 & \text{if } w_0 = 0 \end{cases}$$

Let $F \in \mathcal{M}_\sigma[v]$. There is some integer $k \in \mathbb{Z}$ such that $F[k] \in \mathbf{Coh}^\beta(S)$, but we assumed $\beta < \dfrac{H \cdot v_1}{v_0}$ so k must be even. Hence we can assume that $F \in \mathbf{Coh}^\beta(S)$ with $v(F) = v$. Then we have $\mu_Z(v)$ either goes to 0 of $-\infty$ as α grows. On the other hand, consider the exact sequence (in $\mathbf{Coh}^\beta(S)$)

$$\mathcal{H}^{-1}F[1] \hookrightarrow F \twoheadrightarrow \mathcal{H}^0 F.$$

If $\mathcal{H}^{-1}F[1]$ is not trivial, then it is a torsion-free sheaf concentrated in degree -1 with $\mu_\beta(\mathcal{H}^{-1}F[1]) < 0$, so in particular we have $\mu_Z(\mathcal{H}^{-1}F[1]) \xrightarrow[\alpha \to +\infty]{} +\infty$, so that F cannot be σ-stable. Hence for $\alpha \gg 0$, any σ-stable object with class v is (up to a shift) a sheaf. Moreover, it is easy to see that this sheaf is Gieseker-semistable. If not, let $0 \to E \to F \to T \to 0$ be a Gieseker-destabilizing sequence in $\mathbf{Coh}(S)$. Hence $\mu_\beta(E) \geq \mu_\beta(F) > 0$ so $E \in \mathbf{T}^\beta$. Finally using the existence of HN filtration for both E and T (and since $\mathrm{Hom}(A, B) = 0$ for any Gieseker-stable sheaves with $\mu_\beta(A) > \mu_\beta(B)$) one can prove that $T \in \mathbf{T}^\beta$. We conclude that E would σ-destabilizes F in $\mathbf{Coh}^\beta(S)$, which is a contradiction.

Similarly, any Gieseker-stable sheaf F with $v(F) = v$ satisfies $\mu_\beta(F) > 0$ by assumptions and hence $F \in \mathbf{T}^\beta$. If this sheaf is not σ-semistable, then there is an exact sequence

$$E \hookrightarrow F \twoheadrightarrow T$$

in $\mathbf{Coh}^\beta(S)$. We deduce that $E \simeq \mathcal{H}^0 E$ and we obtain the exact sequences (in $\mathbf{Coh}(S)$)

$$0 \to \mathcal{H}^{-1}T \to E \to I \to 0 \quad (1.24)$$
$$0 \to I \to F \to \mathcal{H}^0 T \to 0 \quad (1.25)$$

with I the image of E in F. But $\mu_Z(E) > \mu_Z(F)$ implies, for $\alpha \gg 0$, $\mu_\beta(E) > \mu_\beta(F)$ (or $\mu_\beta(E) < 0$ if $v_0 = 0$, but this is absurd as $E \in \mathbf{T}^\beta$). But since $\mu_\beta(I) < \mu_\beta(F)$ as F is Gieseker-stable, we obtain $\mu_\beta(E) > \mu_\beta(I)$ and thus $\mu_\beta(\mathcal{H}^{-1}T) > \mu_\beta(E)$, which is absurd because $\mathcal{H}^{-1}T \in \mathbf{F}^\beta$. □

Description of walls

Denote $\mathbb{H} = \{(\beta, \alpha) \in \mathbb{R}^2 \mid \alpha > 0\}$ the open upper halfplane in $\mathbb{R}^2$. Fix $v \in \Lambda$ a primitive Mukai vector. In view of Corollary 1.4.15 and from the wall-and-chamber decomposition of Theorem 1.4.4, we must compute the walls lying in $\mathbb{H}$. To do so, we will use the very useful computations made by Maciocia in [Mac14].

In this thesis, we will focus our attention to a polarized K3 surface (S, H) with Picard rank 1 and the Mukai vector $(2, H, 3)$. For this reason, we assume for simplicity that $\mathrm{NS}(S) \simeq \mathbb{Z} \cdot H$.

We write $Z_{\alpha,\beta}$ for the central charge of a stability condition of the form $\sigma_{\alpha,\beta}$. We drop (α, β) and simply write $Z := Z_{\alpha,\beta}$ when the context is clear.

Let $v = (v_0, v_1, v_2) \in \Lambda$ be a primitive Mukai vector, and let $w := w(E) = (w_0, w_1, w_2) \in \Lambda$ be another Mukai vector. In view of Remark 1.4.13, we want to solve the equation

$$\mu_{Z_{\alpha,\beta}}(v) = \mu_{Z_{\alpha,\beta}}(w) \quad (1.26)$$

with respect to $(\beta, \alpha) \in \mathbb{H}$.

Definition 1.4.17. Given a class $0 \neq w \in \Lambda$, we define the *numerical wall generated by w* as the nonempty subset of $\mathrm{Stab}(X)$ given by

$$W(w) = \{\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(X) \mid \Re Z(v) \cdot \Im Z(w) = \Re Z(w) \cdot \Im Z(v)\}$$

In particular, any *actual* wall of Theorem 1.4.4 for which there exists an inclusion $E_w \hookrightarrow F$ with $v(E_w) = w$ lies in the numerical wall $W(w)$. We say that a point $\sigma \in W(w)$ is an *actual point* if it lies in an actual wall.

Let us focus on the case of stability condition of the form $\sigma_{\alpha,\beta}$. If a numerical wall $W(w) \subset \mathbb{H}$ is actual at a point, then it remains actual on the connected component of $W(w)$, in other words an actual wall is a subset of a numerical wall cut out by two holes (correponding to the existence of spherical objects, as in Proposition 1.4.8), see [MS17], section 6.4.

We make some additional assumptions: we assume $v_0 \neq 0$ and $\gcd(v_0, v_1) = 1$ which is enough for the purpose of this thesis.

Proposition 1.4.18. *Let $w = (w_0, w_1 H, w_2) \in \Lambda$, $w \neq 0$, and consider the associated numerical wall $W(w) \subset \mathbb{H}$. Assume $W(w)$ is not 0-dimensional. Let $(\beta, \alpha) \in W(w)$.*

1. Either $\beta = \frac{v_1}{v_0}$, $\alpha > 0$, or $\beta \neq \frac{v_1}{v_0}$ and (β, α) lies in a semicircle of center $(C, 0)$ and radius R, where

$$C = \frac{v_0 w_2 - v_2 w_0}{H^2(v_0 w_1 - v_1 w_0)} \quad \text{and} \quad R = \sqrt{\left(C - \frac{v_1}{v_0}\right)^2 - Q},$$

with $Q = \frac{\langle v, v\rangle}{H^2 v_0^2}$.

*In the first case, we call $W_v(w) := \{\beta = \frac{v_1}{v_0}\}$ the **vertical wall**, the semicircle in the second case is called a **semicircular wall**.*

2. Assume $Q \geq 0$. If $W(w)$ is a semicircular wall, then the center C satisfies either

$$C < \frac{v_1}{v_0} - \sqrt{Q} \quad \text{or} \quad \frac{v_1}{v_0} + \sqrt{Q} < C.$$

Moreover, $W(w)$ must intersect either the ray $\{\beta = \frac{v_1}{v_0} - \sqrt{Q}\}$ or the ray $\{\beta = \frac{v_1}{v_0} + \sqrt{Q}\}$ depending on its position relative to the vertical wall.

Proof. We will simply gives the main steps, for a complete proof (in a more general setting) see [Mac14], section 2.

1. If $\Im Z(w) = 0$ (i.e. $w_1 = \beta w_0$), we must have either $\Im Z(v) = 0$ or $\Re Z(w) = 0$. If $w_0 = 0$, then necessarily $w_1 = 0$, which leads to $\Re Z(w) \neq 0$. We get $\Im Z(v) = 0$, that is $\beta = \frac{v_1}{v_0}$. If $w_0 \neq 0$, then $\beta = \frac{w_1}{w_0}$. In this case, $\Re(w) \neq 0$ (otherwise we obtain that $W(w)$ is 0-dimensional), so $\Im Z(v) = 0$, that is $\beta = \frac{v_1}{v_0}$.

Now assume $\Im Z(w) \neq 0$. Then $\beta \neq \frac{v_1}{v_0}$ (otherwise $Z(v) = 0$ and $W(v)$ is 0-dimensional). Developing the equality $\mu_Z(w) = \mu_Z(v)$ gives

$$\frac{(v_1 w_2 - w_1 v_2) - \beta(v_0 w_2 - w_0 v_2) + H^2\beta^2(v_0 w_1 - w_0 v_1) + \frac{H^2}{2}(\beta^2 - \alpha^2)(v_1 w_0 - w_1 v_0)}{\alpha H^2 (v_1 - \beta v_0)(w_1 - \beta w_0)} = 0.$$

If $\mu_\beta(v) = \mu_\beta(w)$, simplifying and isolating β gives

$$\frac{v_1 w_2 - w_1 v_2}{v_0 w_2 - w_0 v_2} = \beta.$$

Using $w_1 v_0 = w_0 v_1$, we get $\beta = \dfrac{v_1}{v_0}$.

If $\mu_\beta(v) \neq \mu_\beta(w)$, we can divide by $(v_0 w_1 - v_1 w_0)$ and regrouping the terms we obtain

$$(C-\beta)^2 + \alpha^2 - R^2 = 0$$

which is what we wanted.

2. We see from the definition of the radius R that $R > 0$ exactly when $|C - \dfrac{v_1}{v_0}| > \sqrt{Q}$.

Let us prove the last statement for the case $C < \dfrac{v_1}{v_0}$, the other case is similar. Note that $R \sim_{-\infty} \dfrac{v_1}{v_0} - C$, hence $\lim_{C\to-\infty}(C+R) = \dfrac{v_1}{v_0}$. Moreover,

$$\frac{d}{dC}(C+R) = 1 - \frac{\frac{v_1}{v_0} - C}{\sqrt{(\frac{v_1}{v_0} - C)^2 - Q}} \leq 0.$$

Finally, we have $\lim_{C\to\frac{v_1}{v_0}-\sqrt{Q}}(C+R) = \dfrac{v_1}{v_0} - \sqrt{Q}$, hence

$$\frac{v_1}{v_0} - \sqrt{Q} < C + R < \frac{v_1}{v_0}.$$

Combined with the bound $C < \dfrac{v_1}{v_0} - \sqrt{Q}$, we obtain the desired result.

Remark 1.4.19. Note that a semicircular wall might cross the ray $\{\beta = \frac{v_1}{v_0} - \sqrt{Q}\}$ at a point (β, α) for which $Z_{\alpha,\beta}$ does not satisfy the hypotheses of Proposition 1.4.8. Though, for arithmetical reasons, it is sometimes possible to prove that for specific choices of (β, α) there is no root $\delta \in \Lambda$ with the properties appearing in Proposition 1.4.8, see for instance section 2.4.2.

□

Chapter 2

Study of moduli spaces of sheaves on Fano threefolds and K3 surfaces of genus 9

This chapter, we assume that all varieties and schemes are defined over $\mathbb{C}$. We start with a smooth prime Fano threefold X of genus 9 and index 1 embedded in a projective space $\mathbb{P}^{13}$, and a K3 surface $S \subset X$ obtained as a hyperplane section of X. The pullback by the closed immersion $i_{SX} : S \hookrightarrow X$ gives a map

$$res : \mathcal{M}_X(2,1,7) \to \mathcal{M}_S(2,1,7) \tag{2.1}$$

between the moduli spaces of sheaves on X and S. After recalling section 2.1 some aspect of the geometry of X and a description of $\mathcal{M}_X(2,1,7)$ following [BF13], we study the restriction map (2.1) and prove that the image of $\mathcal{M}_X(2,1,7)$ in $\mathcal{M}_S(2,1,7)$ is a Lagrangian subvariety with finitely many double points (Theorem 2.2.10).

Section 2.3, we fix S and we vary the Fano X containing it. We consider the relative moduli space $\mathcal{M}_{\mathfrak{X}/\mathcal{W}}(2,1,7)$ of sheaves over the family $\mathfrak{X}$ of Fano containing S parametrized by an open $\mathcal{W} \subset \mathbb{P}^3$. The fibrewise restriction maps (2.1) glue to give a birational map

$$\mathcal{M}_{\mathfrak{X}/\mathcal{W}}(2,1,7) \dashrightarrow \mathcal{M}_S(2,1,7).$$

Sending a sheaf of the form $i_{SX}^* F \in \mathcal{M}_S(2,1,7)$, with $F \in \mathcal{M}_S(2,1,7)$, to the class of the Fano $[X] \in \mathcal{W}$ describes a rational Lagrangian fibration

$$\mathcal{M}_S(2,1,7) \dashrightarrow \mathbb{P}^3 \tag{2.2}$$

with general fibre birational to $\mathcal{M}_X(2,1,7)$ (Corollary 2.3.8). Moreover, we construct a birational model $\mathcal{M} \dashrightarrow \mathcal{M}_S(2,1,7)$ such that the rational map (2.2) extends to a Lagrangian fibration

$$\mathcal{M} \to \mathbb{P}^3$$

with general fibre over $[X] \in \mathcal{W}$ an abelian variety obtained as a blow-down of $\mathcal{M}_X(2,1,7)$ (Theorem 2.3.9).

Finally, we study in section 2.4 the possible birational models of $\mathcal{M}_S(2,1,7)$. To do so, we compute the walls-and-chambers decomposition of the space $\mathrm{Stab}(S)$ of Bridgeland stability conditions on S, and identify the chambers corresponding to $\mathcal{M}_S(2,1,7)$ and $\mathcal{M}$. It turns out that these are the only smooth K-trivial birational models of $\mathcal{M}_S(2,1,7)$, and they are related by a flop (Theorem 2.4.1).

2.1 The prime Fano threefold X of genus 9 and the moduli space $\mathcal{M}_X(2,1,7)$

2.1.1 Geometry of the Fano threefold of genus 9

From the work works of Mukai ([Muk89], [Muk88]) any smooth prime Fano threefold of genus g, for $7 \leq g \leq 10$, can be embedded in $\mathbb{P}^{n+g-2}$ as the complete intersection of an homogeneous space X^n_{2g-2} of dimension n and a linear subspace of codimension $n-3$. Let us present with more details the the case $g = 9$, following [IR05].

First, we consider $\Sigma = LG(3,6)$, the Lagrangian Grassmannian of 3-dimensional subspaces of a 6-dimensional vector space V which are isotropic with respect to a symplectic 2-form ω. The manifold Σ embeds in $\mathbb{P}^{13} = \mathbb{P}V_{14}$ as follows. The Plücker embedding

$$G(3,6) \ni \langle u, v, w\rangle \mapsto u \wedge v \wedge w \in \mathbb{P}(\Lambda^3 V)$$

induces an embedding $\Sigma \hookrightarrow \mathbb{P}(\Lambda^3 V)$. The image is contained in the 14-dimensional subspace

$$V_{14} := \ker(c_\omega : \Lambda^3 V \to V) \subset \Lambda^3 V$$

where c_ω denotes the contraction by ω. In fact, $\Sigma = G(3,6) \cap \mathbb{P}V_{14} \subset \mathbb{P}\Lambda^3 V$.

Now we define X as a general 3-codimensional linear section of Σ, that is

$$X := \Sigma \cap \mathbb{P}V_{11}$$

for $V_{11} \subset V_{14}$ a general 11-dimensional subvector space.

The variety X is a smooth Fano threefold of genus 9, with Picard group generated by a hyperplane section, that is $\mathrm{Pic}(X) = \langle H_X\rangle$. Moreover $-K_X = H_X$.

A very general hyperplane section S of X is a smooth K3 surface of genus 9 polarized by the restriction H_S of H_X to S, with $\mathrm{Pic}(S) = \langle H_S\rangle$ thanks to the Moishezon theorem [Moi68] (see also [BN20]).

The manifold Σ is equipped with a tautological homogeneous rank 3 subbundle $\mathcal{U} \subset V \otimes \mathcal{O}_\Sigma$. The isomorphism $V \mapsto V^\vee, v \mapsto \omega(-, v)$ gives an isomorphism between $\mathcal{U}$ and the quotient bundle $(V \otimes \mathcal{O}_\Sigma)/\mathcal{U}$. As we will principally study X, we denote $\mathcal{U}$ again its restriction to X, and $\mathcal{U}_S$ its restriction to S. Hence the bundle $\mathcal{U}$ lies in the exact sequence

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_X \to \mathcal{U}^\vee \to 0 \tag{2.3}$$

and its Chern classes are $c_1(\mathcal{U}) = -1$, $c_2(\mathcal{U}) = 8$, $c_3(\mathcal{U}) = -2$. A way to obtain these values of Chern classes is to note that ω gives a global section of $\Lambda^2\mathcal{U}^\vee_{G(3,6)}$ on $G(3,6)$, and $LG(3,6)$ is the zero locus of this section. Hence, the degree of $c_i(\mathcal{U}_X)$ is $c_i(\mathcal{U}_{G(3,6)}) \cdot c_3(\Lambda^2\mathcal{U}^\vee_{G(3,6)}) \cdot H^{6-i}$. For computations of Chern classes on Grassmannians, we refer to [EH16]. Explicit computations can also be made using the package *Schubert2* of *Macaulay2* ([GS]).

HPD for X

In this section we will only consider a special case of *Homological Projective Duality* (HPD for short) introduced section 1.2.2. We follow [Kuz06]: Kuznetsov considers the embedding

$$f : \Sigma \hookrightarrow \mathbb{P}V_{14} = \mathbb{P}^{13}$$

and the embedding

$$Y \hookrightarrow \mathbb{P}V_{14}^\vee$$

where $Y \simeq \Sigma^\vee \setminus \mathbf{Z}$, and $\Sigma^\vee$ is the (classical) projective dual variety of Σ which is a quartic hypersurface singular along a subvariety $\mathbf{Z} \subset \mathbb{P}V_{14}^\vee$ of codimension 3.

Then he proves (incomplete) HPD for Σ and Y over $\mathbb{P}V_{14}^\vee \setminus \mathbf{Z}$, see section 1.2.2. By Proposition 1.2.7, we obtain the following semiorthogonal decompositions, denoting $\Sigma_j := \Sigma \cap L$, resp. $Y_j = Y \cap L^\perp$, for an admissible linear subspace $L \subset V$ of dimension j:

- $\mathrm{D}^b(\Sigma) = \langle \mathcal{O}_\Sigma(1), \mathcal{U}_\Sigma^*(1), \mathcal{O}_\Sigma(2), \mathcal{U}_\Sigma^*(2), \mathcal{O}_\Sigma(3), \mathcal{U}_\Sigma^*(3), \mathcal{O}_\Sigma(4), \mathcal{U}_\Sigma^*(4)\rangle$
- $\mathrm{D}^b(\Sigma_{11}) = \langle \mathrm{D}^b(Y_{11}), \mathcal{O}_{\Sigma_{11}}(1), \mathcal{U}_{\Sigma_{11}}^*(1)\rangle$
- $\mathrm{D}^b(\Sigma_{10}) = \mathrm{D}^b(Y_{10}, \mathcal{A}_Y)$.

In these cases, Σ_{11} is the Fano threefold X of genus 9 defined above, Y_{11} is a plane quartic curve, and Σ_{10} and Y_{10} are K3 surfaces of degree 16 and 4 respectively.

Beware that $\mathrm{D}^b(Y_{10}, \mathcal{A}_Y)$ is the derived category of $\mathcal{A}_Y$-module with respect to a sheaf of Azumaya algebra $\mathcal{A}_Y$ over Y. For more details about Azumaya varieties, we refer to [Kuz06], Appendix D. Equivalently, we can use the equivalence $\mathrm{D}^b(Y_{10}, \mathcal{A}_Y) \simeq \mathrm{D}^b(Y_{10}, \alpha)$ with the derived category of coherent sheaves *twisted* by a Brauer class $\alpha \in \mathrm{Br}(Y_{10})$ provided by [Č00].

Notations for the HPD

We introduce some notation for the next parts.

- We denote $X = \Sigma_{11}$ the Fano threefold, $\Gamma := Y_{11}$ the plane quartic curve, $S := \Sigma_{10}$ and $S' := Y_{10}$ the K3 surfaces. Note that S, resp. Γ, is a hyperplane section of X, resp. S'.
- We denote $\mathcal{E}$ the object in $\mathrm{D}^b(Q(\Sigma, Y), \mathcal{A})$ which gives the HP-duality and by $\mathcal{E}_{11}$, resp. $\mathcal{E}_{10}$ its restriction to $X \times \Gamma$, resp. $S \times S'$.
- We denote by

  $$\begin{aligned} \phi_{11} &: \mathrm{D}^b(\Gamma) \to \mathrm{D}^b(X) \\ \phi_{10} &: \mathrm{D}^b(S', \alpha) \xrightarrow{\sim} \mathrm{D}^b(S). \end{aligned}$$

  the Fourier-Mukai functors with kernel $\mathcal{E}_{11}$ and $\mathcal{E}_{10}$ respectively obtained by HP-duality. Note that ϕ_{11} is fully faithful and ϕ_{10} is an equivalence.

We need the following lemma which relates the different "paths" between the derived categories $\mathrm{D}^b(X)$ and $\mathrm{D}^b(S)$, as it reads on diagram

$$\begin{array}{ccc} \mathrm{D}^b(\Gamma) & \xrightarrow{\phi_{11}} & \mathrm{D}^b(X) \\ {\scriptstyle (Ri_{\Gamma S'})_*}\downarrow & & \downarrow{\scriptstyle Li_{SX}^*} \\ \mathrm{D}^b(S', \mathcal{A}) & \xrightarrow{\phi_{10}} & \mathrm{D}^b(S) \end{array}$$

Lemma 2.1.1. *We have an isomorphism of functors*

$$Li_{SX}^* \circ \phi_{11} \simeq \phi_{10} \circ (Ri_{\Gamma S'})_*$$

from $\mathrm{D}^b(\Gamma)$ *to* $\mathrm{D}^b(S)$.

Proof. It is a consequence of the adaptation of the next lemma ([Huy06], Exercise 5.12) to the case of twisted sheaves. Let us make a proof in the untwisted case. The reader can verify that each step of the proof lift to the twisted case: the key point for this is that all equivalence between twisted derived categories are of Fourier-Mukai type ([CS07]).

Lemma 2.1.2. *Let W, X, Y, Z be smooth projective variety, let $g : W \to X$, $f : Z \to Y$ be morphisms and $\mathcal{E} \in \mathrm{D}^b(X \times Y)$ be an object. Denote all derived functors with the underived notation. Then*

1. $\phi_{\mathcal{E}} \circ g_* \simeq \phi_{\mathcal{F}}$ *where* $\mathcal{F} \simeq (g \times \mathrm{Id}_Y)^* \mathcal{E} \in \mathrm{D}^b(W \times Y)$
2. $f^* \circ \phi_{\mathcal{E}} \simeq \phi_{\mathcal{G}}$ *where* $\mathcal{G} \simeq (\mathrm{Id}_X \times f)^* \mathcal{E} \in \mathrm{D}^b(X \times Z)$.

Proof. We only prove (1), the proof for (2) is similar. Recall that $g_* \simeq \phi_{\mathcal{O}_{\Gamma_g}}$. Denote $i : W \simeq \Gamma_g \to W \times X$ the inclusion of the graph of g. From [Huy06], Proposition 5.10, using the same notation, the kernel of $\phi_{\mathcal{E}} \circ g_*$ is

$$\begin{aligned} \mathcal{F} &= (\pi_{WY})_* \left(\pi_{WX}^* i_* \mathcal{O}_W \otimes \pi_{XY}^* \mathcal{E}\right) \\ &= (\pi_{WY})_* \left((i \times \mathrm{Id}_Y)_* \mathcal{O}_{W \times Y} \otimes \pi_{XY}^* \mathcal{E}\right) \\ &= (\pi_{WY})_* (i \times \mathrm{Id}_Y)_* (i \times \mathrm{Id}_Y)^* \pi_{XY}^* \mathcal{E}) \\ &= (g \times \mathrm{Id}_Y)^* \mathcal{E} \end{aligned}$$

where $\pi_{XY}, \pi_{WX}, \pi_{XY}$ are the projection from $W \times X \times Y$ to $X \times Y$, $W \times X$ and $W \times Y$ respectively. The equalities follow from projection formula and base change with respect to the commutative diagram

$$\begin{array}{ccc} W \times Y & \xhookrightarrow{i \times \mathrm{Id}_Y} & W \times X \times Y \\ {\scriptstyle p_W}\downarrow & & \downarrow{\scriptstyle \pi_{WX}} \\ W & \xhookrightarrow{\quad i \quad} & W \times X \end{array}$$

□

In our case, we get $Li_{SX}^* \circ \phi_{11} \simeq \phi_{\mathcal{F}}$ with $\mathcal{F} \simeq L(\mathrm{Id}_\Gamma \times Li_{SX})^* \mathcal{E}_{11}$ and $\phi_{10} \circ (Ri_{\Gamma S'})_* \simeq \phi_{\mathcal{G}}$ with $\mathcal{G} \simeq L(i_{\Gamma S'} \times \mathrm{Id}_S)^*(\mathcal{E}_{10})$. But by definition of $\mathcal{E}_{10}$ and $\mathcal{E}_{11}$, $\mathcal{F}$ and $\mathcal{G}$ are both isomorphic to $Lj^*\mathcal{E}$ with $j : \Gamma \times S \to Q(Y, \Sigma)$.

□

2.1.2 Description of the moduli spaces $\mathcal{M}_X(2, 1, c_2)$

We use the notation of section 2.1.1. From 1.1.15, we have

$$\mathcal{M}_X(2, 1, d) = \emptyset \text{ for } c_2 < 6.$$

In this section we summarize the results of Brambilla and Faenzi in [BF11], [BF13].

1. The moduli space $\mathcal{M}_X(2, 1, 6)$ is fine and isomorphic to the HP-dual curve Γ of X (see section 2.1.1). The universal sheaf is locally free and isomorphic, up to a twist by a line bundle on Γ, to the sheaf $\mathcal{E}_{11}$ given by HP-duality.

2. Recall we have a (fully faithful) functor

$$\phi_{11} : \mathrm{D}^b(\Gamma) \to \mathrm{D}^b(X),$$

and we denote $\phi_{11}^! : \mathrm{D}^b(X) \to \mathrm{D}^b(\Gamma)$ its right adjoint. When $F \in \mathcal{M}_X(2, 1, 7)$ is a stable sheaf, the object $\phi_{11}^!$ turns out to be a vector bundle on Γ of rank 1 and degree 2, that is $\phi_{11}^!(F) \in \mathrm{Pic}^2(\Gamma)$. Moreover, when F is not globally generated, its locus of non global generation is a line $L_F \subset X$, and in this case $\phi_{11}^!(F)$ lies in the Brill-Noether locus $W = \{\mathcal{L} \in \mathrm{Pic}^2(\Gamma) \mid h^0(\Gamma, \mathcal{V} \otimes \mathcal{L}) \geq 2\}$, where $\mathcal{V}$ is a specific rank 2 vector bundle on Γ.

In fact, we have $\phi^!_{11}(F) = \phi^!_{11}(\mathcal{O}_L(-1))$ and the map

$$\psi : L \mapsto \phi^!_{11}(\mathcal{O}_L(-1))$$

gives an isomorphism

$$\mathcal{H}^0_1(X) \xrightarrow{\sim} W,$$

where $\mathcal{H}^0_1(X)$ denotes the Hilbert scheme of lines in X, which is a reduced curve.

We have a biregular description of $\mathcal{M}_X(2,1,7)$.

Theorem 2.1.3 ([BF13], Theorem 5.1)**.** *The mapping*

$$\begin{array}{rccc} \varphi : & \mathcal{M}_X(2,1,7) & \longrightarrow & \mathrm{Pic}^2(\Gamma) \\ & F & \longmapsto & \phi^!_{11}(F) \end{array}$$

gives an isomorphism of the moduli space $\mathcal{M}_X(2,1,7)$ to the blow-up of $\mathrm{Pic}^2(\Gamma)$ along the subvariety $W \subset \mathrm{Pic}^2(\Gamma)$. The exceptional divisor consists of the sheaves in $\mathcal{M}_X(2,1,7)$ which are not globally generated.

For X general, the curve $\mathcal{H}^0_1(X)$ is smooth, and hence so is $\mathcal{M}_X(2,1,7)$.

Finally, we will use this additional description of sheaves in $\mathcal{M}_X(2,1,7)$.

Proposition 2.1.4 ([BF13], Lemma 5.2)**.** *Let $F \in \mathcal{M}_X(2,1,7)$ be a sheaf. Then we have*

$$H^k(X,F) = 0 \quad \text{for} \quad k = 1,2, \tag{2.4}$$

$$H^k(X,F(-1)) = 0 \quad \text{for} \quad k = 0,1,2,3 \tag{2.5}$$

$$H^1(X,F(-t)) = 0 \quad \text{for} \quad t \geq 1 \tag{2.6}$$

Moreover, either F is locally free or $F^{**} \in \mathcal{M}_X(2,1,6)$ is a stable vector bundle, and there is a line $M_F \subset X$ and an exact sequence

$$0 \to F \to F^{**} \to \mathcal{O}_{M_F} \to 0.$$

Furthermore, the following statements are equivalent:

(a) the sheaf F is not globally generated,

(b) the vector space $\mathrm{Hom}(\mathcal{U}^\vee, F)$ is non-zero,

(c) Denote $I \hookrightarrow F$ the image of the natural evalutation map $I = \mathrm{Im}(ev : H^0(X,F) \otimes \mathcal{O}_X \to F)$. Then $I \in \mathcal{M}_X(2,1,8,2)$ and we have

$$0 \to I \hookrightarrow F \to \mathcal{O}_{L_F}(-1) \to 0,$$

moreover the sheaf I admits a locally free resolution

$$0 \to \mathcal{O}_X \to \mathcal{U}^\vee \to I \to 0.$$

In particular, we see that when a sheaf $F \in \mathcal{M}_X$ is not locally free (resp. globally generated), it fails to be so on a line.

3. For $c_2 = d \geq 8$, recall (Theorem 1.1.17) that there is a "good" component $M(c_2) \subset \mathcal{M}_X(2,1,c_2)$ generically smooth of dimension $2c_2 - 11$.

Theorem 2.1.5 ([BF13], Theorem 4.1)**.** *For any $d \geq 8$, the mapping*

$$\varphi : F \mapsto \phi^!_{11}(F)$$

gives a birational map from $M(d)$ to a generically smooth $(2d-11)$-dimensional component of the locus

$$\{F \in \mathcal{M}_\Gamma(d-6, d-5) \mid h^0(\Gamma, \mathcal{V} \otimes F) \geq d-6\}.$$

For now on, we will focus our attention to $\mathcal{M}_X(2,1,7)$.

2.2 Restriction of sheaves from the Fano threefold of genus 9 to a K3 surface

For this section, we will need the following technical lemmas.

Lemma 2.2.1. *Let X be a smooth integral projective variety, $S \subset X$ a smooth integral hypersurface. Let $F \in \mathbf{Coh}(X)$ be a coherent pure sheaf with $\mathrm{Supp}(F) \not\subset S$. Let $i : S \hookrightarrow X$ be the closed immersion. Then $L^k i^* F = \mathrm{Tor}_k(F, \mathcal{O}_S) = 0$ for all $k > 0$.*

Proof. Note that $L^k i^* F = 0$ if and only if $Li^* F$ is a sheaf. Since i is a closed immersion, $Ri_* = i_*$ do not need to be derived, therefore $Li^* F$ is a sheaf if and only if $i_* Li^* F$ is a sheaf. By the projection formula, we have $i_* Li^* F \simeq F \otimes^L \mathcal{O}_S$. Tensoring the exact sequence

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

by F, to prove both statements of the lemma we are reduced to show that $m : F(-S) \to F$ is injective. Recall that a sheaf is pure if and only if all its associated points have the same dimension, in particular we can work locally and assume that $\mathcal{O}_X(-S)$ is generated by a global function f vanishing on S. Hence the kernel of

$$F \xrightarrow{\times f} F$$

is a subsheaf whose support Z is stricly contained in $\mathrm{Supp}(F) \cap S$, which have dimension smaller than the dimension of $\mathrm{Supp}(F)$ by assumptions. By purity of F, Z must be empty, so m is injective. □

Now we focus our attention on the constructions of section 2.1.1.

We fix $X = \Sigma \cap \mathbb{P}V_{11}$ the Fano threefold, and we fix $S = X \cap H$ a general hyperplane section. We assume that S does not contain a line. We denote $\mathcal{M}_X := \mathcal{M}_X(2, 1, 7)$, $\mathcal{M}_S := \mathcal{M}_S(2, 1, 7)$, $\phi := \phi_{11} : \mathrm{D}^b(\Gamma) \to \mathrm{D}^b(X)$ and $\phi^! := \phi^!_{11}$ its right adjoint.

We prove that the pullback by $i_{SX} : S \hookrightarrow X$ gives a restriction morphism

$$res : \mathcal{M}_X \to \mathcal{M}_S.$$

From Maruyama's theorem [Mar80], the restriction F_S of a sheaf $F \in \mathcal{M}_X$ to S is stable for S general enough, but in our case the assumption that S does not contain a line suffices.

Proposition 2.2.2. *Let $F \in \mathcal{M}_X$ be a sheaf. Then F is μ-stable, and its restriction F_S to S is also μ-stable.*

Proof. We know that F is (Gieseker)-semistable. Let $G \subset F$ be a subsheaf of rank 1 and with first chern class $c_1(G) = aH$ such that $\mu(F) = \mu(G)$. Then $aH^3 = H^3/2$ which is impossible for $a \in \mathbb{Z}$. Hence F is μ-stable.

Consider the exact (by Lemma 2.2.1) sequence

$$0 \to F(-2) \to F(-1) \to F_S(-1) \to 0.$$

If F is locally free, by Hoppe's criterion 1.1.3 we have $H^0(X, F(-1)) = 0$, and by Proposition 2.1.4, $H^1(X, F(-2)) = 0$. Hence $H^0(S, F_S(-1)) = 0$ so F_S is μ-stable.

If F is not locally free, then by Proposition 2.1.4 F lies in an exact sequence

$$0 \to F \to E \to \mathcal{O}_L \to 0$$

with $E \in \mathcal{M}_X(2,1,6)$ stable *vector bundle* and $L \subset X$ a line. Restricting this sequence to S (using Lemma 2.2.1) we get

$$0 \to F_S \to E_S \to \mathcal{O}_Z \to 0$$

with Z a 0-dimensional subscheme. In particular, F_S is torsion free. One more time by Hoppe's criterion and since E is ACM ([BF14], Proposition 3.4), i.e. $H^k(X, E(t)) = 0$ for all t and all $0 < k < 3$, the sheaf E_S is μ-stable. Hence any destabilizing subsheaf of F_S also destabilizes E_S, which is not possible, so F_S is stable. □

2.2.1 Globally generated sheaves

Consider a sheaf $F \in \mathcal{M}_X$. By [BF13], Lemma 4.3, there is an exact sequence

$$0 \to \mathcal{U}^\vee \to \phi\phi^! F \to F \to 0.$$

As F is torsion-free, in view of Lemma 2.2.1 the restriction to S gives the exact sequence

$$0 \to \mathcal{U}_S^\vee \to i_{SX}^* \phi\phi^! F \to F_S \to 0. \tag{2.7}$$

Proposition 2.2.3. *For F globally generated (in particular, general) we have*

$$\dim \operatorname{Ext}^1(F_S, \mathcal{U}_S^\vee) = 1.$$

Proof. We need to consider two exact sequences, namely

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_X \to \mathcal{U}^\vee \to 0 \tag{2.8}$$
$$0 \to \mathcal{U}^\vee(-1) \to \mathcal{U}^\vee \to \mathcal{U}_S^\vee \to 0, \tag{2.9}$$

where V is the $\mathbb{C}$-vector space of dimension 6 defining $LG(3,6)$.

From Hirzebruch-Riemann-Roch, we can compute $\chi(F_S, \mathcal{U}_S^\vee) = \chi(S, F_S \otimes \mathcal{U}_S)$. We have $\operatorname{ch}(F_S) = (2, H_S, 1)$ and $\operatorname{ch}(\mathcal{U}_S) = (3, -H_S, 0)$, and $\operatorname{td}(S) = (1,0,2)$ as S is a K3 surface. We obtain

$$\chi(S, F_S \otimes \mathcal{U}_S) = \int (2, H_S, 1)(3, -H_S, 0)(1,0,2) = -1.$$

To obtain the result we want, we will prove $\operatorname{Ext}^2(F_S, \mathcal{U}_S^\vee) = 0 = \operatorname{Hom}(F_S, \mathcal{U}_S^\vee)$.

First, we have $\mu(F_S) = 1/2$ and $\mu(\mathcal{U}_S^\vee) = 1/3$. By stability, we get $\operatorname{Hom}(F_S, \mathcal{U}_S^\vee) = 0$.

Lemma 2.2.4. *For any $F \in \mathcal{M}_X$, we have*

$$\operatorname{Hom}_S(\mathcal{U}_S^\vee, F_S) \simeq \operatorname{Hom}_X(\mathcal{U}^\vee, F)$$

Proof. By Serre duality on X and S, the statement is equivalent to $\operatorname{Ext}^2(F_S, \mathcal{U}_S^\vee) \simeq \operatorname{Ext}^3(F, \mathcal{U}^\vee(-1))$. Apply $R\operatorname{Hom}(F, -)$ to (2.9) to get

$$\operatorname{Ext}^2(F, \mathcal{U}^\vee) \to \operatorname{Ext}^2(F_S, \mathcal{U}_S^\vee) \to \operatorname{Ext}^3(F, \mathcal{U}^\vee(-1)) \to \operatorname{Ext}^3(F, \mathcal{U}^\vee).$$

Now applying $R\operatorname{Hom}(F, -)$ to (2.8) and since $H^k(X, F(-1)) = 0\ \forall k$ (Proposition 2.1.4), we have

$$\operatorname{Ext}^k(F, \mathcal{U}^\vee) \simeq \operatorname{Ext}^{k+1}(F, \mathcal{U}) \quad \forall k.$$

We have $\operatorname{Ext}^2(F, \mathcal{U}^\vee) \simeq \operatorname{Ext}^3(F, \mathcal{U}) = \operatorname{Hom}(\mathcal{U}, F(-1)) = 0$ by stability since $\mu(F(-1)) = -1/2$ and $\mu(\mathcal{U}) = -1/3$. Moreover $\operatorname{Ext}^3(F, \mathcal{U}^\vee) \simeq \operatorname{Ext}^4(F, \mathcal{U}) = 0$. Hence we obtain $\operatorname{Ext}^2(F_S, \mathcal{U}_S^\vee) \simeq \operatorname{Ext}^3(F, \mathcal{U}^\vee(-1))$. □

Now, for F globally generated we have $\mathrm{Ext}^3(F,\mathcal{U}^\vee(-1)) \simeq \mathrm{Hom}(\mathcal{U}^\vee, F) = 0$ (Proposition 2.1.4), and hence by Lemma 2.2.4 we conclude the proof. □

From the exact sequence (2.7) we obtain the following corollary.

Corollary 2.2.5. *Let $\widetilde{X} = \Sigma \cap \mathbb{P}\widetilde{V}_{11}$, $\widetilde{V}_{11} \subset V_{14}$ be another Fano threefold constructed as in section 2.1.1. Let $\widetilde{\Gamma}$ be its associated quartic plane curve, and let $\widetilde{\phi}_{11} : \mathrm{D}^b(\widetilde{\Gamma}) \to \mathrm{D}^b(\widetilde{X})$ be the functor obtained by HPD. For $F \in \mathcal{M}_X$ and $\widetilde{F} \in \mathcal{M}_{\widetilde{X}}$ globally generated, we have*

$$F_S \simeq \widetilde{F}_S \Rightarrow i^*_{SX}\phi\phi^! F \simeq i^*_{S\widetilde{X}}\widetilde{\phi}\widetilde{\phi}^! \widetilde{F}.$$

Theorem 2.2.6. *The morphism* res *is injective on the set of globally generated sheaves (in particular, it is generically injective). Moreover, in the notation of Corollary 2.2.5, for any two globally generated sheaves $F \in \mathcal{M}_X, \widetilde{F} \in \mathcal{M}_{\widetilde{X}}$, we have*

$$F_S \simeq \widetilde{F}_S \Leftrightarrow \left(X = \widetilde{X} \text{ and } F \simeq \widetilde{F}\right).$$

Proof. Let $F \in \mathcal{M}_X$, $\widetilde{F} \in \mathcal{M}_{\widetilde{X}}$ be globally generated sheaves over X and $\widetilde{X}$ such that $F_S \simeq \widetilde{F}_S$. By Corollary 2.2.5 and Proposition 2.1.1, we obtain

$$\begin{aligned} F_S \simeq \widetilde{F}_S \quad &\Leftrightarrow \quad i^*_{SX}\phi\phi^! F \simeq i^*_{S\widetilde{X}}\widetilde{\phi}\widetilde{\phi}^! \widetilde{F} \\ &\Leftrightarrow \quad \phi_{10}(i_{\Gamma S'})_* \circ \phi^! F \simeq \phi_{10}(i_{\widetilde{\Gamma} S'})_* \circ \widetilde{\phi}^! \widetilde{F} \text{ by Lemma 2.1.1} \\ &\Leftrightarrow \quad (i_{\Gamma S'})_* \circ \phi^! F \simeq (i_{\widetilde{\Gamma} S'})_* \circ \widetilde{\phi}^! \widetilde{F} \;\text{ since } \phi_{10} \text{ is an equivalence.} \end{aligned} \tag{2.10}$$

We know that $\phi^! F$ and $\widetilde{\phi}^! \widetilde{F}$ are line bundles on Γ and $\widetilde{\Gamma}$ respectively (see 2.1.3). But then $(i_{\Gamma S'})_* \phi^! F$ and $(i_{\widetilde{\Gamma} S'})_* \widetilde{\phi}^! \widetilde{F}$ are isomorphic torsion sheaves of rank one over a curve, hence $\Gamma = \widetilde{\Gamma}$, that is $X = \widetilde{X}$. Finally (2.10) implies that $\phi^! F = \phi^! \widetilde{F}$ because the pushforward by closed immersion is fully faithful. Hence $F \simeq \widetilde{F}$ as $\phi^! : \mathcal{M}_X \to \mathrm{Pic}^2(\Gamma)$ is injective on globally generated sheaves: these are exactly the sheaves which are not in the exceptional divisor of the blow-up $\mathcal{M}_X \to \mathrm{Pic}^2 \Gamma$. □

2.2.2 The non-injectivity locus

Let $F, G \in \mathcal{M}_X$ such that $F_S \simeq G_S$. Assume $F \not\simeq G$. Note that neither F nor G is globally generated: indeed assume F is globally generated, from Proposition 2.1.4 we have $\mathrm{Hom}_X(\mathcal{U}^\vee, F) = 0$, hence Lemma 2.2.4 implies that G is also globally generated, and by Theorem 2.2.6 we obtain $F \simeq G$.

For the next lemmas, we will use the following exact sequences from Proposition 2.1.4.

$$0 \;\to\; \mathcal{O}_X \to \mathcal{U}^\vee \to I \to 0 \tag{2.11}$$
$$0 \;\to\; I \to F \to \mathcal{O}_{L_F}(-1) \to 0. \tag{2.12}$$

Recall that the exact sequence (2.12) is induced by the evaluation map $ev : H^0(F) \otimes \mathcal{O}_X \to F$, that is $\mathcal{O}_{L_F}(-1) = \mathrm{coker}(ev)$.

Proposition 2.2.7. *Both F and G are not locally free.*

Proof. Apply $\mathrm{Hom}(F, -)$ to $0 \to G(-1) \to G \to G_S \to 0$ to get

$$0 \to \mathrm{Hom}(F, G) \to \mathrm{Hom}(F_S, G_S) \to \mathrm{Ext}^1(F, G(-1)). \tag{2.13}$$

We get $\mathrm{Ext}^1(F, G(-1)) \simeq \mathrm{Ext}^2(G, F) \neq 0$, otherwise an isomorphism $F_S \simeq G_S$ would lift to an isomorphism $F \simeq G$ by exactness of (2.13).

Apply $\mathrm{Hom}(G, -)$ to (2.12) to obtain

$$\mathrm{Ext}^2(G, I) \to \mathrm{Ext}^2(G, F) \to \mathrm{Ext}^2(G, \mathcal{O}_L(-1)) \to \mathrm{Ext}^3(G, I)$$

- First we prove $\mathrm{Ext}^2(G, I) = 0$. Indeed, apply $\mathrm{Hom}(G, -)$ to $0 \to \mathcal{O}_X \to \mathcal{U}^\vee \to I \to 0$ to get
$$\mathrm{Ext}^2(G, \mathcal{U}^\vee) \to \mathrm{Ext}^2(G, I) \to \mathrm{Ext}^3(G, \mathcal{O}_X).$$
But $\mathrm{Ext}^3(G, \mathcal{O}_X) = 0$ as $H^k(X, G(-1)) = 0 \quad \forall k$ (Proposition 2.1.4), and from (2.8) $\mathrm{Ext}^2(G, \mathcal{U}^\vee) \simeq \mathrm{Ext}^3(G, \mathcal{U}) \simeq \mathrm{Hom}(\mathcal{U}, G(-1)) = 0$ by stability and comparing slopes. Hence $\mathrm{Ext}^2(G, I) = 0$.

- $\mathrm{Ext}^3(G, I) \simeq \mathrm{Hom}(I, G(-1)) = 0$ comparing slope and by stability.

So we obtain $\mathrm{Ext}^2(G, F) \simeq \mathrm{Ext}^2(G, \mathcal{O}_L(-1))$. Hence for $F \neq G$ with $F_S \simeq G_S$, we have $\mathrm{Ext}^2(G, \mathcal{O}_{L_F}(-1)) \simeq \mathrm{Ext}^1(\mathcal{O}_{L_F}, G)^\vee \neq 0$. Assume we have a non-trivial exact sequence

$$0 \to G \to \mathcal{G} \to \mathcal{O}_{L_F} \to 0. \tag{2.14}$$

In particular, we have a (not surjective) inclusion $G \hookrightarrow \mathcal{G}$, with $c_1(\mathcal{G}) = c_1(G)$ and $\mathrm{rk}(\mathcal{G}) = \mathrm{rk}(G)$. If $\mathcal{G}$ is not torsion free, we can replace it with $\mathcal{G}_f := \mathcal{G}/\mathcal{G}_{tors}$. The induce map $G \xrightarrow{\pi} \mathcal{G}_f$ is still injective as G is torsion free. If $\mathrm{coker}(\pi) = 0$, then $\mathcal{G} \to \mathcal{G}_f \simeq G$ splits the sequence (2.14) which is absurd. Otherwise, quick computations lead to $c_1(\mathcal{G}_f) = c_1(G) = 1$.

Lemma 2.2.8. *Let E be a locally free sheaf on X and $\mathcal{E}$ a torsion-free sheaf with $\mathrm{rk}(E) = \mathrm{rk}(\mathcal{E})$ and $c_1(E) = c_1(\mathcal{E})$. Then any injective map $E \to \mathcal{E}$ is an isomorphism.*

Proof. Consider the exact sequence

$$0 \to E \to \mathcal{E} \to T \to 0. \tag{2.15}$$

Then $c_1(T) = 0$, hence T is supported in codimension 2. Hence

$$\mathcal{E}xt^k(T, E) \simeq \mathcal{E}xt^k(T, \mathcal{O}_X) \otimes E = 0$$

for $k = 0, 1$ ([HL10], Proposition 1.1.6). Consider the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(T, E)) \Rightarrow E^{p+q} = \mathrm{Ext}^{p+q}(T, E).$$

We have $E_2^{p,1-p} = 0$ for all p, hence we get $E^1 = \mathrm{Ext}^1(T, E) = 0$. In particular the extension 2.15 must be trivial, that is $\mathcal{E} = E \oplus T$, which is absurd as $\mathcal{E}$ is torsion-free. □

From 2.2.8, G cannot be locally free.
Finally, we know from Proposition 2.1.4 that there is an exact sequence

$$0 \to G \to G^{**} \to \mathcal{O}_{M_G} \to 0$$

for some line $M_G \subset X$, and G^{**} is locally free. Restricting this sequence to S gives

$$0 \to G_S \to G_S^{**} \to \mathcal{O}_y \to 0$$

with $y = M_G \cap S$ (recall we assumed that S does not contain a line). Note that $G_S^{**} \simeq (G^{**})_S$ because $\mathcal{E}xt^q(\mathcal{O}_y, \mathcal{O}_S) = 0$ for $q = 0, 1$. So G_S is not locally free and $F_S \simeq G_S$ implies that F is not locally free neither.

□

Now, denote L_F, resp. L_G, the loci where F, resp. G, are not globally generated, and M_F, resp. M_G, the loci where F, resp. G, are not locally free. Recall that F^{**} and G^{**} are locally free from Proposition 2.1.4.

Lemma 2.2.9. *We have $L_F = M_G$ and $L_G = M_F$.*

Proof. By symmetry, we only prove $L_F = M_G$.

Recall from the proof of Proposition 2.2.7 that $\mathrm{Ext}^2(F, G) \neq 0$ and hence $\mathrm{Ext}^1(\mathcal{O}_{L_F}, G) \neq 0$. Hence there exists a non-trivial extension

$$0 \to G \to \mathcal{G} \to \mathcal{O}_{L_F} \to 0 \tag{2.16}$$

and computations of chern classes gives

	G	$\mathcal{G}$	$\mathcal{O}_{L_F}$
rk	2	2	0
c_1	1	1	0
c_2	7	6	-1
c_3	0	0	1.

If $\mathcal{G}$ is not torsion free, consider its torsion subsheaf $\mathcal{G}_t$ and the exact sequence

$$0 \to \mathcal{G}_t \to \mathcal{G} \to \mathcal{G}_f \to 0.$$

Note that the composite $G \hookrightarrow \mathcal{G} \to \mathcal{G}_f$ is still injective as G is torsion free. Quick computations give $c_1(\mathcal{G}_t) = 0$. Moreover, $\mathcal{G}_f$ is stable. Indeed, if $K \subseteq \mathcal{G}_f$ with rank 1 and $c_1(K) = c \geq 1$, denote $K^{''}$ the image of K in T. We can consider $0 \to K^{'} \to K \to K^{''} \to 0$ and computing Chern classes we obtain $c_1(K^{'}) = c \geq 1$, but $K^{'} \subseteq G$ as it is the kernel of $K \to K^{''}$, so $K^{'}$ destabilizes G which is absurd.

Another computation gives $c_2(\mathcal{G}_t) = -1$ or 0, hence $c_2(\mathcal{G}_f) = 6$ or 7. We distinguish these two cases. Set $d = c_3(\mathcal{G}_t)$.

1. If $c_2(\mathcal{G}_f) = 7$, we obtain $c_3(\mathcal{G}_f) = 1 - d$. The quotient of the injective map $G \hookrightarrow \mathcal{G}_f$ is a zero dimensional torsion sheaf T with $c_3(T) = 1 - d$, hence $1 - d \geq 0$. Now, from [BF14] Proposition 3.4, either $\mathcal{G}_f$ is locally free or there is an exact sequence

$$0 \to \mathcal{G}_f \to E \to \mathcal{O}_L \to 0$$

with E rank 2 vector bundle with $c_1(E) = 1$, $c_2(E) = 6$. In the latter case, computation of Chern classes gives $1 - d = 0$, hence $\mathcal{G}_f \simeq G$. But the map $\mathcal{G} \to \mathcal{G}_f \simeq G$ splits the exact sequence (2.16) which is absurd. In the former case ($\mathcal{G}_f$ locally free), the inclusion $G \hookrightarrow \mathcal{G}_f$ implies that G is locally free on an open subset of codimension 3, which is absurd as the locus of non locally freeness of G is the line M_G. We conclude that $c_2(\mathcal{G}_t) \neq -1$.

2. If $c_2(\mathcal{G}_f) = 6$, consider the exact sequence

$$0 \to \mathcal{G}_f \to \mathcal{G}_f^{**} \to T \to 0.$$

We know that $\mathcal{G}_f^{**}$ is stable (from the same proof as for $\mathcal{G}_f$) and satisfies $c_1(\mathcal{G}_f^{**}) = 1$. From [BF14] Lemma 3.1 , we must have $c_2(\mathcal{G}_f^{**}) \geq 6$, hence $c_2(\mathcal{G}_f^{**}) = 6$. Moreover, since this sheaf is reflexive it also satisfies $c_3(\mathcal{G}_f^{**}) \geq 0$ (generalization of [Har80], Proposition 2.6). From [BF14] Lemma 3.4 again, $\mathcal{G}_f^{**}$ must be locally free. We deduce that $\mathcal{G}_f$ is locally free on an open subset $U \subset X$ of codimension 3.

The cokernel of the injective map $G \hookrightarrow \mathcal{G}_f^{*}$ is a torsion sheaf T with $c_2(T) = -1$, so it is supported on a line L. In particular, we obtain that G is locally free on $U \smallsetminus L$, so $L = M_G$. The composition $\mathcal{G} \twoheadrightarrow \mathcal{G}_f \twoheadrightarrow T$ factors through $\mathcal{O}_{L_F}$ as shown on the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}_{L_F} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & \mathcal{G}_f & \longrightarrow & T & \longrightarrow & 0.
\end{array}$$

We obtain a surjective map $\mathcal{O}_{L_F} \twoheadrightarrow T$, which gives $L_F = M_G$.

□

To conclude, we remark that we cannot have $L_F = M_F$. Indeed, from [BF13] (arXiv version), Lemma 5.6, each sheaf E in the fibre of $\phi^! : \mathcal{M}_X \to \mathrm{Pic}^2(\Gamma)$ over the point $\phi^! F$ is not globally generated with $L_E = L_F$, and if moreover it is not locally free then E is associated to the *reducible* conic $C = L_E + M_E$. In particular, F must correspond to the conic $L_F + M_F$ which must be reducible.

Theorem 2.2.10. *For X general, the image of $\mathcal{M}_X$ in $\mathcal{M}_S$ via the restriction map is a Lagrangian subvariety with finitely many double points.*

Proof. The assumption X general ensures that $\mathcal{M}_X$ is smooth (recall that $\mathcal{M}_X$ is the blow-up of $\mathrm{Pic}^2(\Gamma)$ along a subscheme isomorphic to the Fano of lines $\mathcal{H}^0_1(X)$ which is smooth for X general). Moreover we know that there exist a sheaf $F \in \mathcal{M}_X$ with $\mathrm{Ext}^2(F, F) = 0$ (Theorem 1.1.17), so if $\mathcal{M}_X$ is smooth the dimension of $\mathrm{Ext}^2(F, F)$ is constant on $\mathcal{M}_X$, hence $\mathrm{Ext}^2(F, F) = 0$ for all $F \in \mathcal{M}_X$.

Lemma 2.2.11. *The restriction $res : \mathcal{M}_X \to \mathcal{M}_S$ induces an immersion (that is, a morphism with injective differential) of $\mathcal{M}_X$ onto a Lagrangian subvariety of $\mathcal{M}_S$.*

Proof. Consider the exact sequence $0 \to K_X \to \mathcal{O}_X \to \mathcal{O}_S \to 0$ and tensor it with $\mathcal{E}nd(F)$. We obtain the exact sequence

$$0 \to \mathcal{E}nd(F) \otimes K_X \to \mathcal{E}nd(F) \to \mathcal{E}nd(F_S) \to 0.$$

Computing the cohomology groups and since we assumed $\mathrm{Ext}^2(F, F) = H^2(X, \mathcal{E}nd(F)) = 0$, we get the exact sequence

$$0 \to H^1(X, \mathcal{E}nd(F)) \to H^1(S, \mathcal{E}nd(F_S)) \to H^2(X, \mathcal{E}nd(F) \otimes K_X) \to 0.$$

Now, we have $H^1(X, \mathcal{E}nd(F)) \simeq T_{[F]}(\mathcal{M}_X)$ and $H^1(S, \mathcal{E}nd(F_S)) \simeq T_{[F_S]}(\mathcal{M}_S)$ (Theorem 1.1.5), so we obtain that res is an immersion.

Recall from section 1.1.3 that the symplectic form on $T_{[F_S]}(\mathcal{M}_S)$ is given by the composition

$$\mathrm{Ext}^1(F_S, F_S) \otimes \mathrm{Ext}^1(F_S, F_S) \to \mathrm{Ext}^2(F_S, F_S) \xrightarrow{tr} H^2(S, \mathcal{O}_S) \simeq \mathbb{C}.$$

Hence, the theorem follows from the commutativity of the diagram:

$$\begin{array}{ccc}
T_{[F]}(\mathcal{M}_X) \otimes T_{[F]}(\mathcal{M}_X) & \longrightarrow & T_{[F]}(\mathcal{M}_X) \otimes T_{[F]}(\mathcal{M}_X) \\
\downarrow & & \downarrow \\
0 = \mathrm{Ext}^2(F, F) & \longrightarrow & \mathrm{Ext}^2(F_S, F_S) \\
{\scriptstyle tr}\downarrow & & \downarrow{\scriptstyle tr} \\
0 = H^2(X, \mathcal{O}_x) & \longrightarrow & H^2(S, \mathcal{O}_S) \simeq \mathbb{C}.
\end{array}$$

□

From the study we made in this section, we know that res is injective on the set of sheaves which are either locally free or globally generated.

Now, consider a singular point of $res(\mathcal{M}_X)$, it corresponds to the image of a sheaf $F \in \mathcal{M}_X$ which is neither globally generated nor locally free. But from the results of this section, the set of sheaves E with $E_S \simeq F_S$ consists exactly in $\{F, G\}$ with G the sheaf for which $M_G = L_F$ and $L_G = M_F$. So this singular point is a double point.

Finally, these two lines L_F and M_F intersects on S: as $F_S \simeq G_S$, we must have $M_F \cap S = M_G \cap S$ and $M_G = L_F$. From [IP99], section 4.2, each line in X intersects a finite number of lines. In particular, the scheme parametrizing couples of intersecting lines in X has dimension 1, and the image of the intersection points of such couples of lines forms a 1-dimensional subscheme of X. This subscheme intersects the general divisor S in a finite number of points. Therefore, there are finitely many double points on $res(\mathcal{M}_S)$. □

2.3 Rational Lagrangian fibrations on $\mathcal{M}_S$

In this section, we adress the question of globalizing the restriction $\mathcal{M}_X \to \mathcal{M}_S$ to the family of moduli spaces of such Fano threefolds. We prove that it gives a *rational* Lagrangian fibration. We find a birational model of $\mathcal{M}_S$ which extend the fibration, and we study this birational model.

2.3.1 Relative moduli spaces and relative HPD

We keep the constructions and notations of section 2.1.1. Let us fix S and vary X. The Fanos X containing S are parametrized by the 11-dimensional vector subspaces W verifying $V_{10} \subset W \subset V_{14}$, so they are parametrized by $\mathbb{P}^3 = \mathbb{P}(V_{14}/V_{10})$. Hence, the corresponding plane curves in $(\mathbb{P}^{13})^\vee$ (see section 2.1.1) are parametrized by the 3-dimensional vector subspaces $W^\perp \subset V_{10}^\perp$, hence parametrized by the same space $\mathbb{P}^3$.

Consider the open subset $\mathcal{W} \subset \mathbb{P}^3$ corresponding to smooth Fanos. Up to shrinking $\mathcal{W}$ a little bit, we can assume that for all $X \in \mathcal{W}$, the space $\mathcal{M}_X$ and the corresponding curve Γ are both smooth (see proof of Theorem 2.2.10).

Denote $\mathfrak{X} \to \mathcal{W}$ this family of Fanos, and $\mathfrak{G} \to \mathcal{W}$ the corresponding family of plane curves. We also have relative moduli spaces (Theorem 1.1.6)

$$\begin{array}{rcl} \mathcal{M}_{\mathfrak{X}/\mathcal{W}} & \to & \mathcal{W} \\ \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W}) & \to & \mathcal{W} \end{array}$$

such that for $w \in \mathcal{W}$ with $\mathfrak{X}_w \simeq X$ and $\mathfrak{G}_w \simeq \Gamma$ we have $(\mathcal{M}_{\mathfrak{X}/\mathcal{W}})_w \simeq \mathcal{M}_X$ and $(\mathrm{Pic}^2(\mathfrak{G}/\mathcal{W}))_w \simeq \mathrm{Pic}^2(\Gamma)$. By generic nature of flatness, up to shrinking $\mathcal{W}$ once again, we can assume that both $\mathcal{M}_{\mathfrak{X}/\mathcal{W}}$ and $\mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})$ are flat, in particular smooth of relative dimension 3, over $\mathcal{W}$.

We omit the proof of the next very useful criterion for flatness.

Proposition 2.3.1 (Critère de platitude par fibres, [Sta18], Tag 039A)**.** *Let S be a scheme, let $f : R \to T$ be a morphism of scheme over S. Assume that*

- *R is flat over S,*
- *$f_s : R_s \to T_s$ is flat for every $s \in S$.*

Then f is flat.

First, fix $w = [X] \in \mathcal{W}$ and $\Gamma = \mathfrak{G}_w$ the corresponding curve.

Recall the definition of the functors $\phi := \phi_{11}, \phi^! := \phi^!_{11}$ in section 2.1.1. Consider the mutation functor

$$\phi\phi^! : \mathrm{D}^b(X) \to \phi\,\mathrm{D}^b(\Gamma) \subset \mathrm{D}^b(X).$$

Lemma 2.3.2. *Let $F \in \mathcal{M}_X$ be a sheaf. Then the sheaves $(\phi\phi^! F)$ and $(\phi\phi^! F)_S$ are μ-stable sheaves.*

Proof. Since the restriction to S of any μ-destabilizing subsheaf of $\phi\phi^! F$ would destabilize $(\phi\phi^! F)_S$, it is enough to prove that $(\phi\phi^! F)_S$ is μ-stable. This is done in [Yos99], Lemma 2.1. □

The next step to study the global restriction between the relative moduli spaces is to consider the HPD functors in family.

Proposition 2.3.3. *The functors studied in section 2.1.1 glue and induce morphisms*

$$\begin{array}{rcl} \Phi^! & : & \mathcal{M}_{\mathfrak{X}/\mathcal{W}} \to \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W}) \\ \Phi & : & \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W}) \to \mathcal{M}_{\mathfrak{X}/\mathcal{W}}(5, 2, 31, 7). \end{array}$$

Proof. We only prove the existence of Φ, the existence of $\Phi^!$ is similar. Denote

$$\mathcal{S}_3 \to G(3, V_{14}^\vee),\ \mathcal{Y}_3 \to G(3, V_{14}^\vee)$$

the universal families of linear section of $\Sigma = LG(3, 6)$ and Y. As the HPD between Σ and Y holds only on $\mathbb{P}V_{14}^\vee \setminus \mathbf{Z}$ (see section 2.1.1), we denote $\mathbf{P}_3 \subset G(3, V^\vee)$ the open subset of linear sections L with $L \cap \mathbf{Z} = \emptyset$. By abuse of notation, we denote again $\mathcal{S}_3$ and $\mathcal{Y}_3$ the universal families of linear section over $\mathbf{P}_3$.

From Remark 1.2.8, we know that there is object $\widetilde{\mathcal{E}}_3 \in \mathrm{D}^b(\mathcal{S}_3 \times_{\mathbf{P}_3} \mathcal{Y}_3)$ which gives a functor

$$\Phi_3 : \mathrm{D}^b(\mathcal{Y}_3) \to \mathrm{D}^b(\mathcal{S}_3).$$

Pick a point $w \in \mathcal{W}$ and denote $X = \mathfrak{X}_w$, $\Gamma = \mathfrak{G}_w$ and $\phi_{11} : \mathrm{D}^b(\Gamma) \to \mathrm{D}^b(X)$. We obtain the diagrams

$$\begin{array}{ccccc} X & \hookrightarrow & \mathfrak{X} & \hookrightarrow & \mathcal{S}_3 \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \xhookrightarrow{w} & \mathcal{W} & \hookrightarrow & \mathbf{P}_r \end{array} \qquad \begin{array}{ccccc} \Gamma & \hookrightarrow & \mathfrak{G} & \hookrightarrow & \mathcal{Y}_3 \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \xhookrightarrow{w} & \mathcal{W} & \hookrightarrow & \mathbf{P}_r \end{array}$$

In the terminology of [Kuz06], the diagrams are *exact cartesian* using Corollaries 2.23 and 2.27. In particular, we can apply [Kuz06] Lemma 2.42: if $L \in \mathrm{Pic}^2(\Gamma) \subset \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W}) \subset \mathrm{D}^b(\mathcal{Y}_3)$, we have

$$\phi_{\widetilde{\mathcal{E}}_{11}}((i_{\Gamma\mathcal{Y}_3})_* L) \simeq (i_{X\mathcal{S}_3})_* \phi_{11}(L).$$

But we know from Lemma 2.3.2 that $\phi_{11}(L) \in \mathcal{M}_X(5, 2, 31, 7) \subset \mathcal{M}_{\mathfrak{X}/\mathcal{W}}(5, 2, 31, 7)$. This finishes the proof.

□

For now on, we will denote $\Phi_w, \Phi^!_w$ the morphisms restricted to $\mathcal{M}_{\mathfrak{X}_w}$ and $\mathrm{Pic}^2(\mathfrak{G}_w)$.

Given $w \in \mathcal{W}$, since $\mathcal{M}_{\mathfrak{X}_w}$ is irreducible, there is an irreducible component $M_w \subset \mathcal{M}_{\mathfrak{X}_w}(5, 2, 31, 7)$ with $\Phi_w \Phi^!_w \mathcal{M}_{\mathfrak{X}_w} \subset M_w$.

Proposition 2.3.4. *For any $w \in \mathcal{W}$, there is an isomorphism $\mathrm{Pic}^2(\mathfrak{G}_w) \simeq M_w$. Moreover, these isomorphisms glue to give an isomorphism of $\mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})$ onto an irreducible component $M_{\mathfrak{X}}$ of $\mathcal{M}_{\mathfrak{X}/\mathcal{W}}(5, 2, 31, 7)$.*

Proof. First, fix $w \in \mathcal{W}$ and denote $X = \mathfrak{X}_w$ and $\Gamma = \mathfrak{G}_w$. Denote $\phi = \Phi_w, \phi^! = \Phi^!_w$ and $M = M_w$.

Lemma 2.3.5. *For any $F \in \mathcal{M}_X$, the space M is smooth at $[\phi\phi^! F]$ and $T_{[\phi\phi^! F]}M$ has dimension 3.*

Proof. From Theorem 1.1.5, we have

$$\begin{aligned} T_{[\phi\phi^! F]}M &\simeq \mathrm{Ext}^1_X(\phi\phi^! F, \phi\phi^! F) \\ &\simeq \mathrm{Ext}^1_\Gamma(\phi^! F, \phi^! F) \\ &\simeq \mathbb{C}^{g(\Gamma)} = \mathbb{C}^3 \end{aligned}$$

because ϕ is fully faithful, and $\phi^! F$ is a line bundle on Γ which is a curve of genus 3.

Similarly, from Theorem 1.1.5 again, $\mathrm{Ext}^2_X(\phi\phi^! F, \phi\phi^! F) \simeq \mathrm{Ext}^2_\Gamma(\phi^! F, \phi^! F) = 0$ so the obstruction space vanishes and M is smooth at $[\phi\phi^! F]$. □

Since ϕ is fully faithful, the morphism $L \in \mathrm{Pic}^2 \Gamma \mapsto \phi L \in M$ is both injective and an immersion as the induced linear map $\mathrm{Ext}^1(L, L) \to \mathrm{Ext}^1(\phi L, \phi L)$ is an isomorphism. Moreover $\mathrm{Pic}^2 \Gamma$ and M are irreducible with same dimension so the morphism is also surjective. This proves the first statement of the proposition.

Now we prove that the isomorphisms glue into a global isomorphism. Consider the morphism $\Phi : \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W}) \to \mathcal{M}_{\mathfrak{X}/\mathcal{W}}(5, 2, 31, 7)$ over $\mathcal{W}$. Recall we assumed that both spaces are flat over $\mathcal{W}$. On each fibre over a closed point $w \in \mathcal{W}$, the morphism Φ_w is an isomorphism. By Proposition 2.3.1, we obtain that Φ is flat. Since $\mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})$ is smooth, Φ is smooth of relative dimension 0, therefore étale, and since it is injective it must be an open immersion.

Moreover $\mathrm{Pic}^2(\mathfrak{G}/\mathcal{W}) \to \mathcal{W}$ is projective, hence the image $\mathrm{Im}(\Phi) \subset \mathcal{M}_{\mathfrak{X}/\mathcal{W}}(5, 2, 31, 7)$ is projective over $\mathcal{W}$. In particular, $\mathrm{Im}(\Phi)$ is universally closed, so the map

$$\mathrm{Im}(\Phi) \times_{\mathcal{W}} \mathcal{M}_{\mathfrak{X}/\mathcal{W}}(5, 2, 31, 7) \simeq \mathrm{Im}(\Phi) \to \mathcal{M}_{\mathfrak{X}/\mathcal{W}}(5, 2, 31, 7)$$

is closed, and we obtain that $\mathrm{Im}(\Phi)$ is a closed subset, and thus an irreducible component, of $\mathcal{M}_{\mathfrak{X}/\mathcal{W}}(5, 2, 31, 7)$.

□

2.3.2 Global restriction to S and birational models

Proposition 2.3.6. *The morphisms $\mathcal{M}_X \to \mathcal{M}_S$ and $\mathrm{Pic}^2 \Gamma \to \mathcal{M}_S(5, 2, 31)$ glue to morphisms*

$$t \; : \; \mathcal{M}_{\mathfrak{X}/\mathcal{W}} \to \mathcal{M}_S, \quad p : \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W}) \to \mathcal{M}_S(5, 2, 31).$$

Moreover, there is open subsets $\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}} \subset \mathcal{M}_{\mathfrak{X}/\mathcal{W}}$ and $\mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})^o \subset \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})$ which induce open immersions

$$t^o \; : \; \mathcal{M}^o_{\mathfrak{X}/\mathcal{W}} \hookrightarrow \mathcal{M}_S, \quad p^o : \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})^o \hookrightarrow \mathcal{M}_S(5, 2, 31).$$

Proof. We split the proof in several steps.

1. *The restriction morphisms glue.*

Recall from 2.1.1 that S is constructed as a linear section $\Sigma \cap \mathbb{P}V_{10}$ for $\Sigma = LG(3, 6) \subset \mathbb{P}V_{14}$ the Lagrangian Grassmanian. Moreover, recall that $\mathcal{W}$ is an open subset of the set of linear subspace $V_{11} \subset V_{14}$ such that $V_{10} \subset V_{11}$.

The embedding $S \times \mathcal{W} \hookrightarrow \Sigma \times \mathrm{Gr}(11, V_{14})$ factors through $\mathfrak{X}$, that is we have an embedding

$$S \times \mathcal{W} \overset{j}{\hookrightarrow} \mathfrak{X}$$

which is a morphism over $\mathcal{W}$.

We use notation of section 1.1.1. Consider the moduli functors $\mathbf{M}_{\mathfrak{X}/\mathcal{W}} := \mathbf{M}_{\mathfrak{X}/\mathcal{W}}(2,1,7)$ and $\mathbf{M}_{S\times\mathcal{W}/\mathcal{W}} := \mathbf{M}_{S\times\mathcal{W}/\mathcal{W}}(2,1,7)$ for the corresponding moduli problems. The pullback by j gives a natural transformation

$$\mathbf{j}^* : \mathbf{M}_{\mathfrak{X}/\mathcal{W}} \to \mathbf{M}_{S\times\mathcal{W}/\mathcal{W}}.$$

Both functors admit coarse moduli spaces $\mathcal{M}_{\mathfrak{X}/\mathcal{W}}$ and $\mathcal{M}_{S\times\mathcal{W}/\mathcal{W}}$, hence we obtain a morphism

$$j^* : \mathcal{M}_{\mathfrak{X}/\mathcal{W}} \to \mathcal{M}_{S\times\mathcal{W}/\mathcal{W}}.$$

Finally, we use the natural projection $\mathcal{M}_{S\times\mathcal{W}/\mathcal{W}} \simeq \mathcal{M}_S \times \mathcal{W} \to \mathcal{M}_S$ (in other words, we "forget" from which Fano X a sheaf on S comes from), and we obtain the desired morphism

$$t : \mathcal{M}_{\mathfrak{X}/\mathcal{W}} \to \mathcal{M}_S.$$

The same argument gives a morphism $\mathcal{M}_{\mathfrak{X}/\mathcal{W}}(5,2,31,7) \to \mathcal{M}_{S\times\mathcal{W}/\mathcal{W}}(5,2,31)$, and we can restrict it to a morphism

$$p : M_{\mathfrak{X}} \simeq \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W}) \to \mathcal{M}_S(5,2,31).$$

2. *Defining the open subsets.*

Let us denote

- $\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}} \subset \mathcal{M}_{\mathfrak{X}/\mathcal{W}}$ the subset of globally generated sheaves,
- $\mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})^o = \Phi^!(\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}})$.

Lemma 2.3.7. *The subspace $\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}} \subset \mathcal{M}_{\mathfrak{X}/\mathcal{W}}$ is open.*

Proof. Let us recall some facts about the construction of moduli spaces of sheaves (see Theorem 1.1.4 or directly [HL10], chapter I section 4). Here we use that semistable sheaves are stable in our case (Proposition 2.2.2). There is an open subscheme

$$\mathcal{R} \subset \mathrm{Quot}_{\mathfrak{X}/\mathcal{W}}(\mathcal{H})$$

over $\mathcal{W}$, where $\mathrm{Quot}_{\mathfrak{X}/\mathcal{W}}(\mathcal{H})$ is a Quot scheme, parametrizing quotients $\mathcal{H}_w \to F_w$ with $F_w \in \mathcal{M}_{\mathfrak{X}_w}$, $w \in \mathcal{W}$. Here, $\mathcal{H} = \mathcal{O}_{\mathfrak{X}}(-m)^{\oplus N}$ for some integers $m, N \geq 0$. Moreover, the relative moduli space of sheaves is constructed as a $\mathrm{SL}_N(\mathbb{C})$-GIT quotient of $\mathcal{R}$, in particular the map (over $\mathcal{W}$)

$$\pi : \mathcal{R} \twoheadrightarrow \mathcal{M}_{\mathfrak{X}/\mathcal{W}}$$

is an open map, so we are reduced to prove that $\pi^{-1}(\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}})$ is open.

Note that $\mathrm{Quot}_{\mathfrak{X}/\mathcal{W}}(\mathcal{H})$ is a fine moduli space, in particular it carries a universal quotient family. Restricting it to $\mathcal{R}$, we obtain a universal quotient family

$$\rho : \mathcal{O}_{\mathcal{R}} \boxtimes \mathcal{H} \to \mathcal{F}$$

on $\mathcal{R} \times_{\mathcal{W}} \mathfrak{X}$. Note that $\mathcal{F}$ is $\mathcal{R}$-flat by definition of the Quot-scheme and since open immersions are flat morphisms. Any sheaf $F_w \in \mathcal{M}_{\mathfrak{X}_w}$ with a given surjective map $\rho_w : \mathcal{H}_w \twoheadrightarrow F_w$ is the pullback of ρ by the base change $\mathrm{Spec}\,\mathbb{C} \to \mathcal{R}, * \mapsto [\rho_w]$.

Denote $p_{\mathcal{R}}, p_{\mathfrak{X}}$ the natural projection from $\mathcal{R} \times_{\mathcal{W}} \mathfrak{X}$. Consider the bundle $\mathcal{U}_{\mathfrak{X}}$ obtained by pullback of $\mathcal{U}_{\Sigma}$ by the composition

$$\mathfrak{X} \hookrightarrow \Sigma \times G(11,\mathbb{C}) \twoheadrightarrow \Sigma.$$

It is easy to see that $(\mathcal{U}_{\mathfrak{X}})_w = \mathcal{U}_{\mathfrak{X}_w}$ for any $w \in \mathcal{W}$. We can thus consider

$$\widetilde{\mathcal{F}} := \mathcal{F} \otimes p_{\mathfrak{X}}^* \mathcal{U}_{\mathfrak{X}}.$$

For any $[\rho_w : \mathcal{H}_w \to F_w] \in \mathcal{R}_w$, the sheaf F_w on $\mathfrak{X}_w$ is globally generated if and only if $H^0(\mathfrak{X}_w, F_w \otimes \mathcal{U}_{\mathfrak{X}_w}) = 0$ (Proposition 2.1.4). But this is equivalent to

$$H^0(\mathfrak{X}_w, \widetilde{\mathcal{F}}_{[\rho_w]}) = 0. \tag{2.17}$$

The subset $\mathcal{R}^0 \subset \mathcal{R}$ where (2.17) holds is open in $\mathcal{R}$ by the semicontinuity theorem (see [Har77], III, 12.8). Since $\mathcal{R}^0 = \pi^{-1}(\mathcal{M}_{\mathfrak{X}/\mathcal{W}})^o$, we conclude.

□

Finally, the map

$$\Phi^{!o} : \mathcal{M}^o_{\mathfrak{X}/\mathcal{W}} \to \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})$$

is fibrewise an open immersion (Theorem 2.1.3). Using again 2.3.1 with a similar argument as in Proposition 2.3.4 we obtain that $\Phi^{!o}$ is an open immersion. In particular, $\mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})^o$ is open. This subset consists only in elements in $\mathrm{Pic}^2(\Gamma)$, $[\Gamma] \in \mathcal{W}$, which are not in the locus blown up by $\phi^!$.

3. *The restriction morphisms are open immersions.*

   Up to replace $\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}}$ with a smaller open subset, we can assume that $t|_{\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}}}$ is smooth of relative dimension 0 ([Har77], Lemma 10.5), hence étale, and since $t|_{\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}}}$ is injective (Theorem 2.2.6) it must be an open immersion. In fact, it suffices to shrink $\mathcal{W}$: for any $[F_w] \in \mathcal{M}^o_{\mathfrak{X}/\mathcal{W}}$, the tangent map $T_{\phi^{!o},[F_w]}$ splits into the direct sum

$$b_w \oplus T_{res,[F_w]} : T_{[F_w]}\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}} \simeq T_w\mathcal{W} \oplus T_{[F_w]}\mathcal{M}_{\mathfrak{X}_w} \to T_{[F_w|_S]}\mathcal{M}_S.$$

   But $T_{res,[F_w]}$ is injective for all $[F_w]$, hence the locus where $T_{\phi^{!o}}$ is not an isomorphism lies in the fibre over a closed subset of $\mathcal{W}$.

   In the same vein, we see that $p^o : \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})^o \to \mathcal{M}_S(5,2,31)$ is dominant since its codomain is irreducible, so we can apply the very same argument to conclude that, up to replacing $\mathcal{W}$ with a smaller open subset, the map $p^o : \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})^o \to \mathcal{M}_S(5,2,31)$ is an open immersion.

□

Corollary 2.3.8. *The morphism $\mathcal{M}^o_S \to \mathcal{W}$ which sends a sheaf of the form $F_S \in \mathcal{M}_S$ with $F \in \mathcal{M}_X$ globally generated to $[X] \in \mathcal{W}$ gives a rational Lagrangian fibration*

$$\mathcal{M}_S \dashrightarrow \mathbb{P}^3,$$

where the fibre over a point $[X] \in \mathcal{W}$ is the open subset $\mathcal{M}^o_X \subset \mathcal{M}_X$ of globally generated sheaves.

Proof. From Proposition 2.3.6, the sheaves of the form $F_S \in \mathcal{M}_S$ with $F \in \mathcal{M}_X$ for some $X \in \mathcal{W}$ and F globally generated form an open subset of $\mathcal{M}_S$, and from Theorem 2.2.6 such a sheaf cannot belong to $\mathcal{M}_Y$ with $[X] \neq [Y] \in \mathcal{W}$, hence the map $\mathcal{M}_S \supset \mathcal{M}^o_S = t(\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}}) \to \mathcal{W}$ is well defined. □

This rational fibration cannot extend to an actual morphism $\mathcal{M}_S \to \mathbb{P}^3$ with fibre $res(\mathcal{M}_X)$ over $[X] \in \mathbb{P}^3$ directly. Indeed, the image of $\mathcal{M}_X$ in $\mathcal{M}_S$ is singular (Theorem 2.2.10).

To conclude this section, we show that $\mathcal{M}_S$ admits a birational model for which there is an actual Lagrangian fibration over $\mathbb{P}^3$ with generic fibre $\mathrm{Pic}^2(\Gamma)$, which can be thought as "filling up" the rational fibration in Corollary 2.3.8.

Theorem 2.3.9. *The map $\mathcal{M}_{\mathfrak{X}/\mathcal{W}} \to \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})$ induces a birational map $\mathcal{M}_S \dashrightarrow \mathcal{M}_S(5,2,31)$. Moreover, the functor ϕ_{10} induces a birational map*

$$\mathcal{M}_{S'}(0,H',2) \dashrightarrow \mathcal{M}_S(5,2,31),$$

and $\mathcal{M}_{S'}(0,H',2) \to \mathbb{P}^3 = |H'|$ is a Lagrangian fibration with fibre over a point $w = [\Gamma] \in \mathcal{W}$ isomorphic to $\mathrm{Pic}^2\Gamma$.

Proof. We have seen in the proof of Proposition 2.3.6 that $\mathcal{M}^o_{\mathfrak{X}/\mathcal{W}} \simeq \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})^o$. Hence from Proposition 2.3.6 we obtain a birational map

$$\mathcal{M}_S \dashrightarrow \mathcal{M}_S(5,2,31)$$

which proves the first statement of the theorem.

The functor ϕ_{10} sends a sheaf of the form $(i_{\Gamma S'})_* L \in \mathcal{M}_{S'}(0,H',2)$, $L \in \mathrm{Pic}^2\Gamma$ to $i^*_{SX}\phi L \in \mathcal{M}_S$ (Proposition 2.1.1). In a similar fashion as in the proof of Proposition 2.3.6, one can prove that the composition

$$\phi_{10}^{-1} \circ p^o : \mathrm{Pic}^2(\mathfrak{G}/\mathcal{W})^o \hookrightarrow \mathcal{M}_{S'}(0,H',2)$$

is an open immersion. In particular, we obtain a birational map

$$\mathcal{M}_S \dashrightarrow \mathcal{M}_{S'}(0,H',2).$$

To conclude, consider the morphism $\mathcal{M}_{S'}(0,H',2) \to \mathbb{P}^3 = |H'|$ defined in [Bea91], called *Beauville integrable system*, which sends a sheaf to its support. Over a point $w = [\Gamma] \in \mathcal{W} \subset \mathbb{P}^3$, the fibre is given by $\mathrm{Pic}^2\Gamma$. Indeed, we claim that a sheaf $E \in \mathcal{M}_{S'}(0,H',2)$ with smooth support Γ is of the form i_*L with $L \in \mathrm{Pic}^2\Gamma$ and $i = i_{\Gamma S'} : \Gamma \subseteq S'$. To prove the claim, it suffices to prove that $\mathcal{O}_{S'}(-H)\cdot E = 0$ ([Sta18], Tag 01QY). Apply $E \otimes -$ to $0 \to \mathcal{O}_{S'}(-H) \to \mathcal{O}_{S'} \to \mathcal{O}_\Gamma \to 0$ to obtain the exact sequence

$$0 \to K \to E \to E_\Gamma \to 0$$

with $K = \ker(E \to E_\Gamma) = \mathrm{Im}(E \otimes \mathcal{O}_{S'}(-H) \to E) = \mathcal{O}_{S'}(-H)\cdot E$. But since $c_1(E) = H'$ and $E_\Gamma := i_*i^*E$ is supported on Γ, we get $c_1(K) = 0$. Hence K must be 0 otherwise it is supported in dimension 0, which contradicts the purity of E. By Grothendieck-Riemann-Roch, the degree of L must be 2.

□

2.4 The birational models of $\mathcal{M}_S$

The goal of this section is to study the different birational models of $\mathcal{M}_S$, in particular $\mathcal{M}_S(5,2,31)$ and $\mathcal{M}_{S'}(0,H',2)$. For now on, we use the notations of sections 1.1.3 and 1.4.2. In particular, $\mathcal{M}_S = \mathcal{M}_S(2,1,7) = \mathcal{M}_S[2,1,3]$, $\mathcal{M}_S(5,2,31) = \mathcal{M}_S[5,2,6]$ and $\mathcal{M}_{S'}(0,H',2) = \mathcal{M}_{S'}[0,H',0]$.

Theorem 2.4.1. *The moduli spaces $\mathcal{M}_S$ and $\mathcal{M}_S[5,2,6]$ are not isomorphic and are related by a flop along a $\mathbb{P}^2$-bundle over S which extends the construction of Theorem 2.3.9. They are the only two smooth K-trivial birational models of $\mathcal{M}_S$. Moreover, $\mathcal{M}_S[5,2,6] \simeq \mathcal{M}_{S'}[0,H',0]$.*

The proof of this theorem splits into the descriptions of section 2.4.3 and Propsitions 2.4.17 and 2.4.18.

2.4.1 Lattice-theoretic description of birational models

Divisors on HK varieties

In this short section we recall some facts, and we refer to Debarre's survey [Deb18] for all the statements.

Let M be a hyperKähler variety. A divisor D on M is called *nef* (for *numerically effective*) if its intersection with all curves $C \subset M$ satisfies $D \cdot C \geq 0$. A divisor D is called *movable* if its base locus has codimension at least 2. We denote $\mathrm{Nef}(M)$, resp. $\mathrm{Mov}(M)$ for the cones generated by nef divisors, resp. the closure of the cone generated by movable divisors.

The cohomology space $H^2(M, \mathbb{Z})$ can be equipped with a natural quadratic form $q : H^2(M, \mathbb{Z}) \to \mathbb{Z}$, called *Beauville-Bogomolov form*, and q induces a bilinear form on $\mathrm{NS}(M)_{\mathbb{R}}$. The cone of divisors $D \in \mathrm{NS}(M)_{\mathbb{R}}$ satisfying $q(D, D) > 0$ has 2 connected component, and we define the *strictly positive* cone $\mathrm{Pos}(M)$ as the component which contains an ample class (its closure is called *positive* cone).

In $\mathrm{NS}(M)_{\mathbb{R}}$, the nef cone $\mathrm{Nef}(M)$ is closed and identifies with the closure of the ample (open) cone $\mathrm{Amp}(M)$. Moreover, we have the inclusions

$$\mathrm{Nef}(M) \subset \mathrm{Mov}(M) \subset \overline{\mathrm{Pos}(M)}.$$

For the next theorem, see [Deb18], Proposition 3.5.

Theorem 2.4.2. *Let M, M' be HK varieties. Any birational map $s : M \overset{\sim}{\dashrightarrow} M'$ induces a Hodge isometry $s^* : H^2(M', \mathbb{Z}) \xrightarrow{\sim} H^2(M, \mathbb{Z})$ with respect to the Beauville-Bogomolov form on M and M'. Moreover, we have*

$$s^*(\mathrm{Mov}(M')) = \mathrm{Mov}(M),$$

and if $s^*(\mathrm{Nef}(M'))$ meets $\mathrm{Amp}(M)$, then s extends to an isomorphism and $s^*(\mathrm{Nef}(M')) = \mathrm{Nef}(M)$.

Moreover, from [HT09], we obtain that $\mathrm{Mov}(M)$ admits a locally polyhedral chambers decomposition

$$\mathrm{Mov}(M) = \overline{\bigcup_{s : M \dashrightarrow M'} s^*(\mathrm{Nef}(M'))}$$

where the sum is taken over all birational models $M \overset{\sim}{\dashrightarrow} M'$, and the various $s^*(\mathrm{Nef}(M'))$ are either equal or have disjoint interiors.

Now we focus on the case of moduli spaces of sheaves on K3 surfaces, that is $M = \mathcal{M}_S[v]$ for some K3 surface S and a primitive Mukai vector $v \in \widetilde{H}(S, \mathbb{Z})$ with $v^2 > -2$.

The next theorem is well-known and due to many authors. We refer to Yoshioka [Yos01] for a proof. See also [Yos06] for its generalization to twisted sheaves.

Proposition 2.4.3. *Assume $v^2 > 0$, then there exists an isomorphism*

$$\theta_v : v^{\perp} \xrightarrow{\sim} \mathrm{NS}(\mathcal{M}_S[v]).$$

Under this isomorphism, the Beauville-Bogomolov form of $\mathrm{NS}(\mathcal{M}_S[v])$ coincide with the Mukai pairing on $v^{\perp}$.

If $v^2 = 0$, the same holds true relacing $v^{\perp}$ by $v^{\perp}/v$.

Birational models as moduli spaces of stable objects

Let us first recall some of the methods, developed by Bayer and Macrì in [BM14b] and [BM14a]. Let S be a K3 surface and v be a primitive Mukai vector with $v^2 \geq -2$. Set $\mathcal{M}_S := \mathcal{M}_S[v]$. Consider the following maps.

1. $\mathcal{Z} : \mathrm{Stab}^+(S) \to \widetilde{H}(S,\mathbb{Z}) \otimes \mathbb{C}$ sends a stability condition $(Z, \mathcal{P})$ to Ω_Z, where $Z(-) = \langle -, \Omega_Z \rangle$. See [Bri08], section 8. In particular, in view of section 1.4.2, $\mathcal{Z}(\sigma_{\alpha,\beta}) = \exp\left((i\alpha + \beta)H\right)$.
2. $I : \widetilde{H}(S,\mathbb{Z}) \otimes \mathbb{C} \to v^\perp$ is defined by $I(\Omega) = \Im \dfrac{\Omega}{-\langle \Omega, v\rangle}$.
3. $\theta_v : v^\perp \to \mathrm{NS}(\mathcal{M}_S)$, which comes from Proposition 2.4.3 (we assume $v^2 > 0$ for simplicity).

The composition $l_0 = \theta_v \circ I \circ \mathcal{Z}$ gives a map $\mathrm{Stab}^+(S) \to \mathrm{NS}(\mathcal{M}_S)$. It turns out that the image of l_0 is contained in the strictly positive cone $\mathrm{Pos}(\mathcal{M}_S)$.

An irreducible divisor $E \subset \mathcal{M}_S$ is called *exceptional* if there is a birational map $M \dashrightarrow M'$ contracting E. For each such divisor E, define the *exceptional reflection at* E, denoted $\rho_E \in \mathrm{Aut}(\mathrm{NS}(\mathcal{M}_S)_\mathbb{Q})$, as the reflection along the hyperplane orthogonal to E (that is, the involution fixing $E^\perp$ and such that $\rho_E(E) = -E$). Set $W_{exc} \subset \mathrm{Aut}(\mathrm{NS}(\mathcal{M}_S)_\mathbb{Q})$ the subgroup generated by exceptional reflections.

By [Mar13], exceptional reflections are integral involutions of $\mathrm{NS}(\mathcal{M}_S)$. Moreover, the cone $\mathrm{Mov}(\mathcal{M}_S) \cap \mathrm{Pos}(\mathcal{M}_S)$ is a fundamental chamber of the action of W_{exc} on $\mathrm{NS}(\mathcal{M}_S)$. In particular, for any class $D \in \mathrm{Pos}(\mathcal{M}_S)$, there is a unique class $R_{exc}(D)$ lying in $\mathrm{Mov}(\mathcal{M}_S)$ in the orbit of D by the action of W_{exc}.

Hence, we can compose l_0 with R_{exc} to obtain a map

$$l : \mathrm{Stab}^+(S) \to \mathrm{Mov}(\mathcal{M}_S). \tag{2.18}$$

Theorem 2.4.4 ([BM14a], Theorem 1.2)**.** *1. The image of l is the cone of big movable divisor* $\mathrm{Mov}(\mathcal{M}_S) \cap \mathrm{Pos}(\mathcal{M}_S)$.

2. For any generic stability condition $\sigma \in \mathrm{Stab}^+(S)$, the image $l(\sigma)$ lies in the chamber of $\mathrm{Mov}(\mathcal{M}_S)$ which correspond to the birational model $\mathcal{M}_\sigma[v]$ of $\mathcal{M}_S$. In particular, all smooth K-trivial birational model of $\mathcal{M}_S$ appears as $\mathcal{M}_\mathcal{C}[v]$ for some chamber $\mathcal{C} \subset \mathrm{Stab}^+(S)$.

3. For any chamber $\mathcal{C} \subset \mathrm{Stab}^+(S)$, we have $l(\mathcal{C}) = \mathrm{Amp}(\mathcal{M}_\mathcal{C}[v])$.

Note that for all generic $\sigma \in \mathrm{Stab}^+(S)$, by Theorem 1.4.14 we can identify $\mathrm{NS}(\mathcal{M}_\sigma[v])$ with $\mathrm{NS}(\mathcal{M}_S)$. Theorem 2.4.4 says that the wall and chamber decomposition of $\mathrm{Mov}(\mathcal{M}_S)$ is, up to a quotient by the action of W_{exc}, given by the chamber decomposition of $\mathrm{Stab}^+(\mathcal{M}_S)$.

It remains to understand what happens on the walls of $\mathrm{Stab}^+(S)$. To do so, consider a wall $W \subset \mathrm{Stab}^+(S)$, consider a stability conditions $\sigma \in W$ generic on the wall (i.e. not contained in any other wall), and let σ_+, resp. σ_- be two generic stability conditions on each side of W near σ_0. In particular, $l(\sigma_0)$ induces nef divisors l_+, resp. l_-, in $\mathcal{M}_{\sigma_+}[v]$, resp. $\mathcal{M}_{\sigma_-}[v]$.

Theorem 2.4.5 ([BM14b], Theorem 1.4(a))**.** *The divisors $l_\pm$ are big and nef on $\mathcal{M}_{\sigma_\pm}[v]$. They induce birational contractions*

$$\pi^\pm : \mathcal{M}_{\sigma_\pm}[v] \to \overline{M}_\pm$$

where $\bar{M}_\pm$ are normal irreducible projective varieties. The curves contracted by $\pi^\pm$ are the curves of objects that are S-equivalent with respect to σ_0.

The walls are classified with respect to the type of contraction they produce.

Definition 2.4.6. The wall W is called:

1. a *fake wall* if there are no curves on $\mathcal{M}_{\sigma_\pm}[v]$ of S-equivalent objects with respect to σ_0,
2. a *totally semistable wall* if there is no stable object with Mukai vector w,
3. a *flopping wall* if we can identify $\overline{M}_+ = \overline{M}_-$ and the induced map $\mathcal{M}_{\sigma_+}[v] \dashrightarrow \mathcal{M}_{\sigma_-}[v]$ induces a flopping contraction,
4. a *divisorial wall* if both morphism $\mathcal{M}_{\sigma_\pm}[v] \to \overline{M}_\pm$ are divisorial contractions.

For some background on flops and the Minimal Model Program, we refer to [HM10].

The wall W can in fact be studied in a lattice-theoretic manner. Assume $v^2 > 0$, and associate to W the set

$$\mathcal{H}_W := \{w \in \widetilde{H}(S,\mathbb{Z}) \mid \Im \frac{Z(w)}{Z(v)} = 0 \text{ for all } \sigma = (Z,\mathcal{P}) \in W\}. \tag{2.19}$$

This subset is rank 2 primitive sublattice of $\widetilde{H}(S,\mathbb{Z})$. It turns out that the type of W with respect to Definition 2.4.6 is completely described by the associated hyperbolic lattice $\mathcal{H}_W$. A class $w \in \widetilde{H}(S,\mathbb{Z})$ is called *isotropic* (resp. *spherical*) if $w^2 = 0$ (resp. $w^2 = -2$).

Theorem 2.4.7 ([BM14a], Theorem 5.7)**.** *The wall W is a* ***totally semistable*** *wall if and only if there exists either an isotropic class $w \in \mathcal{H}_W$ with $\langle v, w\rangle = 1$, or an effective spherical class s (i.e. $s^2 = -2$ and $\Re\frac{Z(s)}{Z(v)} > 0$) with $\langle s, v\rangle < 0$. In addition:*

1. *The wall W is a* ***divisorial*** *wall if one of the three conditions hold:*
   - *(a)* (Brill-Noether): *there exists a spherical class $s \in \mathcal{H}_W$ with $\langle s, v\rangle = 0$, or*
   - *(b)* (Hilbert-Chow): *there exists an isotropic class $w \in \mathcal{H}_W$ with $\langle w, v\rangle = 1$, or*
   - *(c)* (Li-Gieseker-Uhlenbeck): *there exists an isotropic class $w \in \mathcal{H}_W$ with $\langle w, v\rangle = 2$.*
2. *Otherwise, if v can be written as the sum $v = a + b$ with a, b positive (i.e. $a^2 \geq 0$ and $\langle a, v\rangle > 0$, and similarly for b), or if there exists a spherical class $s \in \mathcal{H}_W$ with $0 < \langle s, v\rangle \leq \frac{v^2}{2}$, then W is a* ***flopping*** *wall.*
3. *In all other cases, W is a* ***fake*** *wall.*

For divisorial walls, the names are explained by the nature of the contraction morphism $\pi^\pm : \mathcal{M}_{\sigma_\pm} \to \overline{M}_\pm$ of Theorem 2.4.5. The only one that will appear in the next sections is the LGU case (1c), for which the space $\overline{M}_\pm$ is the Uhlenbeck compactification space (see [Li93]). In particular, the object of $\mathcal{M}_S[v]$ which become strictly semistable on the wall are given by sheaves which are either not *locally free* or not slope-*stable.*

Remark 2.4.8. By straight computations, it is easy to see that the image of a numerical wall $W(w)$ generated by a class $w \in \widetilde{H}(S,\mathbb{Z})$ is contained in the subset $(w^\perp \cap v^\perp) \subset v^\perp$.

2.4.2 Computing the walls

Consider the K3 surface we studied in section 2.3.1. Recall that S is a K3 surface of genus 9, with $\mathrm{Pic}(S) = \mathbb{Z}\langle H\rangle$ where H is an ample divisor of square $H^2 = 16$. We want to study

$$\mathcal{M}_S := \mathcal{M}_S(2,1,7) = \mathcal{M}_S[2,1,3].$$

First, we show that it is sufficient to consider stability conditions of the form $\sigma_{\alpha,\beta}$ as constructed section 1.4.2.

Proposition 2.4.9. *Let $\sigma \in \mathrm{Stab}^+(S)$ be a generic stability condition. Then there is an autoequivalence $\phi \in \mathrm{Aut}(\mathrm{D}^b(S))$ with $\phi^H(v) = v$ such that $\phi(\sigma)$ lies in $U(S)$. Moreover, the moduli space $\mathcal{M}_\sigma[v]$ is isomorphic to $\mathcal{M}_{\sigma_{\alpha,\beta}}[v]$ for some α, β.*

Proof. In view of Corollary 1.4.15, the only thing to prove is that the autoequivalence involved satisfies $\phi^H(v) = v$. The equivalence ϕ is described in the proof of Proposition 1.4.11: it is a composition of of autoequivalences either of the form T_A^2, the square of a spherical twist along a spherical vector bundle A on S, or of the form $T_{\mathcal{O}_C(k)}$, the spherical twist along the structure sheaf of a nonsingular rational curve $C \subset S$. But the latter case cannot occur as $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 16$, and any smooth rational curve satisfies $C^2 = -2$ by adjunction formula. Now the fact $\phi(v) = v$ follows from the remark that T_A^2 acts trivially in cohomology (see section 1.3.3). □

Remark 2.4.10. Note that the only wall we are interested in are actual walls W remaining actual all along the numerical wall W_{num}, except on the holes in W_{num} arising from spherical classes (see Proposition 1.4.8). Indeed, assume $W_{num} = W(w)$ for some class $w \in \widetilde{H}(S,\mathbb{Z})$, and let $\sigma := \sigma_{\alpha,\beta} \in W \setminus W(w)$ be a stability condition. Then σ lies in some chamber $\mathcal{C} \subset \mathrm{Stab}^+(S)$, and its image $l(\sigma)$ lies in the open ample cone $\mathrm{Amp}(\mathcal{M}_\mathcal{C}[v])$. Since $\mathrm{Amp}(\mathcal{M}_\mathcal{C}[v])$ is a cone, the whole ray $\mathbb{R}_{>0} \cdot l(\sigma)$ lies in $\mathrm{Amp}(\mathcal{M}_\mathcal{C}[v])$. But this halfline contains $W(w)$ (Remark 2.4.8), in particular for any point $\sigma_0 \in W$, given two stability condition $\sigma_\pm$ near σ_0 in each adjacent chamber, the corresponding image $l(\sigma_\pm)$ both lie in $\mathrm{Amp}(\mathcal{M}_\mathcal{C}[v])$, and hence $\mathcal{M}_{\sigma_+}[v] = \mathcal{M}_{\sigma_-}[v] = \mathcal{M}_\mathcal{C}[v]$.

Therefore, we can compute the wall and chamber decomposition with respect to $v = (2,1,3)$ thanks to the description of Proposition 1.4.18.

Let $\alpha, \beta \in \mathbb{H}$. Assume $\widetilde{F} \in \mathcal{M}_{\sigma_{\alpha,\beta}}[2,1,3]$. By definition, there is an integer $k \in \mathbb{Z}$ with $\widetilde{F}[k] \in \mathbf{Coh}^\beta(S)$. If k is even, we have $v(\widetilde{F}[k]) = (2,1,3)$, and if k is odd we have $v(\widetilde{F}[k]) = (-2,-1,-3)$.

Vertical wall

By 1.4.18, there is (at least numerically) a vertical wall

$$W_v = \{\beta = 1/2\}$$

in $\mathbb{H}$, given by the class $(2,1,4) \in \Lambda$. In fact, this wall is an actual wall. Indeed, following Proposition 2.1.4, pick $F \in \mathcal{M}_S[2,1,3]$ not locally free, the sheaf $E := F^{**}$ is a stable vector bundle such that $E \in \mathcal{M}_S[2,1,4]$. We have the exact sequence

$$0 \to F \to E \to \mathcal{O}_x \to 0$$

for some point $x \in S$. This induces the exact triangle

$$\mathcal{O}_x \to F[1] \to E[1]. \tag{2.20}$$

We claim that (2.20) induces an exact sequence in $\mathbf{Coh}^{1/2}(S)$. Indeed $E[1], F[1] \in \mathbf{F}^{1/2}[1]$ and $\mathcal{O}_x \in \mathbf{T}^{1/2}$, so that (2.20) lies in $\mathbf{Coh}^{1/2}(S)$.

Left side of the halfplane

Consider a wall left to the vertical wall W_v, given by an exact sequence

$$E \hookrightarrow F \to T \tag{2.21}$$

in $\mathbf{Coh}^\beta(S)$, with $F := \widetilde{F}[k] \in \mathbf{Coh}^\beta(S)$, $v(\widetilde{F}) = (2,1,3)$. From $\beta < 1/2$ we must have k even, that is $v(F) = (2,1,3)$. Denote $v(E) = w = (w_0, w_1, w_2)$ and $v(T) = t = (t_0, t_1, t_2)$. Note that E and T will play a symmetric role in the following, so as $2 = v_0 = w_0 + t_0$ we can assume that $w_0 > 0$.

Using Proposition 1.4.18 in our case, the wall is a semicircular wall with center C and radius R. We get

$$C < \frac{1}{4} \tag{2.22}$$

Moreover, any semicircular wall must intersect the ray $\{\beta = \frac{1}{4}\}$. Note that for $\beta = \frac{1}{4}$, there is no class $\delta = (\delta_0, \delta_1, \delta_2) \in \Lambda$ with $\delta_0 > 0$, $\delta^2 = -2$ and $\mu_{1/4}(\delta) = 0$. Indeed, these conditions on δ give

$$\begin{aligned} 8\delta_1^2 &= \delta_0\delta_2 - 1 \\ \delta_0 &= 4\delta_1, \end{aligned}$$

which is impossible because $\delta_0, \delta_1, \delta_2 \in \mathbb{Z}$. In view of Proposition 1.4.8, we see that $Z := Z_{\alpha,\frac{1}{4}}$ defines a stability condition.

In view of Remark 2.4.10, we can assume that the exact sequence (2.21) hold at $\beta = \frac{1}{4}$. Recall that the imaginary and real part of $Z(-)$ are additive on exact sequence in $\mathbf{Coh}^\beta(S)$, and since $\mu_Z(F) < \infty$ we get $0 < \Im(Zw) < \Im(Zv)$ and $0 < \Im(Zt) < \Im(Zv)$. For $\beta = \frac{1}{4}$, it gives

$$\begin{aligned} & 0 < w_1 - \frac{w_0}{4} < \frac{1}{2} \\ \iff \quad & \frac{w_0}{4} < w_1 < \frac{w_0}{4} + \frac{1}{2} \end{aligned}$$

Since w_1 and w_0 are integers, we obtain $w_0 = 4n+3$, $w_1 = n+1$ for some $n \in \mathbb{Z}$. Moreover we assumed $w_0 > 0$, so $n \geq 0$ and in particular $2w_1 - w_0 = -2n - 1 < 0$.

The inequality $C < \frac{1}{4}$ gives

$$\begin{aligned} & \frac{2w_2 - 3w_0}{16(2w_1 - w_0)} < \frac{1}{4} \\ \iff \quad & 2w_2 - 3w_0 > 4(2w_1 - w_0) \\ \iff \quad & w_2 > 4w_1 - \frac{1}{2}w_0 = 2n + \frac{5}{2} \end{aligned} \tag{2.23}$$

To obtain more bounds, we need to study E with more details.

Lemma 2.4.11. *The object E is a $\sigma_{\alpha,1/4}$-stable object, in particular it satisfies*

$$v(E)^2 = 16w_1^2 - 2w_0w_2 \geq -2. \tag{2.24}$$

Proof. If E is stable, then $\mathrm{Hom}_{\mathbf{Coh}^\beta(S)}(E,E) = \mathrm{Hom}_{\mathrm{D}^b(S)}(E,E) = \mathbb{C}$ because any stable object is simple. Hence by Serre duality we get $v(E)^2 = -\chi(E,E) = -(2 - \mathrm{Ext}^1(E,E)) \geq -2$.

Moreover, note that for $\beta = \frac{1}{4}$, any object of class $a = (a_0, a_1, a_2)$ satisfies

$$\Im Z(a) = \alpha H^2 \frac{1}{4}(4a_1 - a_0) \geq \frac{1}{4}\alpha H^2. \tag{2.25}$$

But we have $\Im Z(v(E)) = \alpha H^2(w_1 - \beta w_0) = \frac{1}{4}\alpha H^2$, so E cannot be semistable, otherwise any of its proper Jordan-Hölder factor A would satisfy $\Im Z(v(A)) < \frac{1}{4}H^2\alpha$ which contradicts (2.25). ☐

Using (2.24) we get

$$\begin{aligned} & 16w_1^2 - 2w_0w_2 \geq -2 \\ \iff \quad & w_2 \leq 8\frac{w_1^2}{w_0} + \frac{1}{w_0} \end{aligned}$$

By straightforward computations, we have

$$8\frac{w_1^2}{w_0} = 8\frac{(n+1)^2}{4n+3} = 2n + \frac{5}{2} + \frac{1}{2(4n+3)}.$$

Combined with (2.23) we get

$$2n + \frac{5}{2} < w_2 \leq 2n + \frac{5}{2} + \frac{3}{2(4n+3)}.$$

Hence the only possibility is $n = 0$, which gives $w_0 = 3$,$w_1 = 1$ and $w_2 = 3$. We get $C = \frac{3}{16}$.

Proposition 2.4.12. *The circular numerical wall $W_l(3, 1, 3)$ with center $(C = \frac{3}{16}, 0)$ and radius $R = \frac{3}{16}$ is an actual wall W_l. It is induced by the following exact sequences*

$$\mathcal{U}_S^\vee \hookrightarrow F_S \twoheadrightarrow \mathcal{I}_x^\vee[1] \quad \text{if} \quad \beta < \frac{1}{3} \tag{2.26}$$

$$(\phi\phi^! F)_S \hookrightarrow F_S \twoheadrightarrow \mathcal{U}_S^\vee[1] \quad \text{if} \quad \frac{1}{3} < \beta \tag{2.27}$$

in $\mathbf{Coh}^\beta(S)$, where $F \in \mathcal{M}_X$ is a non globally generated sheaf, for $[X] \in \mathcal{W}$ a Fano threefold containing S, and $\mathcal{I}_x^\vee$ is the derived dual of the ideal sheaf of the point $x \in S$ (for the definition of $\phi\phi^!$, see section 2.3.1).

Note that for $\frac{1}{3} < \beta$ and any $G \in \mathcal{M}_S$, any extension

$$0 \to \mathcal{U}_S^\vee \to R \to G \to 0$$

is slope-stable by [Yos99], Lemma 2.1. Though, if $G = F_S$ comes from a globally generated sheaf $F \in \mathcal{M}_X$, then $\mathrm{Ext}^1(G, \mathcal{U}_S^\vee) = 1$ and two such sheaves do not become S-equivalent on the wall.

Proof. Recall from Proposition 2.1.4 that for any non globally generated sheaf $F \in \mathcal{M}_X$, there is a map $f : \mathcal{U}_S^\vee \to F_S$ which induces an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{U}_S^\vee \xrightarrow{f} F \to \mathcal{O}_x \to 0. \tag{2.28}$$

Consider the cone $C(f)$. It lies in $\mathbf{Coh}^\beta(S)$ for any $\beta > 0$, and moreover an object of class $t = (-1, 0, 0)$ satisfies $\Im Z(t) = 16\alpha\beta$, so it cannot be destabilized for $0 < \beta < 1$ (a similar argument as for Lemma 2.4.11 proves this claim). Hence $C(f)$ is μ_Z-stable for any $0 < \beta < 1$.

Lemma 2.4.13. *We have* $\mathcal{M}_{\sigma_{\alpha,\beta}}[-1,0,0] = \{\mathcal{I}_x^\vee[1] \mid x \in S\} \simeq S$ *for any* $\beta > 0$.

Proof. Once again, it is easy to prove that for any $0 < \beta < 1$ and $x \in S$, the object $\mathcal{I}_x^\vee[1]$ is μ_Z-stable. By Theorem 1.4.12, $\mathcal{M}_\sigma[-1,0,0]$ is a K3 surface, and since it contains S it must be isomorphic to S. □

Hence $C(f) = \mathcal{I}_x^\vee[1]$ for the point $x \in S$ appearing in (2.28). We obtain the desired exact sequence (2.26) in $\mathbf{Coh}^\beta(S)$ for all $0 < \beta < \frac{1}{3}$.

At the point $\sigma_0 = (\frac{1}{3}, \alpha) \in W_l(3,1,3)$, $\mathcal{U}_S^\vee$ is a spherical object with $Z(v(\mathcal{U}_S^\vee)) = 0$, so σ_0 is *not* a stability condition (in other words, σ_0 is a hole of W_l).

Finally, for $\frac{1}{3} < \beta$ the extension (2.7) give the exact sequence (2.27) in $\mathbf{Coh}^\beta(S)$, and it is easily seen to remain valid all along this part of the numerical wall $W_l(3,1,3)$.

□

Right side of the halfplane

In a similar way as in section (2.4.2), consider a wall given by an exact sequence

$$E \hookrightarrow F \twoheadrightarrow T \tag{2.29}$$

in $\mathbf{Coh}^\beta(S)$, with $\beta > 1/2$. Hence we have $F := \widetilde{F}[k]$ with k odd, so that $v(F) = (-2,-1,-3)$. Denote again $v(E) = w = (w_0, w_1, w_2)$ and $v(T) = t = (t_0, t_1, t_2)$. In this case, we have $w_0 + t_0 = -2$. In particular, either w_0 or t_0 is negative. Moreover, either $t_0 = w_0 = -1$, or $-2 \in \{t_0, w_0\}$, or at least one of these two integers is positive.

Using Proposition 1.4.18 in this case, the wall must intersect the vertical ray $\{\beta = 3/4\}$. Once again, there is no class $\delta = (\delta_0, \delta_1, \delta_2) \in \Lambda$ with $\delta_0 > 0$, $\delta^2 = -2$ and $\mu_{3/4}(\delta) = 0$. Indeed, these conditions on δ give

$$\begin{aligned} 8\delta_1^2 &= \delta_0\delta_2 - 1 \\ 3\delta_0 &= 4\delta_1, \end{aligned}$$

which is impossible because $\delta_0, \delta_1, \delta_2 \in \mathbb{Z}$. Therefore we fix $\beta = 3/4$ for now on.

Assume by symmetry that t_0 is negative. From $0 < \Im(Zt) < \Im(Zv) = -1 + 2 \times \frac{3}{4} = \frac{1}{2}$, we get

$$\frac{3t_0}{4} < t_1 < \frac{3t_0}{4} + \frac{1}{2}.$$

We see that the case $t_0 = -1$ and $t_0 = -2$ are not possible. Hence we can assume that $w_0 > 0$.

In view of Remark 2.4.10, we can assume that the exact sequence (2.29) holds at $\beta = \frac{3}{4}$. Hence we have $0 < \Im(Zw) < \Im(Zv)$ which gives

$$\frac{3w_0}{4} < w_1 < \frac{3w_0}{4} + \frac{1}{2},$$

hence we have $w_0 = 4n + 1$ and $w_1 = 3n + 1$ for some $n \in \mathbb{Z}_{\geq 0}$. Now we use again the computations in (2.4.2), which gives in these settings

$$\frac{3}{4} < C.$$

We have $2w_1 - w_0 = 2n + 1 > 0$, hence the lower bound gives

$$\begin{aligned} & \frac{-(2w_2 - 3w_0)}{-16(2w_1 - w_0)} > \frac{3}{4} \\ \iff\ & 2w_2 - 3w_0 > 24w_1 - 12w_0 \\ \iff\ & w_2 > 12w_1 - \frac{9}{2}w_0 = 18n + \frac{15}{2}. \end{aligned} \tag{2.30}$$

Lemma 2.4.11, and hence equality (2.24) also holds in this case (one can perform the same proof), so we obtain

$$\begin{aligned} & -2 \leq 16w_1^2 - 2w_0w_2 \\ \iff \quad & w_0w_2 \leq 8w_1^2 + 1. \end{aligned}$$

Replacing w_0, w_1 by $4n+1$, $3n+1$ respectively, and since $w_0 > 0$ we obtain

$$w_2 \leq 18n + \frac{15}{2} + \frac{3}{2(4n+1)}.$$

Combined with (2.30), the only possibility is $n = 0$, that is $w_0 = w_1 = 1$ and $w_2 = 8$ or 9. Equality $w_2 = 8$ gives $C_8 = \frac{13}{16}$, $R_8 = \frac{3}{16}$ and $w_2 = 9$ gives $C_9 = \frac{15}{16}$, $R_9 = \frac{\sqrt{33}}{16}$.

Proposition 2.4.14. *The numerical wall of center* $(C_9 = \frac{15}{16}, 0)$ *and radius* $R_9 = \frac{\sqrt{33}}{16}$ *is not an actual wall.*

Proof. Note that the circle of center C_9 and radius R_9 cross the ray $\{\beta = 2/3\}$ at $\alpha^2 = R_9^2 - (\frac{2}{3} - C)^2 = \frac{1}{18}$.

Lemma 2.4.15. *Any vector* $\delta = (\delta_0, \delta_1, \delta_2) \in \Lambda$ *with* $\delta_0 > 0$, $\delta^2 = -2$ *and* $\mu_{2/3}(\delta) = 0$ *satisfies* $\Re Z_{\alpha,2/3}(\delta) > 0$ *whenever* $\alpha^2 > 1/72$.

Proof. Rewriting the equations on δ give

$$\begin{aligned} 8\delta_1^2 &= \delta_0\delta_2 - 1 \\ 2\delta_0 &= 3\delta_1. \end{aligned}$$

In particular, we get $\delta_1, \delta_2 > 0$ and $\frac{\delta_2}{\delta_1} = \frac{16}{3} + \frac{2}{3\delta_1^2}$. We get

$$\begin{aligned} \Re Z(\delta) > 0 \quad \iff \quad & \delta_1(\frac{16}{3} + 12\alpha^2) > \delta_2 && (2.31) \\ \iff \quad & \frac{16}{3} + 12\alpha^2 > \frac{\delta_2}{\delta_1} = \frac{16}{3} + \frac{2}{3\delta_1^2} && (2.32) \\ \iff \quad & \alpha^2 > \frac{1}{18\delta_1^2} && (2.33) \end{aligned}$$

Since $\delta_0 = \frac{3}{2}\delta_1$, we have $\delta_1 \geq 2$. In particular $\alpha^2 > 1/72$ works for all δ's.

□

Consider the exact sequence (2.29) in $\mathbf{Coh}^{3/4}(S)$, let $(\frac{3}{4}, \alpha_0)$ be the intersection of the numerical wall $W(1,1,9)$ with the ray $\{\beta = \frac{3}{4}\}$. We can assume that F is $\sigma_{\alpha_0+\epsilon,\frac{3}{4}}$-stable, and since $W(1,1,9)$ is the largest circular wall crossing $\{\beta = \frac{3}{4}\}$ (on the right side of the vertical wall W_v), we can assume that F is $\sigma_{\alpha,\frac{3}{4}}$-stable for all $\alpha \gg 0$. By a similar argument as for Theorem 1.4.16 (see [MS17], Lemma 6.18), one can prove that $\mathcal{H}^0(F)$ is a torsion sheaf supported in dimension 0 and $\mathcal{H}^{-1}F$ is a slope-semistable torsion-free sheaf. Consider the long exact sequence (in $\mathbf{Coh}(S)$)

$$0 \to \mathcal{H}^{-1}E \to \mathcal{H}^{-1}F \to \mathcal{H}^{-1}T \to \mathcal{H}^0E \to \mathcal{H}^0F \to \mathcal{H}^0T \to 0.$$

We have $\mathrm{rk}(\mathcal{H}^0T) = 0 = \deg(\mathcal{H}^0T)$, hence $\mathrm{rk}(\mathcal{H}^{-1}T) = 3$ and $\deg(\mathcal{H}^{-1}T) = 2$. In particular, any subsheaf $G \subset \mathcal{H}^{-1}T$ satisfying $\mu_H(G) < \frac{3}{4}$ also satisfies $\mu_H(G) < \frac{2}{3}$, thus $\mathcal{H}^{-1}T \in \mathbf{F}^{2/3}$. Moreover, either $\mathcal{H}^{-1}E = 0$ or $\mu_H(\mathcal{H}^{-1}E) \leq \mu_H(\mathcal{H}^0F) = \frac{1}{2}$, thus $\mathcal{H}^{-1}E \in \mathbf{F}^{2/3}$. From this observation, we deduce that the exact (in $\mathbf{Coh}^{3/4}(S)$) sequence (2.29) still holds in $\mathbf{Coh}^{2/3}(S)$. But then at $\beta = 2/3$ we have $Z(t) = 0$ which is absurd. □

We will see next section (see Proposition 2.4.16) that W_r is an actual wall which is the reflection of W_l by the vertical wall W_v. In particular, W_l and W_r induce the same wall in $\mathrm{Mov}(S)$.

2.4.3 Crossing the walls

In this section we study the walls W_v (vertical wall), W_l (circular wall on the left side of W_v) and W_r (circular wall on the right side of W_v). To do so, we study the hyperbolic lattice (2.19) associated to each wall. In view of the proof of Proposition 2.4.9, we see that for any wall W intersecting $U(S)$, a class $w \in \Lambda$ lies in $\mathcal{H}_W$ if and only if $\Im \frac{Z(w)}{Z(v)} = 0$ for any $\sigma \in W \cap \mathbb{H}$. In otherwords, we can focus our attention to stability conditions of the form $\sigma_{\alpha,\beta}$ only.

Let us denote $\mathcal{M}_{\alpha,\beta} := \mathcal{M}_{\sigma_{\alpha,\beta}}[v]$.

The vertical wall W_v

Set $W = W_v$, then

$$\mathcal{H}_W = \mathrm{Span}_{\mathbb{Z}}((2,1,4),(0,0,1)) = \{(2a,a,b) \in \Lambda \mid a,b \in \mathbb{Z}\}.$$

In particular, for any $w \in \mathcal{H}_W$, we have $w^2 = 4a(4a-b)$ so w cannot be spherical, and $\langle v, w\rangle$ is even. Moreover, the class $(2,1,4)$ is an isotropic class lying in $\mathcal{H}_W$. By Theorem 2.4.7, the wall W is of type *Li-Gieseker-Uhlenbeck*. By [BM14a], Lemma 10.1, the wall W is *bouncing wall*, in the sense that the image of the chambers on both side of the wall in $\mathrm{Stab}^+(S)$ are sent to the same chamber in $\mathrm{Mov}(\mathcal{M}_S)$. In particular, for $\beta_- < \frac{1}{2} < \beta_+$ close enough to $\frac{1}{2}$ and admissible α's, both moduli spaces $\mathcal{M}_{\alpha,\beta_\pm}$ are isomorphic.

Note that by Theorem 1.4.16, $\mathcal{M}_{\alpha,\beta_-} \simeq \mathcal{M}_S$, and the birational transformation when hitting the wall contracts the non-locally free sheaves $F \in \mathcal{M}_S$, as we see by (2.20).

The circular wall W_l

Set $W = W_l$, then

$$\mathcal{H}_W = \mathrm{Span}_{\mathbb{Z}}((0,1,3),(1,0,0)) = \{(a,b,3b) \in \Lambda \mid a,b \in \mathbb{Z}\}.$$

Note that there is a whole in W_l at $\beta = \frac{1}{3}$. For $\beta < \frac{1}{3}$, $\mathcal{U}_S^\vee$ is a stable object (see Lemma 2.4.11), so this side of W_v is not totally semistable. But for $\beta > \frac{1}{3}$, $-w = -(3,1,3)$ is an *effective* spherical class with $\langle -(3,1,3), v\rangle = -1 < 0$, so this portion of $\mathcal{W}_l$ is totally semistable. Note that $-(3,1,3)$ corresponds to the σ-semistable object $\mathcal{U}^\vee[1]$.

For any $w = (a,b,3b) \in \mathcal{H}_W$, $w^2 = 16b^2 - 6ab$ and $\langle w, v\rangle = 10b - 3a$. Condition $\langle w, v\rangle = 1$ gives $a = 10k+3$, $b = 3k+1$ with $k \in \mathbb{Z}$. For such a, b, we cannot have $w^2 = 0$. In particular, there is no isotropic class w satisfying $\langle w, v\rangle = 1$. The same argument show that there is no isotropic class w with $\langle w, v\rangle = 2$ nor spherical class s with $\langle s, v\rangle = 0$.

On the otherhand, $v(\mathcal{U}_S^\vee) = (3,1,3)$ is a spherical class such that $0 < \langle (3,1,3), v\rangle = 1 \leq 2 = \frac{v^2}{2}$. Hence W_l is a *flopping* wall. Pick $\sigma_\pm$ on each side of the wall near a stability condition $\sigma_{\alpha,\beta} \in W_l$ with $\beta < \frac{1}{3}$. Recall that on the wall W_l, any stable object of class $(-1,0,0)$ is of the for $\mathcal{I}_x^\vee[1]$ for some $x \in S$. Similarly, by Theorem 1.4.12 the only stable object of class $(3,1,3)$ on W_l is $\mathcal{U}_S^\vee$. Indeed, pick a stability condition $\sigma_{\alpha,\beta}$ on W_l and a stable object E, if $\sigma_{\alpha,\beta}$ is not generic with respect to $(3,1,3)$ then take a nearby generic stability condition $\sigma_{\alpha,\beta+\epsilon}$. By openness stability, E is still $\sigma_{\alpha,\beta+\epsilon}$-stable, hence isomorphic to $\mathcal{U}_S^\vee$.

In the proof of [BM14a], Proposition 9.1, we see that the wallcrossing give a birational transformation $f : \mathcal{M}_{\sigma_+}[v] \dashrightarrow \mathcal{M}_{\sigma_-}[v]$ which contracts the objects E obtained as extensions (in $\mathbf{Coh}^{\beta}(S)$)

$$\mathcal{U}^{\vee} \hookrightarrow E \twoheadrightarrow \mathcal{I}_x[1],$$

for $x \in S$. We have $\mathrm{ext}^1(\mathcal{I}_x[1]^{\vee}, \mathcal{U}^{\vee}) = \hom(\mathcal{I}_x^{\vee}, \mathcal{U}^{\vee}) = \hom(\mathcal{U}, \mathcal{I}_x) = 3$. Hence the birational transformation f is a flop along the $\mathbb{P}^2$-bundle obtained this way.

Note that for $\beta > \frac{1}{3}$, the same phenomenon occurs. In view of Proposition 2.4.12 the objects which are S-equivalent to each other on the wall are the sheaves F for which $\mathrm{ext}^1(F, \mathcal{U}_S^{\vee}) > 1$, or equivalently $\mathrm{Hom}(\mathcal{U}_S^{\vee}, F) \neq 0$. These sheaves are exactly the one described on the other part of W_l.

The circular wall W_r

It turns out that the potential wall W_r on the right side of W_v is the same as the circular wall W_l up to a reflection.

Proposition 2.4.16. *The wall W_r is the image of W_l by the reflection in the vertical wall W_v. In particular, W_r and W_l have the same image in* $\mathrm{Mov}(S)$ *via the map* $l : \mathrm{Stab}^+(S) \to \mathrm{Mov}(S)$ *(see (2.18)).*

Proof. Note that a wall W associated to a destabilizing class w is sent by l_0 to $\theta_v(w^{\perp} \cap v^{\perp})$ (see section 2.4.1). We get

$$\begin{aligned} l_0(W_r) &= v^{\perp} \cap (3,1,3)^{\perp} = \mathbb{R}(16,13,80) \\ l_0(W_l) &= v^{\perp} \cap (1,1,8)^{\perp} = \mathbb{R}(16,3,0). \end{aligned}$$

In the proof of [BM14a], Lemma 10.1, we see that the reflection ρ_D which identifies the chambers in both side of W_v is a multiple of $(2,1,5)$. Direct computations give $\rho_D(16,13,80) = (16,3,0)$. □

Identifying the birational models

Thanks to the description of the wallcrossings, we can complete the proof of Theorem 2.4.1 and identify the birational models of $\mathcal{M}_S$ appearing in section 2.3.

Proposition 2.4.17. *Let σ be a stability condition in the interior chamber cut out by W_l. Then*

$$\mathcal{M}_S[5,2,6] \simeq \mathcal{M}_{\sigma}[2,1,3].$$

Proof. By computations of walls, $\mathcal{M}_S[5,2,6]$ is either isomorphic to $\mathcal{M}_{\sigma}[2,1,3]$ or isomorphic to $\mathcal{M}_S$. By [Yos99], Theorem 2.5 (with $v = (5,2,6)$, $w = (2,1,3)$ and $v_1 = (3,1,3)$ in the author's notations), we see that $\mathcal{M}_S[5,2,6]$ and $\mathcal{M}_S$ are not isomorphic and related by a flop. Moreover, this flop $\mathcal{M}_S \dashrightarrow \mathcal{M}_S[5,2,6]$ is exactly the one given by wallcrossing W_l. □

Proposition 2.4.18. *The functor ϕ_{10} induces an isomorphism $\phi_{10} : \mathcal{M}_{S'}[0,H',0] \xrightarrow{\sim} \mathcal{M}_S[5,2,6]$.*

Proof. The equivalence ϕ_{10} gives an isomorphism $\mathcal{M}_{S'} \xrightarrow{\sim} \mathcal{M}_{\sigma}[5,2,6]$ for some stability condition $\sigma \in \mathrm{Stab}(S)$. By [BM14a], Theorem 2.12, we can assume $\sigma \in \mathrm{Stab}^+(S)$, and once again by Corollary 1.4.15 we can assume that σ is of the form $\sigma = \sigma_{\alpha,\beta}$. In particular, since $\mathcal{M}_{S'}[0,H',0]$ is a smooth K-trivial birational model of $\mathcal{M}_S$, then $\mathcal{M}_{\sigma}[5,2,6]$ is either isomorphic to $\mathcal{M}_S$ or $\mathcal{M}_S[5,2,6]$. We conclude by the following remark. Let $[X] \in \mathcal{W}$ be a Fano threefold, let $[\Gamma] \in \mathcal{W}$ be the corresponding quartic curve and let $F \neq G \in \mathcal{M}_X$ be stable sheaves such that $\phi_{11}^! F \simeq \phi_{11}^! G$. Then F_S and G_S define two different points of $\mathcal{M}_S$, though $(\phi_{11}\phi_{11}^! F)_S = (\phi_{11}\phi_{11}^! G)_S \in \mathcal{M}_S[5,2,6]$. In particular we have $\phi_{10}(i_{\Gamma S'})_*(\phi_{11}^! F) \simeq \phi_{10}(i_{\Gamma S'})_*(\phi_{11}^! G)$. In view of Lemma 2.1.1, since ϕ_{10} is an equivalence, we necessarily have $\mathcal{M}_{\sigma}[5,2,6] \simeq \mathcal{M}_S[5,2,6]$. □

2.4.4 Movable and nef cones of $\mathcal{M}_S$

In this last section, we want to give a precise description of $\mathrm{Pos}(\mathcal{M}_S) \subset \mathrm{NS}(\mathcal{M}_S)$. Recall we have $v = (2, 1, 3)$. Sraight computations give an orthogonal basis $\mathcal{B} := \{e_1 = (0, -1, -8), e_2 = (2, 1, 5)\}$ of $v^\perp$.

For the next proposition, we identify $\mathrm{NS}(\mathcal{M}_S) = v^\perp$ by Proposition 2.4.3.

Proposition 2.4.19. *The positive cone* $\overline{\mathrm{Pos}(\mathcal{M}_S)}$ *is generated by* $e_1 + 2e_2$ *and* $e_1 - 2e_2$*. The big movable cone* $\mathrm{Mov}(\mathcal{M}_S) \cap \mathrm{Pos}(\mathcal{M}_S)$ *is cut out in* $\mathrm{Pos}(\mathcal{M}_S)$ *by the line* $\mathbb{R}e_1$*, and it identifies with the chamber which contains* $21e_1 + 8e_2$*.*

Moreover, $\mathrm{Mov}(\mathcal{M}_S)$ *decomposes into two chambers cut out by the line* $\mathbb{R}(8e_2 + 5e_1)$*. The chamber adjacent to the line* $\mathbb{R}e_1$ *is* $\mathrm{Nef}(\mathcal{M}_S)$ *and correspond to the birational model* $\mathcal{M}_S$*, and the other chamber is the image of* $\mathrm{Nef}(\mathcal{M}_S[5, 2, 6])$ *via the birational map given by crossing the wall* W_l*. See Figure 2.1.*

Proof. First, note that $w = ae_1 + be_2$, for $a, b \in \mathbb{Z}$, satisfies $w^2 = 0$ if and only if $16a^2 = 4b^2 = 0$. Hence the cone $\overline{\mathrm{Pos}(\mathcal{M}_S)}$ is the cone generated, up to a sign, by $\{e_1 + 2e_2, e_1 - 2e_2\}$ and contains either e_1 or its inverse. Pick $\alpha = 1, \beta = 0$. It gives a well-defined generic stability condition $\sigma := \sigma_{1,0}$, and its image $A := l(\sigma)$ is an ample class (Theorem 2.4.4). By computations, we find

$$A = \theta_v(\frac{1}{425}(16, -13, 128)) = \theta_v(\frac{1}{425}(21e_1 + 8e_2)).$$

We deduce that $\overline{\mathrm{Pos}(\mathcal{M}_S)}$ is generated by $\{e_1 + e_2, e_1 - 2e_2\}$. Now the image of the vertical wall W_v is given by $v^\perp \cap (2, 1, 4)^\perp = \mathbb{R}e_1$. Hence $\mathrm{Mov}(\mathcal{M}_S)$ is the upper half-cone of $\overline{\mathrm{Pos}(\mathcal{M}_S))}$ cut out by $\mathbb{R}e_1$.

Finally, the image of the circular wall W_l is given by $v^\perp \cap (3, 1, 3)^\perp = \mathbb{R}(16, 3, 0)$, and we obtain two chambers in $\mathrm{Mov}(\mathcal{M}_S)$. The chamber containing the ample class A is $\mathrm{Nef}(\mathcal{M}_S)$, and the other chamber correspond to the unique other birational model $\mathcal{M}_S[5, 2, 6]$.

□

Figure 2.1 represents the cone $\mathrm{Pos}(\mathcal{M}_S)$ (between the green lines), and Figure 2.2 represents the corresponding walls and chambers in $\mathrm{Stab}^+(S)$. The cone $\mathrm{Mov}(\mathcal{M}_S)$ decomposes in two chambers: the **(a)** part is the nef cone $\mathrm{Nef}(\mathcal{M}_S)$ and the **(b)** part is the image of $\mathrm{Nef}(\mathcal{M}_S[5, 2, 6])$. The wall separating **(a)** and **(b)** (resp. **(a')** and **(b')**), represented by a blue doted line, is W_l (resp. W_r). The wall separating **(a)** and **(a')** (red doted line) is W_v.

Figure 2.1: The positive cone $\mathrm{Pos}(\mathcal{M}_S)$ inside $\mathrm{NS}(\mathcal{M}_S)$.

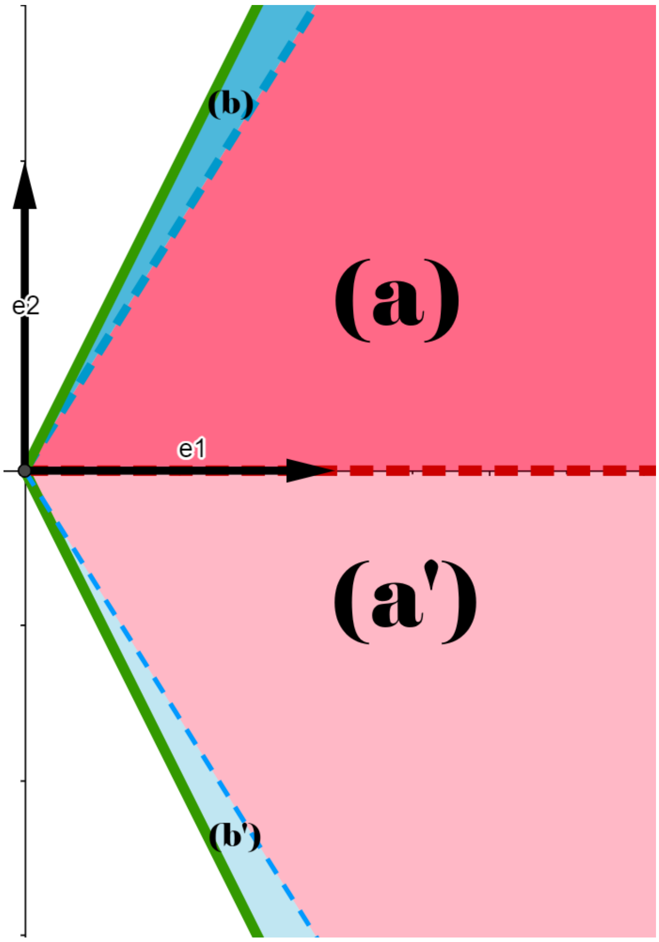
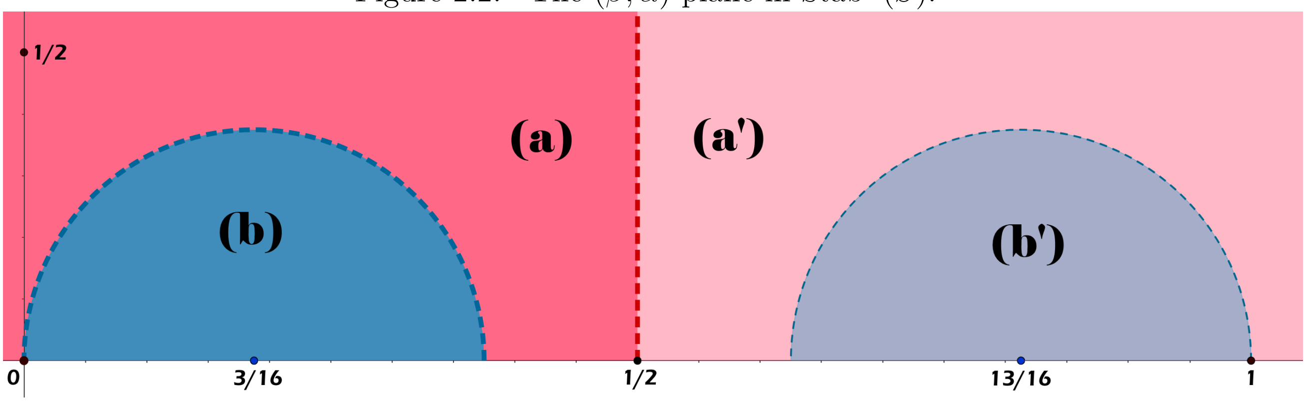


Figure 2.2: The (β, α)-plane in $\mathrm{Stab}^+(S)$.



Chapter 3

Dynamical systems and derived categories

This chapter aims to study dynamical systems in a categorical point of view. The category we are interested in is the derived category $\mathrm{D}^b(X)$ of a smooth projective variety X, that we will always assume to be defined over $\mathbb{C}$ if not stated otherwise. To an endofonctor $F : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ are associated two quantities:

- the *categorical entropy* $h_{cat}(F)$ (defined section 3.1.1), which computes the categorical complexity of the dynamical system $(\mathrm{D}^b(S), F)$, and
- the *generalized topological entropy* $\log \rho(F^H)$, where $\rho(F^H)$ is the spectral radius of the action of F on the Betti cohomology $H^*(X, \mathbb{C})$ of X.

These two quantities coincide in specific cases, but do not in general. We provide section 3.4 a new example of an autoequivalence $\varphi \in \mathrm{Aut}(\mathrm{D}^b(S))$, where S is a smooth projective surface containing a (-2)-curve $C \subset S$, for which the categorical entropy $h_{cat}(\varphi)$ is positive and the generalized topological entropy $\log \rho(\varphi^H)$ is zero.

In sections 3.2 and 3.3, we study more generally the possible values of $\log \rho(\varphi^H)$ when φ ranges in $\mathrm{Aut}(\mathrm{D}^b(S))$ for a smooth projective surface S with mild assumptions. We prove (Theorem 3.2.2 and Proposition 3.3.3) that

$$\{\log \rho(\varphi^H) \mid \varphi \in K \subset \mathrm{Aut}(\mathrm{D}^b(S))\} = \{\log \rho(f^*) \mid f \in \mathrm{Aut}(S)\}$$

where K is a subgroup generated by standard autoequivalences and spherical twists along (-2)-curves in S that we assume to form a disjoint union of configuration of Dynkin type A or $\tilde{A}$. We deduce that if moreover S satisfies $K_S \not\equiv 0$ and admits no minimal elliptic fibration, then the existence of an autoequivalence $\varphi \in \mathrm{Aut}(\mathrm{D}^b(S))$ with $\log \rho(\varphi^H) > 0$ forces S to be rational (Corollary 3.2.7). This is based on a celebrated result of Cantat (Theorem 3.1.12) and Uehara's trichotomy for surfaces according to their group of autoequivalences (see section 3.1.2).

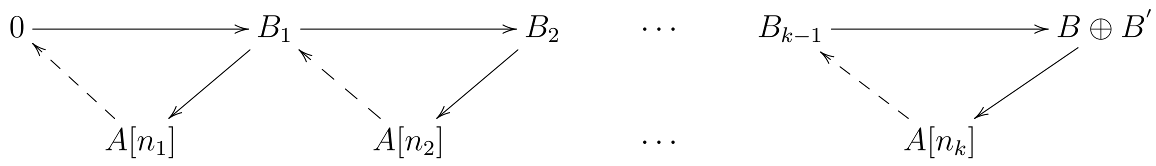
3.1 Categorical entropy and autoequivalences of surfaces

3.1.1 Categorical entropy

Let K be a field and $\mathcal{T}$ be a K-linear triangulated category of finite type, that is for any two objects $A, B \in \mathcal{T}$ we have $\mathrm{Hom}(A, B[i]) = 0$ for $|i| \gg 0$.

Let $A, B \in \mathcal{T}$ be non-zero objects. We denote $\langle A \rangle \subset \mathrm{D}^b(X)$ the smallest triangulated subcategory closed under taking direct summand and isomorphisms. If $B \in \langle A \rangle$, where $\langle - \rangle$

denotes the split closure, we can construct a tower of triangles



(3.1)

for some $B' \in \mathcal{T}$, with $k \geq 0$ and $n_i \in \mathbb{Z}$.

Definition 3.1.1 ([DHKK14], Definition 2.1)**.** We define the *complexity* of A relative to B as the following function: for all $t \in \mathbb{R}$,

$$\delta_t(A, B) := \inf \left\{ \sum_{j=1}^{k} e^{n_j t} \right\} \in \mathbb{R} \cup \{+\infty\}$$

where the infimum is taken over all possible towers as in (3.1).

Note that $\delta_t(A, B) = +\infty$ for all t if and only if $B \notin \langle A \rangle$.

We omit the proof of the next technical proposition.

Proposition 3.1.2 ([DHKK14], Proposition 2.2)**.** *For any non-trivial $A, B, C \in \mathcal{T}$ we have the following:*

- *$\delta_t(A, B)$ depends on A and B only up to isomorphisms,*
- *$\delta_t(A, C) \leq \delta_t(A, B)\delta_t(B, C)$,*
- *If $\mathcal{T}'$ is a triangulated category of finite type and $F : \mathcal{T} \to \mathcal{T}'$ is an exact functor, then $\delta_t(FA, FB) \leq \delta_t(A, B)$.*

Definition 3.1.3 ([DHKK14], Definition 2.4)**.** Let G be a split generator of $\mathcal{T}$ (i.e. $\mathcal{T} = \langle G \rangle$) and $\phi : \mathcal{T} \to \mathcal{T}$ an exact endofunctor such that $\phi^n \neq 0$ for all $n \geq 0$. The *categorical entropy* of ϕ is defined to be the function

$$h_t(\phi) := \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, \phi^n G).$$

In most cases, we are interested in the value at $t = 0$. The following lemma is very useful for computations, as one can adapt the generator depending on the autoequivalence studied.

Lemma 3.1.4 ([DHKK14], Lemma 2.5)**.** *The limit $\lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, \phi^n G)$ exists in $[-\infty, +\infty)$ for every $t \in \mathbb{R}$ and is independant of the choice of the split-generator G. Moreover, if G' is another split-generator, then*

$$h_t(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, \phi^n G').$$

Proof. It follows from Proposition 3.1.2 that

$$\delta_t(G, \phi^{n+m} G) \leq \delta_t(G, \phi^n G)\delta_t(\phi^n G, \phi^{n+m} G) \leq \delta_t(G, \phi^n G)\delta_t(G, \phi^m G).$$

Recall Fekete's Lemma: for any subadditive sequence $(a_n)_{n\geq 1}$ we have $\lim_{n\to\infty} a_n/n = \inf\{a_n/n \mid n \geq 1\}$. We obtain that $h_t(\phi) < +\infty$. The rest of the claim is an easy consequence of Proposition 3.1.2. □

In the case of the derived category of a smooth projective variety X, it turns out that the entropy can be computed as Poincaré polynomials in Ext groups.

Proposition 3.1.5 ([DHKK14], Theorem 2.6)**.** *For any autoequivalence* $\phi : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ *and for any split-generators* G, G' *we have*

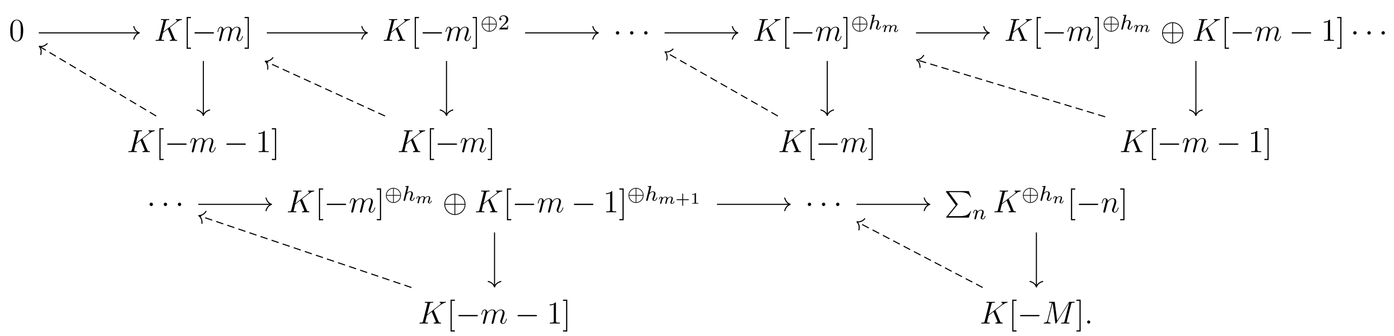
$$h_t(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{j\in\mathbb{Z}} \dim \operatorname{Ext}^j(G, \phi^n G') e^{jt} \right).$$

Proof. First we prove the following lemma.

Lemma 3.1.6. *For any complex of* K*-vector spaces* $V \in \mathrm{D}^b(K)$*, we have*

$$\delta_t(K, V) = \sum_n \dim \mathcal{H}^n(V) e^{-nt}.$$

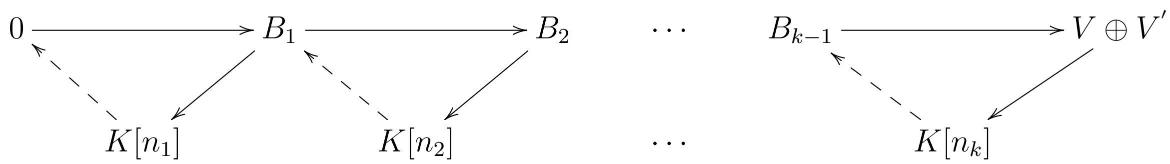
Proof. The complex V decomposes as $V = \bigoplus_n \mathcal{H}^n(V)[-n] = \bigoplus_n K^{\oplus h_n}[-n]$, where $h_n := \dim \mathcal{H}^n(V)$. Denote $M = \max\{n \mid h_n \neq 0\}$ and $m = \min\{n \mid h_n \neq 0\}$. We have a tower of triangles



Hence we have $\delta_t(K, V) \leq \sum_n h_n e^{-nt}$. Note that for any triangle $A \to B \to C$ in $\mathrm{D}^b(K)$ we have

$$\sum_n \dim \mathcal{H}^n B \leq \sum_n \dim \mathcal{H}^n A + \dim \mathcal{H}^n(C).$$

Apply this recursively to any tower



to obtain $\sum h_n e^{-nt} \leq \delta_t(K, V)$. □

Now pick a generator $G \in \mathrm{D}^b(X)$, and set $G_n := \phi^n(G')$. From Lemma 3.1.6 we have

$$\begin{aligned} \delta_t(K, R\operatorname{Hom}(G, G_n)) &\leq \delta_t(K, R\operatorname{Hom}(G, G))\delta_t(R\operatorname{Hom}(G, G), R\operatorname{Hom}(G, G_n)) \\ &\leq \delta_t(K, R\operatorname{Hom}(G, G))\delta_t(G, G_n). \end{aligned}$$

On the otherhand, we have

$$\begin{aligned} \delta_t(G, G_n) &\leq \delta_t(G, G \otimes R\operatorname{Hom}(G, G))\delta_t(G \otimes R\operatorname{Hom}(G, G_n), G_n) \\ &\leq \delta_t(K, R\operatorname{Hom}(G, G))\delta_t(G \otimes R\operatorname{Hom}(G, -), \mathrm{Id}_{\mathrm{D}^b(X)}(-)). \end{aligned}$$

Here, we use Proposition 3.1.2 for the functor $- \otimes G : \mathrm{D}^b(K) \to \mathrm{D}^b(X)$ and $\Phi \mapsto \Phi(G_n), Fun(\mathrm{D}^b(X), \mathrm{D}^b(X)) \to \mathrm{D}^b(X)$. For the latter, we use that $\mathrm{D}^b(X)$ admits a dg-enhancement and that it is smooth as a dg-category. See [DHKK14], Theorem 2.6 and [Hai15], Lecture 10. □

Remark 3.1.7 ([Orl09]). We can point out that if X is a smooth projective variety, then for any very (anti)-ample line bundle $\mathcal{M}$ on X, the vector bundle $\mathcal{M} \oplus \mathcal{M}^{\otimes 2} \oplus \cdots \oplus \mathcal{M}^{\dim X+1}$ is a generator of $\mathrm{D}^b(X)$. This is very useful for computations of categorical entropy.

Example 3.1.8. For X a smooth projective variety, entropy of some functors are easily computed. For instance, given $L \in \mathrm{Pic}(X)$ we have $h_t(- \otimes L) = h_t([k]) = 0$ for all t ([DHKK14]). Also, the Serre functor S (see (1.14)) satisfies $h_t(S) = \dim(X)t$. In [Ouc20], the author shows that the categorical entropy of a spherical twist $T_{\mathcal{E}}$ satisfies $h_t(T_{\mathcal{E}}) = (1 - \dim X)t$ for $t \leq 0$. However, the entropy of the composition of functors is much harder to compute in general, and it can have positive entropy even if the individual functors have not, see for instance Corollary 3.4.6.

In the classical settings, Gromov and Yomdin showed a link between topological entropy and cohomology on compact Käkler manifolds.

Theorem 3.1.9 ([Gro87],[Gro03],[Yom87]). *Let X be a compact Kähler manifold and let $f : X \to X$ be a surjective holomorphic map. Then*

$$h_{top}(f) = \log \rho(f^*)$$

where $f^* : H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$ is the induced map on cohomology, and $\rho(f^*)$ is its spectral radius, that is the largest modulus of eigenvalues.

It is therefore natural to wonder if a similar statement hold in the realm of derived categories. In [KT19], Kikuta and Takahashi proposed the following Gromov-Yomdin type conjecture:

Conjecture 3.1.10. *Let X be a smooth projective variety over $\mathbb{C}$. For any autoequivalence $\phi \in \mathrm{Aut}(\mathrm{D}^b(X))$, we have*

$$h_0(\phi) = \log \rho(\phi^H)$$

where h_0 is the categorical entropy (valued in 0), ϕ^H is the $\mathbb{C}$-linear isomorphism induced by ϕ on the cohomology group $H^*(X, \mathbb{C})$ and ρ denotes the spectral radius.

The lower bound $\log \rho(\phi^H) \leq h_0(\phi)$ is always true ([KST18]). This conjecture is not true in full generality. Indeed, here are the first two known counterexamples:

1. In [Fan18], Fan considers X a strict Calabi-Yau manifold of dimension $d \geq 3$, that is a smooth projective variety X with $\omega_X \simeq \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim X$. In this case, $\mathcal{O}_X$ is a spherical object, and the autoequivalence

$$\phi := T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}_X(-1)),$$

for $\mathcal{O}_X(1)$ a very ample line bundle on X, satisfies $h_0(\phi) > 0$. On the other hand, in the particular cases where X is a hypersurface in $\mathbb{P}^{d+1}$ of degree $(d+2)$, with $d \geq 4$, the autoequivalence ϕ satisfies

$$\log(\rho(\phi^H)) = 0.$$

2. In [Ouc20], based on the proof of Fan, Ouchi considers X a K3 surface, and pick the same autoequivalence

$$\phi = T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}_X(1))$$

for $\mathcal{O}_X(1)$ a very ample line bundle on X. Set $2d = H^2$. Then he proves

$$h_0(\phi) \geq \log(d+2)$$

while on the other hand

$$\rho(\phi^H) = \begin{cases} 1 & \text{if } d = 1,2,3,4 \\ \frac{d-2+\sqrt{d^2-4d}}{2} & \text{if } d \geq 5. \end{cases}$$

However, Conjecture 3.1.10 is known to be true when X is a curve [Kik17], an abelian surface [Yos20], a variety with ample (anti)-canonical bundle [KT19].

In section 3.4, we construct a new counterexample of Conjecture 3.1.10 on any surface S containing a (-2)-curve $C \subset S$.

3.1.2 Generalized topological entropy

Definition 3.1.11. Let X be a smooth projective variety and $\phi : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ be an autoequivalence. Then the value $\log \rho(\phi^H)$ is called the *(generalized) topological entropy* of ϕ.

Theorem 3.1.9 assures that the definitions of topological entropy coincide for $\phi = f^*$, with $f \in \mathrm{Aut}(X)$.

The generalized topological entropy have interesting properties in itself. We will focus ourselves on the case of surfaces, so for now on we let S be a smooth projective surface.

In [Can99], Cantat prove the following.

Theorem 3.1.12. *Assume that S admits an automorphism $f \in \mathrm{Aut}(S)$ of positive topological entropy. Then S is birational to either (i) $\mathbb{P}^2$, (ii) a K3 surface, (iii) a 2-dimensional complex torus or (iv) an Enriques surface. In the case (i), S is a blow up of $\mathbb{P}^2$ at 10 or more points.*

We aim to find an analogue of this theorem relying the birational nature of the surface S with the action on cohomology of its group of autoequivalences $\mathrm{Aut}(\mathrm{D}^b(S))$. One point to adress is the description of the group $\mathrm{Aut}(\mathrm{D}^b(S))$. In [Ueh19], Uehara proposes a trichotomy for surfaces. He conjectures an explicit description of the group of autoequivalences in each cases. Here we summarize his paper.

Define an integer $N_S \in \mathbb{Z}$, called the *Fourier-Mukai support dimension* of S, as follow. For any Fourier-Mukai autoequivalence $\phi_{\mathcal{P}} : \mathrm{D}^b(S) \to \mathrm{D}^b(S)$, consider $\dim_S(\mathcal{P})$ the maximal dimension of irreducible components of $\mathrm{Supp}(\mathcal{P})$ which dominate S by the first projection (that is, the projection to the S for which $\mathrm{D}^b(S)$ is the domain of $\phi_{\mathcal{P}}$). Then we set

$$N_S := \max\{\dim_S(\mathcal{P}) \mid \phi_{\mathcal{P}} \in \mathrm{Aut}(\mathrm{D}^b(S))\}. \tag{3.2}$$

It turns out that N_S is either $2, 3$ or 4.

1. $N_S = 2$ if and only if $K_S \not\equiv 0$ and S has no a minimal elliptic fibration. We develop this case in section 3.2.

2. $N_S = 3$ if and only if $K_S \not\equiv 0$ and S has a minimal elliptic fibration. We develop this case in section 3.3.

3. $N_S = 4$ if and only if $K_S \equiv 0$. In this case, S is either a K3, abelian, bielliptic or Enriques surface. Here are the known studies of $\mathrm{Aut}(\mathrm{D}^b(S))$ in this case.

- If A is an abelian variety over an algebraically closed field K, Orlov proved in [Orl02] that there exist an exact sequence

$$0 \to \mathbb{Z} \oplus (A \times \hat{A})_K \to \mathrm{Aut}(\mathrm{D}^b(A)) \to U(A \times \hat{A}) \to 0,$$

where $U(A \times \hat{A})$ is the group of isometric automorphism of $A \times \hat{A}$, and any point $(a, \alpha) \in (A \times \hat{A})_K$ defines the autoequivalence $T_{a*}(-) \otimes \mathcal{P}_\alpha$ where T_{a*} is induced by the shift automorphism $m(-, a) : A \to A$ and $\mathcal{P}$ is the Poincaré bundle on $A \times \hat{A}$.

- When S is bielliptic, Potter gives in [Pot17] a description of $\mathrm{Aut}(\mathrm{D}^b(S))$. Moreover, if S is of odd type (in the sense of [BPV84], §V.5) the group $\mathrm{Aut}(\mathrm{D}^b(S))$ is generated by standard autoequivalences and relative Fourier-Mukai transforms along the two elliptic fibrations (see section 3.3).
- The case of a K3 surface S with Picard rank $\rho(S) = 1$ have been worked out by Bayer and Bridgeland in [BB17]. Namely, they prove that the group $\mathrm{Aut}(\mathrm{D}^b(S))$ fits in an exact sequence
$$0 \to \mathrm{Aut}^0(\mathrm{D}^b(S)) \to \mathrm{Aut}(\mathrm{D}^b(S)) \to \mathrm{Aut}^+(H^{even}(S,\mathbb{C})) \to 0,$$
where $\mathrm{Aut}^+(H^{even}(S,\mathbb{C}))$ denotes the index 2 subgroup of Hodge isometry preserving the orientation of positive definite 4-planes (see Remark 1.3.22). The kernel $\mathrm{Aut}^0(\mathrm{D}^b(S))$ is in this case generated by the even shift $[2]$ and spherical twist $T_{\mathcal{E}}$ along all spherical objects $\mathcal{E} \in \mathrm{D}^b(S)$.
- Up to the author knowledge, no precise description of the group of autoequivalences of Enriques surfaces exist yet. They seem closely related to autoequivalences of K3 surfaces. Bridgeland and Maciocia have studied equivalences between Enriques surfaces in [BM17] and [BM01]. First, two Enriques surfaces are derived equivalent if and only if they are isomorphic. On the other hand, let S be an Enriques surface and consider $\iota : \widetilde{S} \to S$ the canonical cover of S, that is $\widetilde{S}$ is a K3 surface and ι is an involution. Then any autoequivalence $\phi \in \mathrm{Aut}(\mathrm{D}^b(S))$ lift to an equivariant autoequivalence $\widetilde{\phi} \in \mathrm{Aut}(\mathrm{D}^b(\widetilde{S}))$ such that the diagram
$$\begin{array}{ccc} \mathrm{D}^b(\widetilde{S}) & \xrightarrow{\widetilde{\phi}} & \mathrm{D}^b(\widetilde{S}) \\ {\scriptstyle \iota_*}\downarrow & & \downarrow{\scriptstyle \iota_*} \\ \mathrm{D}^b(S) & \xrightarrow{\phi} & \mathrm{D}^b(S) \end{array}$$
commute. Conversely, any equivariant autoequivalence $\widetilde{\phi} \in \mathrm{Aut}(\mathrm{D}^b(\widetilde{S}))$ descends to an autoequivalence $\phi \in \mathrm{Aut}(\mathrm{D}^b(S))$.

In view of Cantat Theorem 3.1.12, $N_S = 4$ covers all non-rational cases where automorphisms of S can have positive topological entropy.

3.2 Case $N_S = 2$

Assume $N_S = 2$ (see (3.2)). Denote Z the union of all (-2)-curves on S, and set

$$B_Z(S) = \langle T_{\mathcal{E}} \mid \mathcal{E} \in \mathrm{D}^b(S) \text{ spherical object, } \mathrm{Supp}(\mathcal{E}) \subset Z \rangle.$$

Uehara poses the following conjectural description of autoequivalences of S.

Conjecture 3.2.1 ([Ueh19]).

$$\mathrm{Aut}(\mathrm{D}^b(S)) = \langle B_Z(S), \mathrm{Pic}(S) \rangle \rtimes \mathrm{Aut}(S) \times \mathbb{Z}[1].$$

In the same paper, he proves it when Z is a disjoint union of configuration of (-2)-curves of type A, in the sense of Dynkin diagrams.

Recall that on these diagrams, a vertex represents (-2)-curves and two vertices are linked by an edge if the two corresponding curves intersect.

Consider the subset $B := \langle T_{\mathcal{O}_C(a)} | \ C\ (-2)\text{-curve}, \ a \in \mathbb{Z} \rangle \subset B_Z(S)$. The main result of this section is the following.

Theorem 3.2.2. *Let S be a smooth surface for which its union Z of (-2)-curves is a disjoint union of finite configurations of type A-D-E. Let $\varphi \in \langle B, \mathrm{Pic}(S)\rangle \rtimes \mathrm{Aut}(S) \quad \times \mathbb{Z} \cdot [1]$ be an autoequivalence, so that, up to a shift, we have a decomposition*

$$\varphi = b \circ (- \otimes L) \circ f^*$$

with $b \in B, L \in \mathrm{Pic}(S), f \in \mathrm{Aut}(S)$.

Then

$$\rho(\varphi^H) = \rho(f^*).$$

Remark 3.2.3. In [IU05] Corollary 6.10, Ishii and Uehara prove

$$B_Z = B$$

when all configurations are of type A only. This relies on a generalization of Grothendieck theorem: a pure 1-dimensional sheaf supported on a chain Z_0 of (-2)-curves in A_m configuration, $m \in \mathbb{Z}_{\geq 1}$, is a direct sum of line bundles on subtrees of Z_0. For any spherical object $\mathcal{E} \in \mathrm{D}^b(S)$, the authors construct inductively an autoequivalence $\psi \in B_Z(S)$ such that

$$l(\psi(\mathcal{E})) < l(\mathcal{E})$$

where $l(-) := \sum_{i,p} \mathrm{lenght}_{\mathcal{O}_{X,\eta_i}} \mathcal{H}^p(-)_{\eta_i}$, where η_i ranges within the generic points of all (-2)-curves on S. Eventually, the equality $l(\alpha) = 1$ implies $\alpha \simeq \mathcal{O}_C(a)[b]$ for some integer $a, b \in \mathbb{Z}$ and C a (-2)-curve, and Proposition 1.3.10 permits to conclude.

Proof of Theorem 3.2.2. Denote

$$G := \langle B, \mathrm{Pic}(S) \rtimes \mathrm{Aut}(S)\rangle \times \mathbb{Z}[1].$$

Pick $\varphi \in G$. First, we show that φ admits a decomposition as stated.

When ϕ is an autoequivalence belonging to $\mathrm{Pic}(S) \rtimes \mathrm{Aut}(S)$ and C is a (-2)-curve, in view of Proposition 1.3.10 we shall consider, for $a \in \mathbb{Z}$, the image $\phi(\mathcal{O}_C(a))$.

First, for any $L \in \mathrm{Pic}(S)$, we have $L \otimes \mathcal{O}_C(a) = \mathcal{O}_C(a + l)$ with $l := \deg_C(L|_C)$.

Secondly, consider an isomorphism $f : S \to S$. It induces an isomorphism $\bar{f} : C \to C'$ for some (-2)-curve C' as the image of a (-2)-curve must be a (-2)-curve, and it's easy to check, writing i and j the natural inclusion of C and C' respectively, that $f^*(j_*\mathcal{O}_{C'}(a)) \simeq i_*(\bar{f}^*(\mathcal{O}_{C'}(a))) \simeq i_*(\mathcal{O}_C(a))$.

We conclude from this that B is normal in G. In fact, we also have $B \cap \mathrm{Aut}(S) = \{0\}$ ([IU05], Remark 4.17). Hence, up to a shift, φ decomposes as

$$\varphi = b \circ (- \otimes L) \circ f^*,$$

with $b \in B$, $L \in \mathrm{Pic}(S)$, $f \in \mathrm{Aut}(S)$ and such f does not depend on the decomposition.

Note that $\rho((\varphi^H)^{\circ m}) = \rho(\varphi^H)^m$, i.e. the spectral radius of the morphism is totally determined by the spectral radius of its powers. Hence, we can assume that f preserves each (-2)-curve: f acts by permutation on the set of (-2)-curves which is finite, thus some power of f fixes each of them.

We set

$$b = T_{\mathcal{O}_{C_1}(a_1)} \circ \cdots \circ T_{\mathcal{O}_{C_k}(a_k)}$$

with $a_1, \ldots, a_k \in \mathbb{Z}$ and $C_1, \ldots, C_k$ (-2)-curves.

Now, we use Proposition 1.3.10. Since B is normal in G, for any $m \geq 1$ there is a line bundle $L_m \in \mathrm{Pic}(S)$ and equivalences $b_j \in B$, $j = 2, \ldots, m$ such that

$$\varphi^{\circ m} = b \circ b_2 \circ \cdots \circ b_m \circ (- \otimes L_m) \circ (f^*)^{\circ m},$$

where each b_j is given by

$$b_j = T_{\mathcal{O}_{C_1}(a_1')} \circ \cdots \circ T_{\mathcal{O}_{C_k}(a_k')}$$

for some integers $a_1^{'}, \ldots, a_k^{'} \in \mathbb{Z}$ (depending on j). In other words, each b_j, $j = 2, \ldots, m$ is a composition of spherical twists along line bundles over the same curves but with different degrees.

We introduce the following notation: for a morphism $g : H^*(S, \mathbb{Q}) \to H^*(S, \mathbb{Q})$, we denote by $g_2 : H^2(S, \mathbb{Q}) \to H^2(S, \mathbb{Q})$ the restriction to $H^2(S, \mathbb{Q})$ of its composition with the projection $p : H^*(S, \mathbb{Q}) \to H^2(S, \mathbb{Q})$.

Proposition 3.2.4. *We have*

$$\rho((\varphi^H)^{\circ m}) = \max\left(\rho((\varphi^H)_2^{\circ m}), \rho(f^*)^m\right).$$

Proof. We write $1 \in H^0(S, \mathbb{Q})$ the natural generator, $[x] \in H^4(S, \mathbb{Q})$ its dual. We fix a graded basis $(1, e_1, \ldots, e_k, [x])$ of $H^{2*}(X, \mathbb{Q})$ composed by homogeneous elements.

From Remark 1.3.20 we conclude that the matrix of $(-\otimes L_m)^H \circ (f^{*\circ m})^H$ is lower-triangular by blocks, each block corresponding to a graded component of $H^*(S, \mathbb{Q})$.

Fix an integer $a \in \mathbb{Z}$. We shall compute $T^H_{\mathcal{O}_C(a)}$. By Grothendieck-Riemann-Roch, we have $v(\mathcal{O}_C(a)) = [C] + (a+1)[x]$, where $[C]$ denotes the cohomology class of the cycle C.

Now, by section 1.3.3, $T_{\mathcal{O}_C(a)}$ acts as the identity on $H^{odd}(S, \mathbb{Q})$, and for any $w \in H^{2*}(S, \mathbb{Q})$ we have

$$T^H_{\mathcal{O}_C(a)}(w) = w - \langle v(\mathcal{O}_C(a)), w\rangle v(\mathcal{O}_C(a)). \tag{3.3}$$

Denote $w = (w_0, w_2, w_4) \in \mathbb{Q} \oplus H^2(S, \mathbb{Q}) \oplus \mathbb{Q}$. As $c_1(S) \cdot [C] = -K_S \cdot C = 0$, we get

$$\begin{aligned}
\langle v(\mathcal{O}_C(a)), w\rangle &= \int_S (0, -[C], a+1) \cdot (w_0, w_2, w_4) \cdot \exp(c_1(S)/2) \\
&= \int_S \left(0, -w_0[C], -[C] \cdot w_2 + (a+1)w_0\right) \cdot \left(1, \frac{c_1(S)}{2}, \frac{c_1(S)^2}{8}\right) \\
&= (a+1)w_0 - [C] \cdot w_2.
\end{aligned} \tag{3.4}$$

This scalar only depends on w_0 and w_2, and $v(\mathcal{O}_C(a))$ has components only in degree 2 and 4 so by (3.3) we conclude that $T^H_{\mathcal{O}_C(a)}$ acts as identity on $H^4(S, \mathbb{Q})$, and by (3.3) and (3.4) we see that $T^H_{\mathcal{O}_C(a)}(1) = 1 + R$ with $R \in H^{\geq 2}(S, \mathbb{Q})$.

Hence the matrix of $(\varphi^{\circ m})^H$ is triangular by blocks, and both spherical twists and tensors by line bundles have spectral radius 1 on $H^j(S, \mathbb{Q})$, $j \neq 2$. We obtain the result. □

Lemma 3.2.5. *For any (-2)-curve C and any $a \in \mathbb{Z}$, the map*

$$(T^H_{\mathcal{O}_C(a)})_2 : H^2(S, \mathbb{Q}) \to H^2(S, \mathbb{Q})$$

does not depend on a.

Proof. By (3.4), we see that for all $w_2 \in H^2(S, \mathbb{Q})$, we have $(T^H_{\mathcal{O}_C(a)})_2(w_2) = w_2 + ([C]\cdot w_2)[C]$. □

Hence, all the morphisms b^H, b_j^H, $j = 2, \ldots, m$, restrict to the same morphism $(b^H)_2 : H^2(S, \mathbb{Q}) \to H^2(S, \mathbb{Q})$, so we obtain

$$(\varphi^H)_2^{\circ m} = (b^H)_2^{\circ m} \circ (-\otimes \mathcal{L}_m)_2^H \circ (f^*)_2^{\circ m}.$$

From Proposition 1.3.11, the spherical twists along (-2)-curves satisfy the relations.

$$\begin{aligned} T_{\mathcal{O}_{C_1}} \circ T_{\mathcal{O}_{C_2}} \circ T_{\mathcal{O}_{C_1}} &\simeq T_{\mathcal{O}_{C_2}} \circ T_{\mathcal{O}_{C_1}} \circ T_{\mathcal{O}_{C_2}} && \text{if } C_1 \cdot C_2 = 1, \\ T_{\mathcal{O}_{C_1}} \circ T_{\mathcal{O}_{C_2}} &\simeq T_{\mathcal{O}_{C_2}} \circ T_{\mathcal{O}_{C_1}} && \text{if } C_1 \cdot C_2 = 0. \end{aligned}$$

We combine these relations with the fact that $(T^H_{\mathcal{O}_C})^{\circ 2} = \mathrm{Id}$ by (3.3) to conclude that the group $\langle (T^H_{\mathcal{O}_C})_2 \mid C \text{ a } (-2)\text{-curve}\rangle \subseteq \mathrm{Aut}(H^2(S,\mathbb{Q}))$ is a quotient of a finite direct product of Coxeter groups of type A,D and E. In particular, it is finite, thus $(b^H)_2^{\circ m} = \mathrm{Id}$ for some $m \gg 0$. We obtain

$$(\varphi^H)_2^{\circ m} = (- \otimes L_m)_2^H \circ (f^*)_2^{\circ m}.$$

To conclude the proof, note that we can choose a basis for which both $(- \otimes L_m)_2^H$ and $(f^*)_2^{\circ m}$ are lower-triangular, with $(- \otimes L_m)_2^H$ having only $1'$s on the diagonal.

□

From this proof, we remark the following fact, that will be used in section 3.4.

Corollary 3.2.6. *Let S be any smooth projective surface, $C \hookrightarrow S$ a (-2)-curve. Let $L \in \mathrm{Pic}(S)$ be a line bundle. Then*

$$\rho(T^H_{\mathcal{O}_C} \circ (- \otimes L)^H) = 1.$$

We obtain a first step toward a Cantat-like result in the case of generalized topological entropy on surfaces.

Corollary 3.2.7. *Let S be a smooth surface with finitely many (-2)-curve in disjoint A-configurations. Assume $K_S \not\equiv 0$ and that S admits no minimal elliptic fibration. Then, if there is an autoequivalence $\varphi \in \mathrm{Aut}(\mathrm{D}^b(S))$ with $\rho(\varphi^H) > 1$, S is rational.*

In view of section 1.1.2, a minimal surface satisfying $N_S = 2$ is either isomorphic to $\mathbb{P}^2$, ruled or of general type. By Propositions 1.1.8 and 1.1.9, the number of (-2)-curves on this surface is necessarily finite.

3.3 Case $N_S = 3$

Assume $N_S = 3$ (see (3.2)). In this case, $K_S \not\equiv 0$ and S admits a minimal elliptic fibration, that is there exists a morphism $\pi : S \to C$ onto a smooth projective curve such that the general fibre is an elliptic curve, and reducible fibres do not contain any (-1)-curve.

Set λ_S the highest common factor of fibre degrees of sheaves on X, equivalently λ_S is the smallest positive integer such that there is a divisor σ on X with $\sigma \cdot F = \lambda_S$ for F the class of a fibre (such a σ is called a λ_S multisection of π). For integers $a > 0, b$ with $a\lambda_S$ coprime to b, there exists a fine moduli space $J_S(a,b)$ of pure 1-dimensional stable sheaves on S whose general point represents a rank a, degree b stable vector bundle supported on a smooth fibre of π.

The surface $J_S(a,b)$ comes equipped with a map $\tilde{\pi} : J_S(a,b) \to C$ which sends a sheaf supported on $\pi^{-1}(c)$, $c \in C$, to c. It turns out that $\tilde{\pi}$ is also an elliptic surface.

Once again, set

$$B = \langle T_{\mathcal{O}_G} \mid G \subset S \text{ is a } (-2)\text{-curve}\rangle.$$

Moreover, consider the subgroup of autoequivalences

$$K := \langle B, (- \otimes \mathcal{O}_S(D)) \mid D \cdot F = 0,\ F \text{ is a fibre}\rangle \rtimes \mathrm{Aut}(S) \times \mathbb{Z} \cdot [2].$$

Uehara proposes the following conjecture.

Conjecture 3.3.1 ([Ueh16], Conjecture 1.1)**.** *There is a short exact sequence*

$$0 \to K \to \mathrm{Aut}(\mathrm{D}^b(S)) \xrightarrow{\Theta} \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid d \in \lambda_S \mathbb{Z},\ J_S(b) \simeq S \right\} \to 1. \tag{3.5}$$

The map Θ is induced by the action of $\mathrm{Aut}\,\mathrm{D}^b(S)$ on the even integral cohomology $(H^0 \oplus H^2)(F, \mathbb{Z}) \simeq \mathbb{Z}^2$ of a smooth fibre.

Let us be more precise with the definition of Θ. For any autoequivalence $\phi \in \mathrm{Aut}(\mathrm{D}^b(S))$ and any smooth fibre F of S, ϕ induces an equivalence $\phi_f : \mathrm{D}^b(F) \to \mathrm{D}^b(F')$ for F' another smooth fibre. Moreover, two derived equivalent elliptic curves are isomorphic. Indeed, elliptic curves are determined by their Hodge structure which are preserved by equivalences (a similar result as Remark 1.3.21 hold for elliptic curves).

Hence up to fixing F and an isomorphism $F \xrightarrow{\sim} F'$, we can assume that ϕ_f is an autoequivalence of $\mathrm{D}^b(F)$ and hence it induces an automorphism of $H^{even}(F, \mathbb{Z})$.

Remark 3.3.2. The map Θ is always surjective. Pick a matrix $M = \begin{pmatrix} c & a \\ d & b \end{pmatrix}$ which satisfies the assumptions. We can assume $a > 0$. Consider the FM equivalence

$$\phi_{\mathcal{P}} : \mathrm{D}^b(J_S(a, b)) \xrightarrow{\sim} \mathrm{D}^b(S)$$

with kernel $\mathcal{P}$ a universal sheaf associated to the fine moduli space $J_S(a, b)$. Bridgeland studied this equivalence in[Bri98] and proves that, up to twisting $\mathcal{P}$ by a line bundle, the action $\Theta(\phi_{\mathcal{P}})$ on the even cohomology of a smooth fibre acts as the matrix M.

Uehara proves in [Ueh16] that the conjecture is true when each reducible fibre is of type I_n in Kodaira's classification of singular fibres (see [BPV84] §V.7). It means that each reducible fibre is a cycle of (-2)-curves (in $\bar{A}_n$ configuration) with no multiplicities.

Where to go from here

In view of Conjecture 3.3.1, a way to investigate generalized topological entropy of autoequivalences of $\mathrm{D}^b(S)$ is to control the subgroup K and to study how varies the entropy through Θ. A first step is the following proposition. Since Conjecture 3.3.1 is proved for a fibration with I_n-type of singular fibres only, we restrict ourselves to this case.

Proposition 3.3.3. *Assume that the (-2)-curves on S are in $\bar{A}$ configuration. Any autoequivalence $\varphi \in K$ admits, up to a shift, a decomposition*

$$\varphi = b \circ (- \otimes L) \circ f^*$$

with $b \in B$, $L \in \mathrm{Pic}(S)$ and $f \in \mathrm{Aut}(S)$. Moreover

$$\rho(\varphi^H) = \rho(f^*).$$

Proof. We mimic the proof of Theorem 3.2.2 and we keep the notation of its proof. The same arguments give a decomposition

$$\varphi = b \circ (- \otimes L) \circ f^*,$$

where $f \in \mathrm{Aut}(S), L \in \mathrm{Pic}(S)$ and

$$b = T_{\mathcal{O}_{C_1}(a_1)} \circ \cdots \circ T_{\mathcal{O}_{C_k}(a_k)}$$

for some (-2)-curves $C_1, \dots, C_k$ and integers $a_1, \dots, a_k$. Moreover, the element $(b^H)_2$ lies in the group $B_2 := \langle (T^H_{\mathcal{O}_G})_2 \mid G \text{ a } (-2)\text{-curve}\rangle \subseteq \mathrm{Aut}(H^2(S,\mathbb{Q}))$, the restriction of B to $\mathrm{Aut}(H^2(S,\mathbb{Q}))$, which is direct sum of quotients of Coxeter groups of type $\overline{A}$.

Denote $s_i := (T^H_{\mathcal{O}_{C_i}})_2$. We can assume that B_2 is a Coxeter group of type $\overline{A}_n$ for some $n \geq 1$, with generators $s_1, \dots, s_n$ with $\{[C_i], i = 1, \dots, n\}$ linearly independant in $H^2(S,\mathbb{Q})$. By [Jus04], we have a decomposition

$$B_2 = \mathbb{Z}^{n-1} \rtimes G_f$$

where G_f is a finite Coxeter subgroup of type A_{n-1}. More precisely, G_f is (up to reordering) the subgroup generated by $s_2, \dots, s_n$, and $\mathbb{Z}^{n-1}$ is the free subgroup generated by $r_1, \dots, r_{n-1}$ where

$$r_1 = s_1 s_2 \dots s_{n-1} s_n s_{n-1} \dots s_3 s_2 \quad (3.6)$$
$$r_2 = s_2 r_1 s_2 \quad (3.7)$$
$$\dots \quad (3.8)$$
$$r_{n-1} = s_{n-1} r_{n-2} s_{n-1} \quad (3.9)$$

Now $(b^H)_2$ decomposes as $b_2^H = rs$ with $r \in \mathbb{Z}^{n-1}$ and $s \in G_f$. In particular, there is integer $m \gg 0$ such that $(b^H)_2^m = r_m s^m = r_m$ for some $r_m \in \mathbb{Z}^{n-1}$ as G_f is finite. As $\mathbb{Z}^{n-1}$ is free, it is enough to study the matrix form of the generators $r_1, \dots, r_{n-1}$. Let $d = \dim H^2(S,\mathbb{Q})$. Choose the following free family of $H^2(S,\mathbb{Z})$:

$$v_1 = [C_1] + 2[C_n] \quad (3.10)$$
$$v_2 = [C_{n-1}] \quad (3.11)$$
$$\dots \quad (3.12)$$
$$v_{n-2} = [C_3] \quad (3.13)$$
$$v_{n-1} = \sum_i [C_i] \quad (3.14)$$
$$v_n = [C_1] + [C_2]. \quad (3.15)$$

Fill this family into a basis $\mathcal{B}$ of $H^2(S,\mathbb{Q})$. One can check that the matrix of r_1 with respect to this basis $\mathcal{B}$ has the form

$$\begin{pmatrix} J_n & * \\ 0 & \mathrm{Id}_{d-n} \end{pmatrix}$$

where J_n is the Jordan matrix with 1 on the diagonal, 1 in position $(n-1, n)$ and 0 elsewhere. In particular the only eigenvalue of the matrix of r_1 is 1. We deduce the same for r_i, $i = 2, \dots, n-1$. In particular, we obtain that the only eigenvalue of $(b^H)_2^m$ is 1. Note that f^* sends $\mathbb{Q}\langle [C_i]\rangle$ to $\mathbb{Q}\langle [C_i]\rangle$, in particular we conclude that there exists a basis of $H^2(S,\mathbb{Q})$, whose n first vectors lie in $\langle [C_1], \dots, [C_n]\rangle$, such that $(b^H)_2^m$ and $(f^*)_2^{\circ m}$ have respectively the shapes

$$(b^H)_2^m = \begin{pmatrix} T_n^1 & 0 \\ * & \mathrm{Id}_{d-n} \end{pmatrix}, \ (f^*)_2^{\circ m} = \begin{pmatrix} D_n & 0 \\ * & T_{d-n} \end{pmatrix}$$

where T_n^1 is a $n \times n$ lower-triangular matrix with 1's on the diagonal, D_n is a $n \times n$ diagonal matrix and T_{d-n} is a $(d-n) \times (d-n)$ lower-triangular matrix.

Since the matrix of $(- \otimes L)_2^H$ is the identity matrix, we conclude by Proposition 3.2.4.

□

Combining Proposition 3.3.3 and Remark 3.3.2, it seems that the understanding of the dynamical behaviour of $\mathrm{Aut}(\mathrm{D}^b(S))$ lies in the study of the Fourier-Mukai transforms with kernels universal sheaves associated to relative Jacobians on the fibres of the fibration.

3.4 A counter example of the Kikuta-Takahashi conjecture

The goal of this section is to construct an example of autoequivalence on a surface S which contradicts Conjecture 3.1.10. We prove the following theorem.

Theorem 3.4.1 ([Mat19]). *Let S be a smooth projective surface and $C \subseteq S$ a (-2)-curve. Let $\mathcal{L} \in \operatorname{Pic}(S)$ be a line bundle satisfying $\deg_C(\mathcal{L}|_C) < 0$ and consider the autoequivalence $\varphi = T_{\mathcal{O}_C} \circ (- \otimes \mathcal{L})$. Then we have*

$$h_0(\varphi) > 0 = \log \rho(\varphi^H).$$

In other words, Theorem 3.4.1 tells that φ is an autoequivalence with positive *categorical* entropy but zero *topological* entropy.

Note that, if the surface S contains a (-2)-curve C, then $\mathcal{L} := \mathcal{O}_S(C)$ fits the hypothesis. As a consequence, this gives a counterexample of Conjecture 3.1.10 in any birational class of surfaces.

We prove $h_0(\varphi) > 0$ this section. The equality $\log \rho(\phi^H) = 0$ is a consequence of Theorem 3.2.2 (see Corollary 3.2.6).

Let S be a smooth complex projective surface, $C \overset{i}{\hookrightarrow} S$ a (-2)-curve. Let $\mathcal{L} \in \operatorname{Pic}(S)$ be a line bundle verifying $\deg_C(\mathcal{L}|_C) = l < 0$. For instance, $\mathcal{L} = \mathcal{O}_S(C)$ satisfies this assumptions. Consider the autoequivalence

$$\varphi = T_{\mathcal{O}_C} \circ \mathcal{L}.$$

The goal of this section is to show the following:

Theorem 3.4.2. *The categorical entropy of φ verifies*

$$h_0(\varphi) > 0.$$

First, we make some constructions. For any $\mathcal{M} \in \operatorname{Pic}(S)$ we have the distinguished triangle

$$R\operatorname{Hom}(i_*\mathcal{O}_C, \varphi^{n-1}(\mathcal{M}) \otimes \mathcal{L}) \otimes i_*\mathcal{O}_C \to \varphi^{n-1}(\mathcal{M}) \otimes \mathcal{L} \to \varphi^n(\mathcal{M}).$$

Now pick $\mathcal{P} \in \operatorname{Pic}(S)$ and apply $(- \otimes \mathcal{P})$ and $R\operatorname{Hom}(i_*\mathcal{O}_C, -)$ to this triangle. We obtain:

$$\begin{aligned} R\operatorname{Hom}(i_*\mathcal{O}_C,&\varphi^{n-1}(\mathcal{M}) \otimes \mathcal{L}) \otimes R\operatorname{Hom}(i_*\mathcal{O}_C, i_*\mathcal{O}_C \otimes \mathcal{P}) \\ &\to R\operatorname{Hom}(i_*\mathcal{O}_C, \varphi^{n-1}(\mathcal{M}) \otimes \mathcal{L} \otimes \mathcal{P}) \\ &\to R\operatorname{Hom}(i_*\mathcal{O}_C, \varphi^n(\mathcal{M}) \otimes \mathcal{P}). \end{aligned} \tag{3.16}$$

Fix $\deg_C(\mathcal{M}|_C) = m < 0$. We consider the triangle (3.16) depending on the parameter $p := \deg_C(\mathcal{P}|_C)$. For more clarity, we introduce the following notations.

$$\begin{aligned} \widetilde{A}_n &:= R\operatorname{Hom}(i_*\mathcal{O}_C, \varphi^{n-1}(\mathcal{M}) \otimes \mathcal{L}), \\ D(p) &:= R\operatorname{Hom}(i_*\mathcal{O}_C, i_*\mathcal{O}_C \otimes \mathcal{P}), \\ A_n(p) &:= \widetilde{A}_n \otimes D(p), \\ B_n(p) &:= R\operatorname{Hom}(i_*\mathcal{O}_C, \varphi^{n-1}(\mathcal{M}) \otimes \mathcal{L} \otimes \mathcal{P}), \\ C_n(p) &:= R\operatorname{Hom}(i_*\mathcal{O}_C, \varphi^n(\mathcal{M}) \otimes \mathcal{P}). \end{aligned}$$

Thus the triangle (3.16) can be written as:

$$A_n(p) \to B_n(p) \to C_n(p). \tag{3.17}$$

Proposition 3.4.3. *For all $n \geq 1$ and any $\mathcal{P}$ with $p < 0$, we have*

$$\mathcal{H}^j(C_n(p)) = 0 \text{ for } j > n+2$$

and moreover

$$\mathcal{H}^{n+2}(C_n(p)) \simeq \mathcal{H}^{n+1}(C_{n-1}(l)) \otimes \mathcal{H}^2(D(p)) \neq 0.$$

Proof. Let's start with computations for $n = 1$.

By adjunction formula, $\deg_C(i^*\omega_S) = 0$ since $K_S \cdot C = 0$. Now we use the adjunction $i_* \dashv i^*(-) \otimes \omega_C[-1]$ (see [Huy06], Proposition 3.35) and compute $i^* i_* \mathcal{O}_C$ using [Huy06], section 11 again. We obtain:

$$\begin{aligned} \mathrm{Ext}^k_S(i_*\mathcal{O}_C, \mathcal{M} \otimes \mathcal{L} \otimes \mathcal{P}) &= \mathrm{Ext}^k_C(\mathcal{O}_C, \mathcal{O}_C(m+l+p-2)[-1]) \\ &= H^{k-1}(C, \mathcal{O}_C(m+l+p-2)). \end{aligned}$$

$$\begin{aligned} \mathrm{Ext}^k_S(i_*\mathcal{O}_C, i_*\mathcal{O}_C \otimes \mathcal{P}) &= \mathrm{Ext}^k_C(\mathcal{O}_C, i^*i_*(\mathcal{O}_C) \otimes \mathcal{O}_C(p-2)[-1]) \\ &= \mathrm{Ext}^k_C(\mathcal{O}_C, (\mathcal{O}_C \oplus \mathcal{O}_C(2)[1]) \otimes \mathcal{O}_C(p-2)[-1]) \\ &= H^{k-1}(C, \mathcal{O}_C(p-2)) \oplus H^k(C, \mathcal{O}_C(p)). \end{aligned}$$

Since we fixed $m < 0$, $l < 0$, $p < 0$, these Ext groups are non-zero only for $k = 2$ (and possibly $k = 1$ if $p < -1$). Hence we have:

- $\mathcal{H}^j(\widetilde{A}_1) \neq 0$ only for $j = 2$,
- $\mathcal{H}^j(D(p)) \neq 0$ only for $j = 2$ (and $j = 1$ if $p < -1$) and thus $\mathcal{H}^j(A_1(p)) \neq 0$ only for $j = 4$ (and $j = 3$ if $p < -1$),
- $\mathcal{H}^j(B_1(p)) \neq 0$ only for $j = 2$.

Using the long exact sequence in cohomology induced by (3.17) we have

$$\mathcal{H}^j(C_1(p)) \neq 0 \text{ only for } j = 2, 3 \text{ and } \ \mathcal{H}^3(C_1(p)) \simeq \mathcal{H}^4(A_1(p)),$$

as it can be read on the following table:

	$\mathcal{H}^2$	$\mathcal{H}^3$	$\mathcal{H}^4$
A_1	0		$*$
B_1	$*$	0	0
C_1	$*$	$*$	0

where $*$ means that the space does not vanish, and the empty slots are irrelevant to our calculations.

For any $n \geq 1$ we have the identities

$$\widetilde{A}_n \simeq C_{n-1}(l) \text{ and } B_n(p) \simeq C_{n-1}(l+p). \tag{3.18}$$

For $n = 1$, by (3.18) we get

$$\begin{aligned} \mathcal{H}^4(A_1(p)) &\simeq \mathcal{H}^2(\widetilde{A}_1) \otimes \mathcal{H}^2(D(p)) \\ &\simeq \mathcal{H}^2(C_0(l)) \otimes \mathcal{H}^2(D(p)) \\ &\neq 0. \end{aligned}$$

Assume that the lemma is true for all $p < 0$ on rank $n-1$. Since l and p are negative, by induction hypothesis and (3.18) we have

- $\mathcal{H}^j(A_n(p)) = 0$ for $j > n+3$,
- $\mathcal{H}^j(B_n(p)) = 0$ for $j > n+1$,
- $\mathcal{H}^{n+3}(A_n(p)) \simeq \mathcal{H}^{n+1}(\widetilde{A}_n) \otimes \mathcal{H}^2(D(p)) \neq 0$.

Thus using the long exact sequence in cohomology induced by (3.17) we obtain

$$\mathcal{H}^{n+2}(C_n(p)) \simeq \mathcal{H}^{n+3}(A_n(p)).$$

Once again this can be read on the table:

	$\mathcal{H}^{n+1}$	$\mathcal{H}^{n+2}$	$\mathcal{H}^{n+3}$
A_n			$*$
B_n	$*$	0	0
C_n		$*$	0

Finally by the identities (3.18), we obtain

$$\mathcal{H}^{n+2}(C_n(p)) \simeq \mathcal{H}^{n+1}(C_{n-1}(l)) \otimes \mathcal{H}^2(D(p)).$$

□

Corollary 3.4.4. *For any $\mathcal{M}, \mathcal{P}$ with $m, p < 0$ and $n \in \mathbb{Z}_{\geq 1}$, we have*

$$\begin{aligned}\operatorname{Ext}_S^{n+2}(i_*\mathcal{O}_C, \mathcal{P} \otimes \varphi^n(\mathcal{M})) \simeq H^1(C, \mathcal{O}_C(m+l-2)) \otimes H^1(C, \mathcal{O}_C(p-2)) \\ \otimes H^1(C, \mathcal{O}_C(l-2))^{\otimes n-1}.\end{aligned}$$

In particular, $\dim \operatorname{Ext}_S^{n+2}(i_*\mathcal{O}_C, \mathcal{P} \otimes \varphi^n(\mathcal{M})) > (1-l)^{n-1}$.

Proof. By induction on Proposition 3.4.3, we have

$$\begin{aligned}\operatorname{Ext}_S^{n+2}(i_*\mathcal{O}_C, \mathcal{P} \otimes \varphi^n(\mathcal{M})) &\simeq \mathcal{H}^{n+2}(C_n(p)), \\ &\simeq \mathcal{H}^{n+1}(C_{n-1}(l)) \otimes \mathcal{H}^2(D(p), \\ &\simeq \mathcal{H}^2(C_0(l)) \otimes \mathcal{H}^2(D(p)) \otimes \mathcal{H}^2(D(l))^{\otimes n-1}, \\ &\simeq H^1(C, \mathcal{O}_C(m+l-2)) \otimes H^1(C, \mathcal{O}_C(l-2))^{\otimes n-1} \\ &\quad \otimes H^1(C, \mathcal{O}_C(p-2)).\end{aligned}$$

□

Proof of theorem 3.4.2. We make use of Remark 3.1.7. We fix a generator $G = \mathcal{M} \oplus \mathcal{M}^{\otimes 2} \oplus \mathcal{M}^{\otimes 3}$ of $\mathrm{D}^b(S)$ with $\mathcal{M} \in \operatorname{Pic}(S)$ so that $\mathcal{M}^\vee$ is very ample on S. Thus $m := \deg_C(\mathcal{M}|_C) < 0$. Now choose a line bundle $\mathcal{P} \in \operatorname{Pic}(S)$ so that $\mathcal{P}^\vee$ is very ample. Up to taking powers of $\mathcal{P}^\vee$, we can assume that $\mathcal{P}_0^\vee := \mathcal{P}^\vee \otimes \mathcal{O}_S(-C)$ is also very ample (see [Har77], II, ex. 7.5). Then $G_1 := \mathcal{P}^\vee \oplus (\mathcal{P}^\vee)^{\otimes 2} \oplus (\mathcal{P}^\vee)^{\otimes 3}$ and $G_2 := \mathcal{P}_0^\vee \oplus (\mathcal{P}_0^\vee)^{\otimes 2} \oplus (\mathcal{P}_0^\vee)^{\otimes 3}$ are also generator of $\mathrm{D}^b(S)$.

By Corollary 3.4.4, for any $n \geq 1$ we have

$$(1-l)^{n-1} \leq \dim \operatorname{Ext}^{n+2}(i_*\mathcal{O}_C, \mathcal{P} \otimes \varphi^n(\mathcal{M})) \leq \dim \operatorname{Ext}^{n+2}(i_*\mathcal{O}_C, \mathcal{P} \otimes \varphi^n(G)).$$

Write $\delta_0'(F, G) := \sum_{j \in \mathbb{Z}} \dim \operatorname{Ext}^j(F, G)$. Considering the exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to i_*\mathcal{O}_C \to 0,$$

we obtain

$$\begin{aligned}(1-l)^{n-1} \leq \delta_0'(i_*\mathcal{O}_C, \mathcal{P}\otimes\varphi^n(G)) &\leq \delta_0'(\mathcal{O}_S, \mathcal{P}\otimes\varphi^n(G)) + \delta_0'(\mathcal{O}_S(-C), \mathcal{P}\otimes\varphi^n(G)),\\ &\leq \delta_0'(\mathcal{P}^\vee, \varphi^n(G)) + \delta_0'(\mathcal{P}_0^\vee, \varphi^n(G)),\\ &\leq \delta_0'(G_1, \varphi^n(G)) + \delta_0'(G_2, \varphi^n(G)).\end{aligned}$$

Thus, either $\delta_0'(G_1, \varphi^n(G))$ or $\delta_0'(G_2, \varphi^n(G))$ has exponential growth. By Lemma 3.1.4 both terms can be used to compute the categorical entropy $h_0(\varphi)$, hence

$$h_0(\varphi) > 0.$$

□

Remark 3.4.5. The same result is also true with $T_{\mathcal{O}_C(a)} \circ \mathcal{L}$, for a a non-zero integer: one may perform the same proof with the care of choosing a line bundle $\mathcal{P}$ verifying $\deg_C(\mathcal{P}|_C) = p \ll 0$.

Remark 3.4.6. It is interesting to remark that the functor $(-\otimes\mathcal{L})$ can be realized as composition of spherical twists $T_{\mathcal{O}_C(a_1)} \circ \cdots \circ T_{\mathcal{O}_C(a_n)}$ with a nice choice of $a_1, \ldots, a_n \in \mathbb{Z}$. See [IU05] Lemma 4.15 for the claim. In particular, compositions of spherical twists might have positive categorical entropy.

To finish the proof of Theorem 3.4.1, it remains to show that the action of φ on the cohomology of S has spectral radius 1. This has been treated in the previous sections, see Corollary 3.2.6.

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