Graduate Seminar on Topology (S4D2 Master)

Wintersemester 2021

Characteristic Classes

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Tuesdays, 14:15 – 16:00 Uhr in whatever form the Corona pandemic will allow.

Please, observe the registration procedure for seminars.

Content. To many types of bundles $F \to E \xrightarrow{\xi} B$ one can associate *characteristic classes*. These are certain classes $\mathrm{ch}(\xi) \in H^*(B; \mathbb{A})$ in the cohomology of the base space with coefficients in some abelian group $\mathbb{A}$. They are supposed to satisfy only naturality with respect to pull-backs of bundles.

Recall that the important thing in a fibre bundle is the structure group, not the fibre. Fixing a structure group G and a commutative ring $\mathbb{A}$, we can write $\mathrm{CharCl}^*_G(\mathbb{A})$ for the graded ring of all characteristic classes of (principal) G-bundles in $H^*(-; \mathbb{A})$, in other words all natural transformations from the functor $\mathrm{Prin}_G(-)$ of all principal G-bundles to the functor $H^i(-; \mathbb{A})$ for some $i \geq 0$. By the Yoneda Lemma, we have the isomorphism

$$\mathrm{CharCl}^*_G(\mathbb{A}) \;\cong\; H^*(BG; \mathbb{A}), \quad \mathrm{ch} \mapsto \mathrm{ch}(\zeta_G),$$

where BG denotes the classifying space of G and $\zeta_G \colon EG \to BG$ is the universal bundle. So to study characteristic classes means to study the cohomology of classifying spaces.

The typical bundles for which such characteristic classes are defined are vector bundles:

- the *Stiefel-Whitney classes* $w_i(\xi) \in H^i(B; \mathbb{Z}/2)$ real vector bundles with $G = O(n)$,
- the *Chern classes* $c_i(\xi) \in H^{2i}(B; \mathbb{Z})$ for complex vector bundles with $G = U(n)$, and
- the *Pontrjagin classes* $p_i(\xi) \in H^{4i}(B; \mathbb{Z})$ again for real vector bundles via their complexification with $G = SO(2n)$.

Here $i = 0, 1, \ldots, n = \dim(\xi)$. Let us add, that these examples have many more properties than just naturality. In talk 1 we will start with an axiomatic description.

Since a real vector bundle of dimension n determines a sphere bundles of dimension $n-1$, and vice versa, one can associate theses classes also to sphere bundles. An important case is the *Euler class* $e(\xi) \in H^n(X; \mathbb{Z})$ of an orientable $\mathbb{S}^{n-1}$-bundle. Other interesting cases are surface bundles, whose fibre F is a (closed) orientable surface.

The purpose of the seminar is to define and study Stiefel-Whitney classes and Chern classes for vector bundles. We will learn about many applications, in particular for manifolds and their classification up to (co)bordism. There is an axiomatic approach to characteristic classes with which we start. Then we construct the Stiefel-Whitney and the Chern classes; we will actually see several ways to do so. We will see geometric properties, implying or being implied by the vanishing of certain classes. And we will study the importance of universal characteristic classes (that is the characteristic classes of universal bundles) as generators of the cohomology rings of classifying spaces.

Towards the end of the seminar we turn our attention to surface bundles and define

- the *Miller-Morita-Mumford classes* $\kappa_i(\xi) \in H^{2i}(B; \mathbb{Z})$ for surface bundles, derived from the first Chern class of a certain complex line bundle; they are defined for all $i \geq 0$. They are important to study the mapping class group or the moduli space of Riemann surfaces.

For all talks I have sketched the content and listed the literature. The material to belearn is certainly too much for a single talk; so we need – after you have learned the entire material – to make a selection what to present in detail, what to report on, and what to skip.

Prerequisites. The lecture courses *Topologie I* and *Topologie II* are indespensable. The courses *Algebraic Topology I + II* are certainly helpful and for some of the talks necessary; we assume the theory of fundles, in paricular vector bundles; this includes the classifying spaces and the classification theorems. We will use Thom spaces and the Thom isomorphism, the Leray-Hirsch Theorem, as well as the Gysin sequence.

The chapter 16 in Switzers book is an advanced introduction, but worth reading before he seminar starts.

Literature. We will be mostly using the famous book of Milnor and Stasheff [Mi-Sta], but for many talks other sources are useful or necessary as well. For the definition of Stiefel-Whitney classes, Milnor uses the Steenrod squares. One might find this approach somehow surprising, since cohomology operations are more advanced than characteristic classes. Thus we should follow for the first definition other sources, like Hatcher, Cohen or Switzer.

Preparation of talks. The talks are supposed to be 90 minutes long. That means, you should prepare about 70 minutes and allow time for questions during the talk.

You should consult me about the talk at least two weeks before the day of the talk. If a talk is shared by two students, you both prepare together the entire talk and at the beginning of each seminar session we decide by throwing a coin who starts with the first half and who gives the second half of the talk.

Talks

(1) **Stiefel-Whitney classes I** — Teng, Yikai — 12.10.2021

Axioms for Stiefel-Whitney classes $w_i(\xi)$. First consequences of the axioms. Examples. Application: (1) Parallelizability of real projective spaces; (2) Division algebras; (3) Immersions of manifolds into euclidean spaces.

[Mi-Sta, §4].

(2) **Stiefel-Whitney classes II** — Supatashvili, Mari — 19.10.2021

Definition I: using the Leray-Hirsch Theorem. Definition II: using Steenrod operations and the Thom isomorphism. Proof of the axiomatic properties.

[Mi-Sta, §4], [Ha, 3.1], [Bre, VI.17].

(3) **Euler class** — Fourel, Colin — 26.10.2021

Euler class of orientable vector bundles or of orientable sphere bundles. Gysin sequence. Relation to Stiefel-Whitney classes. Examples.

[Mi-Sta, §8+10], [Bre], [Ha], [Coh].

(4) **Real and complex Graßmann manifolds** SCHULTE-GEERS, NIKLAS — 2.11.2021
Graßmann manifolds and Stiefel manifolds. Classifying spaces of the general linear groups or orthogonal resp. unitary groups, universal bundles. Examples.

[Mi-Sta, §5+6], [Ha, 1.2], [Coh].

(5) **Cohomology ring of Graßmann manifolds** FENG, TIANYI — 9.11.2021
Cell structure of Graßmann manifolds, Schubert cells. Mod-2 cohomology ring of real Graßmann manifolds. Uniqueness of Stiefel-Whitney classes.

[Mi-Sta, §6+7],

(6) **Applications for smooth manifolds** FACK, MALCOLM
. — 16.11.2021
Smooth manifolds and their tangent bundle. Normal bundle of an embedding, stable normal bundle. Disk bundle and exponential map. Micro-bundles as generalisations of tangent and disk bundles for topological manifolds. Fundamental class, dual class of a submanifold. Diagonal cohomology class. Poincare duality. Euler class and Euler characteristic. Wu formula.

[Mi-Sta, §11], [Sw, chap. 14], [Ha, 3.2].

(7) **Obstruction theory** WUNDEN, FREDERIK — 23.11.2021
Homology with local coefficients. Stiefel bundles $V_k(\xi)$ of a bundle ξ and its sections. Obstruction class $\mathfrak{o}_j(\xi)$ to extend sections. Theorem $\mathfrak{o}_j(\xi) = w_j(\xi)$. (Combinatorial description of Stiefel-Whitney classes for triangulated manifolds.) Gysin sequence.

[Mi-Sta, §12], [Ste, Part III, §§29+32+38], [Ha, 3.3].

(8) **Chern classes** KOLLY, GABIN + PAPALLO, FILIPPO — 30.11.2021
Oriented bundles, orientation class. Definition of the Euler class. Application: Non-zero sections. Definition of the Chern classes and basic properties (axioms). Complex Stiefel manifolds and Graßmann manifolds; their cohomology ring. Dual and conjugate bundles; tangent bundles of complex projective spaces. Connection to Stiefel-Whitney classes.

[Mi-Sta, §14], [Sw, chap. 16], [Ha, 3.2].

(9) **Pontrjagin classes** BRINGS, TIM + HARICHURN, SHIMAL — 7.12.2021
Complexification of a real vector bundle. Definition of Pontrjagin classes. Properties. Expressing Pontrjagin classes by Chern classes. Oriented real Graßmann manifolds, their cohomology ring.

[Mi-Sta, §15], [Ha, 3.2].

(10) **Characteristic numbers** DAUSER, ADAM + HU, XIANYU — 14.12.2021
Stiefel-Whitney numbers. Examples. Theorem of Pontrjagin. Hint to the converse, the Theorem of Thom. Chern numbers and Pontrjagin numbers. Product formula. Linear indipendence.

[Mi-Sta, §4, p. 50-53], [Mi-Sta, §16].

(11) **Cobordism I** Tyriard, George — 21.12.2021

Cobordant manifolds, Thom spaces. Characteristic numbers of null-bordant manifolds vanish. Bordism as a homology theory, Thom-Pontrjagin construction (both in the easiest case of unoriented manifolds).

[Sw, chap. 12], [Mi-Sta, §17], [Sto, chap. II+III], [Con, §2+4+5+6+11+12].

(12) **Cobordism II** Tigilauri, Giorgi — 11.01.2022

Relation of cobordism to singular homology, Atiyah-Hirzebruch spectral sequence. Computations.

[Mi-Sta, §17], [Con, §7+8+11+12+14], [B-tD, Kap. VI].

(13) **Cobordism III** Mezher, Fadi — 18.01.2022

Steenrod's Problem. Generators for cobordism rings, real projective spaces, Dold manifolds.

[Con, §15], [Do].

(14) **Miller-Morita-Mumford classes for surface bundles I** Dular, Bruno
. — 25.01.2022

Group of orientation preserving diffeomorphisms, mapping class group $\Gamma_g = \pi_0(\mathrm{Diff}^+(S_g))$. Symplectic form on $H_1(S_g;\mathbb{Z})$, symplectic matrix group $\mathrm{Sp}_{2g}(\mathbb{Z})$. Classifying space $B\Gamma_g$ and moduli space $\mathfrak{M}_g$ of surfaces. Universal surface bundle $\zeta_g\colon \mathcal{E}_g := E\mathrm{Diff}(S_g) \times_{\mathrm{Diff}(S_g)} S_g \to B\Gamma_g$, its fibrewise tangent bundle $\tau_g\colon L_g \to \mathcal{E}_g$. The Euler class $e(\tau_g) \in H^2(\mathcal{E}_g;\mathbb{Z})$ of τ_g and its powers. The Mumford-Morita-Miller classes κ_i as the transfer or push-forward of these powers.

[Mo, Chap. 4.1 + 4.2, pages 135-151],

(15) **Miller-Morita-Mumford classes for surface bundles II** Zillinger, Felix
. — 1.02.2022

Ramified covering, non-triviality of κ_1. Iterated surface bundles. Non-triviality of the κ_i and their algebraic independence.

[Mo, Chap. 4.3 + 4.4, pages 151-171],

Literatur

[Bre] **Glen E. Bredon**: *Topology and Geometry.*
Graduate Texts in Mathematics vol. 139, Springer Verlag (1993).

[B-tD] **Theodor Bröcker, Tammo tom Dieck**: *Kobordismentheorie.*
Lecture Notes in Mathematics vol. 178, Springer Verlag (1970).

[Coh] **Ralph L. Cohen**: *Bundles, Homotopy, and Manifolds.*
Lecture Notes (2020). Available on the author's homepage:
`http://math.stanford.edu/~ralph/book.pdf`

[Con] **Pierre E. Conner**: *Differentiable Periodic Maps.*
Lecture Notes in Mathematics vol. 738, Springer Verlag (1979).

[Do] **Albrecht Dold**: *Erzeugende der Thomschen Algebra $\mathcal{N}$.*
Math. Zeitschrift **65** (1956), p. 25-35.

[Ha] **Allen Hatcher**: *Vector Bundles and K-Theory.*
Unfinished book project (2017), available on the author's homepage:
`https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`

[Mi-Sta] **John W. Milnor, James D. Stasheff**: *Characteristic Classes.*
Annals of Mathematics Studies vol. 76, Princeton University Press (1974).

[Mo] **Shigeyuki Morita**: *Geometry of Characteristic Classes.*
Translations of Mathematical Monographs vol. 199, American Mathematical Society (2001).

[Ste] **Norman Steenrod**: *The Topology of Fibre Bundles.*
Princton Mathematical Series vol. 14, Princeton University Press (1951).

[Sto] **Robert E. Stong**: *Notes in Coborsdism Theory.*
Mathematical Notes, Princeton University Press (1968).

[Sw] **Robert Switzer**: *Algebraic Topology.*
Grundlehren der Mathematischen Wissenschaften vol. 212, Springer Verlag (1975).