Stripping and splitting decorated mapping class groups

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Abstract. We study decorated mapping class groups, i.e., mapping class groups of surfaces with marked points and boundary components, and their behaviour under stabilization maps with respect to the genus, the number of punctures and boundary components. Decorated mapping class groups are non-trivial extensions of the undecorated mapping class group, and the first result states that the extension is homologically trivial when one stabilizes with respect to the genus. The second result implies that one also gets splittings of homology groups when stabilizing with respect to the number of punctures and boundary components.

1. Introduction and statement of results

Let $F^k_{g,n+1}$ be an oriented smooth surface of genus g with k marked points and $n+1$ boundary components. The mapping class group $\Gamma^k_{g,n+1}$ is the group of components of the space of diffeomorphisms of $F^k_{g,n+1}$ which fix the marked points and the boundary pointwise. If k and n are zero, they will be dropped from the notation. Gluing a torus with two boundary components to one of the boundary components of $F^k_{g,n+1}$, one gets a surface $F^k_{g+1,n+1}$. Extending diffeomorphisms by the identity induces a map of mapping class groups

$$\sigma\colon \Gamma^k_{g,n+1} \longrightarrow \Gamma^k_{g+1,n+1},$$

and we may define the associated stable mapping class group

$$\Gamma^k_{\infty,n} = \lim_{g\to\infty} \Gamma^k_{g,n+1}.$$

The n boundary components not used in this stabilization process will be called *free*. Consider now diffeomorphisms that may permute the punctures and free boundary components. To be precise, the boundary components should be thought of as having a parametrization and diffeomorphisms have to be compatible with these. The associated mapping class groups will be denoted by $\Gamma^{(k)}_{g,(n),1}$ and $\Gamma^{(k)}_{\infty,(n)}$. They are normal extensions

$$\Gamma^k_{g,n+1} \hookrightarrow \Gamma^{(k)}_{g,(n),1} \twoheadrightarrow \Sigma_k \times \Sigma_n, \qquad \Gamma^k_{\infty,n+1} \hookrightarrow \Gamma^{(k)}_{\infty,(n),1} \twoheadrightarrow \Sigma_k \times \Sigma_n$$

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with quotients the product of the symmetric groups on k and n letters. Similarly, let $\Gamma^{(k)}_{g,n+1}$ and $\Gamma^{k}_{g,(n),1}$ denote the mapping class groups associated to the diffeomorphism groups that may permute the punctures respectively the free boundary components only. These extensions are important for the understanding of surface operads (see for example [G] and [GK]).

Theorem 1.1.
(1) $B(\Gamma^k_{\infty,n})^+ \simeq B\Gamma^+_\infty \times (\mathbb{C}P^\infty)^k$;
(2) $B(\Gamma^{(k)}_{\infty,(n)})^+ \simeq B\Gamma^+_\infty \times B\Sigma^+_n \times B(\Sigma_k \wr S^1)^+$.

These are two special cases of Theorem 3.1. Here $\mathbb{C}P^\infty$ denotes infinite dimensional complex projective space, Σ_k is the symmetric group on k letters, $\Sigma_k \wr S^1$ is the wreath product of the symmetric group Σ_k with the circle group, and X^+ denotes the Quillen plus construction of the space X with respect to the maximal perfect subgroup of its fundamental group. The most important ingredient for the proof of Theorem 1.1 is Harer's homology stability theorem for mapping class groups [H]. It allows one also to deduce results for the mapping class groups of closed surfaces and finite genus g. Theorem 1.1 has the following reinterpretation in homology, the first part of which has also been proved by Looijenga [L] and Morita [M].

Corollary 1.2.
(1) $H_*(\Gamma^k_{g,m}) = H_*(\Gamma_{g,1}) \otimes \mathbb{Z}[x_1, \ldots, x_k]$ *in dimensions* $* \leq g/2$ *when* $k + m \geq 1$*; here each* x_i *has degree 2;*
(2) $H_*(\Gamma^{(k)}_{g,(n),m}; \mathbb{F}) = H_*(\Gamma_{g,1}; \mathbb{F}) \otimes H_*(\Sigma_n; \mathbb{F}) \otimes H_*(\Sigma_k; \mathbb{F}[x_1, \ldots, x_k])$ *in dimensions* $* \leq g/2$ *when* $k + n + m \geq 1$*;* $\mathbb{F}$ *is any field.*

Remark The difficult question of determining the homotopy type of $B\Gamma^+_\infty$ is not addressed in this paper. Some progress has recently been made. In particular, $B\Gamma^+_\infty$ has been shown to be an infinite loop space [T]. Hence, by the above splitting, also $B(\Gamma^k_\infty)^+$ has an infinite loop space structure. It is curious to note that the methods of [T] however do not generalize to the case $k > 0$.

Instead of stabilizing with respect to the genus g, one can also consider stabilizations with respect to the number of boundary components or punctures:

$$\alpha\colon \Gamma^k_{g,n+1} \longrightarrow \Gamma^k_{g,n+2}, \qquad \beta\colon \Gamma^k_{g,n+1} \longrightarrow \Gamma^{k+1}_{g,n+1},$$

$$\tilde{\alpha}\colon \Gamma^{(k)}_{g,n+1} \longrightarrow \Gamma^{(k)}_{g,n+2}, \qquad \tilde{\beta}\colon \Gamma^{(k)}_{g,n+1} \longrightarrow \Gamma^{(k+1)}_{g,n+1},$$

$$\tilde{\tilde{\alpha}}\colon \Gamma^k_{g,(n),1} \longrightarrow \Gamma^k_{g,(n+1),1}, \qquad \tilde{\tilde{\beta}}\colon \Gamma^k_{g,(n),1} \longrightarrow \Gamma^{k+1}_{g,(n),1}.$$

For α, $\tilde{\alpha}$ and $\tilde{\tilde{\alpha}}$ glue a pairs of pants surface to the non-free boundary component of $F^k_{g,n+1}$. Similarly, for β, $\tilde{\beta}$ and $\tilde{\tilde{\beta}}$ glue a cylinder with one puncture to $F^k_{g,n+1}$. α and $\tilde{\alpha}$ have retractions on the group level induced by gluing a disk to one of the boundary components of the pairs of pants surface. β and $\tilde{\tilde{\beta}}$ have retractions on the group level induced by forgetting the $(k+1)$-st puncture. Such splittings on the level of groups cannot be found for $\tilde{\tilde{\alpha}}$ and $\tilde{\beta}$. However, using techniques from configuration spaces we show

Theorem 1.3. *The maps* $\tilde{\tilde{\alpha}}\colon B\Gamma^k_{g,(n),1} \to B\Gamma^k_{g,(n+1),1}$ *and* $\tilde{\beta}\colon B\Gamma^{(k)}_{g,n+1} \to B\Gamma^{(k+1)}_{g,n+1}$ *admit stable retractions.*

In particular, $\tilde{\tilde{\alpha}}$ and $\tilde{\beta}$ have splittings in homology. We note here that for infinite genus Theorem 1.3 follows from Theorem 1.1 and well-known stable splitting of the inclusion of Borel constructions

$$E\Sigma_n \times_{\Sigma_n} X^n \longrightarrow E\Sigma_{n+1} \times_{\Sigma_{n+1}} X^{n+1}$$

for connected X and any n (compare [BöM]).

2. Preliminaries on perfect subgroups and the plus construction

A group P is called *perfect* if every element can be written as a commutator, that is $P = [P, P]$. Any group G has a unique *maximal perfect* subgroup which we will denote by $P(G)$. As the homomorphic image of a perfect group is again perfect, $P(G)$ is a characteristic subgroup of G.

Let X be a connected space and let P be a perfect normal subgroup of its fundamental group $\pi_1(X)$. By attaching 2-cells and 3-cells, one can form a space X^+_P with the properties that the natural inclusion $q_P\colon X \to X^+_P$ induces (i) an epimorphism $\pi_1(X) \to \pi_1(X^+_P)$ with kernel P and (ii) an isomorphism $H_*(X;A) \to H_*(X^+_P;A)$ for any abelian group A with a $\pi_1(X^+_P)$-action. This construction is due to Quillen. We will always consider the plus-construction X^+ with respect to the maximal perfect subgroup of $\pi_1(X)$ and drop the subscript P from the notation. We recall a few functorial properties of the plus construction which will be needed later on [B; Sections 5 and 6].

(2.1) Given a map $f\colon X \to Y$ between two spaces there is a unique homotopy class of maps $f^+\colon X^+ \to Y^+$ making the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle q}\downarrow & & {\scriptstyle q}\downarrow \\ X^+ & \xrightarrow{f^+} & Y^+. \end{array}$$

In particular, X^+ is well-defined up to homotopy relative to X.

(2.2) As $P\pi_1(X \times Y) = P\pi_1(X) \times P\pi_1(Y)$, we have

$$(X \times Y)^+ = X^+ \times Y^+.$$

(2.3) If $F \to E \xrightarrow{p} B$ is a fibre sequence of connected spaces, then so is also $F^+ \to E^+ \to B^+$ provided either (i) $P\pi_1(B) = 1$ or (ii) p is quasi-nilpotent and F^+ is nilpotent.

A space X is *nilpotent* if its fundamental group acts nilpotently on its homotopy groups. A fibration p is *quasi-nilpotent* if the fundamental group $\pi_1(B)$ of the base acts nilpotently on the homology H_*F of the fiber.

3. Stripping decorations: The proof of Theorem 1.1

Let H_k and H_n be any subgroups of the symmetric groups Σ_k and Σ_n on k and n letters. Let $\Gamma^{H_k}_{g,H_n,1}$ be the subgroup of $\Gamma^{(k)}_{g,(n),1}$ that fits into the exact sequence

$$\Gamma^k_{g,n+1} \longrightarrow \Gamma^{H_k}_{g,H_n,1} \xrightarrow{\rho} H_k \times H_n.$$

Extending diffeomorphisms by the identity on an attached disk with an appropriate number of punctures and disks removed, or an attached torus with one boundary component defines the inclusion map

$$\text{incl}\colon \Gamma_{g,1} \longrightarrow \Gamma^{H_k}_{g,H_n,1}$$

and the stabilization map

$$\sigma\colon \Gamma^{H_k}_{g,H_n,1} \longrightarrow \Gamma^{H_k}_{g+1,H_n,1}.$$

These maps are illustrated in Figure 1 where the non-free boundary component of the surfaces is drawn as a rectangle, and gluing is defined by connected sum on this boundary component. (Note that this gluing is homotopy equivalent to the pairs of pants construction but strictly associative.) We may now also fill in the k punctures and n disks, and once again extend diffeomorphisms by the identity. This defines the forgetful map

$$\phi\colon \Gamma^{H_k}_{g,H_n,1} \longrightarrow \Gamma_{g,1}.$$

Clearly $\phi \circ \text{incl}$ is the identity homomorphism. As stabilization commutes with both incl and ϕ, these homomorphisms extend to the stable mapping class groups.

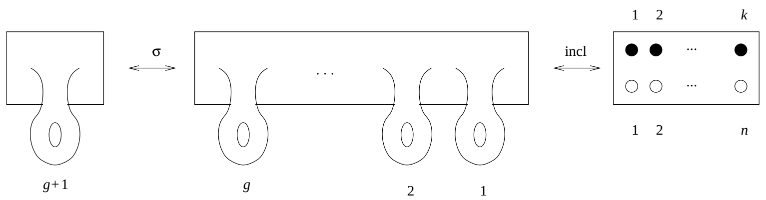

Figure 1: Stabilization and inclusion maps.

Theorem 3.1. $B(\Gamma^{H_k}_{\infty,H_n})^+ \simeq B\Gamma^+_\infty \times BH_n^+ \times B(H_k \wr S^1)^+$.

Proof. Gluing n punctured disks onto the n free boundary components of $F_{g,n+1}$ commutes with stabilization, and hence defines a central extension

$$\mathbb{Z}^n \longrightarrow \Gamma_{\infty,n} \longrightarrow \Gamma^n_\infty.$$

This central extension gives rise to a map $\theta\colon \Gamma^n_\infty \to B\mathbb{Z}^n = (S^1)^n$. Recall that the connected components of the group of orientation preserving diffeomorphisms of a surface $F_{g,1}$ are contractible when $g \geq 2$ [ES]. Hence, identify $\Gamma^n_{g,1}$ up to homotopy with the group of diffeomorphisms of $F_{g,1}$ which fix the boundary as well as n marked points. Then the components θ_i of θ assign to a diffeomorphism ϕ the angle between $t_{\phi(i)}$ and $D\phi(t_i)$ where the t_i's are fixed tangent vectors at the marked points and $D\phi$ is the derivative of ϕ. θ is a group homomorphism and can be extended in an obvious way to a homomorphism $\rho\wr\theta\colon \Gamma^{(n)}_\infty \to \Sigma_n \wr S^1$. We have the following (up to homotopy) commutative diagram of fibrations.

$$\begin{tikzcd}
B\Gamma_{\infty,n} \arrow[r] \arrow[d, equal] & B\Gamma^n_\infty \arrow[r, "\theta"] \arrow[d] & (BS^1)^n \arrow[d] \\
B\Gamma_{\infty,n} \arrow[r] \arrow[d] & B\Gamma^{(n)}_\infty \arrow[r, "\rho\wr\theta"] \arrow[d] & B(\Sigma_n \wr S^1) \arrow[d] \\
* \arrow[r] & B\Sigma_n \arrow[r, equal] & B\Sigma_n.
\end{tikzcd}$$

More generally, there is a fibration

$$B\Gamma_{\infty,k+n} \longrightarrow B\Gamma^{H_k}_{\infty,H_n} \xrightarrow{\rho\wr\theta} B(H_k \wr S^1) \times BH_n. \tag{3.2}$$

$B\Gamma_{\infty,n}$ has perfect fundamental group for any n, and therefore its plus construction is nilpotent. Furthermore, the fundamental group of the base space acts trivially on the homology of the fiber by Lemma 3.3 below. Hence, (3.2) is a quasi-nilpotent fibration. By (2.3) we conclude that therefore also its plus construction is a homotopy fibration:

$$B\Gamma^+_{\infty,k+n} \longrightarrow (B\Gamma^{H_k}_{\infty,H_n})^+ \xrightarrow{\rho\wr\theta} (B(H_k \wr S^1) \times BH_n)^+.$$

The forgetful map ϕ induces a map of fibrations from this into the trivial fibration

$$B\Gamma^+_\infty \longrightarrow B\Gamma^+_\infty \times (B(H_k \wr S^1) \times BH_n)^+ \longrightarrow (B(H_k \wr S^1) \times BH_n)^+.$$

The map of base spaces is the identity. By Harer's stability theorem [H], the homology of the stable mapping class group is independent of the number of free boundary components. Therefore the map of fibers is a homotopy equivalence, and hence so is the map of total spaces. An application of (2.2) finishes the proof. □

Lemma 3.3. *Σ_n acts trivially on $H_* B\Gamma_{g,n}$ for $* \leq g/2$.*

Proof. This is trivially true for $n = 0$ and we may assume $n \geq 1$. By the stability theorem [H], [I], the Σ_n-equivariant map $H_*\Gamma_{g,n+1} \to H_*\Gamma_{g,n}$ induced by gluing a disk to the non-free boundary component induces an isomorphism in degrees $* \leq g/2$. Hence it is enough to show that Σ_n acts trivially on $H_*\Gamma_{g,n+1}$ for $* \leq g/2$.

Let x be an element in the k-th homology of $\Gamma_{g,n+1}$ with $k \leq g/2$. As the inclusion incl: $\Gamma_{g,1} \to \Gamma_{g,n+1}$ induces an isomorphism on the k-th homology [H], [I], x can be represented by a closed chain z in the bar complex of $\Gamma_{g,1}$.

The action of Σ_n on the homology of $\Gamma_{g,n+1}$ is induced by conjugation. Let μ be an element of Σ_n. Then μ has a lift $\tilde{\mu}$ to $\Gamma_{g,(n),1}$ such that it may be represented by a diffeomorphism which is entirely supported on the surface $F_{0,n+1}$ (with $k = 0$) of Figure 1. The mapping classes appearing in the closed chain z, on the other hand, are represented by diffeomorphisms of the surface $F_{g,1}$. Diffeomorphisms with disjoint support commute, and hence $\tilde{\mu}^{-1} z \tilde{\mu} = z$. It follows that μ acts trivially on x. □

The lemma implies the following generalization of the stability theorem. Corollary 1.2 can be deduced immediately from this and the fact that by a theorem of Nakaoka [N], the Leray–Serre spectral sequence for the wreath product collapses at the E^2-term.

Proposition 3.4. $H_*\Gamma^{H_k}_{g,H_n,m} \simeq H_*\Gamma^{H_k}_{g+1,H_n,m}$ *for $* \leq g/2$ and $k + n + m \geq 1$.*

Proof. The two groups are related by group homomorphisms

$$\Gamma^{H_k}_{g,H_n,m} \longleftarrow \Gamma^{H_k}_{g,H_n,m+1} \longrightarrow \Gamma^{H_k}_{g+1,H_n,m+1} \longrightarrow \Gamma^{H_k}_{g+1,H_n,m}.$$

Consider the induced maps of Serre spectral sequences for the fibration of type (3.2) associated to each of these groups. By the above lemma and the stability theorem [H], [I], the induced maps on the second term of the spectral sequences $E^2_{*,q}$ are isomorphisms for $q \leq g/2$. The result thus follows by the Zeeman comparison theorem. □

We finish this section by identifying the maximal perfect subgroup of the decorated mapping class groups.

Proposition 3.5. $P(\Gamma^{H_k}_{g,H_n,1}) = \rho^{-1}(P(H_k \times H_n))$ *for $g > 2$, and in particular contains $\Gamma_{g,1}$.*

Proof. Let $g > 2$. Recall that for all n, $\Gamma_{g,n}$ is perfect [P]. There is a natural surjection $\Gamma_{g,n+k} \to \Gamma^k_{g,n}$. Hence, as the image of a perfect group is perfect, $\Gamma^k_{g,n}$ is perfect for all k and n. Now consider the general case. The kernel of ρ is $\Gamma^k_{g,n}$. As an extension of perfect groups is again perfect, we see $\rho^{-1}(P(H_k \times H_n))$ is perfect. It must be maximal as otherwise its image under ρ would be larger then $P(H_k \times H_n)$ and not perfect. But that is impossible. □

4. Stable retractions: The proof of Theorem 1.3

The purpose of this section is to prove Theorem 1.3. We will give details in the case of $\tilde{\beta}$. We will use a configuration space model for $B\Gamma^{(k)}_{g,n+1}$ and construct a map

$$R\colon \Omega^\infty\Sigma^\infty B\Gamma^{(k+1)}_{g,n+1} \longrightarrow \Omega^\infty\Sigma^\infty B\Gamma^{(k)}_{g,n+1} \tag{4.1}$$

such that $R\circ\Omega^\infty\Sigma^\infty\widetilde{\beta} \simeq \mathrm{Id}$. At the end of the proof, we will indicate the necessary changes to be made in the case of $\tilde{\tilde{\alpha}}$.

Fix a surface $F = F_{g,n+1}$ with a distinguished, non-free boundary curve, and let $C^k(F)$ denote the space of configurations $\mathcal{X} = \{x_1, \dots, x_k\}$ of k distinct, unordered points x_i on the (interior of the) surface F. The group $D = \mathrm{Diff}^+(F;\partial)$ of orientation preserving diffeomorphisms of F which fix the boundary pointwise acts on $C^k(F)$.

Lemma 4.2. $E^k_{g,n+1} = ED\ \times_D C^k(F)$ *is a classifying space for the group* $\Gamma^{(k)}_{g,n+1}$.

Proof. This follows from the fibration $C^k(F) \to E^k_{g,n+1} \to BD \simeq B\Gamma_{g,n+1}$ and from the fact that $C^k(F)$ is a classifying space for the group $\pi_1 C^k(F)$, the group of braids with k strands in $F\times[0,1]$. The last assertion is proved using the covering space $\widetilde{C}^k(F)$ of $C^k(F)$, i.e., the space of ordered configurations of k points on F, which admits an inductive sequence of fibrations

$$\begin{array}{ccccc}
\widetilde{C}^k(F_{g,n+1}) & \longleftarrow\ \dots\ \longleftarrow & \widetilde{C}^2(F^{k-2}_{g,n+1}) & \longleftarrow & \widetilde{C}^1(F^{k-1}_{g,n+1}) = F^{k-1}_{g,n+1}. \\
\Big\downarrow{\scriptstyle P_0} & & \Big\downarrow{\scriptstyle P_{k-2}} & & \\
F_{g,n+1} & & F^{k-2}_{g,n+1} & &
\end{array}$$

Here P_i forgets the last point of a configuration; the total space of P_i is the fibre of P_{i-1}. Since all the base spaces are surfaces with at least one boundary curve, an inductive argument shows that $\widetilde{C}^k(F_{g,n+1})$ and thus $C^k(F_{g,n+1})$ has no higher homotopy. $\square$

The fibration $C^k(F) \to E^k_{g,n+1} \to B\Gamma_{g,n+1}$ now gives the group extension $\pi_1 C^k(F) \to \Gamma^{(k)}_{g,n+1} \to \Gamma_{g,n+1}$. The inclusion $\widetilde{\beta}$ is given by the inclusion of fibres

$$\begin{array}{ccc}
C^k(F) & \xrightarrow{\ b\ } & C^{k+1}(F) \\
\Big\downarrow & & \Big\downarrow \\
E^k_{g,n+1} & \xrightarrow{\ \widetilde{\beta}\ } & E^{k+1}_{g,n+1} \\
\Big\downarrow & & \Big\downarrow \\
B\Gamma^0_{g,n+1} & = & B\Gamma^0_{g,n+1},
\end{array}$$

which adds to a configuration $\mathcal{X} \in C^k(F)$ a new point in the following way. Let $S^1\times[-1,1]$ be a collar along the distinguished boundary curve. Define $b'\colon F \longrightarrow F$

as $b'(y,t) = \big(y, \frac{1}{2}t^2\big)$ for $y \in S^1, 0 \leqslant t \leqslant 1$, and extend b' smoothly to $S^1 \times [-1,0]$ and identically outside the collar. Set $\xi = \big(y_0, \frac{3}{4}\big)$ for some fixed $y_0 \in S^1$, and then define $b(\mathcal{X}) = \{b'(x_1), \ldots, b'(x_k), \xi\}$ for $\mathcal{X} = \{x_1, \ldots, x_k,\} \in C^k(F)$.

$C^k(F)$ is via b a closed subspace of $C^{k+1}(F)$; thus the same is true for $E^k_{g,n+1}$ in $E^{k+1}_{g,n+1}$; we denote the quotient by $\widetilde{E}^{k+1}_{g,n+1} = E^{k+1}_{g,n+1}/E^k_{g,n+1}$; and by $q_k \colon E^{k+1}_{g,n+1} \to \widetilde{E}^{k+1}_{g,n+1}$ the natural map. By V_k we denote the bouquet of these filtration quotients:

$$V_k = \bigvee_{i=1}^{k} \widetilde{E}^i_{g,n+1}.$$

Theorem 4.3. *There is a homotopy equivalence* $\Theta_k \colon \Omega^\infty\Sigma^\infty E^k_{g,n+1} \to \Omega^\infty\Sigma^\infty V_k$.

Proof. We replace (à la Kahn–Priddy, Quillen) $\Omega^\infty\Sigma^\infty Y$ for a connected space Y with base point $*$ by the labelled configuration space $C(\mathbb{R}^\infty; Y)$ of unordered configurations $z_1, \ldots, z_k$ of distinct points $z_i \in \mathbb{R}^\infty$ with (not necessarily distinct) labels $y_1, \ldots, y_k \in Y$; a point in $C(\mathbb{R}^\infty; Y)$ is thus a set $\{(z_1, y_1), \ldots, (z_k, y_k)\}$, where a pair (z_i, y_i) is deleted if $y_i = *$. A point in $E^k_{g,n+1}$ is given as a D-orbit of $\widetilde{e} = (e, \mathcal{X})$, where $e \in ED$ and $\mathcal{X} = \{x_1, \ldots, x_k\}$ a configuration on F. Set $\mathcal{X}_\alpha = \{x_i \in \mathcal{X} \mid i \in \alpha\}$ for any (non-empty) subset $\alpha \subset \{1, \ldots, k\}$. Regard $\mathcal{X}_\alpha$ as a point in $\mathbb{R}^\infty$ by choosing an embedding $\coprod_{l \geq 0} C^l(F) \hookrightarrow \mathbb{R}^\infty$. Then define

$$\Theta'_k \colon E^k_{g,n+1} \longrightarrow C(\mathbb{R}^\infty; V_k)$$

by $\Theta'_k(e, \mathcal{X}) = \{(\mathcal{X}_\alpha, q_l(e, \mathcal{X}_\alpha)) \mid \emptyset \neq \alpha \subset \{1, \ldots, k\},\ l = \#\alpha\}$. This Θ'_k is a continuous map which extends to a map Θ_k on $C\left(\mathbb{R}^\infty; E^k_{g,n+1}\right)$ since the latter is a free object in the category of infinite loop spaces generated by $E^k_{g,n+1}$. To prove that Θ_k is a homotopy equivalence, note first that Θ_1 is homotopic to the identity; then consider the diagram

$$\begin{array}{ccccc}
C\left(\mathbb{R}^\infty;\, E^k_{g,n+1}\right) & \xrightarrow{\ \widetilde{\beta}\ } & C\left(\mathbb{R}^\infty;\, E^{k+1}_{g,n+1}\right) & \xrightarrow{\ q_{k+1}\ } & C\left(\mathbb{R}^\infty;\, \widetilde{E}^{k+1}_{g,n+1}\right) \\
{\scriptstyle \Theta_k}\downarrow & & {\scriptstyle \Theta_{k+1}}\downarrow & & \Big\| \\
C\left(\mathbb{R}^\infty;\, V_k\right) & \xrightarrow{\ \iota\ } & C\left(\mathbb{R}^\infty;\, V_{k+1}\right) & \xrightarrow{\ p_{k+1}\ } & C\left(\mathbb{R}^\infty;\, \widetilde{E}^{k+1}_{g,n+1}\right).
\end{array}$$

Here ι is induced by the inclusion $V_k \hookrightarrow V_{k+1}$, and p_{k+1} is induced by the projection onto the last leaf. It is commutative (up to homotopy): for the left-hand square, all $\big(\mathcal{X}_\alpha, q_l(e, \mathcal{X}_\alpha)\big)$ are deleted: if $k+1 \notin \alpha$ and $l = \#\alpha$, then $q_l(e, \mathcal{X}_\alpha)$ is contained in V_k; if $k+1 \in \alpha$ and $\mathcal{X} = b(\mathcal{X}')$, then $q_l(e, \mathcal{X}_\alpha)$ is the basepoint in V_{k+1} for all $l = 1, \ldots, k+1$. For the right-hand square only $\big(\mathcal{X}_\alpha, q_{k+1}(e, \mathcal{X}_\alpha)\big)$ for $\alpha = \{1, \ldots, k+1\}$ survives the projection q_{k+1}. Because the horizontal sequences are quasi-fibrations and all spaces are of the homotopy type of a CW-complex, it follows by induction that Θ_k is a homotopy equivalence. $\square$

Under these homotopy equivalences Θ_k's, the retraction R of (4.1) corresponds to the projection onto the first $k-1$ leaves of the bouquet V_k. This proves Theorem 1.3 for the case $\tilde{\beta}$.

In the case of $\tilde{\alpha}$, we first note that on the level of mapping class groups, keeping a boundary component fixed is the same as fixing a point and the direction of a tangent vector at that point. Since we always have at least one boundary component the tangent bundle of F is trivial. Thus a tangent direction is nothing else but a label in S^1, and we need to consider the configuration space $C^n(F; S^1)$ of configurations of n points with labels in S^1 of the k-punctured surface $F = F^k_{g,1}$. The group D is now the group of orientation preserving diffeomorphisms $\mathrm{Diff}^+(F;\partial)$ which fix the (non-free) boundary component and the k punctures. The analogous statements of Lemma 4.2 and Theorem 4.3 can now be proved in a similar way.

We remark that the proof relies on the same idea used to stably split loop spaces of suspensions. For details see [Bö] or [CMT]. To reduce the essence of the proof even more: the main idea is that the classifying space of the k-th braid group of a surface (with labels) is a stable retract of the $(k+1)$-st braid group of the surface.

5. Appendix: Group pairs, H-action, and a general splitting result

The splitting of Theorem 3.1 can also be seen in a somewhat different context. The idea is that the retraction induced by the group homomorphisms incl and ϕ on the plus construction of the classifying spaces is *multiplicative*, and hence gives rise to a splitting of spaces. This uses the H-space structure of $B\Gamma^+_\infty$ which will be made explicit below.

Adopting arguments of [W] we consider the general situation: Two groups G and H form a *direct sum pair* if H is a subgroup of G and there is a group homomorphism $\oplus\colon H \times G \to G$. Furthermore, we will assume that for any $g_1, \dots, g_s \in G$ and $h_1, \dots, h_s \in H$ there exist elements $c \in P(G)$ and $d \in P(H)$ such that for all $i = 1, \dots, s$

$$1 \oplus g_i = c \cdot g_i \cdot c^{-1} \quad \text{and} \quad h_i \oplus 1 = d \cdot h_i \cdot d^{-1}. \tag{5.1}$$

Proposition 5.2. *BG^+ admits a left* H*-action by BH^+.*

This means there is a map $\mu\colon BH^+ \times BG^+ \to BG^+$ such that the restriction to the left factor is homotopic to the map induced by the inclusion incl: $H \to G$ and the restriction to the right factor is homotopic to the identity. When H is equal to G, then BG^+ is an H-space.

Proof. Note that, by (2.2), $(BH \times BG)^+ = BH^+ \times BG^+$. Thus the direct sum homomorphism $\oplus$ induces a map

$$m\colon BH^+ \times BG^+ \longrightarrow BG^+.$$

Let $*$ denote the basepoint of BG^+ and BH^+. The map $m(\_\,, *)\colon BH^+ \to BG^+$ is induced by the group homomorphism $\_ \oplus 1$. By property (5.1), $\_ \oplus 1$ factors through H. We want to prove that the induced map $f\colon BH^+ \to BH^+$ is a homotopy equivalence. $BP(H)$ is a regular cover of BH, and hence $BP(H)^+$ is the universal cover of BH^+. By property (5.1), as $d \in P(H)$, the map $BP(H)^+ \to BP(H)^+$ induced by f is the identity on homology (compare Lemma 1.3 [W]). Hence, by the Whitehead theorem it is a homotopy equivalence. But then the map $f\colon BH^+ \to BH^+$ is a homotopy equivalence as well. Similarly, $m(*, \_)$ is a homotopy equivalence of BG^+. Now choose homotopy inverses r and l for these two maps. Then $\mu = m \circ (r \times l)\colon BH^+ \times BG^+ \to BG^+$ defines an H-action. $\square$

Corollary 5.3. *Assume G and H are as above, and that there is a splitting homomorphism $\phi\colon G \to H$. Then there is a splitting of spaces $BG^+ \simeq BH^+ \times F$.*

Proof. Let F be the homotopy fiber of the map $\phi\colon BG^+ \to BH^+$, and let $s\colon F \to BG^+$ denote the inclusion of the fiber. Define $BH^+ \times F \to BG^+$ by mapping (x, y) to $\mu(x, s(y))$. Because μ defines an H-action, this induces an isomorphism on homotopy groups and hence is a homotopy equivalence. $\square$

$\Gamma^{H_k}_{\infty,H_n}$ and Γ_∞ form a direct sum pair with $\oplus\colon \Gamma_\infty \times \Gamma^{H_k}_{\infty,H_n} \to \Gamma^{H_k}_{\infty,H_n}$ defined by letting Γ_∞ act on the odd handles and $\Gamma^{H_k}_{\infty,H_n}$ act on the even handles. The map of the underlying surfaces is illustrated in Figure 2. To check property (5.1), let $h_1, \ldots, h_s \in \Gamma_\infty$. Then they are in the image of $\Gamma_{g,1}$ for some large g. Now choose an appropriate diffeomorphism of the surface $F_{2g,1} \subset F^k_{2g,n+1}$ which moves the even handles over the odd handles to the last g handles. Let $d \in \Gamma_\infty$ be its homotopy class. Similarly, define an element $c \in \Gamma_\infty$. By Proposition 3.5, c and d are in the maximal perfect subgroups of $\Gamma^{H_k}_{\infty,H_n}$ and Γ_∞ respectively. Hence, by the corollary, there exists a space F such that

$$(B\Gamma^{H_k}_{\infty,H_n})^+ \simeq B\Gamma^+_\infty \times F.$$

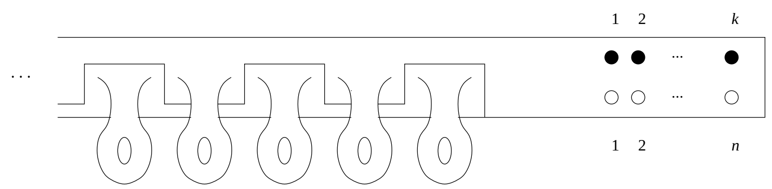

Figure 2: The direct sum homomorphism.

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