STABLE SPLITTINGS OF MAPPING SPACES
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0. Introduction.

In this note we elaborate on two observations concerning configuration spaces; they will lead to a stable splitting of certain mapping spaces into infinite bouquets of simpler spaces.

Let $K$ be a finite complex, $K_0$ a subcomplex, and $X$ a connected CW-complex. Then choose a smooth, compact and parallelizable $m$-manifold $M$ with a submanifold $M_0$ such that the pairs $(K,K_0)$ and $(M,M_0)$ are homotopy equivalent. For the space $\text{map}(K,K_0;S^mX)$ of based maps from $K/K_0$ to $S^mX$ we prove

PROPOSITION 1.

There is a stable equivalence

$$\text{map}(K,K_0;S^mX) \simeq \bigvee_{k=1}^\infty \mathcal{Q}_k;$$

the spaces $\mathcal{Q}_k$ depend on $M,M_0$ and $X$, in particular $\mathcal{Q}_1 = (M\wedge M_0, 3M\wedge M_0)\wedge X$.

Several special cases of this proposition are well-known.

EXAMPLE 1. $K = M = [0,1], K_0 = M_0 = (0,1)$.

The proposition gives a splitting of the suspension spectrum $S^m\Sigma SX$; a refinement of the proof would yield the splitting of $\Sigma SX$ found by Milnor [17], see Remark 3.

EXAMPLE 2. $K = M = D^m, K_0 = M_0 = 3D^m$.

This is the stable splitting of $\alpha^mS^mX$ found by Snaith [20].
EXAMPLE 3. $K = M = S^1, K_o = M_o = \emptyset$.
A stable splitting of the free loop space $\Lambda S X$ of $S X$ has recently been obtained by Goodwillie (unpublished).

EXAMPLE 4. $K = M = S^{m-1} \times [0,1], K_o = M_o = S^{m-1} \times \{0,1\}$.
This example gives a stable splitting of $\Omega^m S^m X$, the space of maps $f : S^m \to S^m X$ such that $f(s_o) = f(-s_o) = \ast$, where $s_o$ and $\ast$ are the basepoints; it is particularly interesting for $\mathbb{Z}/2$- and $S^1$-equivariant homotopy theory, (N.B. $\Omega^m S^m X \simeq \Omega^m S^m M \times \Omega^m S^m X$.)

EXAMPLE 5. $K = D^m, K_o = D^m, M = D^m \times [0,1], M_o = D^m \times [0,1]$.
In this case we obtain - also for non-connected $X$ - a stable splitting of $\Omega^m S^{m+1} X$; it is different but equivalent to the corresponding one replacing $X$ by $S X$ in Example 2.

EXAMPLE 6. $K = M = G$ a compact Lie group of dimension $m, K_o = M_o = \emptyset$.
Here the mapping space is the space of all unbased maps from $G$ to $S^m X$.

EXAMPLE 7. $K = \text{point}, K_o = \emptyset, M = D^m, M_o = \emptyset$.
We have $\text{map}(K,K_o; S^m X) = S^m X = \mathcal{O}_1$, all other $\mathcal{O}_K$ are contractible.

EXAMPLE 8. In general one can choose an embedding $K \subseteq \mathbb{R}^m$ of $K$, a regular neighbourhood $M$, a submanifold $M_o$ with $K_o \subseteq M_o$ and a deformation retraction of pairs $r_t : (M, M_o) \to (K, K_o)$. Hence $\text{map}(K,K_o; S^m X)$ always stably splits into a bouquet, if $m$ is at least the embedding dimension of $K$.

Such splittings are usually obtained by splitting appropriate configuration space models for the mapping spaces. In Section 1 we will define these models. In Section 2 we observe that (under certain connectivity assumptions) they are equivalent to mapping spaces. In Section 3 we ob-
serve that these models split stably, and we conclude Proposition 1. In Section 4 we list some properties of the splittings.

We do not claim any originality. In fact, all the constructions and proofs either can be found in the literature (e.g. [2], [5], [16] and [20]) are well-known to the experts. Only the importance of such splittings may justify the publication of a unified approach.

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1. The Configuration Spaces.

Let \( N \) be a smooth \( m \)-manifold, \( N_0 \) a submanifold (closed as a subspace), and \( X \) a CW-complex with basepoint \( * \). We denote by \( C(N,N_0;X) \) the space of finite configurations of particles in \( N \) with parameters (or labels) in \( X \), which are annihilated in \( N_0 \) or for vanishing; more precisely, let \( \mathcal{C}(N,k) = \{(z_1, \ldots, z_k) \in N^k \mid z_i \neq z_j \text{ for } i \neq j \} \) be the space of ordered (unlabeled) configurations of \( k \) points in \( N \); then \( C(N,N_0;X) \) is the quotient of \( \bigcup_{k=1}^{\infty} \mathcal{C}(N,k) \times X^k \) by the following identifications:

1.1 actions of the symmetric groups \( Y_k \)

\[
(z_1, \ldots, z_k; x_1, \ldots, x_k) \sim (z_s(1), \ldots, z_s(k); x_s(1), \ldots, x_s(k)) \quad \text{for } s \in Y_k.
\]

1.2 annihilation of particles with parameter \( * \)

\[
(z_1, \ldots, z_k; x_1, \ldots, x_k) \sim (z_1, \ldots, z_{k-1}; x_1, \ldots, x_{k-1}) \quad \text{if } x_k = *;
\]

1.3 annihilation of particles in \( N_0 \)

\[
(z_1, \ldots, z_k; x_1, \ldots, x_k) \sim (z_1, \ldots, z_{k-1}; x_1, \ldots, x_{k-1}) \quad \text{if } z_k \in N_0.
\]

Because of (a) we will write a configuration \( \xi \in C = C(N,N_0;X) \) as a formal sum \( \xi = \sum z_i x_i \) bearing in mind that \( C \) is a subspace of the infinite symmetric product \( \text{SP}_\infty((N/N_0) \wedge X) \); then (1.2) and (1.3) can be re-
placed by: \( zx = 0 \) if \( x = \ast \) or \( z \in N_0 \), respectively, where \( O \) denotes the basepoint in \( C \) (which is represented by any \( \xi = \sum z_i x_i \) such that for all \( i, x_i = \ast \) or \( z_i \in N_0 \) holds).

Such configuration spaces have been extensively studied by Fadell-Neuwirth [8] for \( N_0 = \emptyset \) and \( X = S^0 \), by Mc Duff [16] for \( N_0 = \mathbb{R} \) and \( X = S^1 \), and by Cohen-Taylor [2] for \( N_0 = \emptyset \).

**Example 9.** \( C(M;X) = C(M,\emptyset;X) \) are the well known configuration spaces of May [13] and Segal [19]. \( C(M;X) \) is homotopy equivalent to the James construction [9].

The length \( k \) of a configuration \( \xi = \sum_{i=1}^{k} z_i x_i \) induces a natural filtration of \( C \) by closed subspaces \( C_k(N,N_0;X) = (\coprod_{i=1}^{k} C(N,k) \times X^k)/\sim \). The inclusion \( C_{k-1} \to C_k \) is a cofibration, because \( N_0 \to N \) and \( \ast \to X \) are. \( C_0 \) consists of \( \emptyset \) only, and \( C_1 \) is \( (N,N_0) \times X \).

If the pair \( (N,N_0) \) or \( X \) is connected then each particle \( z_i \) of a configuration \( \xi \) can be moved to \( N_0 \) or its parameter \( x_i \) can be moved to \( \ast \); therefore \( \xi \) can be moved to \( O \), i.e. \( C \) is connected. If \( N \) is connected, \( N_0 = \emptyset \) and \( X = S^0 \), then the strata \( C_k - C_{k-1} = C(N,k)/\gamma_k = C(N,k) \) of \( C = C(N) = C(N,\emptyset;S^0) \) are the connected components of \( C \).

So far we have not used that \( N \) is a manifold — indeed \( N \) might have been any space; in particular, \( C(M;X) \) will be of importance to us (see Example 13 and Section 3).

**Example 10.** The connected components of \( C(M) \) are the classifying spaces of the symmetric groups; those of \( C(R^2) \) are the classifying spaces of Artin's braid groups.
EXAMPLE 11. \( C^m \oplus D^m; X \) is homotopy equivalent to \( S^m X \), see [16; p. 95].

The construction \( C \) is a homotopy functor in \( X \), but only an isotopy functor in \( (N, N_0) \).

So, for example, the inclusion \( N \setminus \partial N \to N \) induces a homotopy equivalence \( C(N \setminus \partial N, N_0 \setminus \partial N; X) \to C(N, N_0; X) \). The excision property

\[ C(N, N_0; X) \cong C(N \setminus U, N_0 \setminus U; X) \]

for \( U \subset N_0 \) and \( U \) open in \( N \), and the product property

\[ C(N, N_0; X) \cong C(N', N' \cap N_0; X) \times C(N'', N'' \cap N_0; X) \]

for \( N = N' \cup N'' \) and \( N' \cap N'' \subset N_0 \) follow easily from the definition. The crucial property of \( C \) is contained in the following lemma.

Lemma.

Let \( H \subset N \) be an \( m \)-dimensional submanifold. Then the isotopy cofibration

\[ (H, H \cap N_0) \to (N, N_0) \to (N, H \cup N_0) \]

induces a quasifibration

\[ C(H, H \cap N_0; X) \to C(N, N_0; X) \to C(N, H \cup N_0; X) \]

provided \((H, H \cap N_0)\) or \(X\) is connected.

Proof: Except for the presence of a parameter space \( X \) the proof is that of [16; Proposition 3.1]; we list the various steps.

1. We filter the base space \( B = C(N, H \cup N_0; X) \) by \( B_k = C_k(N, H \cup N_0; X) \), and the total space \( E = C(N, N_0; X) \) by \( E_k = Q^{-1}(B_k) \), and we denote the fibre by \( F = C(H, H \cap N_0; X) \).

2. Observe that for each \( k \) there is homeomorphism

\[ h_k : E_k \setminus E_{k-1} \cong (B_k \setminus B_{k-1}) \times F \quad \text{over} \quad B_k \setminus B_{k-1}. \]

3. A tubular neighbourhood \( U \) of \( H \) defines for each \( k \) a neighbourhood \( U_k \) of \( B_k \) in \( B_{k+1} \), and an isotopy retraction \( r : U \to H \) induces
retractions $r_k : U_k \rightarrow B_k$, and retractions
$I_k : Q^{-1}(U_k) \rightarrow Q^{-1}(B_k) = E_k$ lying over $r_k$.

(4) For every $b \in U_k$ the induced map

\[
\begin{align*}
F & \xrightarrow{\sim} Q^{-1}(b) \xrightarrow{\sim} Q^{-1}(I_k(b)) \xrightarrow{\sim} F \\
h_{k+1} & | r_k | h_k
\end{align*}
\]

is a homotopy equivalence (precisely because $(N, H \cap N_0)$ or $X$ is connected).

It follows from the Dold-Thom criterion [8 ; 2.10, 2.15, 5.2] that $Q$ is a quasifibration. □

2. The Section Spaces

The space $C(N, N_0; X)$ is under certain connectivity conditions equivalent to the space of sections of a certain bundle with fibre $S^m X$, and whence sometimes equivalent to a space of maps into $S^m X$. To make this precise let $W$ be any smooth $m$-manifold without boundary which contains $N$ (for example, $W = N$ if $\exists N = \emptyset$, or $W = N \cup (\exists N \times \{0,1\}$ otherwise); if $\hat{T}(W)$ denotes the fibrewise compactification of the tangent bundle $T(W)$ of $W$, then define $\hat{T}(W; X) = \hat{T}(W) \wedge X$ to be fibrewise smash product of $T(W)$ and $X$; this is a new bundle $\hat{r} : \hat{T}(W; X) \rightarrow W$ with fibre $S^m X$.

The inclusion of the basepoint into each fibre yields a section $g_0$ of $\hat{r}$. For $A_0 \subset A \subset W$ let $\Gamma(A, A_0; X)$ denote the space of sections of $\hat{r}$ which are defined on $A$ and agree with $g_0$ on $A_0$; it is equipped with the (compactly generated topology induced by the) compact-open topology. (For example, if $X = S^0$ then $\hat{T}(W, S^0) = \hat{T}(W)$ and the sections are the vector fields with possible poles.)
The main theorem about configuration spaces on manifolds is the following duality.

**PROPOSITION 2.**

For compact \( N \) there is a map \( \gamma : C(N,N_0;X) \to \Gamma(W \setminus N_0, W \setminus N;X) \), which is a (weak) homotopy equivalence provided \((N,N_0)\) or \( X \) is connected.

**Proof:** The proof is essentially contained in Mc Duff [16; Theorem 1.4] or [15]. For convenience we indicate the various steps.

1. Following ideas of Gromov the map \( \gamma \) is defined as in [16; p. 95], or as in [15; p. 90] using Example 11, we have \( \gamma(0) = g_\infty \).

2. We start to prove the assertion with the case of \((N,N_0)\) being a handle \( (\mathbb{D}^m, \mathbb{D}^k \times S^{m-k-1}) \) of index \( k \). First, the assertion is true for \( k = 0 \) by Example 11. Consider for \( k = 1,2,\ldots,m \) in \( I^k = [0,1]^m \) the subspace \( I^k_\mathbb{R} \) of all \( y = (y^i,\ldots,y^m) \) such that \( y^i = 0 \) or \( y^i = 1 \) for some \( i = k+1,\ldots,m \), or \( y^k = 1 \); set \( H^k = [0,1]^{k-1} \times [0,\frac{1}{2}] \times [0,1]^{m-k} \). In the sequence

3. \( (H_k, H_k \cap I^k_\mathbb{R}) \to (I^k, I^k_k) \to (I^m, I^m_k \cup I^m_k) \) the left hand pair is a handle of index \( k \), the right hand pair is a handle of index \( k-1 \). We apply \( C(\_;X) \) to (3) and obtain by the above lemma a quasifibration for \( k = 1,\ldots,m-1 \) if \( X \) is arbitrary, and in addition for \( k = m \) if \( X \) is arbitrary, and in addition for \( k = m \) if \( X \) is connected. We apply \( \Gamma(\_;X) \) to the complements in \( W = \mathbb{R}^m \) of (3) and obtain a fibration; \( \gamma \) maps the quasifibration to the fibration. Notice that both total spaces are contractible. Hence we conclude by induction the assertion for all handles of index \( k = 0,1,\ldots,m-1 \) if \( X \) is arbitrary, and in addition for the handle of index \( m \) if \( X \) is connected.

4. For the case \((N,3N)\) choose a handle decomposition of \( N \), and if \((N,3N)\) is connected choose one without handles of index \( m \). Attaching a new handle gives a quasifibration for \( C \) and a fibration for \( \Gamma \), \( \gamma \) mapping
one to the other. Induction on the number of handles proves the
assertion for \((N,\exists N)\).

(5) For the case \((N,N_o)\) with \(N_o \subseteq \exists N\), i.e. \(L \cup N_o = \exists N\) and \(L \cap N_o = \exists L = \exists N_o\). We attach a
closed collar to \(N\), \(N = N \cup (N \times [0,1])\), and consider the sequence

\[(\bar{L},\bar{L} \cap \bar{N}_o) \rightarrow (\bar{N},\bar{N}_o) \rightarrow (\bar{N},\bar{L} \cup \bar{N}_o) \text{ with } \bar{L} = L \times [0,1] \text{ and} \]
\(\bar{N}_o = N_o \times [0,1].\) The assertion is true for the right hand pair by
(4) since \((\bar{N},\bar{L} \cup \bar{N}_o) = (\bar{N},\exists \bar{N}) \cong (N,\exists N).\) As before, the assertion
will follow for \((\bar{N},\bar{N}_o) \cong (N,N_o)\) if we can prove it for
\((\bar{L},\bar{L} \cap \bar{N}_o) = (L,\exists \bar{L}) = (L,\exists L) \times [0,1].\)

(7) For this case we use the sequence

\[(L,\exists L) \times [0,1] \rightarrow (L,\exists L) \times ([0,2],\{2\}) \rightarrow (L \times [0,2],\exists (L \times [0,2])).\]

The assertion is true for the right hand pair by (4); it is true
for the middle pair, since this gives contractible spaces. Hence
the assertion follows for the left hand pair.

(9) For the case of an arbitrary submanifold \(N_o \subseteq N\) we replace \(N_o\) by
closed tubular neighbourhood and then remove the interior of this
neighbourhood. By isotopy invariance and excision property both
manipulations leave the homotopy type of \(C\) unaltered. But now we
are in case (5). □

\textbf{Example 12. (Example 8 continued).}\ Under the assumptions of Proposition
1 set \(N = M \setminus M_o\) and \(N_o = \exists M \setminus M_o\), and \(W = M \cup (\exists M \times [0,1])\) if \(\exists M \neq \emptyset\), or
\(W = M\) if \(\exists M = \emptyset\). As a corollary we have

\[\Gamma(M \setminus M_o, \exists M \setminus M_o; X) \cong \Gamma(W \setminus (\exists M \setminus M_o), W \setminus (M \setminus M_o); X)\]
by Proposition 12

\[= \Gamma(W \setminus \exists M) \cup M_o, (W \setminus M) \cup M_o; X)\]

\[= \Gamma(M \setminus \exists M, M_o \setminus \exists M; X)\]
by excision

\[= \Gamma(M, M_o; X)\]
by extension over \(\exists M\)

\[= \text{map}(M, M_o; S^M X)\]
by parallelizability

\[= \text{map}(K, X_o; S^M X)\],
where we should replace \( M_0 \) by an open tubular neighbourhood to ensure compactness of \( M \setminus M_0 \).

**Example 13** (Example 2, 9 and 10 continued). If \( N = D^m \), \( N_0 = \emptyset \) and \( W = \mathbb{R}^m \), then \( \gamma \) is the well-known approximation
\[
C(D^m; X) \simeq C(D^m; X) \to \text{map}(\mathbb{R}^m, \mathbb{R}^m \setminus D^m; S^m X) \simeq \Omega^m S^m X \text{ of May [13] and Segal [19].}
\]
Passing to the limit over \( m \) yields \( \gamma^\infty : C(D^\infty; X) \to \Omega^\infty S^\infty X = (X) \).
See also Vogel [21].

**Remark 1.** For \( C(M \setminus M_0, \mathbb{R}^m \setminus M_0; X) \) to be a model for \( \text{map}(K, K_0; S^m X) \) it is obviously enough that \( (M, M_0) \) is relatively compact and relatively parallelizable; but more important is that \( X \) need not be connected if \( (M \setminus M_0, \mathbb{R}^m \setminus M_0) \) happens to be connected, see e.g. Example 5. In general, \( \gamma \) approximates the homology of the section space, see [16]; so in case \( \mathbb{N} \neq \emptyset \), \( \gamma \) is a completion of homology modules over \( H^* (\Omega \text{map}(\mathbb{N}; S^m X)) \).
An interesting example is \( C(RP^m) \), since \( \Gamma(RP^m) = \Gamma(RP^m; S^0) \) is the space of self-maps of \( S^m \) which are equivariant with respect to the antipodal action.

3. The Stable Splittings.

In [20] Snaith has obtained a stable splitting of \( \Omega^m S^m X \) using the models \( C(RP^m; X) \). Since then several authors have given very elegant proofs of this result, see P. Cohen [5], R. Cohen [6], Cohen-May-Taylor [3], May-Taylor [14], Vogt [22]. Our construction of a stable splitting of \( C = C(N, N_0; X) \) is almost verbatim taken from [5].

Let \( D_k = D_k(N, N_0; X) \) denote the filtration quotients \( C_k/C_{k-1} \) and consider the bouquet \( V = V(N, N_0; X) = \bigvee_{k=1}^\infty D_k \) with the filtration given by \( V_k = \bigvee_{j=1}^k D_j \).

Next we define the "power set map" \( P : C \to C(D^\infty; V) \). Take some
\[ \xi = \sum_i z_i x_i \in C \] and a (non-empty) subset \( \alpha = \{ i_1, \ldots, i_k \} \) of the index set \( I(\xi) \) of \( \xi \). Define \( Z_\alpha \) to be the (unlabeled) configuration
\[ Z_\alpha = \sum_{j=1}^k z_i \] consisting of all \( z_i \) in \( \xi \) such that \( i \in \alpha \); \( Z_\alpha \) is in \( C(N,k) \) which is an \( km \)-manifold; we choose an embedding of their disjoint union \( C(N) = \bigcup_{k=1}^\infty C(N,k) \) into \( \mathbb{R}^\infty \), and let \( \tilde{z}_\alpha \in \mathbb{R}^\infty \) denote the image of \( Z_\alpha \) under this embedding. Correspondingly, define \( \xi_\alpha \) to be the subconfiguration \( \xi_\alpha = \sum_{j=1}^k z_i x_i \) of \( \xi \) consisting of all labeled particles \( z_i x_i \) of \( \xi \) such that \( i \in \alpha \); \( \xi_\alpha \) is in \( C_k = C_k(N,N_0;X) \); using the quotient map \( C_k \to D_k \) and the inclusion \( D_k \to V \) we let \( \tilde{\xi}_\alpha \in V \) denote the image of \( \xi_\alpha \) under the composition of these two maps. Finally, we define \( P(\xi) = \sum_{\alpha} \tilde{z}_\alpha \tilde{\xi}_\alpha \) in \( C(\mathbb{R}^\infty;V) \) where the sum is over all subsets of \( I(\xi) \).

Notice that the \( \tilde{z}_\alpha \) are mutually different since two of the same length \( k \) have already different \( Z_\alpha \) in \( C(N,k) \), and the various \( C(N,k) \) are disjointly embedded into \( \mathbb{R}^\infty \). \( P \) is continuous since it is well-defined: (1.1) is respected because a permutation of \( I(\xi) \) only permutes the new indices \( \alpha \); (1.2) and (1.3) are respected because if \( z_i \in N_0 \) or \( x_i = * \), then, for any \( \alpha \) such that \( i \in \alpha \), \( \tilde{\xi}_\alpha \) is the basepoint in \( D_k \) and in \( V \), hence \( \tilde{z}_\alpha \tilde{\xi}_\alpha = 0 \) in \( C(\mathbb{R}^\infty;V) \).

Now let \( \sigma : S^\infty C \to S^\infty V \) denote the adjoint of the composition
\[ \gamma^\infty \circ P : C \to C(\mathbb{R}^\infty;V) \to Q(V) = \Omega^\infty S^\infty V \] with \( \gamma^\infty \) as in Example 13.

**Proposition 3.**

\( \sigma \) is a stable equivalence \( C(N,N_0;X) \to \bigvee_{k=1}^\infty D_k(N,N_0;X) \) for any \( (N,N_0) \) and \( X \).

**Proof:** \( \sigma \) obviously preserves the filtration and we have a commutative lower square in the diagram
whereas the upper square is only homotopy commutative. Since the vertical sequences are cofibrations and since $C_1 = V_1$, the assertion follows by induction on $k$. 

**Proof of Proposition 1** (Example 7 and 11 continued). The stable splitting of $\map(K, K_0; S^M X)$ now follows from that of $C(M \setminus M_0, 3M \setminus M_0; X)$. The spaces $\mathcal{D}_k$ are $D_k(M \setminus M_0, 3M \setminus M_0; X)$, in particular we have

\[ \mathcal{D}_1 = C_1(M \setminus M_0, 3M \setminus M_0; X) = (M \setminus M_0, 3M \setminus M_0) \wedge X. \]

**EXAMPLE 14** (Example 2, 8 and 12 continued). The splitting we obtain for $K = M = D^m$ and $K_0 = M_0 = 3D^m$ is the Snaith splitting of $[20]$.

**Remark 2.** In the proof of Proposition 3 we did not use that $N$ is a manifold; the proof covers also the case of $C(\mathbb{R}, X)$ which is equivalent to $\mathbb{S}^m X$ if $X$ is connected. A stable splitting of $\mathbb{S}^m X$ was first obtained by Kahn, see [1], [10], [11] and [12]. Furthermore, we did not use that $(N, N_0)$ or $X$ is connected. This and Remark 1 shows that Proposition 1 is more generally true than stated, see e.g. Example 5.

**Remark 3.** A splitting of $\mathbb{S} S X$ is achieved by refining the power set map to a map $P : C = C(\mathbb{R}; X) \rightarrow C(\mathbb{R}; V(\mathbb{R}; X))$; the order of the particles $z_i$ on the real line induces a lexicographic order of the sets $a$, and the hereby induced order of the $\bar{z}_a$ is used to define particles $\bar{z}_a$ in $\mathbb{R}$ instead of $\mathbb{R}^m$. 

\[
\begin{align*}
S^m(C_k/C_{k-1}) & \quad S^m(V_k/V_{k-1}) \\
S^mC_k & \quad S^mV_k \\
S^mC_{k-1} & \quad S^mV_{k-1}
\end{align*}
\]
4. Naturality and Homotopy.

Assume we have two situations as in the introduction, a map

\( f : (K,K_0) \to (K',K'_0) \) together with an embedding

\( F : (M,M_0) \to (M',M'_0) \) making the obvious diagram commutative, \( m = m' \)

and \( X = X' \). Then \( f \) induces \( f^* : \text{map}(K',K'_0;S^mX) \to \text{map}(K,K_0;S^mX) \), while

\( F \) induces \( F^* : C(M' \setminus M'_0,\partial M' \setminus M'_0;X) \to C(M \setminus M_0,\partial M \setminus M_0;X) \) and

\( F'_k : D_k(M' \setminus M'_0,\partial M' \setminus M'_0;X) \to D_k (M \setminus M_0,\partial M \setminus M_0;X) \). The approximation map \( \gamma \) of Proposition 2 and the splitting map \( \sigma \) of Proposition 3 commute with these induced maps.

Examples for such maps \( f \) are the inclusions \( K_0 \to K \) and \( K \to (K,K_0) \),

the inclusion of a bottom cell of \( K \) and the pinch map onto a top cell of \( K \).

\( \gamma \) and \( \sigma \) are natural with respect to the suspension

\( \text{map}(K,K_0;S^mX) \to \text{map}(S(K,K_0);S^{m+1}X) \), which for \( C \) and \( V \) is induced by the equatorial inclusion \( (M,M_0) \to (M,M_0) \times \{[0,1],[0,1]\} \).

An analysis of the splitting map \( \sigma \) reveals that each of the spaces \( \mathcal{D}_k \) is already after a finite number of suspensions a retract of \( C \). An upper bound for the smallest number is given by the embedding dimension of \( C(N,k) \). In our standard situation of Example 7 we have \( N = M \setminus M_0 \) as a submanifold of \( \mathbb{R}^m \), so \( \mathcal{D}_1 = (M \setminus M_0,\partial M \setminus M_0) \times X \) is a retract of \n
\( \text{map}(K,K_0;S^mX) \) after at most \( m \) suspensions.

The (stable) projection onto this first summand

\( S^m \text{map}(K,K_0;S^mX) = S^mC \to S^m\mathcal{D}_1 = S^m(M \setminus M_0,\partial M \setminus M_0) \times X \) induces the homology slant product \( H^j_q(\text{map}(K,K_0;S^mX)) \to \bigoplus_j H^{j-q}(K,K_0;H_{j-m}(X)) \).

For \( X = S^0 \) this homomorphism has been proved by Moore [18] to be an isomorphism if \( q < 2(m - \dim(K,K_0)) \) which is twice the connectivity.
of the mapping space.

Studying the spaces $Q_k$ (which are Thom spaces for $X$ a sphere) is a possible approach to the homology of the mapping spaces; we will return to this in a further article.

References.


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Added in proof:

R. Cohen has independently found a model and a stable splitting for ASX (see his "A Model for the Free Loop Space of a Suspension", to appear).