CYCLIC HOMOLOGY AND MODULI SPACES OF RIEMANN SURFACES

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1 Introduction

The purpose of this note is to report on a certain family of finite complexes and their cyclic structure. The complexes in question arise as compactifications of moduli spaces of directed Riemann surfaces. The cyclic structure is an action of a cyclic group on the cells. The order of this cyclic operator and its behaviour with respect to the face operators is somewhat different from the general theory of cyclic objects. Nevertheless, it enables us to define Hochschild and cyclic homology groups for these complexes and to develop basic properties. Using in a addition a reflection operator one can also define dihedral and quaternionic homology groups.

In the background of all this is the moduli space $\vec{\mathfrak{M}}(g)$ of directed Riemann surfaces of genus g. It consists of conformal equivalence classes of triples $[F, \mathcal{O}, \mathcal{X}]$, where F is a closed Riemann surface and $\mathcal{X}$ is a tangent direction at some point $\mathcal{O}$. Since the mapping class group $\vec{\Gamma}(g) = \pi_0(\mathit{Diff}^+(F, \mathcal{O}, \mathcal{X}))$ acts freely on the contractible Teichmüller space $\vec{\mathfrak{T}}(g)$ of marked directed surfaces, the quotient $\vec{\mathfrak{M}}(g)$ is an orientable, open manifold of dimension $6g-3$ with the homotopy type of $B\vec{\Gamma}(g)$. The group $\vec{\Gamma}(g)$ is better known as the mapping class group of genus g surfaces with one boundary curves.

This moduli space $\vec{\mathfrak{M}}(g)$ can be described as a configuration space $\mathfrak{P}(g)$ of slits in the complex plane; we recall this uniformization in section 3 from [Bö 1]. A compactification $P(g)$ was developed in [Bö 2]. It has a cell structure whose chain complex resembles formally the Hochschild resolution of an algebra, and there is a cyclic operation and an involution on the cells.

This analogy is strong enough to permit the definition of Hochschild homology groups $HH_*(P(g))$ and cyclic homology groups $HC_*(P(g))$ for these

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complexes. They are related to their (topological) homology groups by long exact sequences

$$\cdots \longrightarrow HH_*(P(g)) \longrightarrow H_*(S^\sharp(g)) \longrightarrow H_{*-1}(P(g)) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H_*(S^\sharp(g)) \longrightarrow HC_*(P(g)) \longrightarrow HC_{*-2}(P(g)) \longrightarrow \cdots$$

in which $H_*(S^\sharp(g))$ is the homology of an intermediate complex. Our intension is to use this apparatus to study the spaces $P(g)/W(g)$, which are Poincaré dual to the moduli spaces $\vec{\mathfrak{M}}(g)$, and to study the spaces $\bar{U}(g) = U(g)/(U(g) \cap D(g))$.

Here we merely report on some basic ideas. We have as yet no interpretation available of such important connection between cyclic homology groups and K-Theory, the homology of Lie-algebras, Kähler forms, etc. .

We point out that $\vec{\mathfrak{M}}(g)$ carries a (non-free) S^1-action given by rotation of the tangent vector or $\mathcal{X}$. The quotient is the moduli space of genus g surfaces with one puncture. We have as yet now description of this action on the homeomorphic space $\mathfrak{P}(g)$; but we expect this action to be directly related to the cyclic action on cells. Complex conjugation of conformal structures is another symmetry on $\vec{\mathfrak{M}}(g)$; in this case it will be easy to see that it transforms to the reflection operator on cells.

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2 Moduli and parallel slit domains

We recall the describtion of the moduli space $\vec{\mathfrak{M}}(g)$ which leads to the compactification. The reader is refered to [Bö 1] for more details.

Let $[F, \mathcal{O}, \mathcal{X}] \in \vec{\mathfrak{M}}(g)$ be a directed Riemann surface of genus g. There is a function $u : F \longrightarrow \bar{\mathbf{R}} = \mathbf{R} \cup \infty$ with the following properties: (1) u is harmonic away from $\mathcal{O}$, and (2) $u(z) - Re(1/z)$ is smooth and vanishes at $\mathcal{O}$ for any local parameter z around $\mathcal{O}$ such that $z(\mathcal{O}) = 0$ and $dz(\mathcal{X}) = -dx$. This characterizes u uniquely up to an additve and a positive multiplicative constant.

Let the critical graph K of the gradient flow of u consist of the dipole $\mathcal{O}$, all zeroes of the flow, and of all integral curves which leave zeroes. Since $F_0 = F \backslash K$ is connected and simply-connected, u is the real part of a holomorphic map $w = u + iv : F_0 \longrightarrow \mathbf{C}$; w is unique, up to another additive constant for the harmonic-conjugate v of u. The complement of $w(F_0) \subseteq \mathbf{C}$, — described as a configuration of $2g$ pairs of of slits in the complex plane $\mathbf{C}$, — will comprise all moduli of the conformal class $[F, \mathcal{O}, \mathcal{X}]$.

A slit L_k is a horizontal half-line, starting at some point $z_k = (x_k, y_k) \in \mathbf{C}$, and unbounded to the left. There are always $4g$ slits, paired by a fixed point free involution λ in the symmetric group $\mathfrak{S}_{4g}$, acting on the index set $\mathbf{I} = \{1, \ldots, 4g\}$. A configuration is subject to two conditions:

$$y_k \le y_{k+1} \tag{1}$$

$$x_k = x_{\lambda(k)} \tag{2}$$

No assumption is made so far about the slits being disjoint or different.

To associate a surface $F(L)$ to L we glue, for each pair k and $\lambda(k)$, the upper (resp. lower) bank of L_k to the lower (resp. upper bank of $L_{\lambda(k)}$. As basepoint we choose $\mathcal{O} = \infty$ and $\mathcal{X}$ is the direction of $-dx$ under the local parameter $\zeta \mapsto 1/\zeta$. If $F(L)$ is a (non-singular) surface, it inherits from $\mathbf{C}$ a conformal structure, and thus $[F, \mathcal{O}, \mathcal{X}] \in \vec{\mathfrak{M}}(g)$. In case $F(L)$ has singularities, or if it is a surface of a genus smaller than g, we call L degenerate.

The following conditions (3) and (4) guarantee that $\mathcal{O}$ resp. no finite point of $F(L)$ is singular. Define a new permutation $\sigma = \lambda \circ t$, where t denotes the cyclic rotation $k \mapsto k+1$ $(mod\ 4g)$. Let $\kappa(\lambda) + 1$ denote the number of cycles of σ, which can be any even number between 0 and $4g$. We admit only λ for which

$$\kappa(\lambda) = 0 \tag{3}$$

holds; such a λ is called connected.

The next condition excludes certain subconfigurations.

(4) There are no three indices k, k+1 and k+2 such that :

$$\lambda(k) = k+2, \quad L_k = L_{k+2} \quad \text{and} \quad L_{k+1} \subseteq L_k.$$

In [Bö 2] we examined in detail, what type of singular surfaces occur if (3) or (4) are violated.

It is obvious from the gluing process that two different configurations can lead to conformally equivalent surfaces. Then they are connected by a chain of moves (called Rauzy-moves) of the following type: if $L_{k-1} \subseteq L_k$ then L_{k-1} can jump to the upper bank of the slit $L_{\lambda(k)}$. In its new position it will be contained in $L_{\lambda(k)}$, and all slits overtaken by this move change their index by a cyclic rotation and λ is conjugated accordingly. Such a move leaves $F(L)$ certainly invariant. The equivalence classes generated by Rauzy moves are denoted by $\mathfrak{L} = [L_1, \ldots, L_{4g} | \lambda]$. A class is called non-degenerate, if none of its representatives violates (3) or (4); such classes are in older literatur known as parallel slit domains.

On the space of all parallel slit domains the contractible 3-dimensional group of similarities of **C** acts freely as a group of conformal invariants. It is generated by translations in the x- and y-direction and dilations, whose parameters correspond precisely to the three undetermined constants in the complex potential w. We therefore introduce the following normalisations.

$$y_1 = 0 \tag{5}$$
$$y_{4g} = 1 \tag{6}$$
$$min\{x_k\} = 0 \tag{7}$$

These conditions are invariant under moves, and thus conditions on a class. For a non-degenerate class we always have $y_1 < y_{4g}$, enabling us to normalize as in (5) and (6). In a non-degenerate class the slits can not lie on the same horizontal, but all slits can end on the same vertical. The main result of [Bö 1] is that the space of all non-degenerate, normalized configuration classes is homeomorphic to the moduli space $\vec{\mathfrak{M}}(g)$.

It will be convenient for the compactification to introduce the condition

$$max\{x_k\} < 1. \tag{8}$$

This restricts to a subspace, which is homeomorphic to the entire space by reparametrizing the real parts of the slit end points. We denote this space of all classes satisfying (1) to (8) by $\mathfrak{P}(g)$. What we said above implies that this configuration space $\mathfrak{P}(g)$ is hoeomorphic to the moduli space $\vec{\mathfrak{M}}(g)$ for all g.

3 The cyclic structure of the compactification

One can compactify the space $\mathfrak{P}(g)$ by taking its closure in the space of all normalized classes of configurations: one simply forgets about the condition (4) and replaces (8) by the weak inequality

$$max\{x_k\} \leq 1. \tag{9}$$

This compactification of the moduli space is denoted by $P(g)$. Let $D(g)$ be the subspace of degenerate classes. The subspaces $N(g)$, consisting of all classes with $max\{x_k\} = 1$, is a partial boundary of the manifold $\mathfrak{P}(g)$. The subspace $U(g)$ of all classes such that $max\{x_k\} = 0$ (which we call uni-level surfaces) is a homotopy retract of $P(g)$, see [Bö 2]. $U(g)$ and $N(g)$ are disjoint, but $N(g)$ and $D(g)$ are not. $W(g) = D(g) \cup N(g)$ is called the periphery of $\mathfrak{P}(g)$, because $\mathfrak{P}(g) = P(g) \setminus W(g)$. Since $\mathfrak{P}(g)$ is an orientable manifold of dimension $6g-3$, Poincaré duality implies $H^*(P(g), W(g)) \cong H_{6g-3-*}(\mathfrak{P}(g))$ for all coefficients, see [Bö 2].

It was shown in [Bö 2] that $P(g)$ is a finite cell complex, whose cells are encoded by symbols

$$(10) \qquad E = [a_0, a_1, \ldots, a_{n+1} |\ \lambda\ | B_0, B_1, \ldots B_{m+1}] = [\ a \mid \lambda \mid B\ ].$$

If the slits lie on $n+2$ distinct y-levels – the 0-th being $y = 0$, the $(n+1)$-st being $y = 1$ – then a_i is the number of slits on the i-th horizontal. Thus $0 < a_i < 4g$, $\sum_{i=0}^{n+1} = 4g$ and $0 \le n \le 4g - 2$. Similarily, if the slits start at $m+2$ distinct x-levels – the 0-th being $x = 0$, and the $(m+1)$-st being $x = 1$, although there may be none on this last vertical – then B_j is the subset of indices whose slits start on the j-th vertical. The B_j are a λ-invariant disjoint decompostion of $\mathbf{I}$, non-empty for $j = 0, \ldots, m$; and $0 \le m \le 2g-1$. Taking the distances between these horizontals resp. verticals as barycentric coordinates, the cell E becomes a product of two open simplices, $E \cong \Delta^n \times \Delta^m$. We call n the vertical and m the horizontal dimension of E; and $\ell = n+m$ is its dimension.

Since a Rauzy move changes some of the numbers a_i, some of the sets B_j and conjugates λ, this notation (10) for a cell is not unique, which is indicated by the brackets refering to the equivalence relation generated by Rauzy moves.

But on the other hand, this notation makes it obvious, how similiar this cell structure is to several well-known constructions like the bar-construction or the Hochschild resolution of an algebra, as we shall see by looking at the face operators.

There are face operators ∂_i' for the first factor Δ^n and ∂_j'' for the second factor Δ^m of E, for $i = 0, \ldots, n$ resp. for $j = 0, \ldots, m$:

$$\begin{array}{llcl} (11) & \partial_i'(E) & = & [a_0, \ldots, a_i + a_{i+1}, \ldots, a_{n+1} |\ \lambda\ | B_0, \ldots, B_{m+1}], \\ (12) & \partial_j''(E) & = & [a_0, \ldots, a_{n+1} |\ \lambda\ | B_0, \ldots, B_j \cup B_{j+1}, \ldots, B_{m+1}]. \end{array}$$

The cyclic structure of this cell complex comes from the cyclic operator τ defined by

$$(13) \quad \tau(E) = [a_{n+1}, a_0, \ldots, a_n |\ \tau^{a_{n+1}}\ \lambda\ \tau^{-a_{n+1}}\ |\ \tau^{a_{n+1}} B_0, \ldots, \tau^{a_{n+1}} B_{m+1}],$$

where $\tau \in \mathfrak{S}_{4g}$ is the maximal cyclic permutation $k \mapsto k+1$ used earlier. τ moves the last package of slits on the level $y = 1$ to the bottom to become the first one; it follows that it is well-defined with respect to Rauzy moves. The cycle number of $\sigma' = \tau^{a_{n+1}}\ \lambda\ \tau^{-a_{n+1}}\ \tau$ is the same as that of $\sigma = \lambda\ \tau$, thus $\kappa(t^{a_{n+1}}\ \lambda\ t^{-a_{n+1}}\ t) = \kappa(\lambda) = 0$. The sets B_j are invariant under the new λ.

τ acts essentially on the first factor of E, in accordance with the general philosophy that this factor seems to hold more information.

REMARK. It is not sensible to extend the action of τ to also properly rotate the horizontal entries $B = (B_0, \ldots B_{m+1})$ of E. This would make the subcomplexes $D(g)$, $N(g)$ and $U(g)$ non-invariant. The same remark applies to the reflection operator below.

The order of τ on a cell E is not its dimension, but determined by its vertical dimension,

$$\tau^{n+2} = id. \tag{14}$$

The subspaces $D(g)$, $N(g)$, $U(g)$ and $W(g)$ are subcomplexess under this cell decomposition. The presence of singular subconfigurations as descibed in (4) is independent of the values of barycentric coordinates and therefore a property of a cell E; furthermore, such subconfigurations are then also present in each face of the cell and in the cell $\tau(E)$. $N(g)$ resp. $U(g)$ can be characterized by the properties $B_{m+1} \neq \emptyset$ resp. $m = 0$ of their cells; both properties are invariant under the face operators and the cyclic operator.

The relations between face operators and the cyclic operator are recorded in the following easily proved

LEMMA 1

$$\begin{aligned}
&(15) & \partial'_i \circ \partial'_j &= \partial'_{j-1} \circ \partial'_i \quad \text{for} \quad 0 \leq i < j \leq n, \\
&(16) & \partial'_i \circ \partial'_i &= \partial'_i \circ \partial'_{i+1} \quad \text{for} \quad 0 \leq i \leq n, \\
&(17) & \partial''_i \circ \partial''_j &= \partial''_{j-1} \circ \partial''_i \quad \text{for} \quad 0 \leq i < j \leq m, \\
&(18) & \partial''_j \circ \partial''_j &= \partial''_j \circ \partial''_{j+1} \quad \text{for} \quad 0 \leq j \leq m, \\
&(19) & \partial'_i \circ \partial''_j &= \partial''_j \circ \partial'_i \quad \text{for} \quad 0 \leq i \leq n, \quad 0 \leq j \leq m, \\
&(20) & \tau \circ \partial'_j &= \partial_{i+1} \circ \tau \quad \text{for} \quad 0 \leq i \leq n-1, \\
&(21) & \tau \circ \partial'_n &= \partial'_0 \circ \tau^2 \\
&(22) & \tau \circ \partial''_j &= \partial''_j \circ \tau \quad \text{for} \quad 0 \leq j \leq m.
\end{aligned}$$

It is not clear, how this cyclic structure fits into the general theory of cyclic sets and cyclic spaces as developed in [C], [B], [DHK], [G], [J] and others. There are no degeneracy operators, since the complex $P(g)$ is finite dimensional. The degenaracies seem to be important to put an S^1-action on the geometric realization, see [J]. We point out that the cyclic structure restricted to the subcomplex $U(g)$ is closer to the general theory; only the order of τ is $n+2$ instead of $n+1$, what can be regarded as an effect of our normalization, i.e. the cone of $U(g)$ is a cyclic set in the sense of [C]. On the other hand, certain other cyclic constructions are used, where the order of τ differs from the general theory, e.g. the edgewise subdivision in [BHM].

The S^1-action on the moduli space $\vec{\mathfrak{M}}(g)$ is given by rotating the tangent direction $\mathcal{X}$, i.e. $\alpha \cdot [F, \mathcal{O}, \mathcal{X}] = [F, \mathcal{O}, \alpha\mathcal{X}]$ for an angle $\alpha \in S^1$. It is well-defined, since the tangent bundle of F is a complex vector bundle. This action is not free; whenever $\mathcal{O}$ is a fixed point under some (necessarilly) cyclic automorphism of F order r, then $\mathbf{Z}/r\mathbf{Z} \leq S^1$ is the isotropy group of $[F, \mathcal{O}, \mathcal{X}]$ for any direction $\mathcal{X}$.

But at this point we do not know, how this action transforms to this specific parametrization $\mathfrak{P}(g)$ of the moduli space.

REMARK. There is also a free S^1-action on the "homotopy-type" of $\mathfrak{P}(g)$. The mapping class group $\vec{\Gamma}(g)$ is the central extension of the pointed mapping class group $\Gamma^1(g) = \pi_0(Diff^+(F, \mathcal{O}))$ by an infinite cyclic group generated by a Dehn-twist along a null-homotopic curve enclosing the point $\mathcal{O}$. Thus $B\vec{\Gamma}(g)$ is the total space of an S^1-bundle. But the rotation does not lift to a free flow on the Teichmüller space $\vec{\mathfrak{T}}(g)$; only the isotropy is disjoint from the integral part $\mathbf{Z} \leq \mathbf{R}$. the

4 The chain complex of $P(g)$

For any commutative ring $\mathbf{K}$ with unit let $S(g)$ be the chain complex with $S_\ell(g)$ the free $\mathbf{K}$-module generated by all cells E of $P(g)$ of dimension $\ell = n+m$. The boundary $\partial : S_\ell(g) \longrightarrow S_{\ell-1}(g)$ is given by

$$\partial = \partial' + (-1)^n \partial'' \tag{23}$$

with

$$\partial' = \sum_{i=0}^{n} (-1)^i \, \partial_i' \quad \text{and} \quad \partial'' = \sum_{j=0}^{m} (-1)^j \, \partial_j'' \tag{24}$$

Since ∂' and ∂'' commute by (19), we have $\partial \circ \partial = 0$. We call ∂ the topological boundary operator, and denote its homology $H_*(S(g), \partial)$ by $H_*(P(g)) = H_*(P(g); \mathbf{K})$.

We now exploit the fact that $S(g)$ looks formally similiar to the Hochschild resolution of same algebra A, if we interpret the entry $a = (a_0, \ldots, a_{n+1})$ as a tensor in $A^{\otimes(n+2)}$. The Hochschild boundary operator $b : S_\ell(g) \longrightarrow S_{\ell-1}(g)$ is defined as

$$b = \partial' + (-1)^{n+1} \partial_0' \, \tau + (-1)^n \partial'' \tag{25}$$

Using the commutation relations of Lemma 1 it is straightforward to show $b \circ b = 0$. We denote the complex $S(g)$ with the boundary operator b by $S^\flat(g)$, and call its homology $HH_*(P(g)) = H_*(S(g), b)$ the Hochschild homology of $P(g)$.

Let $T = (-1)^{n+2}\tau$ be the (signed) cyclic operator, denote the invariance operator by $D = id - T$, and the norm operator by $N = id + T + T^2 + \ldots + T^{n+1}$. Then we obtain

LEMMA 2

$$b \circ D = D \circ \partial \tag{26}$$

$$\partial \circ N = N \circ b \tag{27}$$

Proof: The arguments in [LQ] carry over verbatim.

Thus we can form the double complex $C(g)$ with $C_{p,q}(g) = S_q(g)$ and boundary $d : C_{p,q}(g) \longrightarrow C_{p-1,q} \oplus C_{p,q-1}(g)$ given by $d = D - \partial$ for odd p, and $d = N + b$ for even p. Let the cyclic homology of $P(g)$ be the homology of the associated total complex, $HC_*(P(g)) = H_*(Tot(C(g)), d)$.

Different from the classical situation is that the complex $S(g)$ is not acyclic; in fact, its homology is precicely what interests us. Note that the total complex is periodic in dimensions above $6g - 3$, since $S(g)$ vanishes there.

There is an earlier definition of cyclic homology, which uses the complex of τ-coinvariants instead of the double complex $C(g)$. Denote by $\tilde{S}(g)$ the quotient complex with $\tilde{S}_\ell(g) = S_\ell(g)/im(D)$. Because of (26) $\tilde{S}(g)$ inherits from $S(g)$ a boundary operator $\tilde{b}$.

PROPOSITION 1

If $\mathbf{Q} \subseteq \mathbf{K}$, then $\tilde{S}(g)$ is quasi-isomorphic to $Tot(C(g))$.

Proof: One applies the usual argument, that the rows of $C(g)$ form a free resolution of the cyclic groups $\mathbf{Z}/\ell\mathbf{Z}$. In our case, however, their are several groups involved per row. Let the terms $S_\ell(g) = \bigoplus_{n+m=\ell} S_{n,m}(g)$ in the ℓ-th row be decomposed according the vertical and horizontal bigrading of their cells. The summands are no subcomplexes (neither for ∂ nor for b), but they are invariant under both D and N. Thus for each ℓ and n the summands $S_{n,l-n}(g)$ form in the ℓ-th row of $C(g)$ a free, periodic resolution of the group $\mathbf{Z}/\ell\mathbf{Z}$ with alternating differential D and N. The "row spectral sequence" of $C(g)$, which converges to $H_*(\mathbf{Z}/\ell\mathbf{Z}; \mathbf{K})$, is trivial as soon as $\mathbf{Q} \subseteq \mathbf{K}$. It follows that the homology of $Tot(C(g))$ is isomorphic to the homology of $\tilde{S}(g)$.

This means a considerable advantage in terms of actual computations for small g. The orbits of τ tend to be quite large, and so the complex $\tilde{S}(g)$ is much smaller.

REMARK. In search of an algebra behind all this we perhaps may first concentrate on the subspace $U(g)$ or $\bar{U}(g)$. The cells are then all of type $(n, 0)$, and we write $E = [a|\lambda] = [a_0, \ldots, a_{n+1}|\lambda]$. Let $\mathcal{A}$ be the ideal in the polynomial ring $\mathbf{K}[X]$ generated by X. We write X^a for the tensor $X^{a_0} \otimes \cdots \otimes X^{a_{n+1}}$ in $\mathcal{A}_n = \mathcal{A}^{\otimes(n+2)}$. To involve λ, we consider the $\mathbf{K}$-module Λ in the group ring $\mathbf{K}[\mathfrak{S}_{4g}]$ generated by all connected pairings, i.e. free involutions $\lambda \in \mathfrak{S}_{4g}$ with $\kappa(\lambda) = 0$. Consider now the ideal $\mathcal{I}_n \subseteq \mathcal{A}_n \otimes_K \Lambda$ generated by all differences $X^a \otimes \lambda - X^{a'} \otimes \lambda'$ such that $[a|\lambda]$ und $[a'|\lambda']$ are related by a Rauzy-move. If A_n is the quotient by this ideal, then the complex $A = (A_n)$ inherits from the Hochschild resolution of $\mathcal{A}$ a "topological" and a Hochschild boundary ∂ resp. b. This is the complex we study.

5 The Connes-Gysin diagram

To relate the three homologies H, HH and HC we consider the following complexes and chain maps. Let the complex $S^\sharp(g)$ consist of the first two columns of $C(g)$, with d as boundary operator; and let $Tot'(C(g))$ be the total complex associated to the double complex $C(g)$ minus the first column, graded such that $C_{1,0}(g) = S_0(g)$ is the degree zero part. In the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^\flat(g) & \longrightarrow & S^\sharp(g) & \longrightarrow & \Sigma S(g) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^\flat(g) & \longrightarrow & Tot(C(g)) & \longrightarrow & \Sigma Tot'(C(g)) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Sigma^2 Tot(C(g)) & = & \Sigma^2 Tot(C(g)) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

all maps are inclusions, except for two shift maps. $sh : Tot(C(g)) \longrightarrow \Sigma Tot'(C(g))$ is the chain map induced by shifting the columns of the double complex $C(g)$ one column to the left. And $sh^2 : Tot(C(g)) \longrightarrow \Sigma^2 Tot(C(g))$ is then the periodicity self-map of $Tot(C(g))$. Σ denotes the suspension of a complex. The diagram is commutative is commutative, and we obtain

PROPOSITION 2
There is the a diagram of long exact sequences :

$$\begin{array}{cccccccc}
 & & & & \downarrow & & \downarrow & \\
\longrightarrow & HH_*(P(g)) & \longrightarrow & H_*(S^\sharp(g)) & \longrightarrow & H_{*-1}(P(g)) & \longrightarrow & \\
 & \| & & \downarrow & & \downarrow & & \\
\longrightarrow & HH_*(P(g)) & \longrightarrow & HC_*(P(g)) & \longrightarrow & H_{*-1}(Tot'(C(g))) & \longrightarrow & \\
 & & & \downarrow & & \downarrow & & \\
 & & & HC_{*-2}(P(g)) & = & HC_{*-2}(P(g)) & & \\
 & & & \downarrow & & \downarrow & &
\end{array}$$

In the classical situation of an algebra the term $H_*(P(g))$ would vanish and the diagram would reduce to the long exact Connes-Gysin sequence. sh^2_* is then Connes' periodicity operator. The diagram suggests that perhaps $H_*(S^\sharp(g))$ should be named the Hochschild homology.

Since the total complex $Tot(C(g))$ is periodic in high dimensions, the shift induces an isomorphism.

PROPOSITION 3 The double shift

$$sh^2_* : HC_*(P(g)) \longrightarrow HC_{*-2}(P(g))$$

is an isomorphism for $* > 6g - 5$.

As a consequence, the cyclic homology $HC_*(P(g))$ differs from the periodic cyclic homology $HC_*^{per}(P(g)) = \lim_\ell HC_{*+2\ell}(P(g))$ only in dimension $* < 6g - 3$.

It is clear that everything said so far is also true, if we replace the space $P(g)$ by any of the spaces $P(g)/W(g)$, $P(g)/D(g)$, $U(g)$ or $\bar{U}(g)$; note that the dimesion of the last two spaces is $4g - 2$. Recall that $P(g)/W(g)$ is Poincaré dual to $\vec{\mathfrak{M}}(g)$; thus if we work ove a field we obtain results about the homology of the moduli space directly.

6 Dihedral and quaternionic homology

We briefly mention another operator, the reflection $W = (-1)^{\frac{1}{2}(n+1)(n+2)}\omega$, where ω acts on a cell E by

$$\omega(E) = [a_{n+1}, a_n, \ldots, a_1, a_0 |\, \omega\ \lambda\ \omega^{-1} | \omega(B_0), \ldots, \omega(B_{m+1})] \tag{28}$$

Here $\omega \in \mathfrak{S}_{4g}$ is the involution $k \mapsto 4g + 1 - k$.

LEMMA 3

$$W \circ \partial'_i = \partial'_{n-i} \circ W \quad \text{for} \quad 0 \leq i \leq n, \tag{29}$$

$$W \circ \partial''_j = \partial'_j \circ W \quad \text{for} \quad 0 \leq j \leq m. \tag{30}$$

It follows that

$$W \circ \partial' = (-1)^n \partial' \circ W, \tag{31}$$

$$W \circ \partial'' = \partial'' \circ W. \tag{32}$$

One can define dihedral and quaternionic homology groups for the complexes $P(g)$ and the various sub- and quotient complexes, following [L 1]. The easiest case is $U(g)$ or $\bar{U}(g)$. For W it is obvious, that it is the transformed complex conjugation of $\mathfrak{P}(g)$; a conjugation of the conformal structure is for a parallel slit domain just complex-conjugation of its slits in $\mathbf{C}$, and that amounts to reading the slits in reversed order from top to bottom.

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