Representation Theories of the Symmetric Group and the Rook Monoid

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Young Women in Representation Theory
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The special case of the Symmetric Group

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n.º of isoclasses of Irr. Rep. of $\mathcal{S}_n$ = n.º of partitions of n

Isoclasses of Irreducible Representations of $\mathcal{S}_n$

(1) $\qquad$ $n = 1$

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	(1)	$n = 1$
(2)	$(1,1)$	$n = 2$

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	(1)		$n = 1$
(2)		(1,1)	$n = 2$
(3)	(2,1)	(1,1,1)	$n = 3$

Isoclasses of Irreducible Representations of $\mathcal{S}_n$

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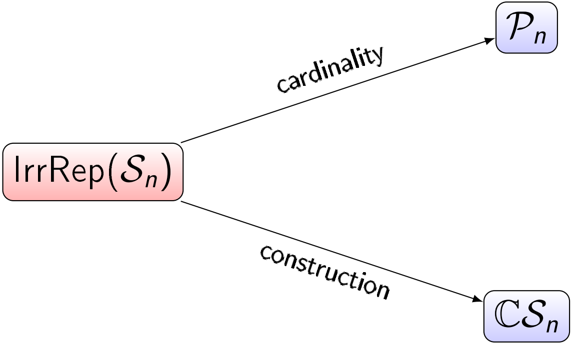
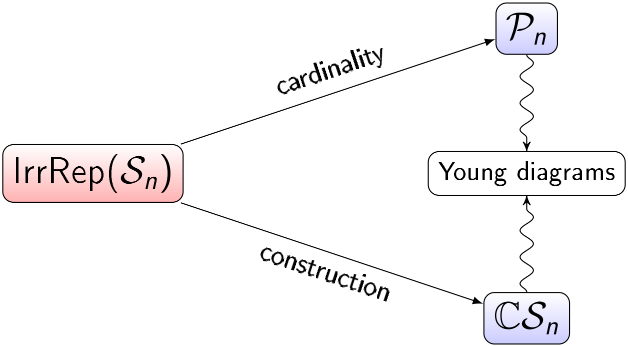
(3) (2,1) (1,1,1) $n = 3$

(4) (3,1) (2,2) (2,1,1) (1,1,1,1) $n = 4$

Isoclasses of Irreducible Representations of $\mathcal{S}_n$

			(1)				$n = 1$
		(2)		(1,1)			$n = 2$
		(3)	(2,1)	(1,1,1)			$n = 3$
	(4)	(3,1)	(2,2)	(2,1,1)	(1,1,1,1)		$n = 4$
(5)	(4,1)	(3,2)	(3,1,1)	(2,2,1)	(2,1,1,1)	(1,1,1,1,1)	$n = 5$


IrrRep($\mathcal{S}_n$)
cardinality
$\mathcal{P}_n$
construction
$\mathbb{C}\mathcal{S}_n$


$\mathcal{P}_n$
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Young diagrams
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$\mathbb{C}\mathcal{S}_n$

Classic Approach

$$\left\{\begin{matrix}\lambda \\ \text{partition of } n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}S^{\lambda} \\ \text{Specht module}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{classe of irreducible} \\ \mathcal{S}_n\text{−representations}\end{matrix}\right\}$$

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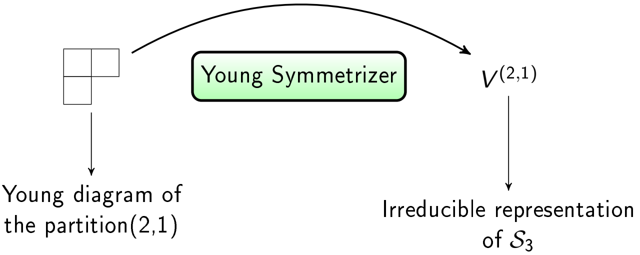

Constructing the Specht Modules of $\mathcal{S}_3$

$\lambda_1 = (3)$ $\lambda_2 = (2,1)$ $\lambda_3 = (1,1,1)$

Constructing the Specht Modules of $\mathcal{S}_3$

$\lambda_1 = (3)$ $\downarrow$

1	2	3

$\lambda_2 = (2, 1)$ $\downarrow$

1	2
3	

$\lambda_3 = (1, 1, 1)$ $\downarrow$

1
2
3

Constructing the Specht Modules of $\mathcal{S}_3$

$\lambda_1 = (3)$

↓

1	2	3

↓

$R_{\lambda_1} = \mathcal{S}_3$

$C_{\lambda_1} = \{(1)\}$

$\lambda_2 = (2, 1)$

↓

1	2
3	

↓

$R_{\lambda_2} = \{(1), (1\ 2)\}$

$C_{\lambda_3} = \{(1), (1\ 3)\}$

$\lambda_3 = (1, 1, 1)$

↓

1
2
3

↑

$R_{\lambda_3} = \{(1)\}$

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$\lambda_1 = (3)$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

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$R_{\lambda_3} = \{(1)\}$

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Young symmetrizers

$$\mathfrak{s}_{\lambda_i} = \left(\sum_{\sigma \in R_{\lambda_i}} \sigma \right) \left(\sum_{\sigma \in C_{\lambda_i}} \operatorname{sign}(\sigma)\sigma \right) \in \mathbb{C}\mathcal{S}_3$$

Constructing the Specht Modules of $\mathcal{S}_3$

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1	2
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$\downarrow$ $\mathbb{C}\mathcal{S}_3 \cdot \mathfrak{s}_{\lambda_1}$

$\mathbb{C} \sum_{\sigma \in \mathcal{S}_3} \sigma$

$\downarrow$ $\mathbb{C}\mathcal{S}_3 \cdot \mathfrak{s}_{\lambda_2}$

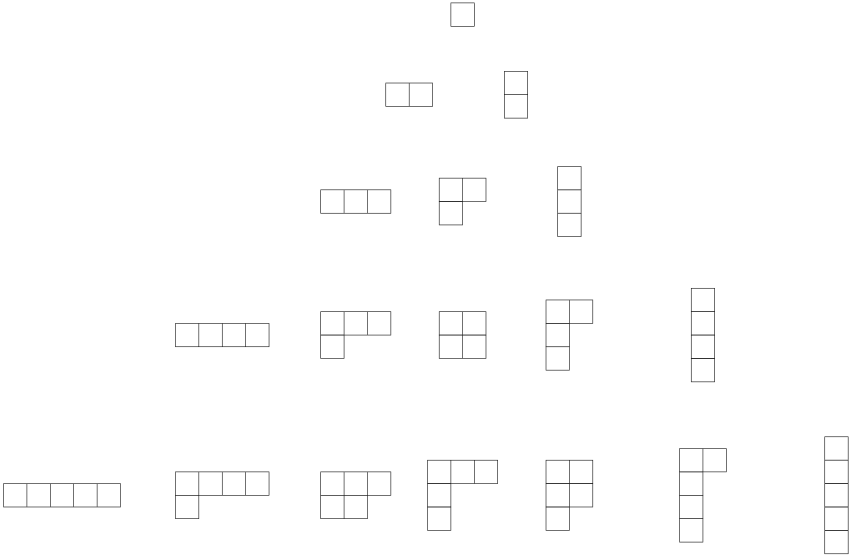
$\mathbb{C}\mathfrak{s}_{\lambda_2} + \mathbb{C}x$

$\downarrow$ $\mathbb{C}\mathcal{S}_3 \cdot \mathfrak{s}_{\lambda_3}$

$\mathbb{C} \sum_{\sigma \in \mathcal{S}_3} \mathrm{sgn}(\sigma)\sigma$

Theorem

When λ ranges over all distinct partitions of n, $\{\mathbb{C}\mathcal{S}_n \cdot \mathfrak{s}_\lambda\}$ is a full set of non-isomorphic simple $\mathbb{C}\mathcal{S}_n$-modules.



Branching Graph

Theorem

Let V be a simple $\mathbb{C}\mathcal{S}_n$-module.
The restriction $V|_{\mathcal{S}_{n-1}}$ is multiplicity-free.

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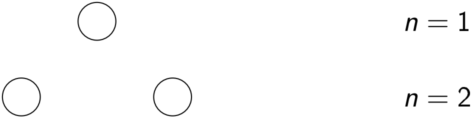

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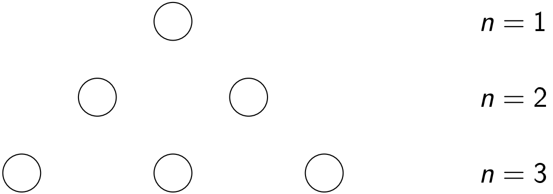

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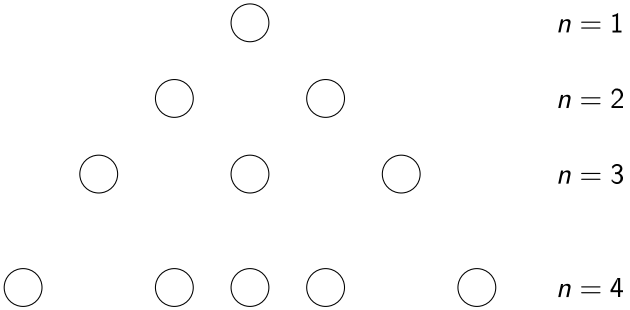

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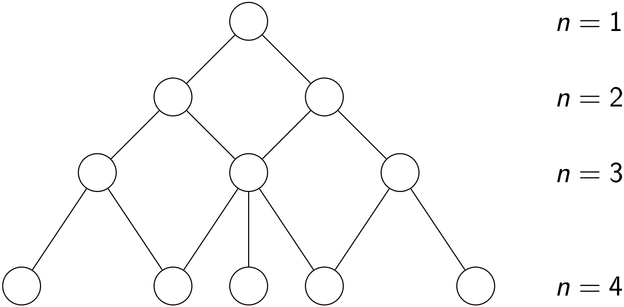
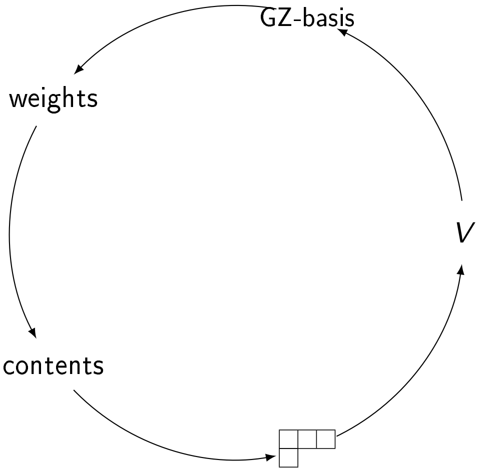


Different Approach



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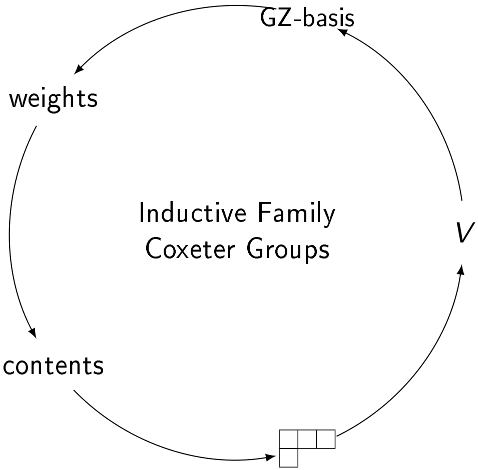

Example: The standard representation of $\mathcal{S}_4$

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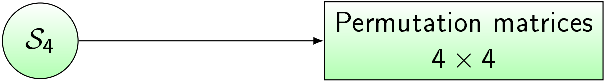

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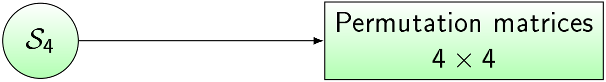

$$\rho : S_4 \to GL(V)$$

where for all $\sigma \in \mathcal{S}_4$

$$\rho(\sigma)(x_1, x_2, x_3, x_4) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}, x_{\sigma^{-1}(4)}).$$

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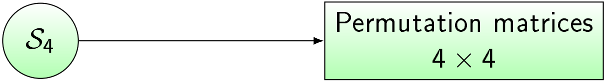

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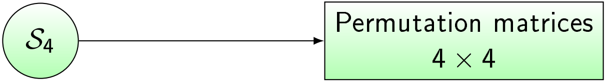

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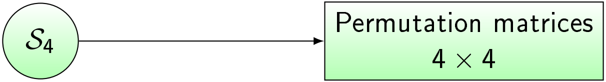

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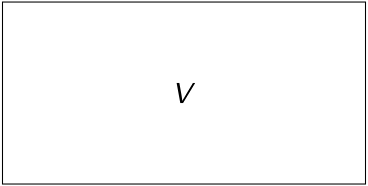

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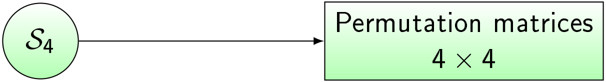

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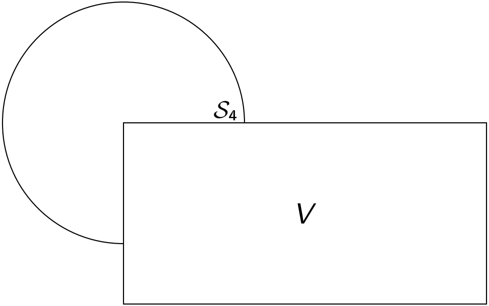

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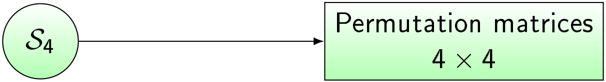

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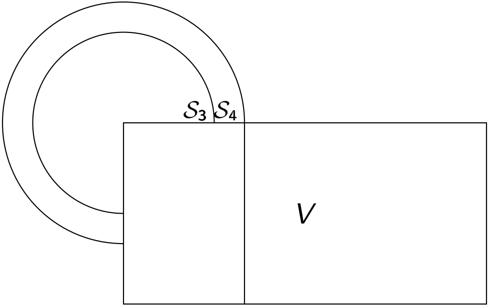

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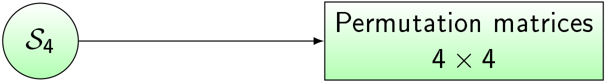

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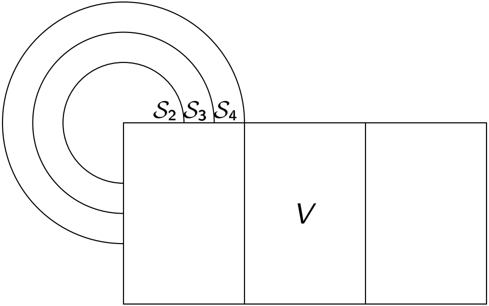

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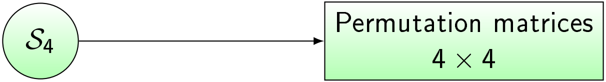

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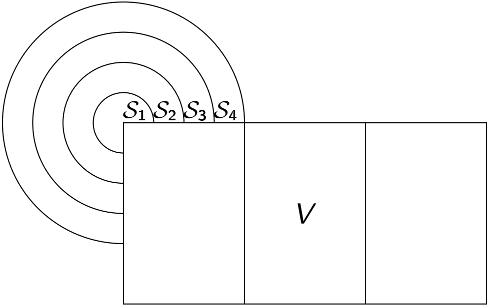

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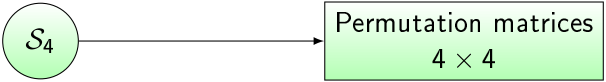

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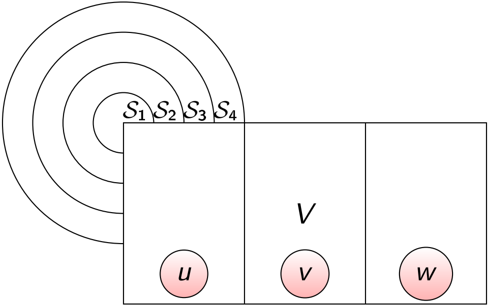
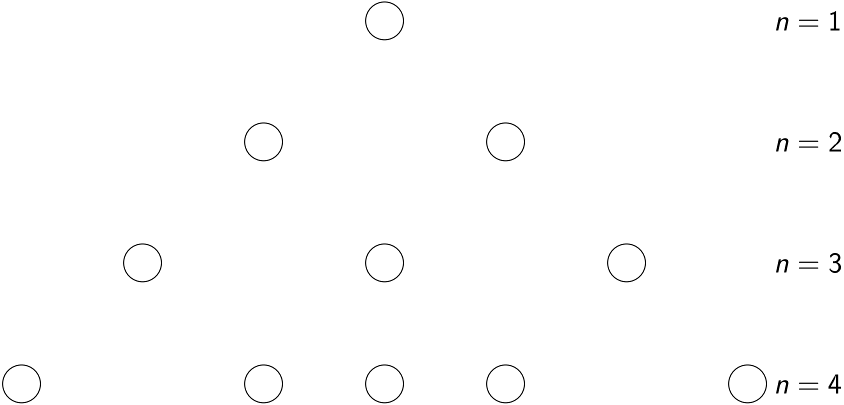


Branching Graph



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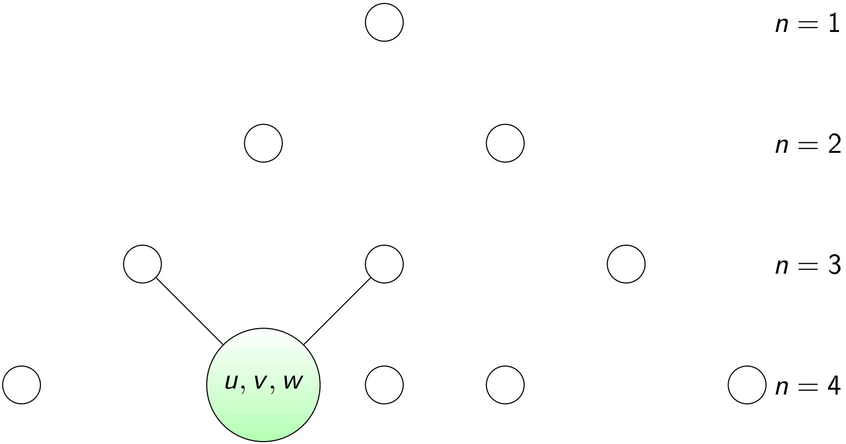

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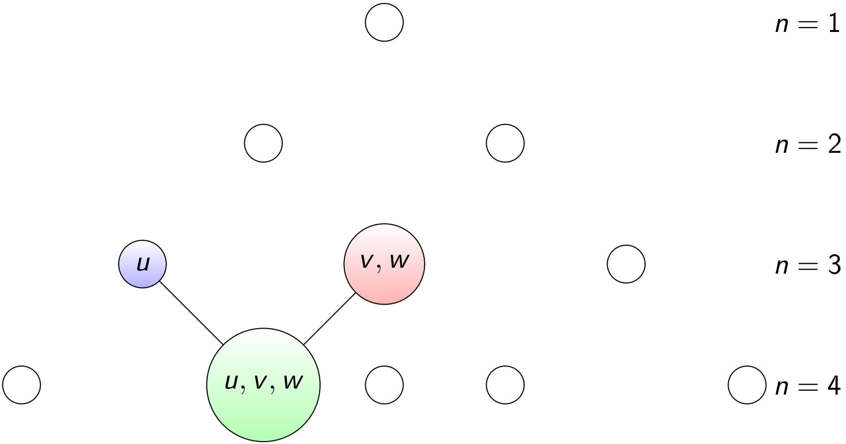

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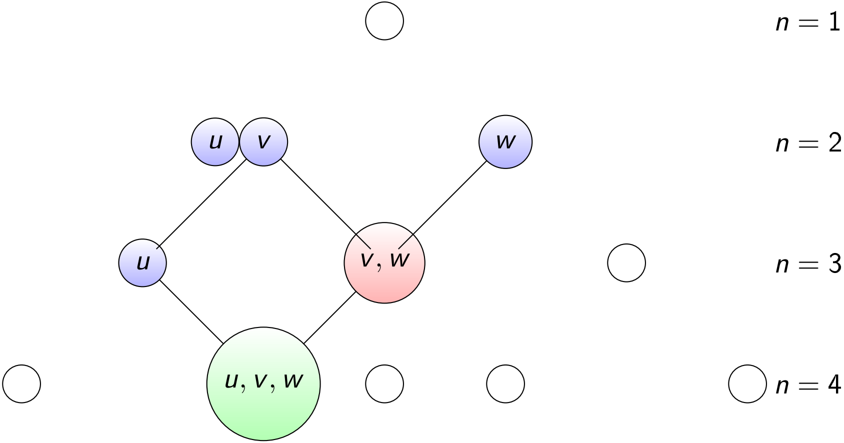

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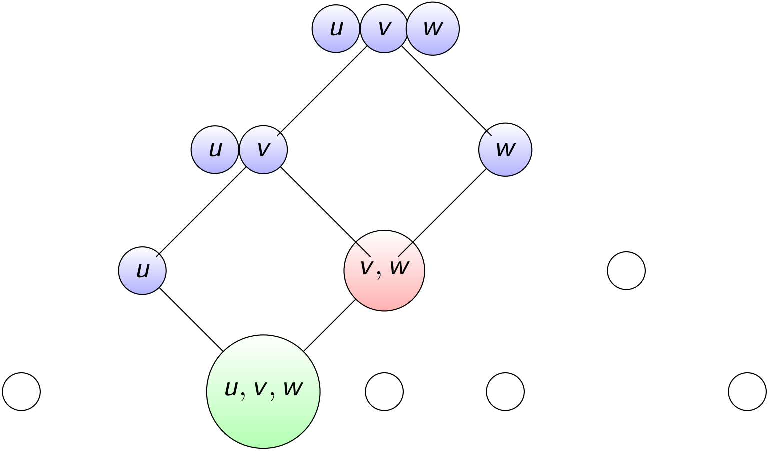

$n = 1$

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Proposition

All simple $\mathbb{C}S_n$-module V has a canonical decomposition

$$V = \bigoplus_T V^T$$

indexed by all possible chains

$$T = \lambda_1 \nearrow \ldots \nearrow \lambda_n$$

where $\lambda_i \in \mathcal{S}_n^\wedge$, $V \in \lambda_n$ and all V^T are simple $\mathcal{S}_1$-modules.

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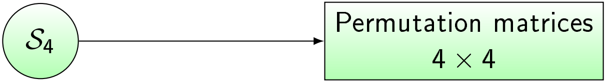
Gelfand-Zetlin Basis

For each T we can choose a vector v_T form each V^T obtaining a basis $\{v_T\}$ of V,which we call **Gelfand-Zetlin basis**.

Example: The standard representation of $\mathcal{S}_4$



$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

$$V = <(1,1,1,-3)> \oplus <(1,1,-2,0)> \oplus <(1,-1,0,0)>.$$

Let $u = (1,1,1,-3)$, $v = (1,1,-2,0)$, $w = (1,-1,0,0)$ and fix the basis $\text{GZ} = \{u, v, w\}$ for V.

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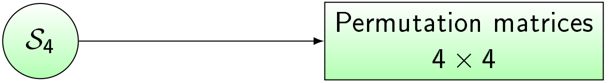

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(1)	(1 2)	(2 3)	(3 4)
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}$.

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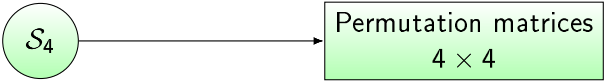

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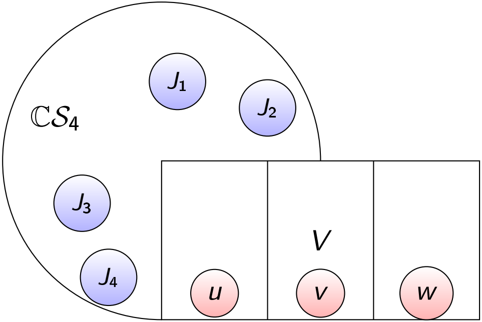


$J_1 = 0$
$J_2 = (1\ 2)$
$J_3 = (1\ 3) + (2\ 3)$
$J_4 = (1\ 4) + (2\ 4) + (3\ 4)$
Jucys-Murphy elements

Example: The standard representation of $\mathcal{S}_4$

0	(1 2)	(1 3) + (2 3)	(1 4) + (2 4) + (3 4)
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$

$$\gamma(u) = (0, 1, 2, -1)$$

$$\gamma(v) = (0, 1, -1, 2)$$

$$\gamma(w) = (0, -1, 1, 2)$$

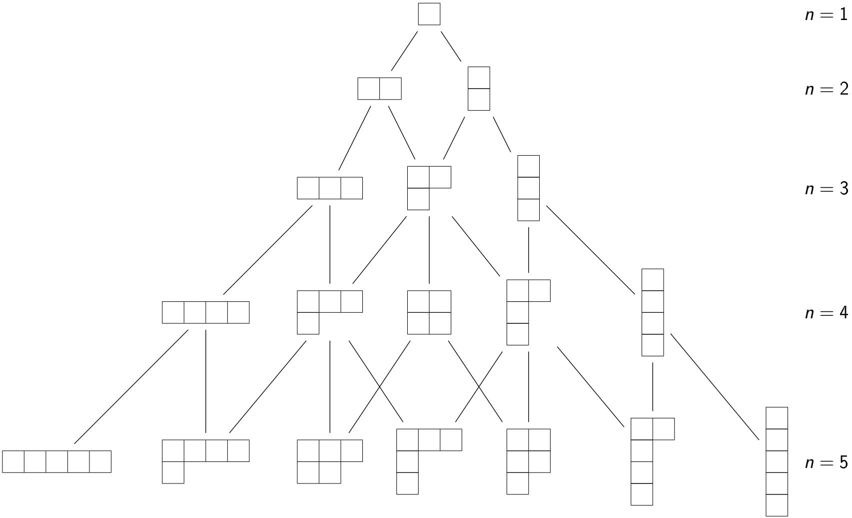
Example: The standard representation of $\mathcal{S}_4$

0	$(1\ 2)$	$(1\ 3)+(2\ 3)$	$(1\ 4)+(2\ 4)+(3\ 4)$
$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$	$\begin{bmatrix}2&0&0\\0&-1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}-1&0&0\\0&2&0\\0&0&2\end{bmatrix}$.

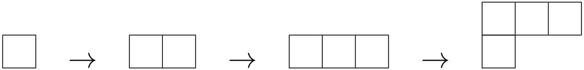
$$\left.\begin{aligned}\gamma(u)&=(0,1,2,-1)\\ \gamma(v)&=(0,1,-1,2)\\ \gamma(w)&=(0,-1,1,2)\end{aligned}\right\}\text{weights}$$

Theorem

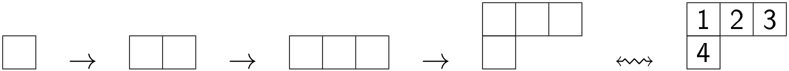
$$\mathbb{B} \simeq \mathbb{Y}$$


$n = 1$
$n = 2$
$n = 3$
$n = 4$
$n = 5$

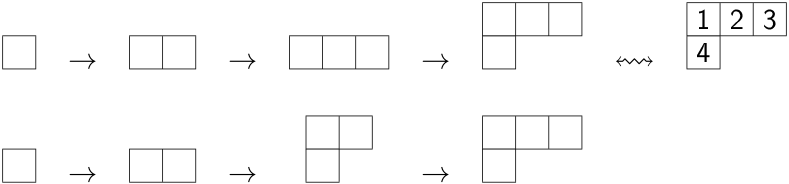
Example



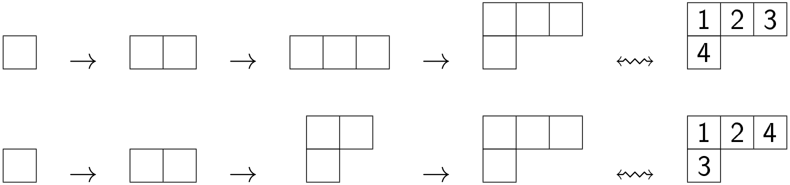
Example



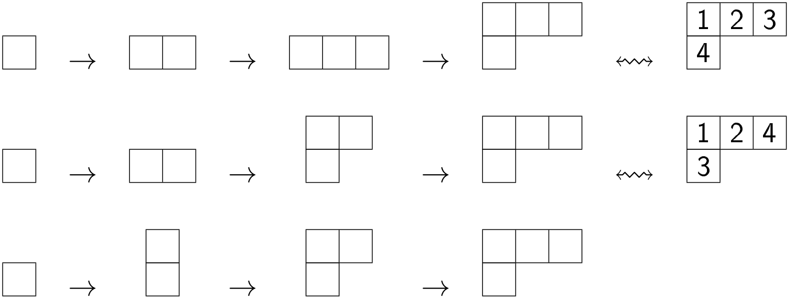
Example



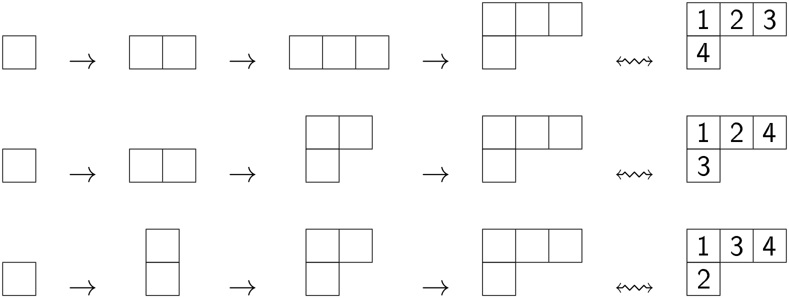
Example



Example



Example



Example

Consider

$$Q_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad Q_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad Q_3 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

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We have:

$$\delta(Q_1) = (0, 1, 2, -1)$$

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Example

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$$Q_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad Q_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad Q_3 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

We have:

$$\left.\begin{array}{l} \delta(Q_1) = (0,1,2,-1) \\ \delta(Q_2) = (0,1,-1,2) \\ \delta(Q_3) = (0,-1,1,2) \end{array}\right\} \text{contents} = \text{weights} \left\{\begin{array}{l} \gamma(u) = (0,1,2,-1) \\ \gamma(v) = (0,1,-1,2) \\ \gamma(w) = (0,-1,1,2) \end{array}\right.$$

The Spectrum and the Resume

Theorem

Let Ω be the set of all elements

$$x = (x_1, \dots, x_n) \in \mathbb{C}^n$$

which verify the following properties:

1. $x_1 = 0$;
2. $\{x_i - 1, x_i + 1\} \cap \{x_1, \dots, x_{i-1}\} \neq \emptyset, \quad \forall i \in \{2, \dots, n\}$;
3. If $x_i = x_j = a$, for some $i < j$, then

$$\{a-1, a+1\} \subseteq \{x_{i+1}, \dots, x_{j-1}\}.$$

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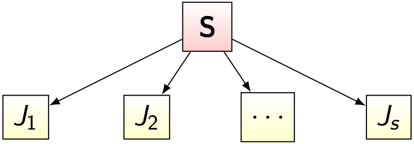
We have

$$\Omega = \mathcal{E}_n = \mathcal{R}_n.$$

Finite Semigroups Representation Theory

S — Finite Semigroup

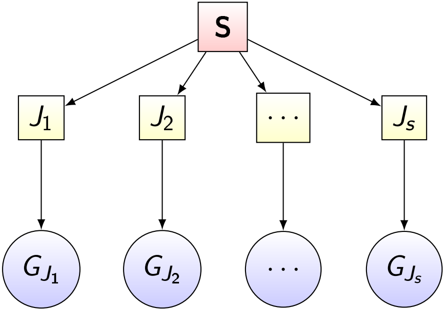
Finite Semigroups Representation Theory



Finite Semigroup

$\mathcal{U}(S)$
Regular $\mathcal{J}$-classes

Finite Semigroups Representation Theory

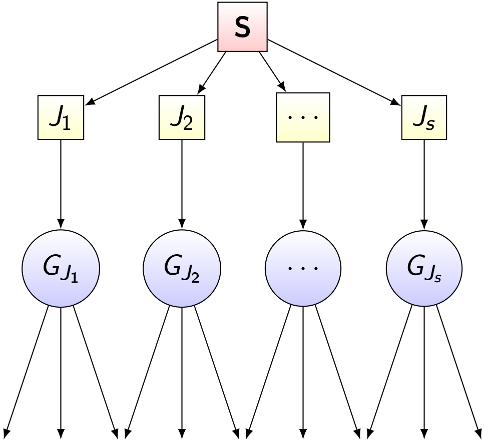


Finite Semigroup

$\mathcal{U}(S)$
Regular $\mathcal{J}$-classes

Maximal Subgroups

Finite Semigroups Representation Theory



Finite Semigroup

$\mathcal{U}(S)$
Regular $\mathcal{J}$-classes

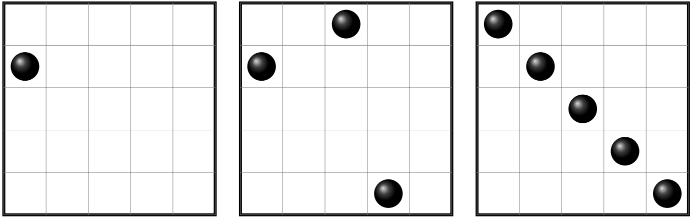
Maximal Subgroups

Irreducible Representations

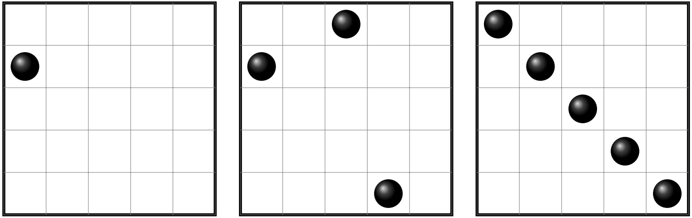
Theorem[Clifford,Munn,Ponizovskii]

The number of irreducible representations of S (up to isomorphism) is equal to the number of irreducible representations of its maximal subgroups G_J, with $J \in \mathcal{U}(S)$.

The Rook Monoid

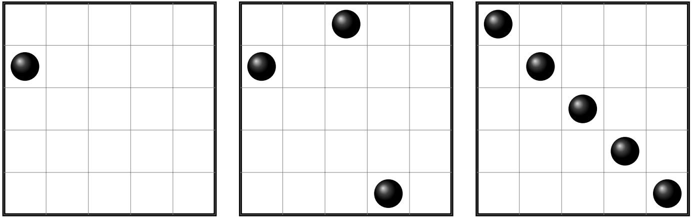


The Rook Monoid

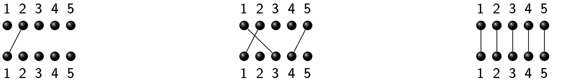


$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

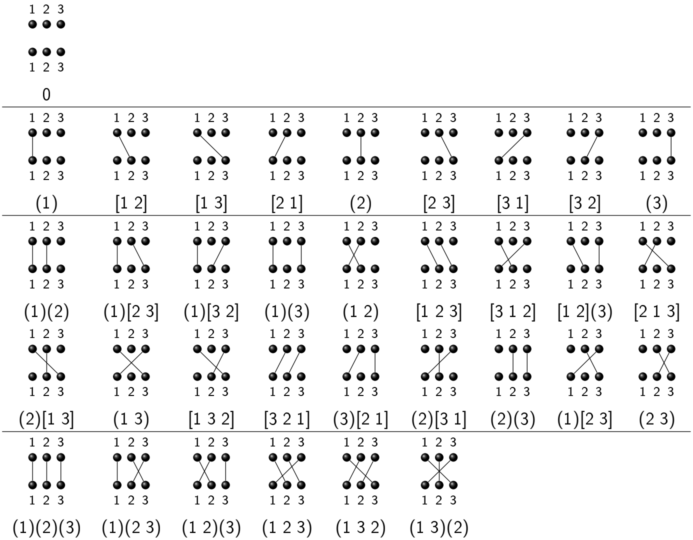
The Rook Monoid



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



[2 1]   [2 1 3][5 4]   (1)(2)(3)(4)(5)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The special case of the Rook Monoid

n.º of isoclasses of Irr. Rep. = sum of the n.º of isoclasses of Irr. Rep of its maximal subgroup G_J

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$$\Downarrow$$

The list of the maximal subgroups G_J of $\mathcal{I}_n$ will be isomorphic to $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n$.

The special case of the Rook Monoid

n.º of isoclasses of Irr. Rep. = sum of the n.º of isoclasses of Irr. Rep of its maximal subgroup G_J

$\Downarrow$

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$\Downarrow$

$$|IrrRep(\mathcal{I}_n)| = \sum_{k=0}^{n} |IrrRep(\mathcal{S}_k)|$$

Isoclasses of Irr. Rep. of $\mathcal{I}_n$

$\emptyset$ $\qquad\qquad$ $n = 0$

Isoclasses of Irr. Rep. of $\mathcal{I}_n$

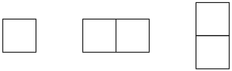
$\emptyset$ $\qquad n = 0$

$\emptyset \quad \square$ $\qquad n = 1$

Isoclasses of Irr. Rep. of $\mathcal{I}_n$

$\emptyset$ $n = 0$

$\emptyset$  $n = 1$

$\emptyset$  $n = 2$

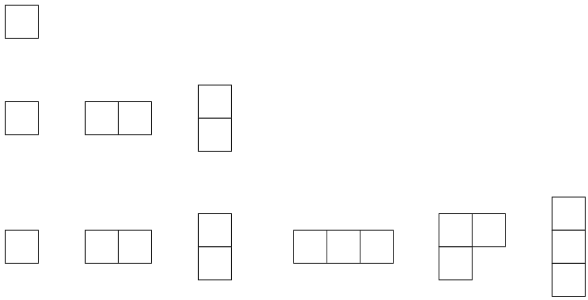
Isoclasses of Irr. Rep. of $\mathcal{I}_n$

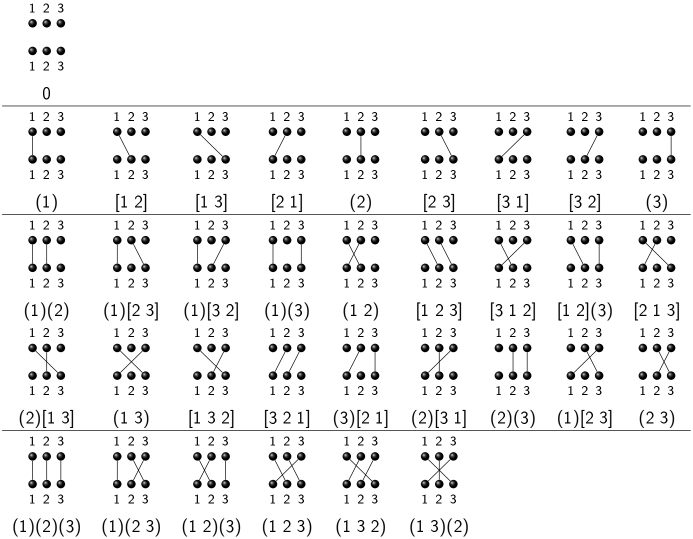
$\emptyset$ $n = 0$

$\emptyset$ $n = 1$

$\emptyset$ $n = 2$

$\emptyset$ $n = 3$

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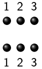
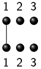
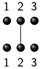
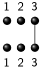
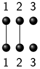
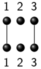
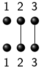
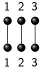
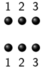
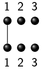
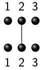
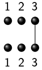
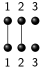
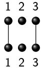
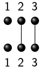
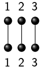
rank 0	rank 1			rank 2			rank 3
							
$\varepsilon_\emptyset$	$\varepsilon_{\{1\}}$	$\varepsilon_{\{2\}}$	$\varepsilon_{\{3\}}$	$\varepsilon_{\{1,2\}}$	$\varepsilon_{\{1,3\}}$	$\varepsilon_{\{2,3\}}$	$\varepsilon_{\{1,2,3\}}$

rank 0	rank 1			rank 2			rank 3
							
$\varepsilon_\emptyset$	$\varepsilon_{\{1\}}$	$\varepsilon_{\{2\}}$	$\varepsilon_{\{3\}}$	$\varepsilon_{\{1,2\}}$	$\varepsilon_{\{1,3\}}$	$\varepsilon_{\{2,3\}}$	$\varepsilon_{\{1,2,3\}}$

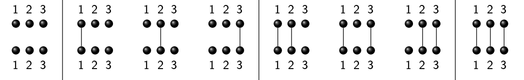
$$\eta_0 = \varepsilon_\emptyset$$

$$\eta_1 = -3\,\varepsilon_\emptyset + \varepsilon_{\{1\}} + \varepsilon_{\{2\}} + \varepsilon_{\{3\}}$$

$$\eta_2 = 3\,\varepsilon_\emptyset - 2\,\varepsilon_{\{1\}} - 2\,\varepsilon_{\{2\}} - 2\,\varepsilon_{\{3\}} + \varepsilon_{\{1,2\}} + \varepsilon_{\{1,3\}} + \varepsilon_{\{2,3\}}$$

$$\eta_3 = -\varepsilon_\emptyset + \varepsilon_{\{1\}} + \varepsilon_{\{2\}} + \varepsilon_{\{3\}} - \varepsilon_{\{1,2\}} - \varepsilon_{\{1,3\}} - \varepsilon_{\{2,3\}} + \varepsilon_{\{1,2,3\}}$$

rank 0	rank 1			rank 2			rank 3
$\varepsilon_{\emptyset}$	$\varepsilon_{\{1\}}$	$\varepsilon_{\{2\}}$	$\varepsilon_{\{3\}}$	$\varepsilon_{\{1,2\}}$	$\varepsilon_{\{1,3\}}$	$\varepsilon_{\{2,3\}}$	$\varepsilon_{\{1,2,3\}}$



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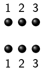
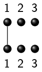
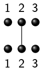
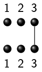
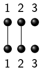
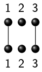
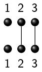
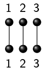
$$\eta_3 = -\varepsilon_{\emptyset} + \varepsilon_{\{1\}} + \varepsilon_{\{2\}} + \varepsilon_{\{3\}} - \varepsilon_{\{1,2\}} - \varepsilon_{\{1,3\}} - \varepsilon_{\{2,3\}} + \varepsilon_{\{1,2,3\}}$$

In this case:

$$\mathbb{C}\mathcal{I}_3 \simeq M_1(\mathbb{C}\mathcal{S}_0) \oplus M_3(\mathbb{C}\mathcal{S}_1) \oplus M_3(\mathbb{C}\mathcal{S}_2) \oplus M_1(\mathbb{C}\mathcal{S}_3)$$

rank 0	rank 1			rank 2			rank 3
							
$\varepsilon_\emptyset$	$\varepsilon_{\{1\}}$	$\varepsilon_{\{2\}}$	$\varepsilon_{\{3\}}$	$\varepsilon_{\{1,2\}}$	$\varepsilon_{\{1,3\}}$	$\varepsilon_{\{2,3\}}$	$\varepsilon_{\{1,2,3\}}$

$$\eta_0 = \varepsilon_\emptyset$$

$$\eta_1 = -3\,\varepsilon_\emptyset + \varepsilon_{\{1\}} + \varepsilon_{\{2\}} + \varepsilon_{\{3\}}$$

$$\eta_2 = 3\,\varepsilon_\emptyset - 2\,\varepsilon_{\{1\}} - 2\,\varepsilon_{\{2\}} - 2\,\varepsilon_{\{3\}} + \varepsilon_{\{1,2\}} + \varepsilon_{\{1,3\}} + \varepsilon_{\{2,3\}}$$

$$\eta_3 = -\varepsilon_\emptyset + \varepsilon_{\{1\}} + \varepsilon_{\{2\}} + \varepsilon_{\{3\}} - \varepsilon_{\{1,2\}} - \varepsilon_{\{1,3\}} - \varepsilon_{\{2,3\}} + \varepsilon_{\{1,2,3\}}$$

In this case:

$$\mathbb{C}\mathcal{I}_3 \simeq M_1(\mathbb{C}\mathcal{S}_0) \oplus M_3(\mathbb{C}\mathcal{S}_1) \oplus M_3(\mathbb{C}\mathcal{S}_2) \oplus M_1(\mathbb{C}\mathcal{S}_3)$$

$$\mathbb{C}\mathcal{I}_n \simeq \mathbb{C}\mathcal{I}_n\eta_0 \oplus \ldots \oplus \mathbb{C}\mathcal{I}_n\eta_n \simeq M_{\binom{n}{0}}(\mathbb{C}\mathcal{S}_0) \oplus \ldots \oplus M_{\binom{n}{n}}(\mathbb{C}\mathcal{S}_n)$$

$\therefore\ \mathbb{C}\mathcal{I}_n$ is semisimple.

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