Moduli spaces of free group representations in reductive groups

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Contents

Introduction

G complex reductive algebraic group

Γ finitely generated group

$$\mathfrak{X}_{\Gamma}(G) := \mathrm{Hom}(\Gamma, G) /\!\!/ G$$

G-character variety of Γ, where the quotient is to be understood in the setting of (affine) geometric invariant theory (GIT), for the conjugation action of G on the representation space $\mathrm{Hom}(\Gamma, G)$.

It arises in hyperbolic geometry, the theory of bundles and connections, knot theory and quantum field theories.

Particular cases

- $\Gamma = \pi_1(X)$ fundamental group of a compact Riemann surface X.
  Character varieties can be identified, up to homeomorphism, with certain moduli spaces of G-Higgs bundles over X (Hitchin 1987, Simpson 1992).
- $\Gamma = \pi_1(M \setminus L)$ where L is a knot (or link) in a 3-manifold M
  Character varieties define important knot and link invariants, such as the A-polynomial. (Cooper-Culler-Gillet-Long-Shalen 1994).
- $\Gamma = F_r$ free group of rank $r \geqslant 1$. The topology of $\mathfrak{X}_r(G) := \mathfrak{X}_{F_r}(G)$, in this generality, was first investigated by Florentino-Lawton, 2009.

Main Goal

With respect to natural Hausdorff topologies, if K is a maximal compact subgroup of G, $\mathfrak{X}_r(G)$ and $\mathfrak{X}_r(K) := \mathrm{Hom}(F_r, K)/K$ are homotopy equivalent and there is a canonical strong deformation retraction from $\mathfrak{X}_r(G)$ to $\mathfrak{X}_r(K)$ (Florentino-Lawton, 2009).

Goal: Extend to the more general case when G is a *real reductive Lie group*

Remark

-It is true when Γ is a finitely generated Abelian group (Florentino-Lawton, 2013), or a finitely generated nilpotent group (Bergeron, 2013).

-It is not true when $\Gamma = \pi_1(X)$, for a Riemann surface X, even in the cases $G = \mathrm{SL}(n, \mathbb{C})$ and $K = \mathrm{SU}(n)$ (Biswas-Florentino, 2011).

Tools

G real reductive algebraic group

The appropriate geometric structure on the analogous GIT quotient

$$\mathfrak{X}_r(G) := \mathrm{Hom}(F_r, G)/\!\!/G$$

was considered by Richardson and Slodowy (1990).

As in the complex case, this quotient parametrizes closed orbits under G, but contrary to that case, even when G is algebraic, the quotient is in general only a semi-algebraic set, in a certain real vector space.

To prove our main result we use the Kempf-Ness theory for real groups developed by Richardson and Slodowy (1990).

Definitions

K compact Lie group.

G is a *real K-reductive Lie group* if:

1. K is a maximal compact subgroup of G;
2. there exists a complex reductive algebraic group $\mathbf{G}$, given by the zeros of a set of polynomials with real coefficients, such that

   $$\mathbf{G}(\mathbb{R})_0 \subseteq G \subseteq \mathbf{G}(\mathbb{R}),$$

   where $\mathbf{G}(\mathbb{R})$ denotes the real algebraic group of $\mathbb{R}$-points of $\mathbf{G}$, and $\mathbf{G}(\mathbb{R})_0$ its identity component (in the Euclidean topology).
3. G is Zariski dense in $\mathbf{G}$.

Remark

1. If $G \neq \mathbf{G}(\mathbb{R})$, then G is not necessarily an algebraic group (Ex: $G = GL(n, \mathbb{R})_0$).
2. One can think of both $\mathbf{G}$ and G as Lie groups of matrices. We will consider on them the usual Euclidean topology which is induced from (and is independent of) an embedding on some $\mathrm{GL}(m, \mathbb{C})$.
3. $\mathbf{G}(\mathbb{R})$ is isomorphic to a closed subgroup of some $\mathrm{GL}(n, \mathbb{R})$ (ie, it is a linear algebraic group).
4. $\mathbf{G}(\mathbb{R})$ is a real algebraic group, hence, if it is connected, $G = \mathbf{G}(\mathbb{R})$ is algebraic and Zariski dense in $\mathbf{G}$. Condition (3) in Definition holds automatically if $\mathbf{G}(\mathbb{R})$ is connected.

Examples

- All classical real matrix groups are in this setting.
- G can also be any complex reductive Lie group, if we view it as a real reductive Lie group in the usual way.
- As an example which is not under the conditions of Definition, we can consider $\widetilde{\mathrm{SL}(n,\mathbb{R})}$, the universal covering group of $\mathrm{SL}(n,\mathbb{R})$, which admits no faithful finite dimensional linear representation (and hence is not a matrix group).

Character varieties

F_r a rank r free group

$\mathbf{G}$ a complex reductive algebraic group defined over $\mathbb{R}$

$\mathbf{G}$-*representation variety of* F_r is $\mathfrak{R}_r(\mathbf{G}) := \mathrm{Hom}(F_r, \mathbf{G})$

$\mathfrak{R}_r(\mathbf{G})$ is endowed with the compact-open topology (as defined on a space of maps, with F_r given the discrete topology)

$\mathbf{G}^r$ with the product topology, there is an homeomorphism

$$\mathfrak{R}_r(\mathbf{G}) \simeq \mathbf{G}^r$$

$\mathbf{G}$ is a smooth affine variety, $\mathfrak{R}_r(\mathbf{G})$ is also a smooth affine variety and it is defined over $\mathbb{R}$.

Consider now the action of $\mathbf{G}$ on $\mathfrak{R}_r(\mathbf{G})$ by conjugation.

This defines an action of $\mathbf{G}$ on the algebra $\mathbb{C}[\mathfrak{R}_r(\mathbf{G})]$ of regular functions on $\mathfrak{R}_r(\mathbf{G})$: $\mathbb{C}[\mathfrak{R}_r(\mathbf{G})]^{\mathbf{G}}$ is the subalgebra of $\mathbf{G}$-invariant functions.

$\mathbf{G}$ is reductive so the affine categorical quotient is

$$\mathfrak{X}_r(\mathbf{G}) := \mathfrak{R}_r(\mathbf{G})/\!/\mathbf{G} = \mathrm{Spec}_{\mathsf{max}}(\mathbb{C}[\mathfrak{R}_r(\mathbf{G})]^{\mathbf{G}}).$$

It is a singular affine variety (irreducible and normal), whose points correspond to unions of $\mathbf{G}$-orbits in $\mathfrak{R}_r(\mathbf{G})$ whose Zariski closures intersect.

It inherits the Euclidean topology, it is homeomorphic to the conjugation orbit space of closed orbits (called the *polystable quotient*).(Florentino, Lawton, 2013)

$\mathfrak{X}_r(\mathbf{G})$, together with that topology, is called the **G**-*character variety*.

K a compact Lie group

G a real K-reductive Lie group

In like fashion, we define the *G-representation variety of F_r*:

$$\mathfrak{R}_r(G) := \mathrm{Hom}(F_r, G).$$

Again, $\mathfrak{R}_r(G)$ is homeomorphic to G^r.

Similarly, as a set, we define the *G-character variety of F_r*

$$\mathfrak{X}_r(G) := \mathfrak{R}_r(G)//G$$

to be the set of closed orbits under the conjugation action of G on $\mathfrak{R}_r(G)$.

And the *K-character variety* of F_r

$$\mathfrak{X}_r(K) := \mathrm{Hom}(F_r, K)/K \cong K^r/K$$

Properties

- $\mathfrak{X}_r(G)$ is an affine real semi-algebraic set when G is real algebraic.
- $\mathfrak{X}_r(G)$ is always Hausdorff because we considered only closed G-orbits.
- $\mathfrak{X}_r(G)$ coincides with the one considered by Richardson-Slodowy (1990).
- $\mathfrak{X}_r(K)$ is a compact and Hausdorff space as the K-orbits are always closed.
- $\mathfrak{X}_r(K)$ can be identified with a semi-algebraic subset of $\mathbb{R}^d$

Cartan decomposition

$\mathfrak{g}$: Lie algebra of G $\quad$ $\mathfrak{g}^{\mathbb{C}}$: Lie algebra of **G**

Fix a Cartan involution $\theta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ which restricts to a Cartan involution

$$\theta : \mathfrak{g} \to \mathfrak{g}, \qquad \theta := \sigma\tau$$

where σ, τ are involutions of $\mathfrak{g}^{\mathbb{C}}$ that commute.

θ lifts to a Lie group involution $\Theta : G \to G$ whose differential is θ.

Our setting: G is embedded in some $\mathrm{GL}(n, \mathbb{C})$ as a closed subgroup, the involutions τ, σ, θ and Θ become explicit: $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}(n, \mathbb{C})$, $G \subset \mathrm{GL}(n, \mathbb{R}) \Rightarrow \tau(A) = -A^*$, where $*$ denotes transpose conjugate, and $\sigma(A) = \bar{A}$. Cartan involution: $\theta(A) = -A^t$, so that $\Theta(g) = (g^{-1})^t$.

$\mathfrak{g} = \mathrm{Fix}(\sigma)$ and $\mathfrak{k}' := \mathrm{Fix}(\tau)$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$ (so that $\mathfrak{k}'$ is the Lie algebra of a maximal compact subgroup, K', of $\mathbf{G}$).

θ yields a Cartan decomposition of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where

$$\mathfrak{k} = \mathfrak{g} \cap \mathfrak{k}', \quad \mathfrak{p} = \mathfrak{g} \cap i\mathfrak{k}'$$

$\theta|_{\mathfrak{k}} = 1$ and $\theta|_{\mathfrak{p}} = -1$.

$\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K of G:

$K = \mathrm{Fix}(\Theta) = \{g \in G : \Theta(g) = g\}$, $K = K' \cap G$, where K' is a maximal compact subgroup of $\mathbf{G}$, with Lie algebra $\mathfrak{k}' = \mathfrak{k} \oplus i\mathfrak{p}$.

$\mathfrak{k}$ and $\mathfrak{p}$ are such that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

We also have a Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$: $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$ with $\theta|_{\mathfrak{k}^{\mathbb{C}}} = 1$ and $\theta|_{\mathfrak{p}^{\mathbb{C}}} = -1$.

Deformation retraction from G to K

Multiplication map: $m : K \times \exp(\mathfrak{p}) \to G$ provides a diffeomorphism $G \simeq K \times \exp(\mathfrak{p})$.

In particular, the exponential is injective on $\mathfrak{p}$.

The inverse $m^{-1} : G \to K \times \exp(\mathfrak{p})$ is defined as

$$m^{-1}(g) = (g(\Theta(g)^{-1}g)^{-1/2}, (\Theta(g)^{-1}g)^{1/2}).$$

If $g \in \exp(\mathfrak{p})$ then $\Theta(g) = g^{-1}$. If we write $g = k\exp(X)$, for some $k \in K$ and $X \in \mathfrak{p}$, then $\Theta(g)^{-1}g = \exp(2X)$.

So define $(\Theta(g)^{-1}g)^t := \exp(2tX)$, for any real parameter t.

The topology of G is determined by K.

There is a K-equivariant strong deformation retraction from G to K:

Consider, for each $t \in [0,1]$, the continuous map $f_t : G \to G$ defined by

$$f_t(g) = g(\Theta(g)^{-1}g)^{-t/2}.$$

Proposition

The map $H : [0,1] \times G \to G$, $H(t,g) = f_t(g)$ is a strong deformation retraction from G to K, and for each t, $H(t,-) = f_t$ is K-equivariant with respect to the action of conjugation of K in G.

So there is a K-equivariant strong deformation retraction from G^r onto K^r with respect to the diagonal action of K. This immediately implies:

Corollary

Let K be a compact Lie group and G be a real K-reductive Lie group. Then $\mathfrak{X}_r(K)$ is a strong deformation retraction of $\mathfrak{R}_r(G)/K$.

The Kempf-Ness set

Fix a compact Lie group K, and a real K-reductive Lie group G.

G acts linearly on a complex vector space $\mathbb{V}$, equipped with a Hermitian inner product $\langle\, , \rangle$. Assume that $\langle\, , \rangle$ is K-invariant, by averaging. $||\cdot||$ is the norm corresponding to $\langle\, , \rangle$.

Definition

A vector $X \in \mathbb{V}$ is a *minimal vector* for the action of G in $\mathbb{V}$ if

$$\|X\| \leqslant \|g \cdot X\|, \forall g \in G.$$

$\mathcal{KN}_G = \mathcal{KN}(G, \mathbb{V}) =$ set of minimal vectors $=$ *Kempf-Ness set* in $\mathbb{V}$ w.r.t. the action of G. (It depends on the choice of $\langle\, , \rangle$.)

For each $X \in \mathbb{V}$, define the smooth real valued function $F_X : G \to \mathbb{R}$ by

$$F_X(g) = \frac{1}{2}\|g \cdot X\|^2.$$

Theorem (Richardson-Slodowy (1990))

Let $X \in \mathbb{V}$. The following conditions are equivalent:

(1) $X \in \mathcal{KN}_G$;

(2) F_X has a critical point at $1_G \in G$;

(3) $\langle A \cdot X, X \rangle = 0$, for every $A \in \mathfrak{p}$.

Since the action is linear and condition (3) above is polynomial, $\mathcal{KN}_G$ is a closed algebraic set in $\mathbb{V}$.

Kempf-Ness theory also works for closed G-subspaces: Y an arbitrary closed G-invariant subspace of $\mathbb{V}$, and define

$$\mathcal{KN}_G^Y := \mathcal{KN}_G \cap Y.$$

Consider the map

$$\eta : \mathcal{KN}_G^Y / K \to Y /\!\!/ G,$$

obtained from the K-equivariant inclusion $\mathcal{KN}_G^Y \hookrightarrow Y$ and the natural map $Y/K \to Y /\!\!/ G$.

Theorem (Richardson-Slodowy (1990))

The map $\eta : \mathcal{KN}_G^Y / K \to Y /\!\!/ G$ is a homeomorphism. In particular, if Y is a real algebraic subset of $\mathbb{V}$, then $Y /\!\!/ G$ is homeomorphic to a closed semi-algebraic set in some $\mathbb{R}^d$. Moreover, there is a K-equivariant deformation retraction of Y onto $\mathcal{KN}_G^Y$.

Kempf-Ness set for character varieties

Apply the Kempf-Ness theorem to our situation:

Embed the G-invariant closed set $Y = \mathfrak{R}_r(G) = \mathrm{Hom}(F_r, G) \cong G^r$ in a complex vector space $\mathbb{V}$.

Commutative diagram of inclusions

$$\begin{array}{ccccccc} \mathrm{O}(n) & \subset & \mathrm{GL}(n,\mathbb{R}) & \subset & \mathrm{GL}(n,\mathbb{C}) & \subset & \mathfrak{gl}(n,\mathbb{C}) \cong \mathbb{C}^{n^2} \\ \cup & & \cup & & \cup & & \\ K & \subset & G & \subset & \mathbf{G}, & & \end{array}$$

The commuting square on the left is guaranteed by the Peter-Weyl theorem.

We obtain the embedding of K^r ($r \in \mathbb{N}$) into the vector space $\mathbb{V}$:

$$\mathfrak{gl}(n, \mathbb{C})^r \cong \mathbb{C}^{n^2 r} =: \mathbb{V}.$$

Adjoint representation of $\mathrm{GL}(n, \mathbb{C})$ in $\mathfrak{gl}(n, \mathbb{C})$ restricts to a representation $G \to \mathrm{Aut}(\mathbb{V})$:

$$g \cdot (X_1, \ldots, X_r) = (gX_1g^{-1}, \ldots, gX_rg^{-1}), \;\; g \in G, \;\; X_i \in \mathfrak{gl}(n, \mathbb{C}).$$

Yields a representation $\mathfrak{g} \to \mathrm{End}(\mathbb{V})$ given by the Lie brackets:

$$A \cdot (X_1, \ldots, X_r) = (AX_1 - X_1A, \ldots, AX_r - X_rA) = ([A, X_1], \ldots, [A, X_r])$$

for every $A \in \mathfrak{g}$ and $X_i \in \mathfrak{gl}(n, \mathbb{C})$.

Choose a inner product $\langle \, , \rangle$ in $\mathfrak{gl}(n,\mathbb{C})$, K-invariant, under the restriction of the representation $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{gl}(n,\mathbb{C}))$ to K.

Obtain a inner product on $\mathbb{V}$, K-invariant by the corresponding diagonal action of K: $\langle (X_1,\ldots,X_r),(Y_1,\ldots,Y_r)\rangle = \sum_{i=1}^{r}\langle X_i, Y_i\rangle$ for $X_i, Y_j \in \mathfrak{gl}(n,\mathbb{C})$.

In $\mathfrak{gl}(n,\mathbb{C})$, $\langle \, , \, \rangle$ can be given explicitly by $\langle A, B\rangle = \mathrm{tr}(A^{*}B)$.

Theorem

The spaces $\mathfrak{X}_r(G) = \mathfrak{R}_r(G)/\!/G$ and $\mathfrak{X}_r(K) = \mathrm{Hom}(F_r, K)/K \cong K^r/K$ have the same homotopy type.

Corollary

The homotopy type of the space $\mathfrak{X}_r(G)$ depends only on the maximal compact subgroup K of G.

Deformation retraction from $\mathfrak{X}_r(G)$ onto $\mathfrak{X}_r(K)$

Proposition

For $Y = \mathfrak{R}_r(G) \cong G^r \subset \mathbb{V}$, the Kempf-Ness set is the closed set given by:

$$\mathcal{KN}_G^Y = \left\{ (g_1, \cdots, g_r) \in G^r : \sum_{i=1}^{r} g_i^* g_i = \sum_{i=1}^{r} g_i g_i^* \right\}.$$

K is the fixed set of the Cartan involution, so we have the inclusion $K^r \cong \mathrm{Hom}(F_r, K) \subset \mathcal{KN}_G^Y$. The Kempf-Ness set is a real algebraic set, when G is algebraic.

A matrix $A \subset \mathrm{GL}(n, \mathbb{C})$ is called *normal* if $A^*A = AA^*$. So, when $r = 1$,

Proposition

$\mathfrak{X}_1(G) = G /\!\!/ G$ is homeomorphic to the orbit space of the set of normal matrices in G, under conjugation by K.

Assume, due to a technical point, that G is algebraic.

Lemma

Assume that G and K are as before, and furthermore that G is a real algebraic set. There is a natural inclusion of finite CW-complexes $\mathfrak{X}_r(K) \subset \mathfrak{X}_r(G)$.

Using the previous results

Theorem

There is a strong deformation retraction from $\mathfrak{X}_r(G)$ to $\mathfrak{X}_r(K)$.

Proof.

The following diagram is commutative:

$$\begin{array}{ccc} K^r/K & \overset{i}{\hookrightarrow} & G^r/K \\ \phi \downarrow & & \| \\ \mathcal{KN}_G^{G^r}/K & \overset{j}{\hookrightarrow} & G^r/K \end{array}$$

The maps i and j induce isomorphisms on all homotopy groups.

Thus, ϕ induces isomorphisms on all homotopy groups as well since $i = j \circ \phi$. Then, the previous Lemma and Whitehead's theorem imply K^r/K is a strong deformation retraction of $\mathcal{KN}_G^{G^r}/K \cong G^r /\!\!/ G$. □

Low rank unitary groups

For any $r, n \in \mathbb{N}$, the following isomorphisms hold:

- $\mathfrak{X}_r(\mathrm{U}(n)) \cong \mathfrak{X}_r(\mathrm{SU}(n)) \times_{(\mathbb{Z}/n\mathbb{Z})^r} \mathrm{U}(1)^r$ [Florentino-Lawton, 2012];
- $\mathfrak{X}_r(\mathrm{O}(n)) \cong \mathfrak{X}_r(\mathrm{SO}(n)) \times (\mathbb{Z}/2\mathbb{Z})^r$, if n is odd.

Consider cohomology with rational coefficients.

We conclude that $H^*(\mathfrak{X}_r(\mathrm{U}(n))) \cong H^*(\mathfrak{X}_r(\mathrm{SU}(n)) \times \mathrm{U}(1)^r)^{(\mathbb{Z}/n\mathbb{Z})^r}$.

For $n = 2$,

Theorem

The action of $(\mathbb{Z}/2\mathbb{Z})^r$ on $H^*(\mathfrak{X}_r(\mathrm{SU}(2)))$ is trivial.

Thus, $H^*(\mathfrak{X}_r(\mathrm{U}(2))) \cong H^*(\mathfrak{X}_r(\mathrm{SU}(2))) \otimes H^*(\mathrm{U}(1)^r)^{(\mathbb{Z}/2\mathbb{Z})^r}$.

The action of $(\mathbb{Z}/2\mathbb{Z})^r$ on $H^*(\mathrm{U}(1)^r)$ is the action of -1 on the circle, which is rotation by 180 degrees. This is homotopic to the identity, and thus the action is trivial on cohomology. I.e.,

$H^*(\mathrm{U}(1)^r)^{(\mathbb{Z}/2\mathbb{Z})^r} = H^*(\mathrm{U}(1)^r)$.

And we conclude that $H^*(\mathfrak{X}_r(\mathrm{U}(2))) \cong H^*(\mathfrak{X}_r(\mathrm{SU}(2))) \otimes H^*(\mathrm{U}(1)^r)$.

The Poincaré polynomial of $\mathfrak{X}_r(\mathrm{SU}(2))$ was calculated by T. Baird (2008), using methods of equivariant cohomology.

Proposition

The Poincaré polynomial of $\mathfrak{X}_r(\mathrm{U}(2))$ is the following:

$$P_t(\mathfrak{X}_r(\mathrm{U}(2))) = (1+t)^{r+1} - \frac{t(1+t+t^3+t^4)^r}{1-t^4} + \frac{t^3}{2}\left(\frac{(1+t)^{2r}}{1-t^2} - \frac{(1-t^2)^r}{1+t^2}\right)$$

$G = \mathrm{U}(p,q)$=group of automorphisms of $\mathbb{C}^{p+q}$ preserving a nondegenerate hermitian form with signature (p,q).

Matrix terms: $\mathrm{U}(p,q) = \{M \in \mathrm{GL}(p+q,\mathbb{C}) \mid M^* I_{p,q} M = I_{p,q}\}$ where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Its maximal compact is $K = \mathrm{U}(p) \times \mathrm{U}(q)$ and it embeds diagonally in $\mathrm{U}(p,q)$:

$$(M,N) \hookrightarrow \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

As a subspace of $\mathfrak{X}_r(\mathrm{U}(p,q))$, $\mathfrak{X}_r(\mathrm{U}(p)\times\mathrm{U}(q))$ is homeomorphic to $\mathfrak{X}_r(\mathrm{U}(p)) \times \mathfrak{X}_r(\mathrm{U}(q))$.

From the main Theorem and the previous Proposition,

Proposition

For any $p, q \geqslant 1$ *and any* $r \geqslant 1$, *there exists a strong deformation retraction from* $\mathfrak{X}_r(\mathrm{U}(p,q))$ *onto* $\mathfrak{X}_r(\mathrm{U}(p)) \times \mathfrak{X}_r(\mathrm{U}(q))$. *In particular, the Poincaré polynomials of* $\mathfrak{X}_r(\mathrm{U}(2,1))$ *and* $\mathfrak{X}_r(\mathrm{U}(2,2))$ *are given respectively by:*

$$P_t(\mathfrak{X}_r(\mathrm{U}(2,1))) = P_t(\mathfrak{X}_r(\mathrm{U}(2)))(1+t)^r$$

and

$$P_t(\mathfrak{X}_r(\mathrm{U}(2,2))) = P_t(\mathfrak{X}_r(\mathrm{U}(2)))^2.$$

Since $\mathrm{U}(2)$ is a maximal compact subgroup of $\mathrm{Sp}(4, \mathbb{R})$ and of $\mathrm{GL}(2, \mathbb{C})$, we have the following:

Proposition

For any $r \geqslant 1$, there exists a strong deformation retraction from $\mathfrak{X}_r(\mathrm{Sp}(4, \mathbb{R}))$ and from $\mathfrak{X}_r(\mathrm{GL}(2, \mathbb{C}))$ onto $\mathfrak{X}_r(\mathrm{U}(2))$. The Poincaré polynomials of $\mathfrak{X}_r(\mathrm{Sp}(4, \mathbb{R}))$ and $\mathfrak{X}_r(\mathrm{GL}(2, \mathbb{C}))$ are such that:

$$P_t(\mathfrak{X}_r(\mathrm{Sp}(4, \mathbb{R}))) = P_t(\mathfrak{X}_r(\mathrm{GL}(2, \mathbb{C}))) = P_t(\mathfrak{X}_r(\mathrm{U}(2))).$$

Low rank orthogonal groups

Proposition

$\mathfrak{X}_r(\mathrm{SU}(2))/(\mathbb{Z}/2\mathbb{Z})^r \cong \mathfrak{X}_r(\mathrm{SO}(3))$

So, $H^*(\mathfrak{X}_r(\mathrm{SO}(3))) \cong H^*(\mathfrak{X}_r(\mathrm{SU}(2)))^{(\mathbb{Z}/2\mathbb{Z})^r}$.

Proposition

The Poincaré polynomials of $\mathfrak{X}_r(\mathrm{SO}(3))$ *and of* $\mathfrak{X}_r(\mathrm{O}(3))$ *are the following:*

$$P_t(\mathfrak{X}_r(\mathrm{SO}(3))) = P_t(\mathfrak{X}_r(\mathrm{SU}(2)))$$

and

$$P_t(\mathfrak{X}_r(\mathrm{O}(3))) = 2^r P_t(\mathfrak{X}_r(\mathrm{SU}(2))).$$

$G = \mathrm{SO}(p,q)$=group of volume preserving automorphisms of $\mathbb{R}^{p+q}$ preserving a nondegenerate symmetric bilinear form with signature (p,q).

Matrix terms: $\mathrm{SO}(p,q) = \{M \in \mathrm{SL}(p+q,\mathbb{R}) \mid M^t I_{p,q} M = I_{p,q}\}$ where

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

If $p+q \geqslant 3$, $\mathrm{SO}(p,q)$ has two connected components. Denote by $\mathrm{SO}_0(p,q)$ the component of the identity.

Maximal compact subgroup of $\mathrm{SO}_0(p,q)$ is $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ and it embeds diagonally in $\mathrm{SO}(p,q)$. So, as before, as a subspace of $\mathfrak{X}_r(\mathrm{SO}_0(p,q))$, $\mathfrak{X}_r(\mathrm{SO}(p) \times \mathrm{SO}(q))$ is homeomorphic to $\mathfrak{X}_r(\mathrm{SO}(p)) \times \mathfrak{X}_r(\mathrm{SO}(q))$.

We have thus the following:

Proposition

For any $p, q \geqslant 1$ and any $r \geqslant 1$, there exists a strong deformation retraction from $\mathfrak{X}_r(\mathrm{SO}_0(p,q))$ onto $\mathfrak{X}_r(\mathrm{SO}(p)) \times \mathfrak{X}_r(\mathrm{SO}(q))$. In particular, the Poincaré polynomials of $\mathfrak{X}_r(\mathrm{SO}_0(2,3))$ and of $\mathfrak{X}_r(\mathrm{SO}_0(3,3))$ are given respectively by

$$P_t(\mathfrak{X}_r(\mathrm{SO}_0(2,3))) = P_t(\mathfrak{X}_r(\mathrm{SU}(2)))(1+t)^r$$

and

$$P_t(\mathfrak{X}_r(\mathrm{SO}_0(3,3))) = P_t(\mathfrak{X}_r(\mathrm{SU}(2)))^2.$$

In the same way, since $\mathrm{SO}(3)$ (resp. $\mathrm{O}(3)$) is a maximal compact subgroup of both $\mathrm{SL}(3,\mathbb{R})$ (resp. $\mathrm{GL}(3,\mathbb{R})$) and $\mathrm{SO}(3,\mathbb{C})$ (resp. $\mathrm{O}(3,\mathbb{C})$), we have the following:

Proposition

For any $r \geqslant 1$, there exists a strong deformation retraction from $\mathfrak{X}_r(\mathrm{SL}(3,\mathbb{R}))$ and $\mathfrak{X}_r(\mathrm{SO}(3,\mathbb{C}))$ onto $\mathfrak{X}_r(\mathrm{SO}(3))$ and from $\mathfrak{X}_r(\mathrm{GL}(3,\mathbb{R}))$ and $\mathfrak{X}_r(\mathrm{O}(3,\mathbb{C}))$ onto $\mathfrak{X}_r(\mathrm{O}(3))$. In particular, the Poincaré polynomials of $\mathfrak{X}_r(\mathrm{SL}(3,\mathbb{R}))$ and $\mathfrak{X}_r(\mathrm{SO}(3,\mathbb{C}))$ are equal and given by:

$$P_t(\mathfrak{X}_r(\mathrm{SL}(3,\mathbb{R}))) = P_t(\mathfrak{X}_r(\mathrm{SO}(3,\mathbb{C}))) = P_t(\mathfrak{X}_r(\mathrm{SU}(2))).$$

The Poincaré polynomials of $\mathfrak{X}_r(\mathrm{GL}(3,\mathbb{R}))$ and $\mathfrak{X}_r(\mathrm{O}(3,\mathbb{C}))$ are:

$P_t(\mathfrak{X}_r(\mathrm{GL}(3,\mathbb{R}))) = P_t(\mathfrak{X}_r(\mathrm{O}(3,\mathbb{C}))) = 2^r P_t(\mathfrak{X}_r(\mathrm{SU}(2))).$