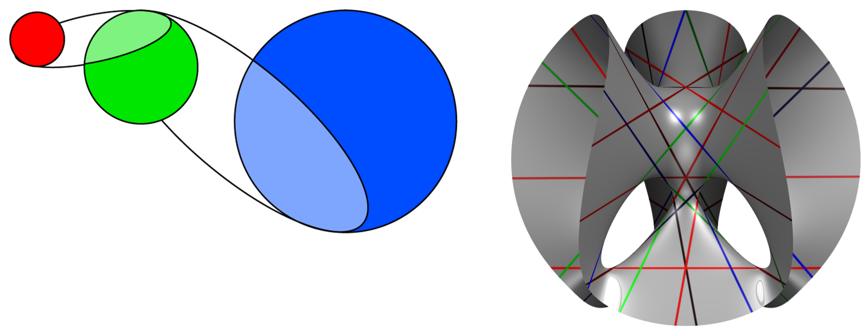
Young Women in Algebraic Geometry



Chow Rings of Irreducible Symplectic Varieties

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Chow Rings

Let X be a smooth quasi-projective variety. A *k-cycle* is a finite sum $\sum n_i Z_i$, where $n_i \in \mathbb{Z}$, and the Z_i are subvarieties (i.e. integral subscheme) of codimension k in X. For any subvariety $V \subseteq X$ of codimension $k-1$ and any $r \in K(V)$, set $\operatorname{div}(r) := \sum \operatorname{ord}_Z(r) \cdot Z$, where the sum is taken over all subvarieties $Z \subseteq V$ of codimension one. The equivalence relation on k-cycles spanned by $\{\operatorname{div}(r) \sim 0\}$ is called *rational equivalence*. One defines:

$$\mathrm{CH}^k(X) := \{k\text{-cycles}\}/\sim_{\mathrm{rat}} \qquad \text{and} \qquad \mathrm{CH}(X) := \bigoplus_{k=0}^{\dim(X)} \mathrm{CH}^k(X).$$

The "intersection product": $\mathrm{CH}^k(X) \times \mathrm{CH}^l(X) \to \mathrm{CH}^{k+l}(X)$ gives $\mathrm{CH}(X)$ the structure of a graded commutative ring, called the *Chow ring* of X.

There exists a natural map $c_X : \mathrm{CH}(X) \to H^*(X, \mathbb{Z})$, named *cycle class map*.

Irreducible Symplectic Varieties

Definition. An *irreducible symplectic variety* (or *projective hyperkähler manifold*) is a simply connected, nonsingular, complex projective variety X with a nowhere degenerate two-form σ generating $H^0(X, \Omega_X^2)$.

Examples:
- Hilbert schemes of points $\mathrm{Hilb}^n(S)$ on a K3 surface S,
- Generalized Kummer varieties $K_n(A)$ associated to an abelian surface A.

Properties:
The existence of a non-degenerate form immediately implies that $\dim X$ is even.

Beauville–Bogomolov–Fujiki form
The second cohomology of an irreducible symplectic variety X is endowed with a primitive integral quadratic form $q : H^2(X, \mathbb{Z}) \to \mathbb{Z}$, called the *Beauville–Bogomolov–Fujiki form*.

Lagrangian fibrations

Definition. Let X be an irreducible symplectic variety. A map $f : X \twoheadrightarrow B$ with connected fibres is called *Lagrangian fibration*, if $\dim(B) = \frac{1}{2}\dim(X)$, and all irreducible components of the fibres are Lagrangian subvarieties of X (with respect to the symplectic structure induced by $\sigma \in H^0(X, \Omega_X^2)$).

An important property of irreducible symplectic varieties is the following:

Proposition 1 ([Mat99]+[Mat00])**.** *Let X be an irreducible symplectic variety, and B be a normal variety with $0 < \dim B < \dim X$. Then every map $f : X \twoheadrightarrow B$ with connected fibres is a Lagrangian fibration.*

Definition. Let X be an irreducible symplectic variety. A rational map $f : X \dashrightarrow B$ is called *rational Lagrangian fibration*, if there exists a birational map $\varphi : X \dashrightarrow X'$ to an irreducible symplectic variety X', and a Lagrangian fibration $f' : X' \twoheadrightarrow B$, such that $f = f' \circ \varphi$.
A line bundle $L \in \mathrm{Pic}(X)$ is said to *induce a rational Lagrangian fibration*, if there exists a rational Lagrangian fibration $f : X \dashrightarrow B$, such that some multiple of L corresponds to an ample line bundle on B.

First Results

Proposition 2. *Let X and X' be birational irreducible symplectic varieties. Then there exist families of algebraic spaces $\mathscr{X}$ and $\mathscr{X}'$ over T (T smooth quasi-projective one-dimensional variety), and a closed point $0 \in T$, such that*
(a) $\mathscr{X}_0 = X$ and $\mathscr{X}'_0 = X'$, and
(b) there is an isomorphism $\Psi : \mathscr{X}_{T\setminus\{0\}} \cong \mathscr{X}'_{T\setminus\{0\}}$ over T.

This proposition is an algebraic version of [Huy99, Theorem 4.6].

Notation. In the situation of Proposition 2, let $Z \in \mathrm{CH}^{\dim X}(X \times X')$ be the degeneration of the graphs of isomorphisms. More precisely, if $\Gamma \subseteq \mathscr{X}_{T\setminus\{0\}} \times_{T\setminus\{0\}} \mathscr{X}'_{T\setminus\{0\}}$ is the graph of Ψ, and $\overline{\Gamma} \subseteq \mathscr{X} \times_T \mathscr{X}'$ its closure, we define $Z \subseteq X \times X'$ as the special fibre of $\overline{\Gamma}$.
Furthermore, let p and p' be the projections from $X \times X'$ to X and X' respectively.

Definition. Define $Z_* : \mathrm{CH}(X) \to \mathrm{CH}(X')$ as the correspondence with kernel Z, i.e. as the map given by $Z_*(\alpha) := p'_*(Z.p^*\alpha)$ for all $\alpha \in \mathrm{CH}(X)$.

The first important result of my work was the following (see [Rie14a]):

Theorem 3. *Let X and X' be birational irreducible symplectic varieties. Then $Z_* : \mathrm{CH}(X) \to \mathrm{CH}(X')$ is an isomorphism of graded rings. Its inverse is the correspondence $Z^t_* : \mathrm{CH}(X') \to \mathrm{CH}(X)$ with kernel Z in the opposite direction.*

The most remarkable part of this theorem is the multiplicativity of Z_*, which has not even been known in the most basic examples.
The proof of the theorem relies on Proposition 2, the specialization map for Chow rings, and uses the fact that for algebraic spaces the intersection theory as presented in [Ful84, Chapters 1-6] still applies.

Weak Splitting Property

The following conjectural "weak splitting property" was the central motivation for my research.

Let X be a nonsingular complex projective variety. Define $\mathrm{DCH}(X) \subseteq \mathrm{CH}_{\mathbb{Q}}(X)$ as the subalgebra generated by $\mathrm{CH}^1_{\mathbb{Q}}(X)$.
In [Bea07], Beauville formulated the following conjecture:

Conjecture 4 (Weak splitting property)**.** *For any irreducible symplectic variety X the restriction*

$$c_X\big|_{\mathrm{DCH}(X)} : \mathrm{DCH}(X) \hookrightarrow H^*(X, \mathbb{Q})$$

of the cycle class map to the subalgebra $\mathrm{DCH}(X)$ is injective.

Remark. This conjecture is called "weak splitting property", because it would be a consequence of some conjectural filtration on the Chow rings (Bloch–Beilinson filtration).

There are stronger versions of this conjecture by Voisin (see [Voi08], and [Voi15]).

Main Theorem

The main result presented here, deduces the weak splitting property under a certain condition from an other conjecture (Conjecture 5):

Definition. For an irreducible symplectic variety X, denote its Kähler cone by $\mathscr{K}_X$. Define its *birational Kähler cone* as

$$\mathscr{BK}_X := \bigcup_f f^*(\mathscr{K}_{X'}) \subseteq H^{1,1}(X, \mathbb{R}),$$

where the union is taken over all birational maps $f : X \dashrightarrow X'$ from X to another irreducible symplectic variety X'. Denote its closure by $\overline{\mathscr{BK}}_X \subseteq H^{1,1}(X, \mathbb{R})$.

Conjecture 5. *Let X be an irreducible symplectic variety. Suppose $0 \neq L \in \mathrm{Pic}(X) \cap \overline{\mathscr{BK}}_X$ satisfies $q(L) = 0$. Then L induces a rational Lagrangian fibration.*

The following is the main result on this poster (it can be found in [Rie14b]):

Theorem 6. *Fix an irreducible symplectic variety X with a line bundle $L \in \mathrm{Pic}(X)$ satisfying $q(L) = 0$. If Conjecture 5 holds for X then the weak splitting property (Conjecture 4) holds for X.*

The advantage of this reduction is, that Conjecture 5 has intensively been studied during the last few years. A long sequence of articles (amongst others by Markushevich, Sawon, Bayer–Macrì, Markman) culminated in the following:

Theorem 7 ([Mat13, Corollary 1.1])**.** *Let X be an irreducible symplectic variety which is deformation equivalent to $\mathrm{Hilb}^n(S)$ for a K3-surface S, or to a generalized Kummer variety $K_n(A)$ for an abelian surface A. Then Conjecture 5 holds for X.*

As a direct consequence from Theorem 6 and Theorem 7 one obtains:

Corollary 8. *Let X be an irreducible symplectic variety which is deformation equivalent to $\mathrm{Hilb}^n(S)$ for a K3-surface S, or to a generalized Kummer variety $K_n(A)$ for an abelian surface A. Suppose that there exists $0 \neq L \in \mathrm{Pic}(X)$ with $q(L) = 0$. Then the weak splitting property (Conjecture 4) holds for X.*

Remark. Until now, all known cases of the weak splitting property heavily relied on explicit geometric descriptions of the examples. This result is particularly remarkable, because it provides examples, where we do not know any such description.
Since the weak splitting property is not yet known for all $\mathrm{Hilb}^n(S)$ (where S is a K3-surface), Corollary 8 even provides new examples for these.

Some of the techniques used to prove Theorem 6, can be used to observe the following reduction:

Proposition 9. *Suppose that the weak splitting property (Conjecture 4) holds for all irreducible symplectic varieties Y satisfying $\rho(Y) = 2$. Then the weak splitting property holds for all irreducible symplectic varieties.*

Main Ingredients of the Proof

The first step is an observation of Beauville (in [Bea07]), based on a result of Bogomolov determining relations in cohomology: In order to prove the weak splitting property, it is enough to show:

$$\forall \alpha \in \mathrm{CH}^1_{\mathbb{C}}(X) \qquad q(\alpha) = 0 \Rightarrow \alpha^{n+1} = 0, \qquad \text{where } n := \frac{1}{2}\dim(X).$$

Use Huybrechts' description of $\overline{\mathscr{BK}}_X$, Markman's result that certain cohomological reflections are monodromy operators, and a refined version of Theorem 3, along with various computations, to reduce to the following:

$$\forall \alpha \in \mathrm{CH}^1_{\mathbb{Q}}(X) \cap \overline{\mathscr{BK}}_X \qquad q(\alpha) = 0 \Rightarrow \alpha^{n+1} = 0.$$

This follows from Conjecture 5, using that the dimension of the base of the Lagrangian fibration is n.

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