Young Women in Algebraic Geometry

Graphing Congruences of Newforms

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Newforms

Definition 1 *Let N be a positive integer. The* **modular group** $\Gamma_0(N)$ *is the group:*

$$\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) \middle| \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}.$$

Definition 2 *Let $\mathbb{H}$ denote the* **complex upper half plane**. *The modular group acts on the complex upper half plane via linear fractional transforms i.e. for $\gamma \in \Gamma_0(N)$ and for $z \in \mathbb{H}$,*

$$\gamma \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty \\ \frac{a}{c} & \text{if } z = \infty \end{cases}.$$

If $c = 0$ or $cz + d = 0$, then $\gamma \cdot z = \infty$

- The **cusps** of $\Gamma_0(N)$ is the set of $\Gamma_0(N)$-orbits in $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$.

Definition 3 *A* **modular form** *of* **weight k** *and* **level N** *for the group $\Gamma_0(N)$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which is holomorphic at all cusps. For all $z \in \mathbb{H}$ and $\gamma \in \Gamma_0(N)$, f satisfies:*

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

A modular form can be expressed as a power series (called q-expansion),

$$f(z) := \sum_{n=0}^{\infty} a_n(f) \cdot q^n, \quad \text{where } q = e^{2\pi i z}, \quad a_n(f) \in \mathbb{C}.$$

If f vanishes at all of the cusps then we call f a **cuspform**. The vector space of all weight k cuspforms for $\Gamma_0(N)$ is denoted by $S_k(N)$.

- Let M be a proper divisor of N. There are canonical maps $S_k(M) \to S_k(N)$. Let $S_k^{\text{old}}(N)$ be the subspace spanned by the images of these maps for all proper divisors M of N.
- The **new subspace $S_k^{\text{new}}(N)$** is the orthogonal complement to $S_k^{\text{old}}(N)$ with respect to the Peterson inner-product.
- The space $S_k^{\text{new}}(N)$ is endowed with an action of commuting Hecke operators T_p and the **newforms** are a simultaneous eigenbasis for these Hecke operators.
- The Hecke eigenvalues of a newform f are the coefficients of its q-expansion, $a_n(f)$, and its **Hecke eigenvalue field** is the number field $\mathbb{Q}(a_1(f), a_2(f), \dots)$.

Congruences of Newforms

Definition 4 *Let $f \in S_{k_1}^{\text{new}}(N_1)$, and let $g \in S_{k_2}^{\text{new}}(N_2)$. We say that f and g are* **congruent modulo $\mathfrak{p}$**, *if there exists an ideal $\mathfrak{p}$ above p in the compositum of the Hecke eigenvalue fields such that $a_n(f) \equiv a_n(g) \bmod \mathfrak{p}$ for all n.*

- Given level N and weight k, the **Sturm Bound** is:

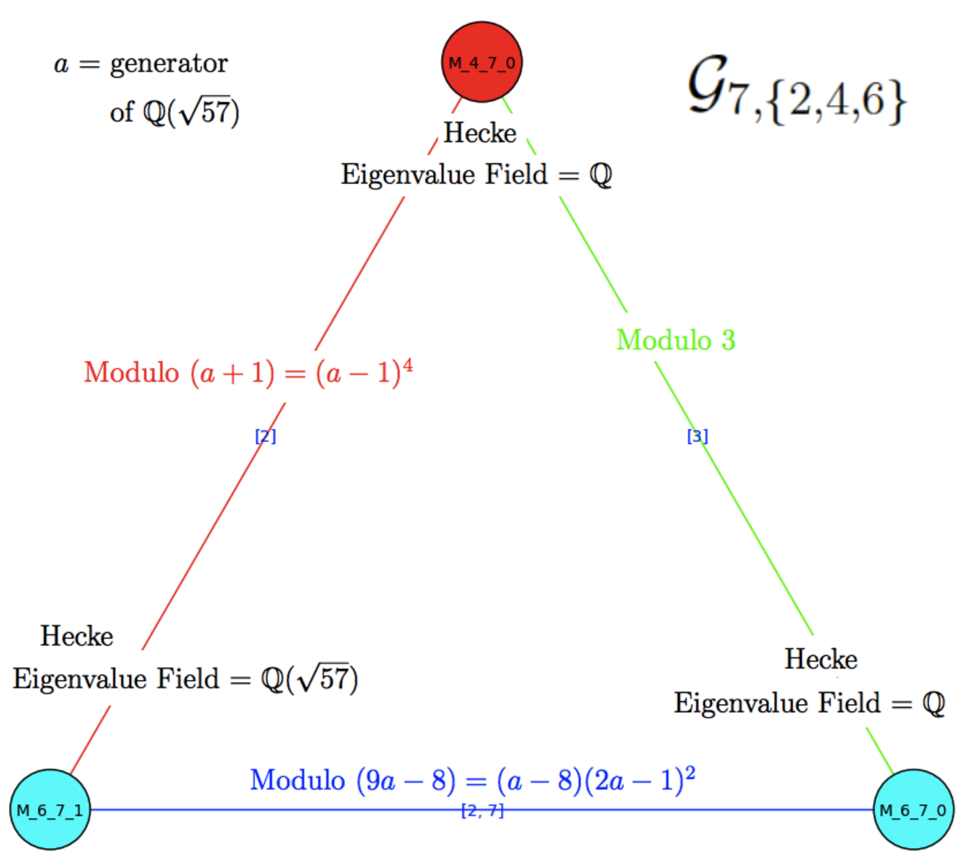
$$SB(N, k) := \frac{k}{12} \cdot N \prod_{p \mid N} \left(1 + \frac{1}{p}\right).$$

Theorem 1 (Sturm, 1987) *Let $f, g \in S_k^{\text{new}}(N)$. Then $f \equiv g \mod p$ if and only if there exist an ideal $\mathfrak{p}$ above p such that $a_n(f) \equiv a_n(g) \bmod \mathfrak{p}$ for all $n \le SB(N, k)$.*

Theorem 2 (Kohnen, 2004) *Let $f \in S_{k_1}^{\text{new}}(N_1)$ and $g \in S_{k_2}^{\text{new}}(N_2)$. Then $f \equiv g \mod p$ if and only if there exist an ideal $\mathfrak{p} \mid p$ such that:*

$$a_n(f) \equiv a_n(g) \mod \mathfrak{p} \text{ for all } n \le SB(\text{lcm}(N_1, N_2), \max\{k_1, k_2\}).$$

A Missing Mod 2 Congruence ...?



Setup

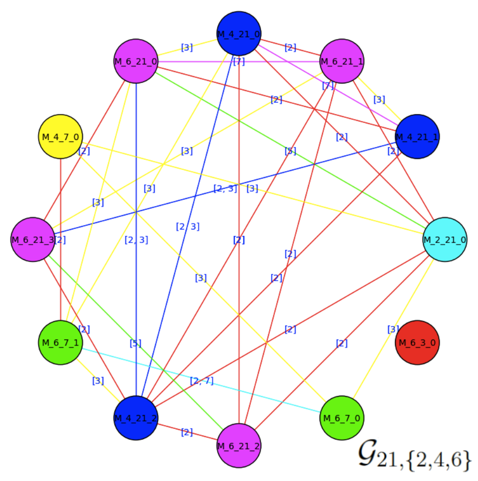
Let $\mathcal{G}_{N,W}$ be the graph with the following sets of vertices and edges:

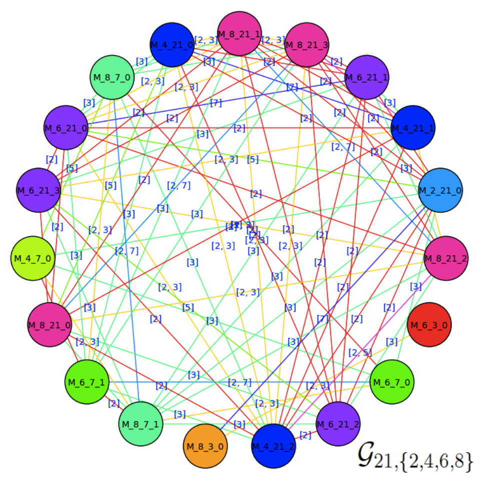
- **Vertices** - newforms which have levels dividing N and have weights belonging to a finite set W.
- **Edges** - are drawn if there is a congruence relation between two newforms see Theorem 2.

Vertex Labeling: we label vertices by $M_{k,N',r}$ where k is the weight of the newform, $N' \mid N$ is the level of the newform, and r is the index that SageMath gives to the newform.
Edge Labeling: we label the edge connecting f and g by $[p]$ if there is an ideal $\mathfrak{p} \mid p$ such that $f \equiv g \pmod{\mathfrak{p}}$.

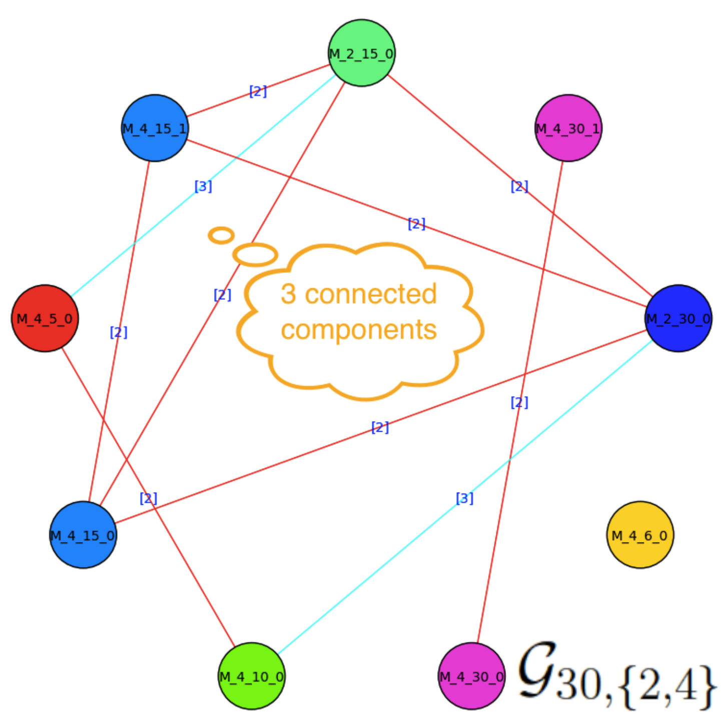
Motivation

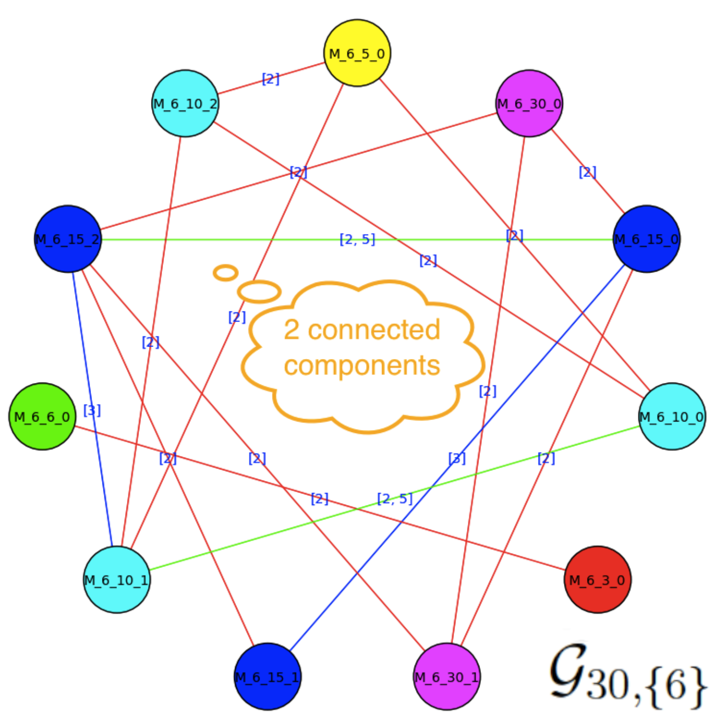
- The **Modularity theorem** due to Wiles, Breuil, Conrad, Diamond and Taylor states that every elliptic curve over $\mathbb{Q}$ corresponds to a weight 2 newform with Hecke eigenvalue field $\mathbb{Q}$.
- Weight 2 newforms are now known, thanks to the proof of **Serre's modularity conjecture** by Khare and Wintenberger, to correspond to isogeny classes of abelian varieties of GL_2-type.
- Congruences between the newforms describe relations between the torsion subgroups of these abelian varieties. These relations are exploited in the **modularity switching** steps in the proofs of the Modularity theorem and Serre's modularity conjecture.
- We expect that the graphs will lead to a better understanding of modularity switching.





Disconnected Examples





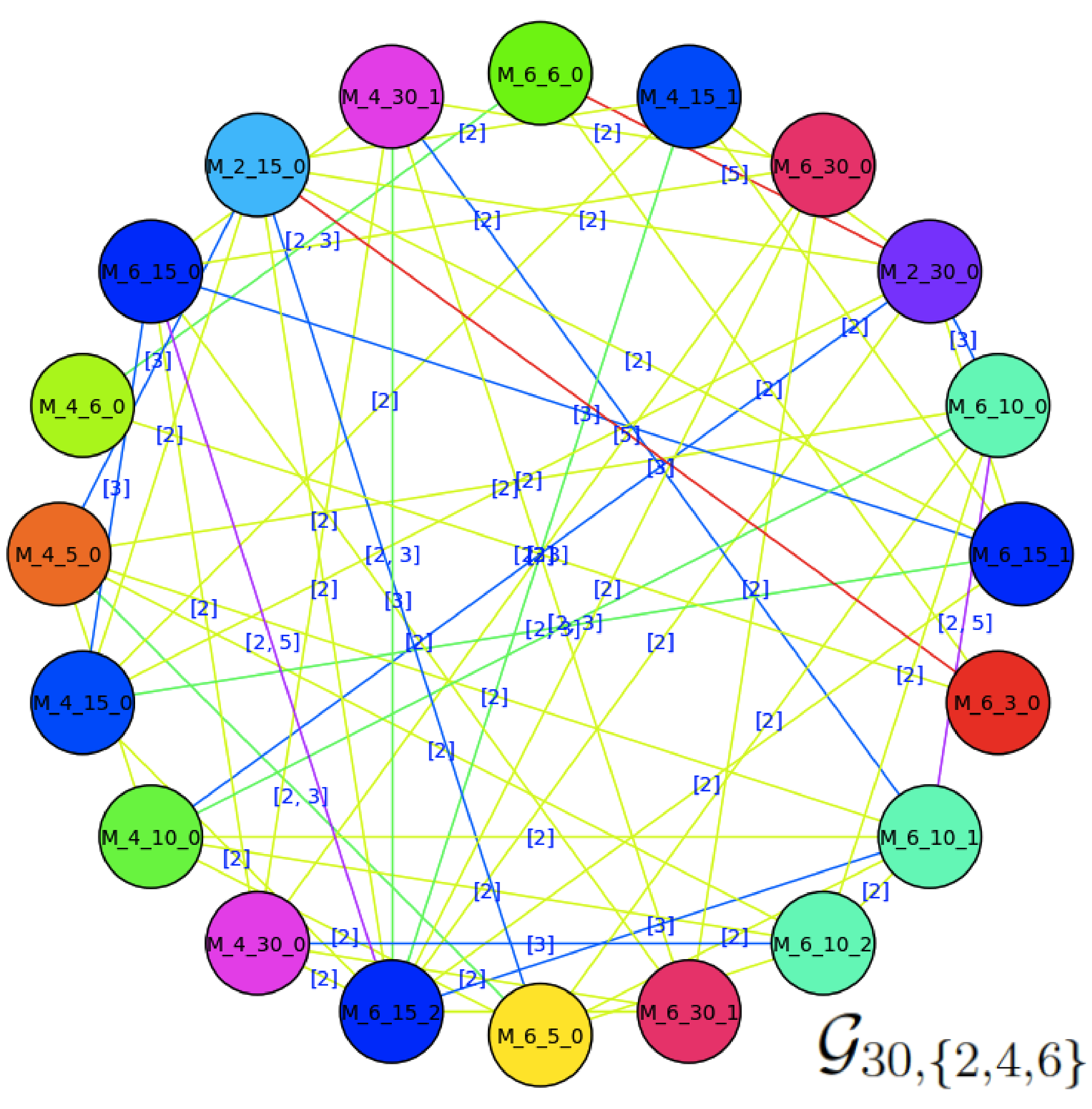
Conjecture

Conjecture 1 (Anni and Patel) *Given N and W, there exists finite $W' \supseteq W$ such that the graph $\mathcal{G}_{N,W'}$ is connected.*

References

- Kohnen, W. (2004). On Fourier coefficients of modular forms of different weights. Acta Arithmetica, 113, 57-67.
- *Sage Mathematics Software (Version 6.6)*, The Sage Developers, 2015, `http://www.sagemath.org`.
- Sturm, J. (1987). On the congruence of modular forms. In Number theory (pp. 275-280). Springer Berlin Heidelberg.

A Connected Example



We now take $W' = \{2, 4, 6\}$. Notice that $\mathcal{G}_{30,W'}$ is connected, illustrating our conjecture.

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