The quotient map on the equivariant Grothendieck ring

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Young Women in Algebraic Geometry

The equivariant Grothendieck ring

Let S be a separated scheme with a good group action of a finite group G. Denote by $(\mathrm{Sch}_{S,G})$ the category of separated schemes of finite type over S with a good G-action compatible with that on S.

Definition. The *equivariant Grothendieck ring of S-varieties* $K_0^G(\mathrm{Var}_S)$ is defined as follows: as abelian group, it is generated by isomorphism classes $[X]$ of elements $X \in (\mathrm{Sch}_{S,G})$. These generators are subject to the following relations:

- $[X]=[Y]+[X \setminus Y]$, whenever Y is a closed G-equivariant subscheme of X (scissors relation).
- $[V]=[W]$, whenever $V \to B$ and $W \to B$ are two G-equivariant affine bundles of rank d over $B \in (\mathrm{Sch}_{S,G})$ with affine G-action.

For all $X, Y \in (\mathrm{Sch}_{S,G})$, set $[X].[Y] := [X \times_S Y]$. Let $\mathscr{M}_S^G$ be the localization of $K_0^G(\mathrm{Var}_S)$ with respect to $\mathbb{L} := [\mathbb{A}_S^1]$.

The quotient map

How can the equivariant Grothendieck ring be connected to the usual one? A natural thing to do is to send the class of $X \in (\mathrm{Sch}_{S,G})$ to the class of its quotient. By [1, Lemma 3.2] such a map is well defined if G acts freely on S. We can show:

Theorem ([4]). *Let G be a finite abelian group. Assume that the residue field F_s of any point $s \in S/G$ contains all $|G|$-th roots of unity. Then there is a well defined group homomorphism*

$$K_0^G(Var_S) \to K_0^*(Var_{S/G})$$

sending the class of $X \in (Sch_{S,G})$ to $[X/G] \in K_0^*(Var_{S/G})$.

Here $K_0^*(\mathrm{Var}_{S/G})$ is the usual Grothendieck ring of varieties over S/G if G acts on S tamely, and its quotient by purely inseparable maps if the action of G on S is wild.

On the proof

The hardest part of the proof is to show that the quotient of a G-equivariant affine bundle $\varphi : V \to B$ only depends on its rank and base. We show this by computing the fibers of $\varphi_G : V/G \to B/G$ separately and using spreading out to conclude. For every point $b = \mathrm{Spec}(F_b) \in B/G$ we have

$$[\varphi_G^{-1}(b)] = [\varphi^{-1}(b')/G_b] \in K_0^*(\mathrm{Var}_{F_b}),$$

where $b' \in B$ is a point in the inverse image of b and G_b its stabilizor. Now we compute $\varphi^{-1}(b')/G'$ using the following proposition, which was already shown in [3, Lemma 1.1] for tame actions:

Proposition ([4]). *Let G be a finite abelian group with quotient Γ. Let k be a field, and let K/k be a Galois extension with Galois group Γ. Assume that the Galois action on K lifts to a k-linear action of G on a finite dimensional K-vector space V. If k contains all $|G|$-th roots of unity, then*

$$[V/G] = \mathbb{L}^{dim_K V} \in K_0^*(Var_k).$$

Equivariant motivic integration

Using the equivariant Grothendieck ring of varieties as value ring for motivic integration allows us to also encode some group action on a scheme X, as done for example with the monodromy action on the motivic Zeta function, see [2]:

Definition. Let X be a smooth irreducible algebraic variety of dimension $m+1$ over $\mathbb{C}$, and $f : X \to \mathbb{A}_{\mathbb{C}}^1$ a non-constant map. Set $X_0 := f^{-1}(0)$. Let $\mathscr{X}_{d,1}$ be the $\mathbb{C}$-scheme whose $\mathbb{C}$-points are given by

$$\{\psi : \mathrm{Spec}(\mathbb{C}[[t]]/(t^{d+1})) \to X \mid f(\psi(t)) = t^d \bmod t^{d+1}\}.$$

Let the profinite group $\hat{\mu}$ of roots of unity act on $\mathscr{X}_{d,1}$ by multiplication with a primitive d-th root of unity. *Denef and Loeser's motivic Zeta function* is defined as

$$Z(f;T) = \sum_{d>0} [\mathscr{X}_{d,1}] \mathbb{L}^{(m+1)d} T^d \in \mathscr{M}_{X_0}^{\hat{\mu}}[[T]].$$

Formal schemes

There exists also a theory of motivic integration for formal schemes. An equivariant version is done in [5]. In particular we can recover Denef and Loeser's motivic Zeta function:

Theorem ([5]). *Let X_∞ be the formal completion of X along X_0. Assume that there exists a global gauge form ω on the generic fiber X_η. Set $X_\infty(d) := X_\infty \otimes_{\mathbb{C}[[t]]} \mathbb{C}[[\sqrt[d]{t}]]$, on which $\hat{\mu}$ acts by multiplication of $\sqrt[d]{t}$ with a primitive d-th root of unity. Then*

$$S(f;T) := \sum_{d>0} \int_{X_\infty(d)} |\omega'(d)|\, T^d = \mathbb{L}^{-m} Z(f; \mathbb{L}T) \in \mathscr{M}_{X_0}^{\hat{\mu}}[[T]],$$

where $\int_{X_\infty(d)} |\omega'(d)| \in \mathscr{M}_{X_0}^{\hat{\mu}}$ is the equivariant motivic integral of the pullback $\omega'(d)$ of $\omega' := \frac{\omega}{df}$ to $X_\infty(d)$.

An application of the quotient map

Definition. The *motivic Nearby fiber $\mathscr{S}_f$ / equivariant motivic volume $\mathscr{S}_{X_\infty}$* is the limit of $-Z(f;T)$ / $-S(f;T)$ for $T \to \infty$.

As the quotient map is well defined, $\mathscr{S}_f/\hat{\mu}$ and $\mathscr{S}_{X_\infty}\mathbb{L}^m/\hat{\mu}$ are well defined invariant with values in $\mathscr{M}_{X_0}$. Explicit formulas for $\mathscr{S}_f$ and $\mathscr{S}_{X_\infty}$ yield that these quotients are modulo $\mathbb{L}$ equal to the class of the special fiber $h^{-1}(X_0)$ of any embedded resolution $h : Y \to X$ of f.

References

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