Young Women in Algebraic Geometry

Analogue of Hilbert's 1888 Theorem for Symmetric and Even Symmetric forms

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Abstract

Sums of squares representations of polynomials are of fundamental importance in real algebraic geometry. In 1888, Hilbert [3] gave a complete characterisation of the pairs $(n, 2d)$ for which a n-ary $2d$-ic form non-negative on $\mathbb{R}^n$ can be written as sums of squares of other forms. This poster presents our recent results [1, 2] giving the analogue of Hilbert's characterisation *under the additional assumptions* of symmetry and even symmetry on the given form.

Definitions & Notations

- *n-ary $2d$-ic form*: $\mathcal{F}_{n,2d}$:= homogenous polynomials in n variables and of degree $2d$
- *Positive semidefinite (psd) form*: $\mathcal{P}_{n,2d} := \{f \in \mathcal{F}_{n,2d} | f(\underline{x}) \geq 0 \; \forall \; \underline{x} \in \mathbb{R}^n\}$
- *Sum of squares (sos) form*: $\Sigma_{n,2d} := \{f \in \mathcal{F}_{n,2d} | f = \sum_i h_i(\underline{x})^2, h_i \in \mathcal{F}_{n,d}\}$
- *Symmetric form*: $f \in \mathcal{F}_{n,2d}$ such that $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n) \; \forall \; \sigma \in S_n$
- $S\mathcal{P}_{n,2d} := \{f \in \mathcal{F}_{n,2d} | f$ is symmetric & psd$\}$
- $S\Sigma_{n,2d} := \{f \in \mathcal{F}_{n,2d} | f$ is symmetric & sos$\}$
- *Even symmetric form*: $f \in \mathcal{F}_{n,2d}$ such that f is symmetric and all its monomials appear with even exponents
- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} | f$ is even symm & psd$\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} | f$ is even symm & sos$\}$

Hilbert's 1888 theorem

Every sos form is automatically psd (by definition), but not the converse:

- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
  - ▶ The arguments for $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ for $n = 2$ & $d = 1$ were already known in the late 19th century and Hilbert proved $\mathcal{P}_{3,4} = \Sigma_{3,4}$
  - ▶ Conversely, Hilbert proved $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$, and demonstrated that **Proposition [Reduction to Basic cases]:** If $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$, then $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d} \; \forall \; n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$.
    Idea of Proof: If $f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d}$, then $f \in \mathcal{P}_{n+j,2d} \setminus \Sigma_{n+j,2d} \; \forall \; j \geq 0$ and $x_1^{2i} f \in \mathcal{P}_{n,\,2d+2i} \setminus \Sigma_{n,\,2d+2i} \; \forall \; i \geq 0$.
- **(Motzkin, 1967):** $M(x, y, z) := z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$
- **(Choi-Lam, 1976):** $Q(x, y, z, w) := w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw \in \mathcal{P}_{4,4} \setminus \Sigma_{4,4}$

Test Sets for positivity

- $\Omega \subseteq \mathbb{R}^n$ is a *test set* for f if f is psd if and only if $f(\underline{x}) \geq 0$ for all $\underline{x} \in \Omega$.
- Choi, Lam, Reznick and Harris gave test sets for symmetric quartics and even symmetric sextics, octics and ternary decics.
- **Theorem (Timofte, 2003):** A symmetric real polynomial of degree $2d$ in n variables is nonnegative on $\mathbb{R}^n \Leftrightarrow$ it is nonnegative on the subset $\Lambda_{n,k} := \{\underline{x} \in \mathbb{R}^n \mid$ number of distinct components in $\underline{x}$ is $\leq k\}$, where $k := \max\{2, d\}$.

Analogue of Hilbert's Theorem for Symmetric Forms

- **Theorem (Choi-Lam, 1976):** $S\mathcal{P}_{n,2d} = S\Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
  - ▶ If $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$, then $S\mathcal{P}_{n,2d} = S\Sigma_{n,2d}$ follows by Hilbert's Theorem
  - ▶ Conversely, enough to find $f \in S\mathcal{P}_{n,2d} \setminus S\Sigma_{n,2d}$ for the pairs $(3, 6)$ and $(n, 4) \; \forall \; n \geq 4$, using: **Proposition [Reduction to Basic cases]:** If $S\Sigma_{3,6} \subsetneq S\mathcal{P}_{3,6}$ and $S\Sigma_{n,4} \subsetneq S\mathcal{P}_{n,4} \; \forall \; n \geq 4$, then $S\Sigma_{n,2d} \subsetneq S\mathcal{P}_{n,2d} \; \forall \; n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$.

  Proof: By Lemma 1 (below), $f \in S\mathcal{P}_{n,2d} \setminus S\Sigma_{n,2d} \Rightarrow \left(\sum_{j=1}^n x_j\right)^{2i} f \in S\mathcal{P}_{n,2d+2i} \setminus S\Sigma_{n,2d+2i} \; \forall \; i \geq 0$.
- **(Robinson, 1969):** $R(x, y, z) := \sum^3 x^6 - \sum^6 x^4y^2 + 3x^2y^2z^2 \in S\mathcal{P}_{3,6} \setminus S\Sigma_{3,6}$
- **(Choi-Lam, 1976):** $f(x, y, z, w) := \sum^6 x^2y^2 + \sum^{12} x^2yz - 2xyzw \in S\mathcal{P}_{4,4} \setminus S\Sigma_{4,4}$
  [The summations in the above two examples denote the full symmetric sums]
- To construct explicit $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$, we [1] consider the following symmetric quartic in $n \geq 4$ variables:
$$L_n(\underline{x}) := m(n-m) \sum_{i<j} (x_i - x_j)^4 - \Big(\sum_{i<j} (x_i - x_j)^2\Big)^2; \text{ where } m = \lfloor \tfrac{n}{2} \rfloor$$
- **Proposition:** L_n is psd for all n and sos for even n.

Results 1

- **Lemma 1:** Let $f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d}$ and p an irreducible indefinite form of degree r in n variables. Then $p^2 f \in \mathcal{P}_{n,2d+2r} \setminus \Sigma_{n,2d+2r}$.
- **Theorem:** If $n \geq 5$ is odd, then L_n is not sos.
- **Theorem:** For $m \geq 2$, $C_{2m}(x_1, \ldots, x_{2m}) := L_{2m+1}(x_1, \ldots, x_{2m}, 0) \in S\mathcal{P}_{2m,4} \setminus S\Sigma_{2m,4}$.

Extension of Hilbert's Theorem for Even Symm. forms

- $(\mathcal{Q})$: For what pairs $(n, 2d)$ is $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?
- Let $\Delta_{n,2d} := S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ and $M_r(x_1, \ldots, x_n) := x_1^r + \ldots + x_n^r$ for an integer $r \geq 1$
- Known: $\Delta_{n,2d} = \emptyset$ if $n = 2, d = 1, (n, 2d) = (n, 4)_{n \geq 4}, (3, 8)$, and
  $\Delta_{n,2d} \neq \emptyset$ for $(n, 2d) = (n, 6)_{n \geq 3}, (3, 10), (4, 8)$
- To get a complete answer to $(\mathcal{Q})$, we need to look at the pairs $(3, 2d)_{d \geq 6}, (n, 8)_{n \geq 5}, (n, 2d)_{n \geq 4, d \geq 5}$

Results 2

- **Lemma:** For $n \geq 3$, the even symmetric real forms $p_n := 4\sum_{j=1}^n x_j^4 - 17\sum_{1 \leq i<j \leq n} x_i^2x_j^2$ and $q_n := \sum_{j=1}^n x_j^6 + 3\sum_{1 \leq i \neq j \leq n} x_i^4x_j^2 - 100\sum_{1 \leq i<j<k \leq n} x_i^2x_j^2x_k^2$ are irreducible over $\mathbb{R}$.
- **Theorem [Degree Jumping Principle]:** Suppose $f \in \Delta_{n,2d}$ for $n \geq 3$, then
  (i) for any integer $r \geq 2$, the form $p_n^{2a} q_n^{2b} f \in \Delta_{n,2d+4r}$, where $r = 2a + 3b$; $a, b \in \mathbb{Z}_+$,
  (ii) $(x_1 \ldots x_n)^2 f \in \Delta_{n,2d+2n}$.
- **Proposition [Reduction to Basic cases]:** If $\Delta_{n,2d} \neq \emptyset$ for $(n, 8)_{n \geq 4}, (n, 10)_{n \geq 3}$ and $(n, 12)_{n \geq 3}$, then $\Delta_{n,2d} \neq \emptyset$ for $(n, 2d)_{n \geq 3, d \geq 7}$.
- **Theorem:** For $m \geq 2$, $D_{2m} := C_{2m}(x_1^2, \ldots, x_{2m}^2) \in \Delta_{2m,8}$ and $G_{2m+1} := L_{2m+1}(x_1^2, \ldots, x_{2m+1}^2) \in \Delta_{2m+1,8}$.
- **Theorem:** For $n \geq 4$, $T_n(x_1, \ldots, x_n) := M_2\Big(M_2^3 - 5M_2M_4 + 6M_6\Big) \in \Delta_{n,8}$.
- **Theorem:** For $n \geq 4$, $P_n(x_1, \ldots, x_n) := (nM_4 - M_2^2)(M_2^3 - 5M_2M_4 + 6M_6) \in \Delta_{n,10}$.
- **Theorem:** For $n \geq 3$, $R_n(x_1, \ldots, x_n) := (M_2^3 - 3M_2M_4 + 2M_6)(M_2^3 - 5M_2M_4 + 6M_6) \in \Delta_{n,12}$.
- **Theorem [Analogue of Hilbert's Theorem for even symmetric forms]:**
  $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2, d = 1$ or $(n, 2d) = (n, 4)_{n \geq 4}, (3, 8)$.

Summarising

▶ $(n, 2d)$ for which $S\mathcal{P}_{n,2d} \subseteq S\Sigma_{n,2d}$:

$2d \setminus n =$ ‖	2	3	4	5	6	…
2	✓	✓	✓	✓	✓	…
4	✓	✓	×	×	×	…
6	✓	×	×	×	×	…
8	✓	×	×	×	×	…
⋮	⋮	⋮	⋮	⋮	⋮	⋱

▶ $(n, 2d)$ for which $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$:

$2d \setminus n$	2	3	4	5	6	…
2	✓	✓	✓	✓	✓	…
4	✓	✓	✓	✓	✓	…
6	✓	×	×	×	×	…
8	✓	✓	×	×	×	…
10	✓	×	×	×	×	…
12	✓	×	×	×	×	…
⋮	⋮	⋮	⋮	⋮	⋮	⋱

References

[1] C. Goel, S. Kuhlmann, B. Reznick, On the Choi-Lam Analogue of Hilbert's 1888 theorem for symmetric forms, arXiv:1505.08145, 2015.
[2] C. Goel, S. Kuhlmann, B. Reznick, The Analogue of Hilbert's 1888 theorem for even symmetric forms, arXiv:1509.07482, 2015.
[3] D. Hilbert, Über die Darstellung definiter Formen als Summe von Formenquadraten, Math. Ann., 32 (1888), 342-350.