Young Women in Algebraic Geometry

On the Selmer group associated to a modular form

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Birch-Swinnerton-Dyer conjecture

Mordell-Weil: Let E be an elliptic curve over a number field K. Then

$$E(K) \simeq \mathbb{Z}^r + E(K)_{tor}$$

where

- $r =$ the algebraic rank of E
- $E(K)_{tors} =$ the finite torsion subgroup of $E(K)$.

Questions arising:

- When is $E(K)$ finite?
- How do we compute r?
- Could we produce a set of generators for $E(K)/E(K)_{tors}$?

Modularity (Wiles-BCDT): For $K = \mathbb{Q}$, $L(E/K, s)$ has analytic continuation to all of $\mathbb{C}$ and satisfies

$$L^*(E/K, 2-s) = w(E/K)L^*(E/K, s).$$

The analytic rank of E/K is defined as

$$r_{an} = ord_{s=1}L(E/K, s).$$

Birch and Swinnerton-Dyer conjecture:

$$r = r_{an}.$$

Exact sequence of G_K modules: Consider the short exact sequence

$$0 \longrightarrow E_p \longrightarrow E \xrightarrow{p} E \longrightarrow 0.$$

Local cohomology: For a place v of $K = Q(\sqrt{-D})$, $K \hookrightarrow K_v$ induces $Gal(\overline{K_v}/K_v) \longrightarrow Gal(\overline{K}/K)$.

$$\begin{tikzcd}
0 \arrow[r] & E(K)/pE(K) \arrow[r, "\delta"] \arrow[d] & H^1(K, E_p) \arrow[r] \arrow[d] \arrow[dr, "\rho"] & H^1(K, E)_p \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \prod_v E(K_v)/pE(K_v) \arrow[r, "\delta"] & \prod_v H^1(K_v, E_p) \arrow[r] & \prod_v H^1(K_v, E)_p \arrow[r] & 0
\end{tikzcd}$$

Definition

- $Sel_p(E/K) = ker(\rho)$
- $Ш(E/K)_p = ker(h)$

Importance of the Selmer group

Information on the algebraic rank r:

$$0 \longrightarrow E(K)/pE(K) \xrightarrow{\delta} Sel_p(E/K) \longrightarrow Ш(E/K)_p \longrightarrow 0$$

relates r to the size of $Sel_p(E/K)$.

Shafarevich-Tate conjecture: The Shafarevich group $Ш(E/K)$ is finite. In particular,

$$Sel_p(E/K) = \delta(E(K)/pE(K))$$

for all but finitely many p.

Gross-Zagier:

$$L'(E/K, 1) = *\ height(y_K),$$

where $y_K \in E(K) \rightsquigarrow$ Heegner point of conductor 1. Hence,

$$r_{an} = 1 \implies r \geq 1.$$

Kolyvagin: If y_K is of infinite order in $E(K)$ then $Sel_p(E/K)$ has rank 1 and so does $E(K)$. Hence,

$$r_{an} = 1 \implies r = 1.$$

Combined with results of Kumar and Ram Murty, this can be used to show

$$r_{an} = 0 \implies r = 0.$$

Generalization:

$$E \rightsquigarrow f, \quad T_p(E) \rightsquigarrow T_p(f)$$

- f normalized newform of level $N \geq 5$ and even weight $2r$.
- $T_p(f) = p$-adic Galois representation associated to f, higher-weight analogue of the Tate module $T_p(E)$
- $K = \mathbb{Q}(\sqrt{-D})$ imaginary quadratic field (with odd discriminant) satisfying the Heegner hypothesis with $|\mathcal{O}_K^\times| = 2$.

Beilinson-Bloch conjecture

Definition: The Selmer group

$$Sel_p \subseteq H^1(K, T_p(f))$$

consists of the cohomology classes whose localizations at a prime v of K lie in

$$\begin{cases} H^1(K_v^{ur}/K_v, T_p(f)) \text{ for } v \text{ not dividing } Np \\ H^1_f(K_v, T_p(f)) \text{ for } v \text{ dividing } p \end{cases}$$

where

- K_v is the completion of K at v
- and $H^1_f(K_v, T_p(f))$ is the *finite part* of $H^1(K_v, T_p(f))$.

p-adic Abel-Jacobi map:

- $W_r =$ Kuga-Sato variety of dimension $2r-2$.
- $T_p(f)$ is realized in the middle cohomology $H^{2r-1}_{et}(W_r \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$ of W_r.
- $E(K) \rightsquigarrow CH^r(W_r/K)_0 = r$-th Chow group of W_r over K.
- transition map $\delta \rightsquigarrow p$-adic Abel Jacobi map ϕ

$$CH^r(W_r/K)_0 \to H^1(K, H^{2r-1}_{et}(W_r \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p(r))) \to H^1(K, T_p(f))$$

Beilinson-Bloch conjectures:

$$dim_{\mathbb{Q}_p}(Im(\phi) \otimes \mathbb{Q}_p) = ord_{s=r}L(f, s)$$

$$Ker(\Phi) = 0 \ \ \& \ \ Im(\Phi) \otimes \mathbb{Q}_p = Sel_p.$$

Heegner point of conductor 1:

- $\rightsquigarrow$ there is an ideal $\mathcal{N}$ of $\mathcal{O}_K$ with $\mathcal{O}_K/\mathcal{N} \simeq \mathbb{Z}/N\mathbb{Z}$
- $\rightsquigarrow$ point x_1 of $X_0(N) =$ modular curve with $\Gamma_0(N)$ level structure
- $\rightsquigarrow$ x_1 is defined over the Hilbert class field K_1 of K
- $\rightsquigarrow$ $y_1 = \phi(x_1)$, for $\phi : X_0(N) \to E$ modular parametrization

Kolyvagin: If $Tr_{K_1/K}(y_1)$ is of infinite order in $E(K)$ then $Sel_p(E/K)$ has rank 1 and so does $E(K)$.

Modular forms of higher even weight. Consider the elliptic curve E corresponding to x_1

- $\rightsquigarrow$ **Heegner cycle of conductor 1:** $\Delta_1 = e_r\ graph(\sqrt{-D})^{r-1}$
- $\rightsquigarrow$ Δ_1 belongs to $CH^r(W_r/K_1)_0$.

Nekovar: Assuming $\Phi(\Delta_1)$ is not torsion,

$$rank(Im(\Phi)) = 1.$$

Results

Modular forms twisted by a ring class character

- $H =$ ring class field of conductor c for some c, and $e =$ exponent of $G = Gal(H/K)$
- $F = \mathbb{Q}(a_1, a_2, \cdots, \mu_e)$ where the a_i's are the coefficients of f
- $\hat{G} = \text{Hom}(G, \mu_e)$ the group of characters of G
- $e_\chi = \frac{1}{|G|}\sum_{g \in G} \chi^{-1}(g)g$ the projector onto the χ-eigenspace

Theorem 1 Let $\chi \in \hat{G}$ be such that $e_{\overline{\chi}}\Phi(\Delta_c)$ is not divisible by p. Then the χ-eigenspace of the Selmer group Sel_p^χ is of rank 1 over $\mathcal{O}_{F,\wp}/p$.

Modular forms twisted by an algebraic Hecke character

- $\psi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ unramified Hecke character of K of infinity type $(2r-2, 0)$
- A elliptic curve defined over the Hilbert class field K_1 of K with CM by $\mathcal{O}_K$
- $F = \mathbb{Q}(a_1, a_2, \cdots, b_1, b_2, \cdots)$, where the a_i's and b_i's are the coefficients of f and θ_ψ.

Galois representation associated to f and ψ

$$V = V_f \otimes_{\mathcal{O}_F \otimes \mathbb{Z}_p} V_\psi(2r-1).$$

Generalized Heegner cycle of conductor 1: Consider (φ_1, A_1) where

- A_1 is an elliptic curve defined over K_1 with level N structure and CM by $\mathcal{O}_K$ and
- $\varphi_1 : A \longrightarrow A_1$ is an isogeny over $\overline{K}$.

$\rightsquigarrow$ GHC $e_r\Upsilon_{\varphi_1} = Graph(\varphi_1)^{2r-2} \subset (A \times A_1)^{2r-2} \simeq (A_1)^{2r-2} \times A^{2r-2}$.

p-adic Abel-Jacobi map

$$\phi : CH^{2r-1}(X/K)_0 \longrightarrow H^1(K, V)$$

where

- $X = W_{2r-2} \times A^{2r-2}$ and $CH^{2r-1}(X/K)_0 = 2r-1$-th Chow group of X over K.

Theorem 2 Under certain technical assumptions, if

$$\Phi(\Delta_{\varphi_1}) \neq 0,$$

then the Selmer group Sel_p has rank 1 over $\mathcal{O}_{F,\wp_1}/p$, the localization of $\mathcal{O}_F$ at $\wp_1$ mod p.

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