Young Women in Algebraic Geometry

Around Hodge, Tate and Mumford-Tate conjectures on abelian varieties

Victoria Cantoral-Farfán
Advisor: Prof. Marc Hindry

Preliminaries

Definition. An abelian variety is a projective algebraic variety which is also an algebraic group with a group law which is commutative.

Example. An elliptic curve is an abelian variety of dimension 1 with a group law which is explained here:

Let define \( V = H_1(X, \mathbb{Q}) \) and \( V_l = H_l(X, \mathbb{Z}/l\mathbb{Z}) \) where the \( l \)-adic Tate module is defined as follows \( H_l(X, \mathbb{Z}/l\mathbb{Z}) = \lim_{\rightarrow} H^i(X, \mathbb{Z}/l^i\mathbb{Z}) \) and where \( X^{[l]} \) is the kernel of the multiplication by \( l \). By the comparison theorem we have \( V_l \cong V \otimes \mathbb{Q} \).

We recall that the cohomology group \( H^p(X, \mathbb{Q}) \) of \( X \) is naturally endowed with a Hodge structure of weight \( 2p \).

\[ H^p(X, \mathbb{Q}) \otimes \mathbb{C} = H^p(X, \mathbb{C}) \cong \bigoplus_{i+j=p} H^{i,j}(X). \]

Because \( X \) is an a.u.v. we have that \( H^p(X, \mathbb{C}) = \bigoplus_{i+j=p} H^{i,j}(X, \mathbb{C}) \). The Tate conjecture (T) follows from by Hodge (H) and Mumford-Tate (MT).

If \( \rho \) is the \( G \)-representation:

\[ \rho : G \rightarrow \text{Aut}_d(\mathbb{Q}) \]

We define the algebraic group \( G = G(\mathbb{Q}) \) as the group of homotheties and \( S \) be the restriction of scalars of \( G_{\mathbb{Q}} \) from \( G \) to \( \mathbb{Q} \). Thanks to Hodge decomposition we have \( V_l = H_l(X, \mathbb{C}) = V^{1,0} \oplus V^{0,1} \) and we can consider therefore the morphism:

\[ A : \mathbb{C} \rightarrow \mathbb{C} \] \( \forall x \in A \mathbb{C} \) is the \( \mathbb{Q} \)-linear combinations of algebraic classes.

Definition. The group of Hodge classes of codimension \( p \) is defined by:

\[ B^p(X, \mathbb{Q}) = \bigoplus_{i+j=p} H^{i,j}(X, \mathbb{Q}) \]

We define \( G_{\mathbb{Q}}(\mathbb{Q}) \) as the group of homotheties and \( S \) be the restriction of scalars of \( G_{\mathbb{Q}} \) from \( G \) to \( \mathbb{Q} \).

Motivations

Let \( X_0 \) be an abelian variety over a number field \( K \) of dimension \( g \) and let us fix an embedding \( \sigma : K \hookrightarrow \mathbb{C} \) such that \( X_0 = X_0,_{K,\sigma} \). Let \( p \in \mathbb{Z}, \) \( p \neq 0 \) and \( l \) be a prime number.

\[ H^p(X_0, \mathbb{Q}_l) \otimes \mathbb{Q}_l = H^p(X_0, \mathbb{Q}_l) \]

One can remark that the first equality of Hodge classes is actually a theorem whereas the second equality is, as a matter of fact, the definition of Tate classes.

Definition. The group of algebraic classes of codimension \( p \) is defined as follows:

\[ C^p(X, \mathbb{Q}) = \bigoplus_{i+j=p} H^{i,j}(X, \mathbb{Q}) \]

Example. Divisors are algebraic classes of codimension 1. Moreover due to Lefschetz and Faltings theorems Hodge and Tate conjecture holds for divisors.

Some links and results for the three conjectures

Pink and Shafarevic proved the following equivalence, the first implication is easier to explain, it follows directly by definitions:

\[ (H) \Rightarrow (MT) \Rightarrow (T) \]

For instance, in these two examples Tate conjecture (T) follows from Hodge (H) and Mumford-Tate (MT) conjectures.

Example.

- \( X \) an a.v. of prime dimension
  - Faltings 83
  - Ch. 92

- \( X \) an a.v. of type I or II on the Albert’s classification
  - Marty & Hama 84
  - Banaszak, Goyt & Kreuss, 93

Some others results

\[ X \] an elliptic curve

\[ \text{Serre 72} \]

\[ X \] an a.v. such that \( \text{dim} X \leq 3 \)

\[ \text{Lefschetz 24} \]

\[ \text{Faltings 83} \]

Introduction

The aim of this poster is to present Hodge, Tate and Mumford-Tate conjectures focusing on abelian varieties and to describe the links between them. Furthermore we are going to illustrate some known results.

Hodge conjecture was introduced in 1950, its main goal is to establish a bridge between Algebraic Geometry and Differential Geometry. Hodge was inspired by Lefschetz’s theorems. This conjecture is stated for abelian varieties which are projective and smooth. Nevertheless it is easier to describe and even to establish some results about this conjecture for complex abelian varieties. One of the reasons is the simple decompositions of the singular cohomology of an abelian variety and its Hodge decomposition. Moreover we can consider if an equivalent of this conjecture exists in Arithmetic Geometry. The answer is the Tate conjecture, stated in 1963.

Serre’s conjecture was introduced in 1972, its main goal is to establish a bridge between Algebraic Geometry and Number Theory. Serre was inspired by the work of Faltings and theorems of Lefschetz. This conjecture is stated for abelian varieties which are projective and smooth. Nevertheless it is easier to describe and even to establish some results about this conjecture for complex abelian varieties. One of the reasons is the simple decompositions of the singular cohomology of an abelian variety and its Hodge decomposition. Moreover we can consider if an equivalent of this conjecture exists in Arithmetic Geometry. The answer is the Tate conjecture, stated in 1963.

Some results on (H) have been proved concerning the product of elliptic curves not isogenous and abelian fourfold (except some special cases).


Some results on (MT) with conditions on the type of the endomorphism ring \( \text{End}(X) \) and the dimension of \( X \).

Example. Serre ‘76, Chi ‘92, Pink ‘98 and Banaszak, Goyt & Kreuss, 93.

Hodge conjecture for abelian varieties.


Conjecture. Each Hodge classes are \( \mathbb{Q} \)-linear combinations of algebraic classes.

- Hodge classes = \( H^p(X_0, \mathbb{Q}) \)

- Tate classes = \( H^p(X_0, \mathbb{Q}_l) \)

Example. Divisors are algebraic classes of codimension 1. Moreover due to Lefschetz and Faltings theorems Hodge and Tate conjecture holds for divisors.

<table>
<thead>
<tr>
<th>Conjecture</th>
<th>Hodge conjecture (H)</th>
<th>Tate conjecture (T)</th>
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<tbody>
<tr>
<td>for a.v. over ( \mathbb{C} )</td>
<td>1950</td>
<td>1963</td>
</tr>
<tr>
<td>for a.v. over ( k )</td>
<td>1950</td>
<td>1963</td>
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Conjecture. Each Tate classes are \( \mathbb{Q} \)-linear combinations of algebraic classes.

- Tate conjecture (MT) 1966

Example. Serre ‘76 and Hindry & Raynaud ‘15.

Finally, some results of (T) can be deduced thanks to this previous equivalence.

References


2015 Institut de Mathématiques de Jussieu - Paris Rive Gauche (IMJ-PRG), Paris, France victoria.cantoral-farf@imj-prg.fr