Young Women in
Algebraic Geometry

Frobenius structures on spaces of stability conditions

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Space of stability conditions

- Given $\mathcal{C} = D^b(\mathcal{B})$ a bounded derived category of an abelian category $\mathcal{B}$, $\mathcal{K}(\mathcal{C})$ its numerical Grothendieck group, a *stability condition* ([B]) is a pair

$$(\mathcal{A}, Z) \quad \text{for} \quad \begin{array}{l} \mathcal{A} \text{ a } \textit{heart} \text{ (an abelian category contained in } \mathcal{C}), \\ Z \in \operatorname{Hom}(\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}(\mathcal{A}), \mathbb{C}) \text{ a } \textit{central charge}. \end{array}$$

  $Stab(\mathcal{C})$ is the *space of stability conditions* of $\mathcal{C}$. It is a complex manifold with local coordinate Z.

- Fixed any heart $\mathcal{A}$, we say $U(\mathcal{A}) \subset Stab(C)$ the open subset of stability conditions supported on $\mathcal{A}$.

- On an abelian category $\mathcal{A}$, for any $\alpha \in \mathcal{K}(\mathcal{A})$ there is a well-defined *Joyce function*

$$f^\alpha : Stab(\mathcal{A}) \to g_{\mathcal{K}(\mathcal{A})}$$

  encoding *Donaldson-Thomas* invariants counting $Z-$semistable elements. It is valued in an infinite dimension Lie-algebra $g_{\mathcal{K}(\mathcal{A})}$, [J].

- We can extend these functions to the case of $\mathcal{C} = D^b(\mathcal{A})$ as formal power series in an extra variable t, whose degree keeps track of the "length" of an object.

Motivations

DT invariants are somehow related to GW invariants. Let (M, ω) be a compact symplectic manifold, and $\Phi_A(\alpha, \beta, \gamma)$s denote the GW invariants of M. A GW potential is a holomorphic generating function $\mathcal{S} : H^{ev}(M, \mathbb{C}) \to \mathbb{C}$, given by a formal power series with coefficients the $\Phi_A(\alpha, \beta, \gamma)$s. Identities on the $\Phi_A(\alpha, \beta, \gamma)$ imply that $\mathcal{S}$ satisfies a p. d. e., the WDVV equation, that can be interpreted as the flatness of a 1-parameter family of connections, which make $H^{ev}(M, \mathbb{C})$ into a Frobenius manifold.

The setup we are in is formally quite similar and, in this spirit, it seems natural to extend Joyce's functions to the triangulated case and then try to endow its space of stability conditions with a Frobenius structure.

Moreover, it has been shown (Bridgeland) that, for $\mathcal{A} = \mod(\bullet \to \bullet)$, the space $Stab(D^b(\mathcal{A}))$ is "isomorphic" to the space of deformations of the A_2 singularity, which is known to be a Frobenius manifold.

Result & Open questions

Under suitable hypoteses on the Euler form on $\mathcal{K}(\mathcal{C})$ & up to cubic terms in t, any open subset $U(\mathcal{A}) \subset Stab(\mathcal{C})$ is a semisimple Frobenius manifold.

Open problems are:
- lift to all orders in t
- extend to a global structure on $Stab(\mathcal{C})$
- in high dimension the "suitable hypoteses" could be quite restrictive, what is their geometrical meaning?

Frobenius structures

A *Frobenius structure with Euler field* consists of

$$(M, g, \circ, E, \nabla)$$

where M is a manifold whose tangent bundle $\mathcal{T}_M$ is equipped with

g : a (pseudo)metric constant on the subsheaf of commuting vector fields,

$\circ$: an associative multiplication,

E : an Euler field, that is a vector field such that

$$\mathrm{Lie}_E(g) = Dg, \quad \mathrm{Lie}_E(\circ) = d\circ, \quad D, d \in \mathbb{C}$$

and a flat connection ∇ on $\pi^*(\mathcal{T}_M) \to M \times \mathbb{P}^1_z$ defined by

$$\begin{cases} \nabla_X(\cdot) = \nabla^g_X(\cdot) - \frac{1}{z} X \circ (\cdot), \\ \nabla_{z\frac{\partial}{\partial z}}(\cdot) = \frac{1}{z} E \circ (\cdot) - [E, \cdot] \end{cases}$$

It is the so-called *extended structure connection*.

Example: dimension 2

Let $U \subset Stab(D^b(\mathbb{C}(\bullet \to \bullet)))$ open;
$\mathcal{K}(\mathcal{C}) \simeq \mathbb{Z}\xi \oplus \mathbb{Z}\gamma$, where ξ and γ are classes of simple objects in a heart;
$\langle \cdot, \cdot \rangle$ the Euler form on $\mathcal{K}(\mathcal{C})$;
$\alpha_i \in \mathcal{K}(\mathcal{C})$, $i = 1, 2$, such that $\alpha_1 - \alpha_2 = \xi$.

With the strategy below we obtain the following Frobenius structure on U:

- local coordinates $u_i = Z(\alpha_i)$, $i = 1, 2$,
- diagonal metric

$$g \propto \sum_i (u_1 - u_2)^{2f^\xi \langle \alpha_1, \alpha_2 \rangle} \, \mathrm{d}\, u_i^{\otimes 2},$$

- (semisimple) multiplication $\frac{\partial}{\partial u_i} \circ \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}$
- Euler field

$$E = u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2},$$

  with constants $D = 2 \pm 2\,\mathrm{i}\, f^\xi \langle \alpha_1, \alpha_2 \rangle$ and $d = 1$,

- extended structure connection

$$\nabla_X(\cdot) = \nabla^g_X(\cdot) - \frac{1}{z} X \circ (\cdot)$$

$$\nabla_{z\partial_z}(\cdot) = \frac{1}{z} E \circ (\cdot) - [E, \cdot]$$

Idea of the construction

The idea of the construction passes through a Frobenius type structure, a concept due to Hertling [H], and goes as follow on $U = U(\mathcal{A}) \subset Stab(\mathcal{C})$.

Let $K = g_{\mathcal{K}(\mathcal{A})} \times U \to U$ be the trivial bundle with fiber the infinite dimensional Lie algebra $g_{\mathcal{K}(\mathcal{A})} = \{e_\alpha \mid \alpha \in \mathcal{K}(\mathcal{C})\}$ over $\mathbb{C}[[t]]$, with bracket $[e_\alpha, e_\beta] = \langle \alpha, \beta \rangle e_{\alpha+\beta}$, where $\langle \cdot, \cdot \rangle$ is the Euler form on $\mathcal{K}(\mathcal{A})$.

Extend the central charge to

$$Z \in \mathrm{End}(g_{\mathcal{K}(\mathcal{A})}), \quad Z(e_\alpha) = Z(\alpha) e_\alpha.$$

Recall Joyce functions f^α and define

a Higgs field	$C = -\mathrm{d}\, Z,$
endomorphisms	$\mathcal{U} = Z,$ and $\mathcal{V} = -\operatorname{ad} \sum_\alpha f^\alpha e_\alpha,$
a flat connection	$\nabla^r = \mathrm{d} - \operatorname{ad} \sum_\alpha f^\alpha \frac{\mathrm{d}\, Z(\alpha)}{Z(\alpha)}.$

These data define a *Frobenius type structure* on K. That is, on the holomorphic vector bundle $H = \pi^* K \to U \times \mathbb{P}^1_z$, the connection

$$\nabla = \nabla^r + \frac{1}{z} C + \left(\frac{1}{z} \mathcal{U} - \mathcal{V} + \frac{\omega}{2} \mathrm{Id} \right) \frac{\mathrm{d}\, z}{z}, \quad \omega \in \mathbb{Z}$$

is flat.
Moreover there is a (essentialy unique) symmetric non-degenerate ∇^r-flat pairing g on K with

$$g(C_X a, b) = g(a, C_X b), \quad g(\mathcal{U} a, b) = g(a, \mathcal{U} b), \quad g(\mathcal{V} a, b) = -g(a, \mathcal{V} b),$$

for $X \in \mathcal{T}_U$, $a, b \in O(K)$.

Now push forward the Frobenius type structure on a finite rank sub-bundle $K(\zeta) \simeq T_U$, the isomorphism being given by $\nu = -C_\bullet \zeta$, for a section $\zeta : U \to K$.
We have checked that, under suitable hypoteses on the Euler form and up to cubic terms in t, the Frobenius type structure on $K(\zeta)$ can be pulled back to a genuine Frobenius structure on the tangent bundle T_U.

References

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