Young Women in Algebraic Geometry

On a Conjecture on Linear Systems

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IN PURSUIT OF KNOWLEDGE

Aspects of a curve

Let C be a smooth projective curve

Aspects of a curve

Clifford Index (Geometric Aspect) $\longleftrightarrow$ **Green's Conjecture** $\longleftrightarrow$ Koszul Cohomology (Algebraic Aspect)

Definition. H.H.Martens has defined the Clifford index γ_C of C as

$$\gamma_C = min\{deg(D) - 2dim|D|\}$$

where D runs over all divisors of C such that $h^0(C, \mathcal{O}_C(D)) \geq 2$ and $h^1(C, \mathcal{O}_C(D)) \geq 2$.

Koszul Cohomology

$$\begin{cases} k & \text{a field,} \\ V & \text{a finite-dimensional vector space over } k, \\ S(V) & \text{the symmetric algebra on V,} \\ B = \bigoplus_{q\in\mathbb{Z}} B_q & \text{a graded S(V)- module.} \end{cases}$$

There is a Koszul complex

$$\cdots \to \wedge^{p+1}V \otimes B_{q-1} \xrightarrow{d_{p+1,q-1}} \wedge^{p}V \otimes B_{q} \xrightarrow{d_{p,q}} \wedge^{p-1}V \otimes B_{q+1} \xrightarrow{d_{p-1,q+1}} \wedge^{p-2}V \otimes B_{q+2} \to \cdots$$

Definition. The Koszul cohomology groups of B are

$$\mathcal{K}_{p,q}(B,V) = \frac{kerd_{p,q}}{Imd_{p+1,q-1}}$$

The above construction will be of interest to us primarily in the case

$$\begin{cases} X & \text{a projective variety over } \mathbb{C}, \\ L \to X & \text{an analytic line bundle,} \\ V = H^0(X,L) & \text{a finite dimensional vector space} \end{cases}$$

$$B = \bigoplus_{q\in\mathbb{Z}} H^0(X, L^q)$$

Denote $\mathcal{K}_{p,q}(X,L) = \mathcal{K}_{p,q}(B,V)$

Green's Conjecture

Proposition [Gre84] Let L be a line bundle such that $h^0(C,L) \geq 2$, $h^1(C,L) \geq 2$. Then

$$\mathcal{K}_{p,1}(C, K_C) \neq 0 \text{ for all } p \leq g - \gamma_L - 2.$$

The strongest non vanishing result of this type is obtained by taking the minimal value of γ_L. This gives the implication

$$p \leq g - \gamma_C - 2 \implies \mathcal{K}_{p,1}(C, K_C) \neq 0. \quad (1)$$

Green [Gre84] conjectures that the converse of (1) holds.

Conjecture (Green). Let C be a smooth projective curve. Then

$$\mathcal{K}_{p,1}(C, K_C) = 0 \Leftrightarrow p \geq g - \gamma_C - 1.$$

Equivalent Formulation of Green's Conjecture [PR88]. Let E_K be the pullback by the canonical map $\Phi : C \to \mathbb{P}^{g-1}$ of universal quotient bundle on $\mathbb{P}^{g-1}$.

The map $\wedge^i\Gamma(C, E_K) \to \Gamma(C, \wedge^i E_K)$ *is surjective* $\forall\ i \leq \gamma_C$.

Result [PR88]. All sections of $\wedge^i E_K$ which are locally decomposable are in the image of $\wedge^i\Gamma(E_K)\ \forall\ i \leq \gamma_C$.

Let $\sum_{i,K}$ be the cone of locally decomposable sections of $\wedge^i E_K$.

Conjecture [HPR92]. $\sum_{i,K}$ spans $\Gamma(\wedge^i E_K)\ \forall\ i$ and for all curves.

Result [HPR92]. Proved for curves with Clifford index 1 (trigonal curves and plane quintics)

A Remark on a conjecture of Paranjape and Ramanan [ES12].

Conjecture [ES12]. $\Gamma(\wedge^i N)$ is spanned by locally decomposable sections holds for every (stable) globally generated vector bundle N on every curve C.

Conjecture for General Linear Systems

Conjecture on General Linear Systems [Ana] Let C be a smooth curve of genus $g \geq 1$ and let L be a globally generated line bundle on C of degree $d \geq g+3$. The evaluation map gives rise to an exact sequence

$$0 \to E^* \to \Gamma(L)_C \to L \to 0 \quad (2)$$

where E^* is locally free of rank $h^0(L) - 1$. Let $\sum_i$ be the cone of locally decomposable sections of $\wedge^i E$. We state

Conjecture. $\sum_i$ spans $\Gamma(\wedge^i E)\ \forall\ i$ and for all curves.
Result. Prove for hyperelliptic curves.

Highlights of proof

- Geometry of hyperelliptic curve will provide a 2:1 map $\pi : C \to \mathbb{P}^1$.
- Set $T := \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ and consider $W := \pi_* L$.
- In order to prove the conjecture, we want to relate the sections of $\wedge^i E$ to the sections of a suitable vector bundle on $\mathbb{P}^1$.
- **Lemma.** Let F be a vector bundle on $\mathbb{P}^1$ that is globally generated. Then the evaluation sequence is
  $$0 \to \Gamma(F(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \Gamma(F)_{\mathbb{P}^1} \to F \to 0$$
  Apply this lemma to W and by series of commutative diagrams found a vector bundle L'_i on $\mathbb{P}^1$ such that $\Gamma(\wedge^i E) \cong \Gamma(L'_i)$.
- The bundle E and its exterior powers $\wedge^i E$ are the main object. $\sum_i \subset \Gamma(\wedge^i E)$, the set of locally decomposable sections is a scheme defined by requiring that at each point the section satisfy the equation of Grassmannian cone in its Plucker embedding. These are obtained in the following way: Let F be a subbundle of E of rank i and $s \in \Gamma(detF)$, then s can be treated as a section of $\wedge^i E$ as well, where it is locally decomposable.
- Construction of a subbundle F of E for a fixed point $a \in \mathbb{P}^1$
- Globalize this construction.
- **Lemma [HPR92]** Let $\mathcal{O}_{\mathbb{P}^1}(-n) \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(n))^*_{\mathbb{P}^1}$ be a non-zero $Sl_2(\mathbb{C})$- equivariant morphism. Then this morphism defines an embedding of $\mathbb{P}^1$ into $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^1}(n)^*))$ as a rational normal curve of degree n.
- Making use of above lemma, prove the result for $i = 2$.
- Prove it for all i.

Progress on Green's conjecture till date

- In 1984, Mark L. Green introduced Koszul cohomology and stated the conjecture in the appendix [Gre84].
- In 1988, K.Paranjape and S. Ramanan gave an equivalent definition of Green's conjecture [PR88].
- In 1992, K. Hulek, K. Paranjape and S. Ramanan proved stronger version of Green's conjecture for curves with Clifford Index 1 [HPR92]
- In 1998, A. Hirschowitz, S. Ramanan gave new evidence for Green's conjecture on syzygies of canonical curves [HR98].
- In 2002, Claire Voisin proved Green's conjecture for curves of even genus lying on a K3 surface [Voi02].
- In 2005, Claire Voisin proved it for curves of odd genus lying on a K3 surface [Voi05].
- In 2011, Marian Aprodu and Gavril Farkas coupled with results of Voisin and Hirschowitz - Ramanan provides a complete solution to Green's conjecture for smooth curves on arbitrary K3 surfaces [AF11].
- In 2012, Eusen and Schreyer gave a remark on Green's Conjecture [ES12].

References

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