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Enveloping Algebras and Geometric Representation Theory

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Abstract. The workshop brought together experts investigating algebraic Lie theory from the geometric and categorical viewpoints.

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Introduction by the Organisers

This workshop continues a series of conferences on enveloping algebras, as the first part of the title suggests, but the organisers and also the focus of these meetings changed over the years to reflect the newest developments in the field of algebraic Lie theory. This year the main focus was on geometric and categorical methods.

The meeting was attended by over 50 participants from all over the world, a lot of them young researchers. We usually had three talks in the morning and two in the afternoon, and with one exception all talks were given on the blackboard. Wednesday afternoon was reserved for a walk to Sankt Roman, and Thursday we had three shorter talks by younger mathematicians.

A particular highlight seemed to us the solution by Vera Serganova to a question of Pierre Deligne, asking for certain universal abelian tensor categories, which was solved using supergroups. Very remarkable was also Shrawan Kumar's solution, joint with Belkale, of the multiplicative eigenvalue problem for general compact groups, which in the case of the unitary groups reduces to the question of giving the possible eigenvalues for the product of two unitary matrices in terms of the eigenvalues of the factors. Many lecturers presented work that involved categorification or cluster combinatorics, with examples arising across a broad spectrum of Lie theoretic representation theory. A big theme was also sheaves and in particular intersection cohomology sheaves and theirs variants with modular coefficients on the varieties appearing in geometric representation theory.

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Workshop: Enveloping Algebras and Geometric Representation Theory

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Abstracts

Geometric Satake, Springer correspondence and small representations

Anthony Henderson
(joint work with Pramod N. Achar, Simon Riche, Daniel Juteau)

Let $G \supset B \supset T$ denote a connected reductive group over $\mathbb{C}$, with a chosen Borel subgroup and maximal torus. Let $\check{G} \supset \check{B} \supset \check{T}$ denote the dual connected reductive group over an algebraically closed field k of characteristic $p \geq 0$, with the induced choice of Borel subgroup and maximal torus. Let W denote the Weyl group of (G, T), or equivalently that of $(\check{G}, \check{T})$ (the two are canonically identified).

Write $\check{Q}^+$ for the set of dominant weights for $\check{G}$ that belong to the root lattice of $\check{G}$ (= dominant coweights for G that belong to the coroot lattice of G). For every $\lambda \in \check{Q}^+$, let $L(\lambda)$ be the irreducible rational representation of $\check{G}$ with highest weight λ. It is a classical problem to describe the zero weight space $L(\lambda)_0 = L(\lambda)^{\check{T}}$ as a representation of W over k.

As an example, consider the case where $\check{G} = GL_n(k)$ and $W = S_n$. Then

$$\check{Q}^+ = \{(a_1, \cdots, a_n) \in \mathbb{Z}^n \mid a_1 \geq \cdots \geq a_n,\ a_1 + \cdots + a_n = 0\}.$$

Suppose that $\lambda = (a_i) \in \check{Q}^+$ has the special property that $a_n \geq -1$. In this case it follows from the work of Schur that

$$L(\lambda)_0 \cong \mathrm{Hom}_{GL_n(k)}((k^n)^{\otimes n} \otimes \det{}^{-1}, L(\lambda)),$$

where S_n acts on the representation $(k^n)^{\otimes n} \otimes \det^{-1}$ by permutation of the tensor factors composed with the sign character. When $p = 0$, $L(\lambda)_0$ is an irreducible representation of S_n, and every irreducible representation of S_n arises in this way for a unique such λ. When $p > 0$, the situation is the same except that $L(\lambda)_0 = 0$ for non-p-restricted λ.

Following [4] we say that λ, or the corresponding representation $L(\lambda)$, is *small* if it is not true that $\lambda \geq 2\alpha$ for any dominant root α of $\check{G}$, where $\geq$ denotes the usual partial order on $\check{Q}^+$. The small elements of $\check{Q}^+$ form a finite lower order ideal. When $\check{G} = GL_n(k)$, an element $\lambda = (a_i) \in \check{Q}^+$ is small if and only if either $a_n \geq -1$ or $a_1 \leq 1$. In [9, 10], Reeder determined $L(\lambda)_0$ as a representation of W for all small λ in the $p = 0$ case, by case-by-case computations using Broer's covariant restriction theorem for small representations [4].

The aim of our work was to give a geometric approach to these problems, capable of handling all characteristics p at once. We use the *geometric Satake equivalence* of [8] to translate rational representations of $\check{G}$ into $G[[t]]$-equivariant perverse k-sheaves on the affine Grassmannian $\mathsf{Gr} = G((t))/G[[t]]$. It is well known that the $G[[t]]$-orbits in the base-point connected component Gr° are parametrized by $\check{Q}^+$. Write Gr^λ for the orbit labelled by $\lambda \in \check{Q}^+$, and $\mathsf{Gr}^{\mathrm{sm}}$ for the closed (possibly reducible) subvariety $\bigsqcup_{\lambda \text{ small}} \mathsf{Gr}^\lambda$. Under the geometric Satake equivalence, the category $\mathrm{Rep}(\check{G}, k)_{\mathrm{sm}}$ of representations of $\check{G}$ with small constituents is equivalent to the category of $G[[t]]$-equivariant perverse k-sheaves on $\mathsf{Gr}^{\mathrm{sm}}$, and the irreducible representation $L(\lambda)$ corresponds to the simple perverse sheaf $\mathrm{IC}(\overline{\mathsf{Gr}}^{\lambda}, k)$.

We also have the Springer functor from G-equivariant perverse k-sheaves on the nilpotent cone $\mathcal{N}$ to representations of W over k. This functor is represented by the Springer sheaf $\underline{\mathsf{Spr}}$, on which W acts; we use the action defined by Juteau [5] in the analogous étale setting, which by the main result of [2] differs from that used by Mautner [7] by a sign twist. For a simple G-equivariant perverse sheaf $\mathrm{IC}(\mathcal{O}, \mathcal{E})$ on $\mathcal{N}$, $\mathrm{Hom}(\underline{\mathsf{Spr}}, \mathrm{IC}(\mathcal{O}, \mathcal{E}))$ is either an irreducible representation of W or zero, and every irreducible representation of W arises in this way for a unique $(\mathcal{O}, \mathcal{E})$; this *Springer correspondence* was described explicitly by Springer, Lusztig, Shoji *et al.* in the $p = 0$ case, and by Juteau [5] and Juteau–Lecouvey–Sorlin [6] in the $p > 0$ case.

To find a geometric interpretation of the zero weight space functor, we need to relate the geometry of $G[[t]]$-orbits in Gr° to the geometry of G-orbits in $\mathcal{N}$. Let Gr_0 denote the $G[t^{-1}]$-orbit of the base-point of Gr; for any $\lambda \in \check{Q}^+$, the intersection $\mathsf{Gr}_0^{\lambda} = \mathsf{Gr}^{\lambda} \cap \mathsf{Gr}_0$ is a G-stable open subset of Gr^{λ}. Let $G[t^{-1}]_1$ denote the first congruence subgroup, i.e. $\ker(G[t^{-1}] \to G)$. We have an isomorphism $G[t^{-1}]_1 \xrightarrow{\sim} \mathsf{Gr}_0$ defined by acting on the base-point, which we use to identify Gr_0 and $G[t^{-1}]_1$; the G-action on the latter is by conjugation. We can regard the Lie algebra $\mathfrak{g}$ as $\ker(G(\mathbb{C}[t^{-1}]/(t^{-2})) \to G)$, so we have a natural G-equivariant map $\pi : \mathsf{Gr}_0 \to \mathfrak{g}$. We also have an involution $\iota : \mathsf{Gr}_0 \to \mathsf{Gr}_0$ sending $g(t^{-1})$ to $g(-t^{-1})^{-1}$, which preserves every fibre of π and maps Gr_0^{λ} to $\mathsf{Gr}_0^{-w_0\lambda}$ where w_0 is the longest element of W.

In [1] we give a geometric interpretation of the condition of smallness. Namely, for $\lambda \in \check{Q}^+$, the following are equivalent:

(1) $\pi(\mathsf{Gr}_0^{\lambda}) \subset \mathcal{N}$;
(2) G has finitely many orbits in Gr_0^{λ};
(3) λ is small.

Define $\mathcal{M} = \mathsf{Gr}^{\mathrm{sm}} \cap \mathsf{Gr}_0 = \bigsqcup_{\lambda \text{ small}} \mathsf{Gr}_0^{\lambda}$, a dense affine open subvariety of $\mathsf{Gr}^{\mathrm{sm}}$. It is shown in [1] that the restriction $\pi|_{\mathcal{M}} : \mathcal{M} \to \mathcal{N}$ is a finite morphism of affine varieties. In fact, if G has an irreducible root system, each fibre of $\pi|_{\mathcal{M}}$ is an orbit of ι and hence consists of either one or two points.

If $G = GL_n(\mathbb{C})$ for $n \geq 3$, then $\mathcal{M}$ has two irreducible components (corresponding to the two kinds of small weights), each of which is mapped isomorphically onto $\mathcal{N}$ by π; this is a new way of viewing Lusztig's embedding of the nilpotent cone into the affine Grassmannian in type A. In other types, $\pi(\mathcal{M})$ is a proper closed subvariety of $\mathcal{N}$, described case-by-case in [1]. For example, if $G = SO_8(\mathbb{C})$, then $\mathcal{M} = \overline{\mathsf{Gr}}^{(2,1,1,0)} \cap \mathsf{Gr}_0$ and $\pi(\mathcal{M}) = \overline{\mathcal{O}}_{(3,3,1,1)}$, where we use a standard combinatorial labelling for coweights and nilpotent orbits; over the orbit $\mathcal{O}_{(3,3,1,1)}$, the map $\pi|_{\mathcal{M}}$ is a nontrivial double covering, so we do not have an embedding of $\mathcal{O}_{(3,3,1,1)}$ in Gr as we did in the type A case.

Since direct image through a finite map preserves perversity, we have a functor from $G[[t]]$-equivariant perverse sheaves on $\mathsf{Gr}^{\mathrm{sm}}$ to G-equivariant perverse sheaves on $\mathcal{N}$, sending A to $\pi_*(A|_{\mathcal{M}})$. We have now defined the following diagram of abelian categories and functors between them, where the right-hand arrow is the functor of taking the zero weight space:

$$\begin{array}{ccc} \mathrm{Perv}_{G[[t]]}(\mathsf{Gr}^{\mathrm{sm}}, k) & \xrightarrow{\sim} & \mathrm{Rep}(\check{G}, k)_{\mathrm{sm}} \\ \downarrow & & \downarrow \\ \mathrm{Perv}_G(\mathcal{N}, k) & \longrightarrow & \mathrm{Rep}(W, k) \end{array}$$

The main result of [3] is that this diagram commutes up to a canonical natural isomorphism. In particular, this gives a geometric interpretation of the zero weight space of any small representation in terms of the Springer functor: for any small $\lambda \in \check{Q}^+$, we have

$$L(\lambda)_0 \cong \mathrm{Hom}(\underline{\mathsf{Spr}}, \pi_*\mathrm{IC}(\overline{\mathsf{Gr}}_0^{\lambda}, k)).$$

In [2] we use this formula to compute $L(\lambda)_0$ explicitly in all characteristics for G of type A, C and in all characteristics except $p = 2$ for G of types B, D and exceptional types. This recovers the results of Reeder in the $p = 0$ case. The reason for excluding $p = 2$ is that, as we have seen, the map π is sometimes 2-to-1. For the same reason, in the $p \neq 2$ case $L(\lambda)_0$ is sometimes the direct sum of two irreducibles, but is otherwise either irreducible or zero.

Finally, by taking equivariant cohomology we can deduce from our theorem a new geometric proof of Broer's covariant theorem for small representations.

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Topology of two-row Springer fibers for the even orthogonal group

Arik Wilbert

In the following it is explained how to construct an explicit topological model for every two-row Springer fiber associated with the even orthogonal group (similar to the type A topological Springer fibers appearing in work of Khovanov [4] and Russell [6]).

Fix an even positive integer $n = 2m$. Let β be a nondegenerate symmetric bilinear form on $\mathbb{C}^n$ and let $O(\mathbb{C}^n, \beta)$ be the corresponding orthogonal group with Lie algebra $\mathfrak{o}(\mathbb{C}^n, \beta)$. The group $O(\mathbb{C}^n, \beta)$ acts on the affine variety of nilpotent elements $\mathcal{N} \subseteq \mathfrak{o}(\mathbb{C}^n, \beta)$ by conjugation and it is well known that the orbits under this action are in bijective correspondence with partitions of n in which even parts occur with even multiplicity, cf. e.g. [2] for details. The parts of the partition associated to the orbit of an endomorphism $x \in \mathcal{N}$ encode the sizes of the Jordan blocks of x in Jordan normal form.

Given a nilpotent endomorphism $x \in \mathcal{N}$, the associated *Springer fiber* $\mathcal{F}l^x$ is defined as the projective variety consisting of all full isotropic flags

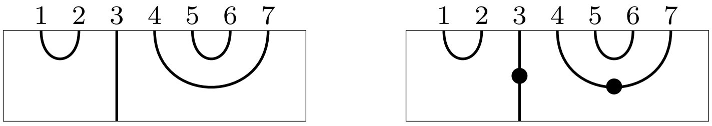
$$\{0\} = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_m$$

in $\mathbb{C}^n$ which satisfy the condition $xF_i \subseteq F_{i-1}$ for all $i \in \{1, \ldots, m\}$. These varieties naturally arise as the fibers of a resolution of singularities of $\mathcal{N}$, see e.g. [1]. In general they are not smooth and decompose into many irreducible components.

Goal. *Understand the topology of the irreducible components of $\mathcal{F}l^x$ and their intersections explicitly and provide a combinatorial description.*

In general this is a very difficult problem. Thus, we restrict ourselves to *two-row Springer fibers*, i.e. we assume that $x \in \mathcal{N}$ has Jordan type $(n-k, k)$ for some $k \in \{1, \ldots, m\}$. Note that this means that either $k = m$ or k is odd. Since $\mathcal{F}l^x$ depends (up to isomorphism) only on the $O(\mathbb{C}^n, \beta)$-conjugacy class of x it makes sense to speak about the $(n-k, k)$ Springer fiber, which we denote by $\mathcal{F}l^{n-k,k}$, without further specifying the nilpotent endomorphism x.

In order to reach the above goal in the case of two-row Springer fibers we introduce some combinatorial tools. Consider a rectangle in the plane with m vertices evenly spread along the upper horizontal edge of the rectangle. The vertices are labelled by the consecutive integers $1, 2, \ldots, m$ in increasing order from left to right. A *cup diagram* is a non-intersecting diagram inside the rectangle obtained by attaching lower semicircles called *cups* and vertical line segments called *rays* to the vertices. We require that every vertex is incident with precisely one endpoint of a cup or ray. Moreover, a ray always connects a vertex with a point on the lower horizontal edge of the rectangle. Additionally, any cup or ray for which there exists a path inside the rectangle connecting this cup or ray to the right edge of the rectangle without intersecting any other part of the diagram may be equipped with one single dot. Here are two examples:



If the cups and rays of two given cup diagrams are incident with exactly the same vertices (regardless of the precise shape of the cups) and the distribution of dots on corresponding cups and rays coincides in both diagrams we consider them as equal. We write $\mathbb{B}^{n-k,k}$ to denote the set of all cup diagrams with $\lfloor \frac{k}{2} \rfloor$ cups.

Let $\mathbb{S}^2 \subseteq \mathbb{R}^3$ be the standard two-dimensional unit sphere on which we fix the points $p = (0,0,1)$ and $q = (1,0,0)$. Given a cup diagram $\mathbf{a} \in \mathbb{B}^{n-k,k}$, we define $S_{\mathbf{a}} \subseteq \left(\mathbb{S}^2\right)^m$ as the manifold consisting of all $(x_1, \dots, x_m) \in \left(\mathbb{S}^2\right)^m$ which satisfy the relations $x_i = -x_j$ (resp. $x_i = x_j$) if the vertices i and j are connected by an undotted cup (resp. dotted cup). Moreover, we impose the relations $x_i = p$ if the vertex i is connected to a dotted ray and $x_i = -p$ (resp. $x_i = q$) if i is connected to an undotted ray which is the rightmost ray in $\mathbf{a}$ (resp. not the rightmost ray). The *topological Springer fiber* $\mathcal{S}^{n-k,k}$ is defined as the union

$$\mathcal{S}^{n-k,k} := \bigcup_{\mathbf{a} \in \mathbb{B}^{n-k,k}} S_{\mathbf{a}} \subseteq \left(\mathbb{S}^2\right)^m .$$

Theorem. *There exists a homeomorphism $\mathcal{S}^{n-k,k} \cong \mathcal{F}l^{n-k,k}$ such that the images of the $S_{\mathbf{a}}$ are irreducible components of $\mathcal{F}l^{n-k,k}$ for all $\mathbf{a} \in \mathbb{B}^{n-k,k}$.*

Remark. This proves a conjecture by Ehrig and Stroppel [3] on the topology of Springer fibers corresponding to partitions with two equal parts and extends the result to all two-row Springer fibers. The analogous conjecture for Springer fibers of type A corresponding to partitions with two equal parts goes back to Khovanov [4] and was proven independently in [5] and [7]. The results were generalized to all two-row Springer fibers of type A by Russell in [6].

Given a cup diagram $\mathbf{a} \in \mathbb{B}^{n-k,k}$, we write $\overline{\mathbf{a}}$ to denote the diagram obtained by reflecting $\mathbf{a}$ in the horizontal middle line of the rectangle. If $\mathbf{b} \in \mathbb{B}^{n-k,k}$ is another cup diagram let $\overline{\mathbf{a}}\mathbf{b}$ denote the diagram obtained by sticking $\overline{\mathbf{a}}$ on top of $\mathbf{b}$, i.e. we glue the two diagrams along the horizontal edges of the rectangles containing the vertices (thereby identifying the vertices in $\overline{\mathbf{a}}$ and $\mathbf{b}$ pairwise from left to right). Note that in general the diagram $\overline{\mathbf{a}}\mathbf{b}$ consists of several connected components each of which is either closed (i.e. it has no endpoints) or a line segment.

The following proposition is a straightforward consequence of the definitions. In combination with the theorem it enables us to explicitly determine the topology of intersections of the irreducible components of $\mathcal{F}l^{n-k,k}$.

Proposition. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{B}^{n-k,k}$ be cup diagrams.*

- *We have $S_{\mathbf{a}} \cap S_{\mathbf{b}} \neq \emptyset$ if and only if the following two conditions hold:*
  (1) Every connected component of $\overline{\mathbf{a}}\mathbf{b}$ contains an even number of dots.
  (2) The rightmost ray in $\mathbf{a}$ and the rightmost ray in $\mathbf{b}$ are part of the same connected component of $\overline{\mathbf{a}}\mathbf{b}$.

- *If $S_{\mathbf{a}} \cap S_{\mathbf{b}} \neq \emptyset$, then there is a homeomorphism $S_{\mathbf{a}} \cap S_{\mathbf{b}} \cong \left(\mathbb{S}^2\right)^c$, where c denotes the number of closed connected components in $\overline{\mathbf{a}}\mathbf{b}$.*

The results outlined above are part of the author's PhD thesis and will appear in [8].

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Diagonalization in categorical representation theory

Ben Elias

(joint work with Matt Hogancamp)

Suppose that f is an endomorphism of a finite-dimensional vector space V, satisfying $\prod_{\lambda \in \mathbb{S}}(f - \lambda I) = 0$. In other words, f is *diagonalizable*, and we know its spectrum $\mathbb{S}$. Linear algebra has the machinery to extract a great deal out of this small amount of information. Namely, one gets "for free" a canonical splitting of V into eigenspaces V_λ, with an explicit construction of the operators p_λ which project from V to V_λ.

In this talk we discuss how this machinery lifts to the categorical setting, to *categorically diagonalizable* functors acting on graded triangulated categories, and we provide some conjectural applications. Our machinery is currently limited to the case when the eigenvalues of our functor, acting on the Grothendieck group, are $\pm v^n$, or more precisely, when the "categorical eigenvalue" is just a grading and a homological shift.

Our motivating examples are these.

Example 1. It is known that representations of $U_q(\mathfrak{sl}_2)$ over $\mathbb{Q}(q)$ are semisimple, so that any finite dimensional representation splits canonically into isotypic components. This splitting can be realized as the eigenspace decomposition for the Casimir operator c, which is diagonalizable. Chuang and Rouquier [1] defined the notion of a categorical representation of $\mathfrak{sl}_2$, and proved that any "finite-dimensional" categorical representation has a canonical filtration by isotypic components, where the component with maximal highest weight is a subrepresentation, and the component with minimal highest weight is a quotient. Beliakova-Khovanov-Lauda [5] constructed a complex of functors C which acts on the homotopy category of any categorical representation, lifting the Casimir element. An eventual goal is to use the machinery of categorical diagonalization to explicitly construct the canonical filtration using the complex C. Our machinery is not yet up to the task, because the eigenvalues are too complicated.

This illustrates a basic principle in categorical diagonalization: the spectrum $\mathbb{S}$ of a diagonalizable functor is a partially ordered set, and one expects the category it acts on to have a filtration by eigencategories.

Example 2. Inside the Hecke algebra of the symmetric group S_n, there is a family of commuting operators ft_i, $1 \le i \le n$. These are the images of the full twist in the braid group of $S_i \subset S_n$; they are also the products of the first i Young-Jucys-Murphy elements. The full twist ft_n is central and diagonalizable, and its spectrum is in bijection with partitions of n, modulo some equivalence. It almost splits a representation into isotypic components, except that it can not tell certain components apart. The joint spectrum of the simultaneously diagonalizable family $\{ft_i\}$ is in bijection with standard tableaux with n boxes. Thus this family can distinguish isotypic components, and split irreducibles further into one-dimensional pieces.

One can categorify the Hecke algebra of S_n using Soergel bimodules [6], or its diagrammatic description due to Elias-Khovanov [2]. Rouquier [7] constructed complexes of Soergel bimodules which categorify elements of the braid group, whence one can construct the functors FT_i. We conjecture below that FT_n is categorically diagonalizable (and the family is simultaneously so) with spectrum given by partitions (resp. standard tableaux). Note that there is no equivalence class; the categorical full twist can distinguish between every isotypic component. Moreover, given this conjecture, we have the technology to construct explicit projection functors P_λ for each partition of n, projecting to an isotypic component, and to construct a filtration of the identity functor by these P_λ.

Let us briefly describe how categorical diagonalization works, in the setting for which our machinery applies. In linear algebra, the formula for projection to the λ_i eigenspace of an operator f is

$$p_i = \prod_{j \neq i} \frac{f - \lambda_j}{\lambda_i - \lambda_j}.$$

Suppose that $\lambda_i = (-1)^{k_i} v^{n_i}$. If we wish to categorify this formula, we must find an operator which categorifies $(f - \lambda_j)$, and then find a way to make sense of the fraction. By ordering the eigenvalues based on the value of k_i which appears (and for simplicity we assume that all k_i are distinct, though this is false for the full twist), we can express $\frac{1}{\lambda_i - \lambda_j}$ as a power series in the v-adic topology. This power series will eventually lead to the construction of an infinite complex lifting p_i, but satisfying certain boundedness conditions related to the choice of topology. We will ignore questions of boundedness for now.

Consider a graded additive category $\mathcal{A}$ and its homotopy category. For sake of simplicity, we assume that $\mathcal{A}$ is monoidal, so that objects of $\mathcal{A}$ can also be viewed as functors $\mathcal{A} \to \mathcal{A}$ under tensor product. Similarly, complexes (with some boundedness condition) can also be viewed as endofunctors of the homotopy category, and the tensor product $\otimes$ agrees with composition.

Let $\mathcal{F}$ be a bounded complex in the homotopy category of $\mathcal{A}$. A *potential (forward) eigenmap* for $\mathcal{F}$ is a natural transformation $\alpha\colon \mathbb{1}(n)\langle k\rangle \to \mathcal{F}$, where (n) is a grading shift and $\langle k\rangle$ is a homological shift. Its corresponding eigenvalue or *eigenshift* is $\mathbb{1}(n)\langle k\rangle$. A complex M is an *eigencomplex* if $\alpha M\colon M(n)\langle k\rangle \to \mathcal{F}M$ is an isomorphism. Eigencomplexes form a full subcategory of the homotopy category, the *eigencategory* of α. Let Λ_α denote the cone of α, the *eigencone*. It is a categorification of $(f - \lambda I)$ in the traditional setting. Note that M is an eigencomplex if and only if $\Lambda_\alpha M = 0$. We say that $\mathcal{F}$ is *categorically diagonalizable* with spectrum $\mathbb{S} = \{\alpha_i\}$ if $\bigotimes \Lambda_{\alpha_i} = 0$. These operators commute, so it does not matter what order the tensor product is taken in.

Theorem 1. *Let $\mathcal{F}$ be categorically diagonalizable with spectrum $\{\alpha_i\}$, having associated eigenshift $\mathbb{1}(n_i)\langle k_i\rangle$. For simplicity we make the assumption that all homological shifts k_i are distinct (otherwise the result is more technical). Place a partial order on the eigenmaps where $\alpha_i < \alpha_j$ when $k_i < k_j$. Then one can construct (potentially infinite) complexes P_i which project to eigencategories, and a filtration (in the triangulated sense) of the identity functor with subquotients isomorphic to P_i.*

Now we state a conjecture which is the main application of this theorem. Note that Soergel bimodules and Rouquier's complexes are defined for all Coxeter groups.

Conjecture 3. *Let FT denote the Rouquier complex attached to the full twist of any finite Coxeter group W. Then FT is categorically diagonalizable, and its spectrum is in bijection with the 2-sided cells of W (with known eigenshifts).*

So far, we can prove the following.

Theorem 2. *This conjecture holds for finite dihedral groups W, as well as in type A_n for $n \leq 5$.*

The work in progress [3] contains a general discussion of categorical diagonalization, together with the proof in the dihedral case. Hogancamp [4] also has related work on constructing the projection functor P_λ attached to the one-row partitions.

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Coideal subalgebras, categorification and branching of Brauer algebras

Catharina Stroppel
(joint work with M. Ehrig)

Let $\mathfrak{g}$ be a reductive complex Lie algebra and $\theta : \mathfrak{g} \to \mathfrak{g}$ an involution. Then its fixed point Lie algebra $\mathfrak{g}^\theta$ is a reductive Lie subalgebra and we have an embedding of Hopf algebras $U(\mathfrak{g}^\theta) \subset U(\mathfrak{g})$ of the corresponding universal enveloping algebras. For the quantum groups $U_q(\mathfrak{g}^\theta)$ and $U_q(\mathfrak{g})$ there is however no embedding as Hopf subalgebras. Hence there is the obvious question how one should quantize the symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\theta)$. Letzter, [5], gave a possible definition of a quantum symmetric pair as a subalgebra $\mathbb{B}$ of $U_q(\mathfrak{g})$ which specializes to $U(\mathfrak{g}^\theta)$ and which forms a *coideal subalgebra*.

In this talk we consider the special cases where $\mathfrak{g} = \mathfrak{gl}_{2n}$ or $\mathfrak{g} = \mathfrak{gl}_{2n+1}$ and $\mathfrak{g}^\theta = \mathfrak{gl}_n \times \mathfrak{gl}_n$ respectively $\mathfrak{gl}_n \times \mathfrak{gl}_{n+1}$ which is the type AIII in the classification of symmetric pairs. Let $\mathcal{B}_m$ for $m = 2n$ or $m = 2n + 1$ be this coideal. We recall the basic properties and the Serre type presentation (from [5] and [6]), and then state a PBW type theorem.

As a main result we show that this coideal subalgebra naturally acts via exact translation functors on parabolic category $\mathcal{O}$ of type (D, A), more precisely on the graded version of the BGG category $\mathcal{O}$ for the classical semi-simple Lie algebra of type D with parabolic of type A: Denote by $\mathfrak{g} = \mathfrak{so}_{2n}$ the even orthogonal complex Lie algebra. By $\mathcal{O}(\mathfrak{g})$ we denote the graded *integral* BGG category $\mathcal{O}$ of $\mathfrak{g}$, [1]. It decomposes into blocks $\mathcal{O}(\mathfrak{g}) = \bigoplus_\chi \mathcal{O}_\chi$. In this case W acts by permutations and even number of sign changes on the elements (written in the standard ϵ-basis) in the weight lattice $X = \mathbb{Z}^n \cup (\frac{1}{2} + \mathbb{Z})^n$. Hence weights in a fixed block are are either integral or half-integral in this basis. Moreover they have either an even or an odd number of negative coefficients.

Theorem 1. *On the direct sum of the blocks in $\mathcal{O}(\mathfrak{g})$ for half-integral weights there is an action of the coideal subalgebra $\mathcal{B}_{2n}$ where Letzter's generators act via certain graded translation functors. These functors are graded lifts of functors of the form $\mathrm{pr}_{\leq m} \circ (? \otimes L(\omega_1))$ which first take the tensor product with the vector* *representation of $\mathfrak{so}(2n)$ and afterwards projects onto some specific blocks. Similarly, on the direct sum of the blocks in $\mathcal{O}(\mathfrak{g})$ for integral weights there is an action of the coideal subalgebra $\mathcal{B}_{2n+1}$.*

Hereby, one of the involved translation functors actually defines an equivalence of categories which just switches the parity of negative coefficients.

The categorification result is based on a the following observation which should give an indication why category $\mathcal{O}$ for type D or at least some category attached to Hecke algebras should play a role:

Theorem 2. *(Generalized Schur-Weyl duality)*
Let X be the d-fold tensor product of the quantized vector representation of $\mathfrak{g} = \mathfrak{gl}_{2n}$ with its restricted action to the coideal subalgebra. Then X has the double centralizer property between the coideal action and the action of the 2-parameter Hecke algebra H_{q_1,q_2} of type B with either $q_1 = q_2 = q$ or $q_1 = 1$ $q_2 = q$, depending on the parameter appearing in the definition of the coideal subalegbra.

Note that in case $q_1 = 1$, $q_2 = q$, the Hecke algebra of type D embeds as a subalgebra. Using Zuckerman functors acting on graded category $\mathcal{O}$, the above categorification contains a categorification of this duality. The special generator for $q_1 = 1$ is then just an involution. In the categorification above it corresponds to the parity switching functor mentioned.

The second part of the talk deals with another occurence of the action of this coideal algebra in the context of Brauer algebras. We present the folllowing result:

Theorem 3. *The restriction/induction functors for the Brauer algebras $\mathrm{Br}_d(\delta)$ for fixed parameter $\delta \in \mathbb{Z}$ and varying rank d satisfy the relations of the coideal subalgebra $\mathcal{B}_m$. Hence the Grothendieck group of representation of all $\mathrm{Br}_d(\delta)$ (defined in an appropriate way) becomes a module for the coideal subalgebra $\mathcal{B}_m$. Here $m = 2n$ in case δ is odd and $m = 2n+1$ in case δ is even, for $n = \infty$.*

Recalling that the Brauer algebras are cellular (or in case $\delta \neq 0$ even quasi-herditary algebras) with cell/standard modules labelled by certain partitions, we see that the Grothendieck group has a basis given by partitions, hence has the size of a Fock space. We show that the combinatorics of the coideal action by functors induces on the Grothendieck group the usual Fock space action of $\mathfrak{gl}_m$, where $m = 2\infty$ or $m = 2\infty + 1$ restricted to our coideal subalgebra in infinitely many generators. Hence this can be viewed as a non-trivial generalization of the famous Ariki-Grojnowski categorification theorem for thh symmetric groups.
The connection between the two categorifications is explained in [4]

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Asymptotical representations, spectra of quantum integrable systems and cluster algebras

David Hernandez

We explain how the existence of the asymptotical prefundamental representations ([HJ] *joint with M. Jimbo) can be used to prove a general conjecture on the spectra of quantum integrable systems (*[FH] *joint with E. Frenkel) and to establish new monoidal categorifications of cluster algebras (*[HL2] *joint with B. Leclerc).*

The partition function Z of a (quantum) integrable system is crucial to understand its physical properties. For a quantum model on an $M \times N$ lattice it may be written in terms of the eigenvalues of the transfer matrix T:

$$Z = \mathrm{Tr}(T^M) = \sum_j \lambda_j^M.$$

Therefore, to find Z, one needs to find the spectrum of T. Baxter tackled this question in his seminal 1971 paper [B] for the (ice) 6- and 8-vertex models. Baxter observed moreover that the eigenvalues λ_j of T have a very remarkable form

$$\lambda_j = A(z)\frac{Q_j(zq^2)}{Q_j(z)} + D(z)\frac{Q_j(zq^{-2})}{Q_j(z)}, \tag{1}$$

where q, z are parameters of the model (quantum and spectral), the functions $A(z)$, $D(z)$ are universal (in the sense that there are the same for all eigenvalues), and Q_j is a polynomial. The above relation is now called Baxter's relation (or Baxter's TQ relation) and Q_j are called Baxter's polynomials.

In 1998, Frenkel-Reshetikhin conjectured [FR] that the spectra of more general quantum integrable systems (more precisely, generalizing the XXZ model of quantum spin chains, whose spectrum is the same as that of the 6-vertex model) have an analog form. Let us formulate this conjecture in terms of representation theory. Let $\mathfrak{g}$ be an untwisted affine Kac-Moody algebra and $q \in \mathbb{C}^*$ which is not a root of 1. For simplicity of notations, we suppose that $\mathfrak{g}$ is simply-laced, but our results hold in general. Consider the corresponding quantum affine algebra $U_q(\mathfrak{g})$. Its completed tensor square contains the universal R-matrix $\mathcal{R}$ satisfying the Yang–Baxter relation. Given a finite-dimensional representation V of $U_q(\mathfrak{g})$, we have the (twisted) transfer-matrix

$$t_V(z) = \mathrm{Tr}_V(\pi_{V(z)} \otimes \mathrm{Id})(\mathcal{R}) \in U_q(\mathfrak{g})[[z]],$$

where $V(z)$ is a twist of V by a "spectral parameter" z for the natural grading of $U_q(\mathfrak{g})$ and Tr_V is the (graded) trace on V. As a consequence of the Yang-Baxter equation we have $[t_V(z), t_{V'}(z')] = 0$ for all V, V' and z, z'. Therefore these transfer-matrices give rise to a family of commuting operators on any finite-dimensional representation W of $U_q(\mathfrak{g})$.

The finite-dimensional representation V has also a q-character defined in [FR] and which is a Laurent polynomial $\chi_q(V) \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times}$ where I is the set of vertices of the underlying finite type Dynkin diagram. For example, if $\mathfrak{g} = \hat{sl}_2$ and V is a simple two-dimensional representation, then there is $a \in \mathbb{C}^*$ such that

$$\chi_q(V) = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

Conjecture 1 [Frenkel-Reshetikhin, 1998] *The eigenvalues λ_j of $t_V(z)$ on W are obtained from $\chi_q(V)$ by replacing each variable $Y_{i,a}$ by a quotient*

$$\frac{f_i(azq^{-1})Q_{i,j}(zaq^{-1})}{f_i(azq)Q_{i,j}(zaq)}$$

where the functions $f_i(z)$ are universal (in the sense that there are the same for all eigenvalues),and $Q_{i,j}$ is a polynomial.

For $\mathfrak{g} = \hat{sl}_2$ and V of dimension 2, the conjecture is the Baxter's formula. In general, there are more than 2 terms and no hope for explicit computations. However:

Theorem 1 [Frenkel-Hernandez, 2013] *The conjecture 1 is true.*

Our proof [FH] of the conjecture 1 for arbitrary untwisted affine types is based on the study of the "prefundamental representations" which I had previous constructed with M. Jimbo [HJ]. They are obtained as asymptotical limits of the Kirillov-Reshetikhin modules $W_{k,a}^{(i)}$ ($a \in \mathbb{C}^*$, $k \geq 0$, $i \in I$) which form a family of simple finite-dimensional representations of $U_q(\mathfrak{g})$. More precisely, we have constructed an inductive system

$$W_{1,a}^{(i)} \subset W_{2,a}^{(i)} \subset W_{3,a}^{(i)} \subset \cdots$$

By using one of the main result of [H], we prove that the inductive limit $W_{\infty,a}^{(i)}$ has a structure of representation for the Borel subalgebra $U_q(\mathfrak{b}) \subset U_q(\mathfrak{g})$:

Theorem 2 [Hernandez-Jimbo, 2011] *The action of $U_q(\mathfrak{b})$ on the inductive system "converges" to a simple infinite-dimensional $U_q(\mathfrak{b})$-module $L_{i,a} = W_{\infty,a}^{(i)}$.*

Such a representation was constructed explicitly in the case of $\mathfrak{g} = \hat{sl}_2$ by Bazhanov-Lukyanov-Zamolodchikov. In general, we have proved moreover in [HJ] that these prefundamental representations $L_{i,a}$ (and their duals) form a family of fundamental representations for a category $\mathcal{O}$ of $U_q(\mathfrak{b})$-modules extending the category of finite-dimensional representations.

Our proof [FH] of the conjecture 1 has two main steps. First we establish the following relation in the Grothendieck ring of $\mathcal{O}$, generalizing the Baxter relation: for any finite-dimensional representation V of $U_q(\mathfrak{g})$, take its q-character and replace each $Y_{i,a}$ by the ratio of classes $[L_{i,aq^{-1}}]/[L_{i,aq}]$ (times the class a one-dimensional representation, that we omit here for clarity). Then this expression is equal to the class of V in the Grothendieck ring of $\mathcal{O}$ (viewed as a representation of $U_q(\mathfrak{b})$). For example, if $\mathfrak{g} = \hat{sl}_2$ and V of dimension 2, we get a categorified Baxter's relation :

$$[V \otimes L_{1,aq}] = [L_{1,aq^{-1}}] + [L_{1,aq^3}].$$

The second step is that the transfer-matrix associated to $L_{i,a}$ is well-defined and all of its eigenvalues on any irreducible finite-dimensional representation W of $U_q(\mathfrak{g})$ are *polynomials* up to one and the same factor that depends only on W. Combining these two results, we obtain the proof of the conjecture 1.

In a work in progress with B. Leclerc [HL2], we also use this category $\mathcal{O}$ and such asymptotical representations to obtain new monoidal categorification of (infinite rank) cluster algebras. The program of monoidal categorifications of cluster algebras was initiated in [HL1]. The cluster algebra $\mathcal{A}(Q)$ attached to a quiver Q is a commutative algebra with a distinguished set of generators called cluster variables and obtained inductively by relations called Fomin-Zelevinsky mutations. When the quiver Q is finite, the cluster algebra is said to be of finite rank.

A monoidal category $\mathcal{C}$ is said to be a monoidal categorification of $\mathcal{A}(Q)$ if there exists a ring isomorphism with its Grothendieck ring $K_0(\mathcal{C})$:

$$\phi : \mathcal{A}(Q) \to K_0(\mathcal{C})$$

which induces a bijection between cluster variables and isomorphism classes of simple modules which are prime (without non trivial tensor factorization) and real (whose tensor square is simple). Various examples of monoidal categorifications have been established in terms of quantum affine algebras (Hernandez-Leclerc), perverse sheaves on quiver varieties (Nakajima, Kimura-Qin, Qin) and Khovanov-Lauda-Rouquier algebras (Kang-Kashiwara-Kim-Oh).

Consider $\mathcal{O}^+$ the subcategory of the category $\mathcal{O}$ of $U_q(\mathfrak{b})$-modules generated by finite-dimensional representations and prefundamental representations.

Theorem 3 [Hernandez-Leclerc 2015] *There is an infinite rank cluster algebra $\mathcal{A}(Q)$ such that there is a ring isomorphism $\mathcal{A}(Q) \simeq K_0(\mathcal{O}^+)$, with prefundamental representations corresponding to cluster variables.*

In particular, Baxter's relations get interpreted as mutation relations.

Conjecture 2 [Hernandez-Leclerc 2015] *The category $\mathcal{O}^+$ is a monoidal categorification of $\mathcal{A}(Q)$.*

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Categorical representations and finite classical groups

Eric Vasserot

(joint work with Olivier Dudas, Michela Varagnolo)

Given an abelian artinian category $\mathcal{C}$ (linear over a field K) and a symmetrizable Kac-Moody algebra $\mathfrak{g}$, Rouquier and Khovanov-Lauda have introduced the notion of an integrable representation of $\mathfrak{g}$ on $\mathcal{C}$. Such a representation is the datum of some bi-adjoint pairs (E_i, F_i) of endofunctors of $\mathcal{C}$, where i runs over the set I of simple roots of $\mathfrak{g}$, and of some endomorphisms of the nth power E^n for all $n \in \mathbb{Z}_{>0}$ of the direct sum $E = \bigoplus_{i \in I} E_i$ satisfying explicit relations. In particular, the complexified Grothendieck group $[\mathcal{C}]$ of $\mathcal{C}$ has the structure of an integrable representation over $\mathfrak{g}$, and the set $Irr(\mathcal{C})$ of isomorphism classes of simple objects in $\mathcal{C}$ has a crystal structure in the sense of Kashiwara, which is isomorphic to the crystal of the integrable module $[\mathcal{C}]$.

An example is the representation of $\widehat{\mathfrak{sl}}_e$ on the category O of the rational double affine Hecke algebras associated with the complex reflexion groups $G(l, 1, n)$ for e, l fixed positive integers and n which varies in $\mathbb{N}$. This representation has been defined by P. Shan in [3]. The parameters of the rational double affine Hecke algebras are fixed in a very particular way.

Another example is the representation of $\widehat{\mathfrak{sl}}_e$ on a category $\bigoplus_{n \in \mathbb{N}} \mathcal{U}(KGL_n(\mathbb{F}_q))$ of unipotent modules over $KGL_n(\mathbb{F}_q)$ where n varies in $\mathbb{N}$ and K is a field of positive characteristic ℓ prime to q such that q has the order e modulo ℓ. This was introduced by R. Rouquier and J. Chuang in [1] to construct some derived equivalences between unipotent blocks of $KGL_n(\mathbb{F}_q)$.

Our goal is to apply categorical representations to modular unipotent representations of finite groups of classical type. In this case, the partition of simple modules into Harish-Chandra series is not known and can be (partially) recovered from our work.

We define a representation of some affine Lie algebra $\mathfrak{g}$ on the category $\mathcal{C}$ given by $\mathcal{C} = \bigoplus_{n \in \mathbb{N}} \mathcal{U}(KG_n(\mathbb{F}_q))$, where $G_n(\mathbb{F}_q)$ is a finite Chevalley group of rank n and of (fixed) classical type. We identify the $\mathfrak{g}$-module $[\mathcal{C}]$ explicitely. It is a direct sum of Fock spaces whose highest weights are computed. Thus, the crystal $Irr(\mathcal{C})$ is known combinatorialy, up to some automorphism. Then, we prove that the connected components of $Irr(\mathcal{C})$ are exactly the (weak) Harish-Chandra series.

If G_n is the unitary group, we can fix the crystal isomorphism in a canonical way. We deduce an explicit parametrization of all (weakly) cuspidal unipotent modules which was conjectured by Gerber-Hiss-Jacob recently. We also deduce some explicit conditions for two unipotent blocks to be derived equivalent, giving a partial answer to an old conjecture of Broué.

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Quantum cluster algebras, quantum groups, and monoidal categorification

Fan Qin

Cluster algebras were invented by Sergey Fomin and Andrei Zelevinsky around the year 2000 in their seminal work [2]. These are commutative algebras with generators defined recursively called cluster variables. The quantization was later introduced in [1]. Fomin and Zelevinsky aimed to develop a combinatorial approach to the canonical bases in quantum groups (discovered by Lusztig [11] and Kashiwara [7] independently) and the theory of total positivity in algebraic groups (by Lusztig [12][13]). They conjectured that the cluster structure should serve as an algebra framework for the study of the "dual canonical bases" in various coordinate rings and their q-deformations. In particular, they proposed the following conjecture.

Conjecture 1. All monomials in the variables of any given cluster (the cluster monomials) belong to the dual canonical basis.

As a new approach to cluster algebras, David Hernandez and Bernard Leclerc found a cluster algebra structure on the Grothendieck ring of finite dimensional representations of quantum affine algebras in [4]. They proposed the following monoidal categorification conjecture.

Conjecture 2. All cluster monomials belong to the basis of the simple modules.

Partial results were due to [4][5][10][14][9]. By using quiver Hecke algebras, Conjecture 1 could also be viewed as a monoidal categorification conjecture, cf. [8][15][17][16].

In this talk, we consider the quantum cluster algebras which are *injective-reachable* and introduce a *common triangular basis*. This basis, if exists, is unique. It is parametrized by tropical points as expected in the Fock-Goncharov conjecture.

As applications, we prove the existence of the common triangular basis in the following cases.

Theorem 3. (1) Assume that $\mathcal{A}$ arises from the quantum unipotent subgroup $A_q(\mathfrak{n}(w))$ associated with a reduced word w (called type (i)). If the Cartan matrix is of Dynkin type ADE, or if the word w is inferior to an adaptable word under the left or right weak order, then the dual canonical basis gives the common triangular basis after normalization and localization at the frozen variables.

(2) Assume that $\mathcal{A}$ arises from representations of quantum affine algebras (called type (ii)). Then, after localization at the frozen variables, the basis of the simple modules gives the common triangular basis.

This result implies the above conjectures in the corresponding cases.

We remark that the Fock-Goncharov conjecture was recently reformulated and proved by Gross-Hacking-Keel-Kontsevich [3]. And the monoidal categorification conjecture for type (i) quantum cluster algebras was recently proved by Kang-Kashiwara-Kim-Oh [6].

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Cacti and cells

Ivan Losev

Let W be a Weyl group. We have three equivalence relations on W: $\sim_L, \sim_R, \sim_{LR}$ coming from the Kazhdan-Lusztig theory. Their equivalences classes are called left, right and two-sided cells, respectively. On the other hand, to W one can assign the so called cactus group Cact_W that should be thought as a crystal analog of the braid group. The group is given by generators τ_D, where D is a connected subdiagram in the Dynkin diagram of W and an explicit set of relations.

The main construction in the talk is that of a Cact_W-action on W that preserves right cells and permutes left cells. In type A, the right cells are precisely the orbits of the action, but outside of type A this is not the case, in general.

The action arises as a crystallization of the categorical braid group action of the braid group Br_W on the BGG category $\mathcal{O}$. The key observation needed to perform the crystallization is that the functors corresponding to the longest elements in the standard parabolic subgroups are perverse in the sense of Rouquier, so they give rise to self-bijections of the set of the irreducible objects that is identified with W.

I expect that this construction generalizes to a wider representation theoretic context, e.g., to quantizations of symplectic resolutions.

Quantum K-theoretic geometric Satake correspondence

Joel Kamnitzer
(joint work with Sabin Cautis)

Let G be a semisimple complex group. Let $G^\vee$ be its Langlands dual group and let $Gr = G^\vee((t))/G^\vee[[t]]$ be the affine Grassmannian. The geometric Satake correspondence gives an equivalence of categories between the representations of G and the spherical perverse sheaves on Gr. Bezrukavnikov-Finkelberg [BF] proved a derived version of this equivalence which relates the derived category of $G^\vee$-equivariant constructible sheaves on Gr with the category of G-equivariant coherent sheaves on $\mathfrak{g}$.

The goal of our project was to develop a version of derived geometric Satake involving the quantum group $U_q\mathfrak{g}$. We define a category $KConv(Gr)$ whose morphism spaces are given by the $G^\vee \times \mathbb{C}^\times$-equivariant K-theory of certain fibre products defined using Gr. We conjecture that $KConv(Gr)$ is equivalent to a full subcategory of the category of $U_q\mathfrak{g}$-equivariant $\mathcal{O}_q(G)$ modules.

We prove this conjecture when $G = SL_n$. The main tool in our proof is a combinatorial description, called the SL_n spider, of the category of $U_q\mathfrak{g}$ representations; this was developed in a previous work with S. Morrison [CKM]. Using the machinery of horizontal trace, we prove that the annular SL_n spider describes the category of $U_q\mathfrak{sl}_n$-equivariant $\mathcal{O}_q(SL_n)$ modules.

The annular SL_n spider is closely related to the quantum loop algebra $U_qL\mathfrak{gl}_m$. Thus in order to complete the proof of the conjecture, we use a map from $U_qL\mathfrak{gl}_m$ to $KConv(Gr)$ which we studied in previous work with A. Licata [CKL].

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Towards a Riemann-Hilbert correspondence for $\widehat{\mathcal{D}}$-modules

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(joint work with Simon Wadsley)

1. Background

Let R be a complete discrete valuation ring with uniformiser π, residue field $k := R/\pi R$ and field of fractions $K := R[\frac{1}{\pi}]$.

Definition. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over K.*

(a) A Lie lattice *in $\mathfrak{g}$ is a finitely generated R-submodule L of $\mathfrak{g}$ which satisfies $[L, L] \subset L$ and which spans $\mathfrak{g}$ as a K-vector space.*

(b) Let L be a Lie lattice in $\mathfrak{g}$. The affinoid enveloping algebra *of L is*

$$\widehat{U(L)_K} := \left(\varprojlim U(L)/(\pi^a)\right) \otimes_R K.$$

(c) The Arens-Michael envelope *of $U(\mathfrak{g})$ is*

$$\widehat{U(\mathfrak{g})} := \varprojlim \widehat{U(L)_K}$$

where the inverse limit is taken over all possible Lie lattices L in $\mathfrak{g}$.

For any Lie lattice L in $\mathfrak{g}$, its set of π-power multiples is cofinal in the set of all Lie lattices, so that

$$\widehat{U(\mathfrak{g})} \cong \varprojlim \widehat{U(\pi^n L)_K}.$$

Example. Suppose that $\mathfrak{g} = Kx$ is a one-dimensional Lie algebra, spanned by an element x. If $L = Rx$ then $U(L) = R[x]$ is just a polynomial ring in one variable over R, the π-adic completion $\widehat{U(L)} = \widehat{R[x]}$ can be identified with the following subset of $R[[x]]$:

$$\widehat{R[x]} = \left\{ \sum_{i=0}^{\infty} \lambda_i x^i \in R[[x]] : \lim_{i\to\infty} \lambda_i = 0 \right\}.$$

The affinoid enveloping algebra $\widehat{U(L)_K}$ consists of power series in $K[[x]]$ satisfying the same convergence condition:

$$\widehat{U(L)_K} = K\langle x\rangle := \left\{ \sum_{i=0}^{\infty} \lambda_i x^i \in K[[x]] : \lim_{i\to\infty} \lambda_i = 0 \right\}.$$

Similarly, $\widehat{U(\pi^n L)_K} = K\langle \pi^n x\rangle$ can be identified with the set of formal power series $\sum_{i=0}^{\infty} \lambda_i x^i \in K[[x]]$ satisfying the stronger convergence condition

$$\lim_{i\to\infty} \lambda_i/\pi^{ni} = 0 \quad \text{for all} \quad n \geq 0.$$

It follows that the Arens-Michael envelope $\widehat{K[x]}$ of $K[x]$ consists of formal power series $\sum_{i=0}^{\infty} \lambda_i x^i \in K[[x]]$ whose sequence of coefficients (λ_i) is *rapidly decreasing*:

$$\widehat{K[x]} = K\{x\} := \left\{ \sum_{i=0}^{\infty} \lambda_i x^i \in K[[x]] : \lim_{i\to\infty} \lambda_i/\pi^{ni} = 0 \quad \text{for all} \quad n \geq 0 \right\}$$

Motivation. Let G be a p-adic Lie group, and suppose that the ground field K is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. In number theory [13], we study *admissible locally analytic K-representations of G*. This is an abelian category which is anti-equivalent to the category of *co-admissible $D(G,K)$-modules*. We do not recall the definition of the *locally analytic distribution algebra $D(G,K)$* here, but simply note that it is a particular K-Fréchet-space completion of the abstract group ring $K[G]$. This completion is large enough to contain the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, and the closure of $U(\mathfrak{g})$ in $D(G,K)$ turns out to be isomorphic to its Arens-Michael envelope $\widehat{U(\mathfrak{g})}$.

Unfortunately, Arens-Michael envelopes are non-Noetherian rings whenever $\mathfrak{g}$ is non-zero. To get around this, Schneider and Teitelbaum introduced the following

Definition.

(a) Suppose that $A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots$ is a tower of Noetherian K-Banach algebras such that
  - *A_{n+1} has dense image in A_n for all $n \geq 0$, and*
  - *A_n is a flat right A_{n+1}-module for all $n \geq 0$.*

  Then $A := \varprojlim A_n$ is said to be a Fréchet-Stein algebra.

(b) A left A-module M is said to be co-admissible *if $A_n \otimes_A M$ is a finitely generated A_n-module for all $n \geq 0$.*

(c) We let $\mathcal{C}_A$ denote the full subcategory of left A-modules consisting of the co-admissible A-modules.

Schneider and Teitelbaum proved that $\mathcal{C}_A$ is always an abelian category whenever A is a Fréchet-Stein algebra. They also proved that the locally analytic distribution algebras $D(G,K)$ and the Arens-Michael envelopes $\widehat{U(\mathfrak{g})}$ are Fréchet-Stein.

Example. The algebras $A_n = K\langle \pi^n x\rangle := \widehat{R[\pi^n x]} \otimes_R K$ satisfy the conditions above, so their inverse limit $\widehat{K[x]} = \varprojlim A_n$ provides an example of a (commutative) Fréchet-Stein algebra.

2. $\widehat{\mathcal{D}}$-modules on rigid analytic spaces

Suppose now that $\mathfrak{g}$ is a split semisimple Lie algebra over K. Prompted by a desire to establish an analogue of the Beilinson-Bernstein localisation theorem for co-admissible modules over $\widehat{U(\mathfrak{g})}$, we introduced the sheaf $\widehat{\mathcal{D}}$ of infinite-order differential operators on rigid analytic spaces in [1]. For the necessary background on rigid analytic geometry, we refer the reader to the survey paper [14].

Definition. *Let X be an affinoid variety over K, and let $\mathcal{T}(X) := \mathrm{Der}_K\, \mathcal{O}(X)$.*

(a) A Lie lattice *on X is any finitely generated $\mathcal{O}(X)^\circ$-submodule L of $\mathcal{T}(X)$ such that $[L,L] \subset L$ and L spans $\mathcal{T}(X)$ as a K-vector space.*

(b) For any Lie lattice L on X we have the Noetherian Banach algebra

$$\widehat{U(L)_K} := \left(\varprojlim U(L)/(\pi^a)\right) \otimes_R K.$$

(c) $\widehat{\mathcal{D}}(X) := \varprojlim \widehat{U(L)_K}$, the inverse limit being taken over all possible Lie lattices L in $\mathcal{T}(X)$.

Any Lie lattice L on X can be viewed as a *Lie-Rinehart algebra* over $(R, \mathcal{O}(X)^\circ)$, and as such has an enveloping algebra $U(L)$. These concepts were introduced by George Rinehart in [11].

Example. If $X = \mathrm{Sp}\, K\langle x\rangle$ is the closed unit disc, then

$$\widehat{\mathcal{D}}(X) = K\langle x\rangle\{\partial\} := \left\{ \sum_{i=0}^{\infty} a_i \partial^i \in K\langle x\rangle[[\partial]] : \lim_{i\to\infty} a_i/\pi^{ni} = 0 \quad \text{for all} \quad n \geq 0 \right\}$$

is a particular K-Fréchet-space completion of the Weyl algebra $K[x;\partial]$.

Theorem 1 ([2])**.** *Let X be a smooth rigid analytic space.*

(1) $\widehat{\mathcal{D}}$ extends to a sheaf of K-Fréchet algebras on X.

(2) If X is affinoid and $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module, then $\widehat{\mathcal{D}}(X)$ is a Fréchet-Stein algebra.

This basic result makes the following definition meaningful.

Definition. *Let X be a smooth rigid analytic space. A sheaf of $\widehat{\mathcal{D}}$-modules $\mathcal{M}$ on X is* co-admissible *if there is an admissible covering $\{X_i\}$ of X such that $\mathcal{T}(X_i)$ is a free $\mathcal{O}(X_i)$-module, and $\mathcal{M}(X_i) \in \mathcal{C}_{\widehat{\mathcal{D}}(X_i)}$ for all i. We denote the category of all co-admissible $\widehat{\mathcal{D}}$-modules on X by $\mathcal{C}_X$.*

Co-admissible $\widehat{\mathcal{D}}$-modules form a stack on smooth rigid analytic spaces. More precisely, we have the following analogue of Kiehl's Theorem in rigid analytic geometry.

Theorem 2 ([2])**.** *If X is a smooth affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module, then the global sections functor induces an equivalence of categories*

$$\Gamma : \mathcal{C}_X \xrightarrow{\cong} \mathcal{C}_{\widehat{\mathcal{D}}(X)}.$$

We can now formulate our version of the Beilinson-Bernstein equivalence.

Theorem 3 ([4])**.** *Let $\mathbf{G}$ be a connected, simply connected, split semisimple algebraic group over K, let $\mathfrak{g}$ be its Lie algebra and let $\mathcal{B} := (\mathbf{G}/\mathbf{B})^{\mathrm{an}}$ be the rigid-analytic flag variety. Then $\mathcal{C}_{\mathcal{B}} \cong \mathcal{C}_{\widehat{\mathcal{D}(\mathcal{B})}}$ and $\widehat{\mathcal{D}(\mathcal{B})} \cong \widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g})} K$.*

3. Holonomicity and $\widehat{\mathcal{D}}$-module operations

Let us recall the classical Riemann-Hilbert correspondence.

Theorem 4 (Kashiwara-Mebkhout)**.** *Let X be a smooth complex algebraic variety. Then the de Rham functor is an equivalence of categories*

$$\mathrm{DR} : D^b_{\mathrm{rh}}(\mathcal{D}_X) \longrightarrow D^b_c(\mathbb{C}_{X^{\mathrm{an}}}).$$

It sends regular holonomic $\mathcal{D}_X$-modules to perverse sheaves on X.

We are still rather far away from a perfect analogue of this theorem in the world of $\widehat{\mathcal{D}}$-modules on rigid analytic spaces! Nevertheless, there are some mildly encouraging signs that *some* such analogue exists. Let us recall some necessary ingredients of the proof of Theorem 4.

(1) DR gives an equivalence between integrable connections and local systems,
(2) a classification theorem for holonomic $\mathcal{D}$-modules,
(3) preservation of holonomicity under f_+, f^+ and $\mathbb{D}$.

We will not say anything in the direction of (1), except point out that there is a very well-developed theory of p-adic differential equations, which in part seeks to find an appropriate generalisation of (1) in the rigid-analytic setting. See for example [6], [10] and [12, Theorem 7.2]. It follows from [3, Theorem B] that integrable connections on smooth rigid analytic spaces can be naturally identified with co-admissible $\widehat{\mathcal{D}}$-modules that are $\mathcal{O}$-coherent.

In the direction of (2), a currently unresolved problem is to develop a good theory of characteristic varieties for co-admissible $\widehat{\mathcal{D}}$-modules. Nevertheless, we can make the following

Definition. *Let X be a smooth affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module, and let M be a co-admissible $D := \widehat{\mathcal{D}(X)}$-module.*

(1) The grade *of M is $j(M) = \min\{j \in \mathbb{N} : \mathrm{Ext}^j_D(M, D) \neq 0\}$.*
(2) The dimension *of M is $d(M) := 2\dim X - j(M)$.*
(3) M is weakly holonomic *if $d(M) = \dim X$.*

These are reasonable definitions because (a slight modification of) the theory in [13, §8] can be applied to co-admissible D-modules. This is permissible because of the following theorem, whose proof uses Hartl's result [7] on the existence of regular formal models for smooth rigid analytic spaces.

Theorem 5 ([5]). *Let X be a smooth affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module. Then*

(1) There is a Fréchet-Stein structure $\widehat{\mathcal{D}(X)} \cong \varprojlim A_n$ where each A_n is Auslander-Gorenstein with injective dimension bounded above by $2\dim X$.

(2) $d(M) \geq \dim X$ for every non-zero co-admissible $\widehat{\mathcal{D}(X)}$-module M.

Weakly holonomic $\widehat{\mathcal{D}}$-modules need not have finite length, as the following example shows.

Example. Let $\theta_n(t) = \prod_{m=0}^{n}(1-\pi^m t)$ and define

$$\theta(t) := \lim_{n\to\infty} \theta_n(t) = \prod_{m=0}^{\infty}(1-\pi^m t) \in \widehat{K[t]}.$$

Let $X = \operatorname{Sp} K\langle x\rangle$ be the closed unit disc, let $D := \widehat{\mathcal{D}(X)}$ and define

$$M := D/D\theta(\partial).$$

Then $d(M) = 1$ so M is weakly holonomic. However for every $n \geq 0$, M surjects onto $D/D\theta_n(\partial)$ which is a direct sum of $n+1$ integrable connections of rank 1 on X. Hence M has infinite length.

In the direction of (3), there is an analogue of Kashiwara's Equivalence:

Theorem 6 ([3, Theorem A]). *Let $\iota : Y \hookrightarrow X$ be a closed embedding of smooth rigid analytic spaces. Then the $\widehat{\mathcal{D}}$-module push-forward functor*

$$\iota_+ : \mathcal{C}_Y \to \mathcal{C}_X$$

is fully faithful, and its essential image consists of the co-admissible $\widehat{\mathcal{D}}_X$-modules $\mathcal{M}$ supported on $\iota(Y)$.

It is straightforward to check that ι_+ preserves weakly holonomic $\widehat{\mathcal{D}}$-modules. However, the following examples show that weakly holonomic $\widehat{\mathcal{D}}$-modules are too large to be preserved $\widehat{\mathcal{D}}$-module pushforwards and pullbacks, in general.

Example.

(1) Consider the weakly holonomic D-module M on $X = \operatorname{Sp} K\langle x\rangle$ from the previous Example, and let $\iota : Y := \{0\} \hookrightarrow X = \operatorname{Sp} K\langle x\rangle$ be the inclusion of a point. It is natural to define the pull-back $\iota^+ M$ of M along ι to be

$$\iota^+ M := M/xM.$$

However this is *not* a finite dimensional K-vector space, because M admits surjections onto integrable connections of arbitrarily large rank. Thus $\iota^+ M$ is not weakly holonomic.

(2) Let $U = X\backslash\{0\}$ and let $N := \widehat{\mathcal{D}(U)}/\widehat{\mathcal{D}(U)}\theta(1/x)$. Then N is the global sections of a weakly holonomic $\widehat{\mathcal{D}}$-module on the smooth quasi-Stein variety U, but it can be shown that N is not even co-admissible as a $D = \widehat{\mathcal{D}(X)}$-module.

In this last example, the problem is caused by the fact that $\mathrm{Supp}(N)$ is not a proper subset of X, and already in the classical setting of $\mathcal{D}$-modules on complex analytic manifolds, holonomicity is not preserved under $\mathcal{D}$-module pushforwards along open embeddings. However, we do have the following positive result, whose proof relies on Temkin's rigid-analytic version [15] of Hironaka's theorem on the embedded resolution of singularities of complex analytic spaces.

Theorem 7 ([5])**.** *Let $j : U \hookrightarrow X$ be a Zariski open embedding of smooth rigid analytic spaces, and let $\mathcal{M}$ be an integrable connection on U. Then $\mathbf{R}^i j_*(\mathcal{M})$ is a co-admissible weakly holonomic $\widehat{\mathcal{D}}_X$-module for all $i \geq 0$.*

Corollary ([5])**.** *Let Z be a closed analytic subset of the smooth rigid analytic space X, and let $\mathcal{M}$ be an integrable connection on X. Then the local cohomology sheaves with support in Z*

$$H^i_Z(\mathcal{M})$$

are co-admissible $\widehat{\mathcal{D}}_X$-modules for all $i \geq 0$.

These results give new examples of interesting weakly holonomic $\widehat{\mathcal{D}}$-modules. Local cohomology sheaves in rigid analytic geometry were originally considered by Kisin in [9]; note that $H^i_Z(\mathcal{M})$ is not in general a coherent $\mathcal{D}$-module.

We end with expressing the hope that there is some full subcategory $\mathcal{H}$ of weakly holonomic $\widehat{\mathcal{D}}$-modules containing all integrable connections, whose objects have finite length and have well-defined characteristic varieties, whose simple objects admit a classification similar to [8, Theorem 3.4.2], and which are stable under all appropriate $\widehat{\mathcal{D}}$-module pushforwards and pullbacks.

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The affine Grassmannian and the (Langlands dual) Springer resolution in positive characteristic

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(joint work with Pramod Achar)

1. Introduction

A well known result of Arkhipov–Bezrukavnikov–Ginzburg relates constructible sheaves on the affine Grassmannian, coherent sheaves on the Springer resolution, and representations of the quantum group (notation to be defined)

$$(*) \qquad \mathrm{D}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, \mathbb{C}) \xrightarrow{\sim} \mathrm{DCoh}^{G^\vee \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \xleftarrow{\sim} \mathrm{D}^{\mathrm{mix}}(\mathrm{Rep}(U_q)_{\mathbf{0}}).$$

My goal in this talk is to explain recent progress towards a modular version of this theorem.

Let G be a connected reductive algebraic group over $\mathbb{C}$, and let k be an algebraically closed field. In what follows, we consider the k-linear category of perverse sheaves on the affine Grassmannian $\mathcal{G}r$ of G. We stratify $\mathcal{G}r$ in two ways: by spherical orbits and by Iwahori orbits.

Spherical Orbits. The $G_{\mathbf{o}} = G(\mathbb{C}[[t]])$ orbits of $\mathcal{G}r$ are labeled by dominant coweights $\mathbb{X}^+$ for G. The category of spherical perverse sheaves $\mathrm{Perv}_{G_{\mathbf{o}}}(\mathcal{G}r, k)$ has a convolution $* : \mathrm{Perv}_{G_{\mathbf{o}}}(\mathcal{G}r, k) \times \mathrm{Perv}_{G_{\mathbf{o}}}(\mathcal{G}r, k) \to \mathrm{Perv}_{G_{\mathbf{o}}}(\mathcal{G}r, k)$.

The geometric Satake theorem [9, 12, 7] describes the tensor category of representations of the Langlands dual group $G^\vee$ in terms of spherical perverse sheaves on the affine Grassmannian. The proof due to Mirković and Vilonen works for integral coefficients, and hence, they deduce the theorem for any field k.

Theorem (Geometric Satake, Mirković–Vilonen [12])**.** *We have an equivalence of tensor categories:*

$$\mathcal{S} : (\mathrm{Rep}(G^\vee, k), \otimes) \xrightarrow{\sim} (\mathrm{Perv}_{G_{\mathbf{o}}}(\mathcal{G}r, k), *).$$

Iwahori Orbits. A spherical orbit labeled by $\lambda \in \mathbb{X}^+$ is a union of $|W\lambda|$-many Iwahori orbits. This stratifies $\mathcal{G}r$ into cells indexed by all coweights $\mathbb{X}$ for G. We denote an Iwahori orbit by $I\lambda$ for $\lambda \in \mathbb{X}$.

In [6], Finkelberg–Mirković suggest another equivalence relating Iwahori constructible perverse sheaves on $\mathcal{G}r$ to the block of $\mathrm{Rep}(G^\vee, k)$ containing the trivial representation, denoted $\mathrm{Rep}_{\mathbf{0}}(G^\vee, k)$.

Conjecture (Finkelberg–Mirković). *There is an equivalence of abelian categories*

$$\mathcal{P} : \mathrm{Perv}_{(I)}(\mathcal{G}r, k) \xrightarrow{\sim} \mathrm{Rep}_{\mathbf{0}}(G^\vee, k).$$

Characteristic 0 and the quantum group. Let $U_q = U_q(\mathfrak{g}^\vee)$ be the quantum group at an odd root of unity (of order bigger than the Coxeter number, coprime to 3 in type G_2). Arkhipov–Bezrukavnikov–Ginzburg proved in [4] the following characteristic 0 version of the Finkelberg–Mirković Conjecture

$$\mathrm{Perv}_{(I)}(\mathcal{G}r, \mathbb{C}) \cong \mathrm{Rep}(U_q)_{\mathbf{0}}.$$

The main idea of their proof is relating both categories to coherent sheaves on the Springer resolution $\tilde{\mathcal{N}}$ for $G^\vee$.

2. Progress in positive characteristic

In recent years, Soergel's bimodules have inspired the discovery of a remarkable collection of objects in the constructible derived category. These are Juteau–Mautner–Williamson's parity sheaves (definition in our setting below).

Theorem (Juteau–Mautner–Williamson [8]). *For each $\lambda \in \mathbb{X}$, there is a unique indecomposable $\mathcal{E}_\lambda \in \mathrm{D}_{(I)}(\mathcal{G}r, k)$ such that*

- *the support of $\mathcal{E}_\lambda \subset \overline{I\lambda}$;*
- *$\mathcal{E}_\lambda|_{I\lambda} = \underline{k}[\mathrm{dim} I\lambda]$; and*
- *the stalks and costalks of $\mathcal{E}_\lambda$ are concentrated in degrees congruent to $\dim(I\lambda)$ mod 2.*

Definition. [8] The chain complex $\mathcal{E} \in \mathrm{D}_{(I)}(\mathcal{G}r, k)$ is called a parity sheaf if $\mathcal{E} \cong \bigoplus_i \mathcal{E}_{\lambda_i}[d_i]$ for some finite collection $\lambda_i \in \mathbb{X}$ and $d_i \in \mathbb{Z}$. Let $\mathrm{Parity}_{(I)}(\mathcal{G}r)$ be the additive category of parity sheaves. (Note: The definition of parity sheaves does not require an affine stratification, but they may not exist in general.)

Some evidence that parity sheaves are interesting geometric objects to consider is the following theorem due to Juteau–Mautner–Williamson with some mild assumptions on $\mathrm{char}(k)$ recently improved in [11] by Mautner–Riche to good characteristic for $G^\vee$.

Theorem (Juteau–Mautner–Williamson). *For $\lambda \in \mathbb{X}^+$, the indecomposable parity sheaf $\mathcal{E}_\lambda$ is perverse and corresponds under geometric Satake to the indecomposable tilting $G^\vee$ representation of highest weight λ.*

This theorem turns out to be equivalent to the Mirković–Vilonen conjecture in [12] (suitably modified by Juteau):

Theorem 1 ([2]). *The $\mathbb{Z}$-stalks of spherical intersection cohomology sheaves on $\mathcal{G}r$ have no p-torsion for p a good prime.*

A key component of our proof of this fact (see [2]) is an extension of JMW's theorem when the derived group of $G^\vee$ is simply connected, and its Lie algebra admits a nondegenerate $G^\vee$-invariant bilinear form which we assume from now on.

Theorem 2 ([2]). *$\mathcal{S}$ extends to an equivalence of additive categories*

$$\mathrm{Parity}_{(G_o)}(\mathcal{G}r) \cong \mathrm{Tilt}(\mathcal{N}).$$

Here, $\mathcal{N}$ is the nilpotent cone for $G^\vee$ and $\mathrm{Tilt}(\mathcal{N})$ is the category of tilting perverse coherent sheaves on $\mathcal{N}$. We interpret Ext between two parity sheaves in a manner similar to Ginzburg.

Another application of parity sheaves is Achar–Riche's definition of the mixed modular derived category of sheaves on a flag variety (see [1]). This technology gives one interpretation of a modular version of the first equivalence in $(*)$, which is our main theorem.

Theorem 3 ([3]). *Assume* $\mathrm{char}(k)$ *is good for $G^\vee$. We have an equivalence of triangulated categories:*

$$\mathcal{P} : \mathrm{D}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, k) \xrightarrow{\sim} \mathrm{DCoh}^{G^\vee \times \mathbb{G}_m}(\tilde{\mathcal{N}})$$

such that $\mathcal{P}(F\langle 1\rangle) \cong \mathcal{P}(F)\langle -1\rangle[1]$. Furthermore, $\mathcal{P}$ is compatible with geometric Satake, i.e. for $V \in \mathrm{Rep}(G^\vee, k)$, $\mathcal{P}(F * \mathcal{S}(V)) \cong \mathcal{P}(F) \otimes V$.

Our proof of Theorem 3 requires stronger restrictions on the characteristic of k, but [11] implies the argument is valid in good characteristic.

The category $\mathrm{DCoh}^{G^\vee \times \mathbb{G}_m}(\tilde{\mathcal{N}})$ has a t-structure defined by Bezrukavnikov in [5]. The heart of this t-structure is denoted $\mathrm{ExCoh}(\tilde{\mathcal{N}})$, and objects therein are called exotic coherent sheaves. The category $\mathrm{ExCoh}(\tilde{\mathcal{N}})$ is a graded highest weight category ([10]) with indecomposable tilting objects in bijection with $\mathbb{X} \times \mathbb{Z}$.

Corollary 1 ([3]). *$\mathcal{P}$ restricts to an equivalence of additive categories:*

$$\mathrm{Parity}_{(I)}(\mathcal{G}r) \cong \mathrm{Tilt}(\tilde{\mathcal{N}}).$$

In our proof of Theorem 3, we rely heavily on so-called *Wakimoto sheaves.* These are the constructible analogue of line bundles on $\tilde{\mathcal{N}}$. We prove that Wakimoto sheaves lift to the mixed category. We then construct two functors: the correct functor which is triangulated, and another naive functor which is only additive. However, the naive functor has the advantage that we can compute the output. We then use weight arguments to show that the images of these functors match for nice enough objects. One of the reasons we expected this method to work is that we knew that tilting exotic coherent sheaves, a priori chain complexes of coherent sheaves, should be coherent. This also becomes a consequence of our theorem and the study of the transport of the exotic t-structure on $\mathrm{DCoh}^{G^\vee \times \mathbb{G}_m}(\tilde{\mathcal{N}})$ to $\mathrm{D}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, k)$.

Corollary 2 ([3]). *Let $\mathcal{T} \in \mathrm{Tilt}(\tilde{\mathcal{N}})$. Then $\mathcal{T}$ is a coherent sheaf on $\tilde{\mathcal{N}}$.*

Remark 3. In [11], Mautner–Riche have proven Theorem 3 by very different methods. They utilize braid group actions on both categories. They first prove Corollary 1 and obtain Theorem 3 as a consequence. Note that their proof does *not* rely on geometric Satake. Their theorem has fewer restrictions on characteristic of k, but they do not prove that their equivalence is compatible with geometric Satake. Mautner–Riche also prove Corollary 2 in [10] without relying on the geometry of the affine Grassmannian.

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Finite parabolic conjugation and commuting varieties

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(joint work with Michaël Bulois)

In Algebraic Lie Theory, a common topic is the study of algebraic group actions on affine varieties, for example of the adjoint action of a reductive algebraic group on its Lie algebra and numerous variants thereof.

Let $\mathcal{N}$ be the nilpotent cone of all nilpotent complex matrices of square size n. Consider a standard parabolic subgroup $P \subset \mathrm{GL}_n(\mathbf{C})$ of block sizes $(b_1, ..., b_p)$. We examine the conjugation action of P on the variety $\mathcal{N}_{\mathfrak{p}} := \mathcal{N} \cap \mathfrak{p}$, where $\mathfrak{p}$ is the Lie algebra corresponding to P.

The main question posed is:

"For which P are there only finitely many orbits?"

One motivation why to consider this question comes up in the context of commuting varieties:

Consider the nilpotent commuting variety $\mathcal{C}(\mathcal{N}_\mathfrak{p}) := \{(x,y) \in \mathcal{N}_\mathfrak{p} \times \mathcal{N}_\mathfrak{p} \mid [x,y] = 0\}$ of $\mathfrak{p}$. The parabolic P acts diagonally on $\mathcal{C}(\mathcal{N}_\mathfrak{p})$, and the study of certain cases [4] suggests the following conjecture of M. Bulois to be true:

"$\mathcal{C}(\mathcal{N}_\mathfrak{p})$ is equidimensional if and only if the number of P-orbits in $\mathcal{N}_\mathfrak{p}$ is finite."

Note that there is a connection to nested punctual Hilbert schemes, which is examined in [4] for certain cases of the above stated setup. Thus, answering the main question and examining the stated conjecture should provide further insights into nested punctual Hilbert schemes.

There is a number of articles which fit into the given context and deal with similar problems; we sum them up briefly.

To start, a criterium for P-conjugation on the nilradical $\mathfrak{n}_\mathfrak{p}$ of $\mathfrak{p}$ to admit only finitely many orbits is given in [5]. The classification is rather beautiful, since it depends only on the number of blocks of P: The number of P-orbits in $\mathfrak{n}_\mathfrak{p}$ is finite if and only if P is a parabolic of at most 5 blocks.

The study of the enhanced nilpotent cone [1] yields that the number of orbits for the above action is finite, if P is maximal of block sizes $(1, n-1)$.

Given $x \leq n$, let us define the variety $\mathcal{N}^{(x)}$ of all x-nilpotent matrices, that is, of matrices $N \in \mathbf{C}^{n\times n}$, such that $N^x = 0$. A criterion for P-conjugation on $\mathcal{N}^{(x)}$ to admit only finitely many orbits can be found in [2].

Clearly, the study of spherical varieties [3]; in particular of spherical nilpotent G-orbits in semi-simple Lie algebras $\mathfrak{g} = \mathrm{Lie}\ G$ [7] fit into this context, as well.

We make use of methods from Representation Theory by translating the given setup via an associated fibre product. Therefore, let us define the quiver $\mathcal{Q}_p$ given by

$$\mathcal{Q}_p := \quad \begin{tikzcd} \underset{1}{\bullet} \arrow[loop above, "\beta_1"] \arrow[r, "\alpha_1"] & \underset{2}{\bullet} \arrow[loop above, "\beta_2"] \arrow[r, "\alpha_2"] & \underset{3}{\bullet} \arrow[loop above, "\beta_3"] \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{p-3}"] & \underset{\mathrm{p}-2}{\bullet} \arrow[loop above, "\beta_{p-2}"] \arrow[r, "\alpha_{p-2}"] & \underset{\mathrm{p}-1}{\bullet} \arrow[loop above, "\beta_{p-1}"] \arrow[r, "\alpha_{p-1}"] & \underset{\mathrm{p}}{\bullet} \arrow[loop above, "\beta_p"] \end{tikzcd}$$

together with the admissible ideal I generated by commutativity relations $\alpha_i\beta_i = \beta_{i+1}\alpha_i$ (where $i \in \{1, ..., p-1\}$) and nilpotency conditions $\beta_i^n = 0$ at the loops (where $i \in \{1, ..., p\}$).

Let $A_p := \mathbf{C}\mathcal{Q}_p/I$ be the corresponding finite-dimensional path algebra with relations. We define the dimension vector $\underline{d}_P := (b_1, b_1 + b_2, ..., b_1 + ... + b_p = n)$ and denote the category of (complex) representations of A_p of dimension vector $\underline{d}_P$ by $\mathrm{rep}A_p(\underline{d}_P)$. Let us consider the full subcategory defined by the condition that all maps at the arrows α_i are injective and denote it by $\mathrm{rep}^{\mathrm{inj}}A_p(\underline{d}_P)$.

Then, by constructing an associated fibre product, it can be shown that the P-orbits in $\mathcal{N}_\mathfrak{p}$ correspond bijectively to the isomorphism classes in $\mathrm{rep}^{\mathrm{inj}}A_p(\underline{d}_P)$.

The main question can, thus, be approached with known methods from Representation Theory (of finite-dimensional algebras); for example Auslander-Reiten techniques, in particular by making use of coverings, are available.

Let us denote the universal covering quiver of A_p by $\hat{Q}_p$ and the subquiver of n rows by $\hat{Q}_p(n)$; we denote its vertices by (i,j) for $1 \leq i \leq n$ and $1 \leq j \leq p$; increasing from top to bottom and from left to right. For each such vertex, let $D_{i,j}$ be the module of $\hat{Q}_p(n)$, given by vector spaces

$$D(i,j)_{k,l} = \begin{cases} \mathbf{C} & \text{if } i = k \text{ and } l \geq j \\ 0 & \text{otherwise} \end{cases}$$

together with the usual zero and identity maps. Let Δ be the set of all these representations $D(i,j)$, then the category of Δ-filtered modules is denoted by $\mathcal{F}(\Delta)$.

We can show that the number of isomorphism classes in $\mathrm{rep}^{\mathrm{inj}}(A_p)(\underline{d}_P)$ is finite if and only if the number of isomorphism classes in $\mathcal{F}(\Delta)$ of expanded dimension vectors (via the covering map) summing up to $\underline{d}_P$ is finite; we have thus found another translation of our main question.

In this context, boxes and, in particular, the box reduction algorithm may be used to decide the remaining finite cases via an equivalence of categories introduced in [6]. The calculation is not finished yet, but the progress suggests our methods to work in order to prove finiteness in certain cases.

We show two kinds of reductions: Assume that P acts on $\mathcal{N}_{\mathfrak{p}}$ with infinitely many orbits. Let Q be a parabolic subgroup of $\mathrm{GL}_n(\mathbf{C})$ with $\mathfrak{q} := \mathrm{Lie}Q$, such that one of the following holds:

(1) $Q \subseteq P$
(2) The block sizes of P build a subpartition of the block sizes of Q

Then Q acts on $\mathcal{N}_{\mathfrak{q}}$ with infinitely many orbits, as well.

The criterion is not finished yet, but it progresses and we expect the described methods to work. We, thus, provide an overview of the actual status and so far results.

Combining known results of [5] and one of our reductions, we can show that there are infinitely many orbits, if the parabolic has more than 5 blocks and admits finitely many orbits, if it is a Borel of size $n \leq 5$.

If the block sizes of P are $(1, n-1)$, then P admits only a finite number of orbits in $\mathcal{N}_{\mathfrak{p}}$, which is shown in [4] and also follows from [1]. The cases $(2, n-2)$ and $(3, n-3)$ admit only finitely many orbits, as well.

Many infinite families are displayed, these can (amongst others) be found by adapting one parameter families of type $\tilde{D}_4$ and $\tilde{E}_6$. We can show that there are infinitely many P-orbits in $\mathcal{N}_{\mathfrak{p}}$ as soon as the block sizes of P contain $(6,6)$, $(2,2,2)$, $(1,4,6)$, $(4,1,4)$, $(1,2,1,4)$, $(1,4,4,1)$ or $(1,2,1,2,1)$ as subpartitions.

Combined with the above mentioned reductions, we expect all open cases to admit only a finite number of orbits - and, thus, assume to be able to prove finiteness by box reduction.

There are many questions, which should be considered after finishing the criterion, first of all the concrete classification of the orbits in all finite cases and the translation to commuting varieties. Generalizations come to mind, for example to different Lie types. A first possible way how to manage the latter, might be to try to adapt methods used in [5].

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Three regimes for torsion in Schubert and quiver varieties

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The study of torsion in the stalks and costalks of interesection cohomology sheaves of Schubert varieties and (Lusztig) quiver varieties is a problem that is of fundamental importance in modular representation theory. We identify and discuss three different qualitative types of behaviour which can occur, and discuss examples of each such type of behaviour. These possibilities are for there to be no torsion, tamely controlled torsion and exponential growth of torsion.

Let $\mathfrak{g}$ be a symmetrisable Kac-Moody Lie algebra and w an element of its Weyl group. We let $A_q(\mathfrak{n}(w))$ be the quantised coordinate ring which is a quantisation of the coordinate ring of the unipotent group whose Lie algebra is the span of the positive roots α such that $w\alpha$ is negative. For each prime p (including $p = 0$) this algebra has a basis, called the dual p-canonical basis, obtained via categorification. This categorification either uses indecomposable summands of Lusztig sheaves with coefficients in a field of characteristic p, or representations of KLR algebras (with a diagram automorphism if necessary) over a field of characteristic p.

Building on recent work of Kang, Kashiwara, Kim and Oh [KKKO1, KKKO2], who prove the $p = 0$ version of the below theorem when $\mathfrak{g}$ is symmetric, we are able to prove the following:

Theorem 4. [Mc] The quantised coordinate ring $A_q(\mathfrak{n}(w))$ has the structure of a quantum cluster algebra in which the cluster monomials lie in the dual p-canonical basis.

The quantum cluster algebra structure on $A_q(\mathfrak{n}(w))$ is defined independently of p, in terms of an initial seed consisting of quantum generalised minors. This theorem thus provides a region where the dual p-canonical bases for different p all agree, or equivalently that certain decomposition numbers for KLR algebras are trivial. By [W1], this is connected to the nonexistence of torsion in the intersection cohomology of Lusztig quiver varieties.

We now turn our example to examples of torsion. For primes p dividing the off-diagonal entries of the Cartan matrix, the existence of p-torsion in Schubert varieties has been known since near the beginning of the theory of intersection cohomology. More interesting are the examples of 2-torsion in the A_7 and D_4 flag varieties discovered by Braden [B], and p-torsion in the A_{4p-1} flag variety discovered by Polo (unpublished).

We provide some examples of tame families of torsion, which are smoothly equivalent to well-understood examples of torsion in quiver varieties of affine type. These families provide examples of p-torsion in quiver varieties of type A_5 and D_4 (the former generalising the famous singularity of Kashiwara-Saito) and p-torsion within a single left cell in the flag variety of type A_{6p-1} (generalising an example of Williamson [W3]).

Now we turn to examples of examples of torsion which grow exponentially, the first families of which were constructed by Williamson [W2] using the diagrammatic calculus of Soergel bimodules. In joint work with Kontorovich and Williamson, we are able to use recent advances in analytic number theory (the affine sieve, or the recent progress on Zaremba's conjecture) to prove the theorem below. We say there exists p-torsion in a variety X if there exists p-torsion in a stalk or costalk of the intersection cohomology complex $IC(X;\mathbb{Z})$.

Theorem 5. [KMW] There exists a constant $c > 1$ such that the set

$$\{p > c^N \mid p \text{ is prime and there exists } p\text{-torsion in a Schubert variety in } SL_N\}$$

is nonempty for all sufficiently large N.

There is a more precise version with a lower bound on the growth of the size of this set in [KMW].

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Towards a cluster structure on trigonometric zastava

Michael Finkelberg

(joint work with Alexander Kuznetsov, Leonid Rybnikov, Galina Dobrovolska)

1.1. **Zastava and euclidean monopoles.** Let G be an almost simple simply connected algebraic group over $\mathbb{C}$. We denote by $\mathcal{B}$ the flag variety of G. Let us also fix a pair of opposite Borel subgroups B, B_- whose intersection is a maximal torus T. Let Λ denote the cocharacter lattice of T; since G is assumed to be simply connected, this is also the coroot lattice of G. We denote by $\Lambda_+ \subset \Lambda$ the sub-semigroup spanned by positive coroots.

It is well-known that $H_2(\mathcal{B}, \mathbb{Z}) = \Lambda$ and that an element $\alpha \in H_2(\mathcal{B}, \mathbb{Z})$ is representable by an effective algebraic curve if and only if $\alpha \in \Lambda_+$. Let $\overset{\circ}{Z}{}^\alpha$ denote the space of maps $C = \mathbb{P}^1 \to \mathcal{B}$ of degree α sending $\infty \in \mathbb{P}^1$ to $B_- \in \mathcal{B}$. It is known [5] that this is a smooth symplectic affine algebraic variety, which can be identified with the hyperkähler moduli space of framed G-monopoles on $\mathbb{R}^3$ with maximal symmetry breaking at infinity of charge α [7], [8].

The monopole space $\overset{\circ}{Z}{}^\alpha$ has a natural partial compactification Z^α (*zastava* scheme). It can be realized as the moduli space of based *quasi-maps* of degree α; set-theoretically it can be described in the following way:

$$Z^\alpha = \bigsqcup_{0 \le \beta \le \alpha} \overset{\circ}{Z}{}^\beta \times \mathbb{A}^{\alpha - \beta},$$

where for $\gamma \in \Lambda_+$ we denote by $\mathbb{A}^\gamma$ the space of all colored divisors $\sum \gamma_i x_i$ with $x_i \in \mathbb{A}^1$, $\gamma_i \in \Lambda_+$ such that $\sum \gamma_i = \gamma$.

The zastava space is equipped with a *factorization* morphism $\pi_\alpha : \ Z^\alpha \to \mathbb{A}^\alpha$ whose restriction to $\overset{\circ}{Z}{}^\alpha \subset Z^\alpha$ has a simple geometric meaning: for a based map $\varphi \in \overset{\circ}{Z}{}^\alpha$ the colored divisor $\pi_\alpha(\phi)$ is just the pullback of the colored Schubert divisor $D \subset \mathcal{B}$ equal to the complement of the open B-orbit in $\mathcal{B}$. The morphism $\pi_\alpha : \ \overset{\circ}{Z}{}^\alpha \to \mathbb{A}^\alpha$ is the *Atiyah-Hitchin* integrable system (with respect to the above symplectic structure): all the fibers of π_α are Lagrangian.

A system of étale birational coordinates on $\overset{\circ}{Z}{}^\alpha$ was introduced in [5]. Let us recall the definition for $G = SL(2)$. In this case $\overset{\circ}{Z}{}^\alpha$ consists of all maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree α which send ∞ to 0. We can represent such a map by a rational function $\frac{R}{Q}$ where Q is a monic polynomial of degree α and R is a polynomial of degree $< \alpha$. Let $w_1, \ldots, w_\alpha$ be the zeros of Q. Set $y_r = R(w_r)$. Then the functions $(y_1, \ldots, y_\alpha, w_1, \ldots, w_\alpha)$ form a system of étale birational coordinates on $\overset{\circ}{Z}{}^\alpha$, and the above mentioned symplectic form in these coordinates reads $\Omega_{\mathrm{rat}} = \sum_{r=1}^{\alpha} \frac{dy_r \wedge dw_r}{y_r}$.

For general G the definition of the above coordinates is quite similar. In this case given a point in $\overset{\circ}{Z}{}^\alpha$ we can define polynomials R_i, Q_i where i runs through the set I of vertices of the Dynkin diagram of G, $\alpha = \sum a_i\alpha_i$, and

(1) Q_i is a monic polynomial of degree a_i,

(2) R_i is a polynomial of degree $< a_i$.

Hence, we can define (étale, birational) coordinates $(y_{i,r}, w_{i,r})$ where $i \in I$ and $r = 1, \dots, a_i$. Namely, $w_{i,r}$ are the roots of Q_i, and $y_{i,r} = R_i(w_{i,r})$. The Poisson brackets of these coordinates with respect to the above symplectic form are as follows: $\{w_{i,r}, w_{j,s}\}_{\mathrm{rat}} = 0$, $\{w_{i,r}, y_{j,s}\}_{\mathrm{rat}} = \check{d}_i\delta_{ij}\delta_{rs}y_{j,s}$, $\{y_{i,r}, y_{j,s}\}_{\mathrm{rat}} = (\check{\alpha}_i, \check{\alpha}_j)\frac{y_{i,r}y_{j,s}}{w_{i,r}-w_{j,s}}$ for $i \neq j$, and finally $\{y_{i,r}, y_{i,s}\}_{\mathrm{rat}} = 0$. Here $\check{\alpha}_i$ is a simple root, $(,)$ is the invariant scalar product on $(\mathrm{Lie}\, T)^*$ such that the square length of a short root is 2, and $\check{d}_i = (\check{\alpha}_i, \check{\alpha}_i)/2$.

Now recall that the standard *rational* r-matrix for $\mathfrak{g} = \mathrm{Lie}\, G$ gives rise to a Lie bialgebra structure on $\mathfrak{g}[z^{\pm 1}]$ corresponding to the Manin triple $\mathfrak{g}[z], z^{-1}\mathfrak{g}[z^{-1}]$, $\mathfrak{g}[z^{\pm 1}]$. This in turn gives rise to a Poisson structure on the affine Grassmannian $\mathrm{Gr}_G = G[z^{\pm 1}]/G[z]$. The transversal slices $\mathcal{W}^\lambda_\mu$ from a $G[z]$-orbit Gr^μ_G to another orbit Gr^λ_G (here $\mu \leq \lambda$ are dominant coweights of G) are examples of symplectic leaves of the above Poisson structure. According to [1], the zastava spaces are "stable limits" of the above slices. More precisely, for $\alpha = \lambda - \mu$ there is a birational Poisson map $s^\lambda_\mu :\ \mathcal{W}^\lambda_\mu \to Z^\alpha$.

1.2. **Trigonometric zastava and monopoles.** We have an open subset $\mathbb{G}^\alpha_m \subset \mathbb{A}^\alpha$ (colored divisors not meeting $0 \in \mathbb{A}^1$), and we introduce the open subscheme of *trigonometric zastava* ${}^\dagger Z^\alpha := \pi_\alpha^{-1}\mathbb{G}^\alpha_m \subset Z^\alpha$, and its smooth open affine subvariety of *trigonometric monopoles* ${}^\dagger\overset{\circ}{Z}{}^\alpha := {}^\dagger Z^\alpha \cap \overset{\circ}{Z}{}^\alpha$. These schemes are solutions of the following modular problems.

Let $C^\dagger$ be an irreducible nodal curve of arithmetic genus 1 obtained by gluing the points $0, \infty \in C = \mathbb{P}^1$, so that $\pi :\ C \to C^\dagger$ is the normalization. Let $\mathfrak{c} \in C^\dagger$ be the singular point. The moduli space $\mathrm{Bun}^0_T(C^\dagger)$ of T-bundles on $C^\dagger$ of degree 0 is canonically identified with the Cartan torus T itself. We fix a T-bundle $\mathcal{F}_T$ which corresponds to a *regular* point $t \in T^{\mathrm{reg}}$. Then ${}^\dagger\overset{\circ}{Z}{}^\alpha$ is the moduli space of the following data:

(a) a trivialization $\tau_{\mathfrak{c}}$ of the fiber of $\mathcal{F}_T$ at the singular point $\mathfrak{c} \in C^\dagger$;

(b) a B-structure ϕ in the induced G-bundle $\mathcal{F}_G = \mathcal{F}_T \overset{T}{\times} G$ of degree α which is transversal to $\mathcal{F}_B = \mathcal{F}_T \overset{T}{\times} B$ at $\mathfrak{c}$.

The scheme ${}^\dagger Z^\alpha$ is the moduli space of the similar data with the only difference: we allow a B-structure in (b) to be *generalized* (i.e. to acquire defects at certain points of $C^\dagger$), but require it to have no defect at $\mathfrak{c}$.

As a regular Cartan element t varies, the above moduli spaces become fibers of a single family. More precisely, we consider the following moduli problem:

(t) a T-bundle $\mathcal{F}_T$ of degree 0 on $C^\dagger$ corresponding to a regular element of T;

(a,b) as above;

(c) a trivialization $f_{\mathfrak{c}}$ at $\mathfrak{c}$ of the T-bundle ϕ_T induced from the B-bundle ϕ in (b). This moduli problem is represented by a scheme $\overset{\circ}{Y}{}^\alpha \subset Y^\alpha$ (depending on whether the B-structure in (b) is genuine or generalized). Note that Y^α is equipped with an action of $T \times T$ changing the trivializations in (a,c). We prove that $\overset{\circ}{Y}{}^\alpha$ is a smooth affine variety equipped with a natural projection $\varpi : \overset{\circ}{Y}{}^\alpha \to {}^\dagger\overset{\circ}{Z}{}^\alpha$, and we construct a nondegenerate bivector field on $\overset{\circ}{Y}{}^\alpha$ arising from a differential in a spectral sequence involving the tangent and cotangent bundles of $\overset{\circ}{Y}{}^\alpha$ (this construction is a trigonometric degeneration of the construction [4] for elliptic curves; its rational analogue was worked out in [5]). This bivector field descends to ${}^\dagger\overset{\circ}{Z}{}^\alpha$ under the projection $\varpi : \overset{\circ}{Y}{}^\alpha \to {}^\dagger\overset{\circ}{Z}{}^\alpha$. The Poisson brackets of the coordinates in the corresponding symplectic structure of ${}^\dagger\overset{\circ}{Z}{}^\alpha$ are as follows: $\{w_{i,r}, w_{j,s}\}_{\mathrm{trig}} = 0$, $\{w_{i,r}, y_{j,s}\}_{\mathrm{trig}} = \check{d}_i \delta_{ij} \delta_{rs} w_{j,s} y_{j,s}$, $\{y_{i,r}, y_{j,s}\}_{\mathrm{trig}} = (\check{\alpha}_i, \check{\alpha}_j) \frac{(w_{i,r}+w_{j,s}) y_{i,r} y_{j,s}}{2(w_{i,r}-w_{j,s})}$ for $i \neq j$, and finally $\{y_{i,r}, y_{i,s}\}_{\mathrm{trig}} = 0$. In particular, the projection $\pi_\alpha : {}^\dagger\overset{\circ}{Z}{}^\alpha \to \mathbb{G}_m^\alpha$ is the *trigonometric Atiyah-Hitchin* integrable system (for $G = SL(2)$ this system goes back at least to [3]).

Now recall that the standard *trigonometric* r-matrix for $\mathfrak{g}$ gives rise to a Lie bialgebra structure on $\mathfrak{g}((z^{-1}))$ which in turn gives rise to a Poisson structure on the affine flag variety $\mathcal{F}\ell_G$ (the quotient of $G[z^{\pm 1}]$ with respect to an Iwahori subgroup). The symplectic leaves of this Poisson structure are the intersections $\mathcal{F}\ell_y^w$ of the opposite Iwahori orbits (aka *open Richardson varieties*). Here w, y are elements of the affine Weyl group $W_a = W \ltimes \Lambda$. For dominant coweights $\mu \leq \lambda \in \Lambda$, and the longest element w_0 of the finite Weyl group W, the projection $\mathrm{pr} : \mathcal{F}\ell_{w_0 \times \mu}^{w_0 \times \lambda} \to \mathcal{W}_\mu^\lambda$ is an open embedding, and the composition $s_\mu^\lambda \circ \mathrm{pr} : \mathcal{F}\ell_{w_0 \times \mu}^{w_0 \times \lambda} \to Z^\alpha$ is a *symplectomorphism* onto its image ${}^\dagger\overset{\circ}{Z}{}^\alpha \subset Z^\alpha$ equipped with the trigonometric symplectic structure.

1.3. **Cluster aspirations.** It seems likely that the construction due to B. Leclerc [9] extends from the open Richardson varieties in the finite flag varieties to the case of the affine flag varieties, and provides $\mathcal{F}\ell_y^w$ with a cluster structure. This structure can be transferred from $\mathcal{F}\ell_{w_0 \times \mu}^{w_0 \times \lambda}$ to ${}^\dagger\overset{\circ}{Z}{}^\alpha$ via the above symplectomorphism. In case of $G = SL(2)$ the resulting cluster structure on the moduli space of trigonometric monopoles was discovered in [6], which served as the starting point of the present note. It seems likely that for general G the Gaiotto-Witten superpotential on Z^α (see e.g. [2]) restricted to ${}^\dagger\overset{\circ}{Z}{}^\alpha$ is totally positive in the above cluster structure.

Remark. Implicit in the above discussion when α is *dominant* (as a coweight of G) is an affine open embedding ${}^\dagger\overset{\circ}{Z}{}^\alpha \subset \overset{\circ}{Z}{}^\alpha \hookrightarrow \mathbb{A}^{2|\alpha|}$ into an affine space. Here is a modular interpretation: $\mathbb{A}^{2|\alpha|}$ is the moduli space of B-bundles ϕ_B on $\mathbb{P}^1$ equipped with a trivialization $(\phi_B)_\infty \xrightarrow{\sim} B$ of the fiber at $\infty \in \mathbb{P}^1$, such that the induced T-bundle (under projection $B \twoheadrightarrow T$) has degree α.

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Minimal rational curves on reductive group compactifications

Michel Brion

(joint work with Baohua Fu)

The talk began with a brief survey of rational curves on algebraic varieties; a standard reference for this topic is Kollár's book [9]. Let X be a smooth projective algebraic variety over the field of complex numbers. A *rational curve on X* is the image of a non-constant morphism $f : \mathbb{P}^1 \to X$; in particular, such a curve is reduced and irreducible, possibly singular. A parameter space for rational curves has been constructed in [9]; it is a (generally infinite) disjoint union of irreducible quasiprojective varieties, called *families of rational curves*. Given such a family $\mathcal{K}$ and a point $x \in X$, the subvariety of curves through x is denoted by $\mathcal{K}_x$. The family $\mathcal{K}$ is called *covering* if $\mathcal{K}_x$ is non-empty for a general point x; if in addition $\mathcal{K}_x$ is projective, then $\mathcal{K}$ is called *minimal*.

The *tangent map* sends each rational curve which is smooth at x to its tangent direction at that point; this defines a rational map $\tau : \mathcal{K}_x \dashrightarrow \mathbb{P}(T_xX)$ (the projectivization of the tangent space to X at x). For a minimal family $\mathcal{K}$ and a general point x, the tangent map is a finite morphism, birational over its image (see [8], [6]). This image is called the *variety of minimal rational tangents* (VMRT) of X and denoted by $\mathcal{C}_x \subset \mathbb{P}(T_xX)$. The VMRT is an intrinsic invariant of the algebraic variety X; it plays an important rôle in questions of global rigidity, as shown by Mok and Hwang (see e.g. [7]).

Next, assume that X is *almost homogeneous*, i.e., it is equipped with an action of a connected linear algebraic group G having an open orbit X^0. Choose $x \in X^0$ and denote by $H \subset G$ its isotropy group. This identifies X^0 with the homogenous space G/H, and T_xX with the quotient of Lie algebras $\mathfrak{g}/\mathfrak{h}$, the *isotropy representation* of H. One easily shows that a family of rational curves $\mathcal{K}$ on X is covering if and only if $\mathcal{K}_x$ is non-empty; then $\mathcal{K}_x$ is smooth and its components are permuted transitively by H (in particular, $\mathcal{K}_x$ is connected if so is H). The description of the minimal families and their associated VMRTs is an open question in general.

When X is homogeneous, that is, H is a parabolic subgroup of G, the minimal rational curves are just lines, and their families have been determined by Hwang and Mok in a series of articles (see [4], [6], [7]); in all cases, the VMRT consists of 1 or 2 orbits of P (see also [10]). On the other hand, when X is toric, the VMRTs have been described by Chen, Fu and Hwang in [2]; all of them are linear subspaces of $\mathbb{P}(T_xX)$.

We recently treated the case where X is an *equivariant compactification* of a connected reductive algebraic group G, i.e., X is equipped with an action of $G \times G$ having an open orbit isomorphic to $G \cong (G \times G)/\operatorname{diag}(G)$; then the base point x is the neutral element of G, and the isotropy representation is the adjoint representation $\mathfrak{g}$.

When G is semisimple of adjoint type, a canonical equivariant compactification has been constructed by De Concini and Procesi in [3]; the boundary $\partial X := X \setminus G$ of this *wonderful compactification* is the union of ℓ smooth irreducible divisors with normal crossings, where ℓ denotes the rank of G, and the $G \times G$-orbit closures in X are exactly the partial intersections of the boundary divisors.

For an arbitrary reductive group G, an equivariant compactification is called *toroidal* if the homomorphism $G \to G/Z = G_{\mathrm{ad}}$ (the adjoint quotient) extends to a morphism $\varphi : X \to X_{\mathrm{ad}}$ (the wonderful compactification of G_{ad}). Then the boundary ∂X is still a divisor with smooth normal crossings. Also, the fiber of φ at x is smooth, and its component through x is a toric variety F under the action of the connected center Z^0; all components of the fiber are isomorphic to F.

Our main results can be stated as follows.

Theorem 1. *Let X be the wonderful compactification of a simple algebraic group G of adjoint type. Then X admits a unique family of minimal rational curves. Moreover, the corresponding VMRT $\mathcal{C}_x \subset \mathbb{P}\mathfrak{g}$ is the projectivization of the minimal nilpotent orbit, unless G is of type A_ℓ with $\ell \geq 2$; then $\mathcal{C}_x \cong \mathbb{P}^\ell \times (\mathbb{P}^\ell)^*$.*

More specifically, if $G = \mathrm{PGL}_{\ell+1}$, then $\mathfrak{g}$ is the quotient of the space of matrices of size $\ell + 1$, by the scalar matrices. The matrices of rank at most 1 form an affine cone over $\mathbb{P}^\ell \times (\mathbb{P}^\ell)^*$, which is sent isomorphically to its projection. The projectivization of the minimal nilpotent orbit is identified with the incidence variety in $\mathbb{P}^\ell \times (\mathbb{P}^\ell)^*$.

Theorem 2. *Let X be a toroidal compactification of a connected reductive group G, and $\mathcal{K}$ a family of minimal rational curves on X. Then either $\mathcal{K}_x$ consists of curves in the toric variety F, or it is isomorphic to the family of minimal rational curves of a simple factor of G. Moreover, distinct simple factors yield disjoint families, and all of them consist of smooth curves.*

In particular, $\mathcal{K}$ consists of the deformations of a rational curve in F, or of the closure in X of the image of an additive one-parameter subgroup $u_\theta : \mathbb{C} \to G$, where θ denotes a highest root of G (so that the G-orbit of any point of the corresponding root subspace $\mathfrak{g}_\theta$ is a minimal nilpotent orbit in $\mathfrak{g}$). This assertion follows readily from Borel's fixed point theorem; also, one easily shows that the closure $C := \overline{u_\theta(\mathbb{C})}$ is isomorphic to the projective line, and intersects ∂X at its point at infinity.

To complete the proof of the theorem, we determine the dimension of the space of rational curves on X at its point corresponding to the highest root curve C. For this, a natural approach would be to describe the normal bundle to C in X, but this seems to be out of reach at this stage. We rather compute the intersection numbers of C with the boundary divisors of X, by using the decomposition

$$G(\mathbb{C}((t))) = G(\mathbb{C}[[t]])\, P^\vee\, G(\mathbb{C}[[t]]),$$

where the coweight lattice $P^\vee$ is identified with the lattice of cocharacters of a maximal torus T of G; an argument of reduction to $\mathrm{SL}(2)$ shows that

$$u_\theta(t^{-1}) \in G(\mathbb{C}[[t]])\, \theta^\vee(t)\, G(\mathbb{C}[[t]]),$$

where of course, $\theta^\vee$ denotes the coroot of the highest root θ. Moreover, it is easy to determine the intersection numbers of the closure in X of an arbitrary multiplicative one-parameter subgroup $\eta : \mathbb{C}^* \to T$, with the boundary divisors (see [1, Sec. 3] for details).

It is a natural problem to extend these results to the class of *complete symmetric varieties* introduced in [3], i.e., to the toroidal compactifications of symmetric spaces G/H, where G is reductive. Work in progress with Baohua Fu and Nicolas Perrin yields partial results in this direction. They are based on ad hoc considerations for special classes of symmetric spaces; a unified approach is needed to complete the picture.

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On the center of quiver Hecke algebras

Peng Shan
(joint work with Michela Varagnolo, Eric Vasserot)

The quiver Hecke algebras is a family of $\mathbb{Z}$-graded algebras $R(\alpha)$ associated with a Kac-Moody algebra $\mathfrak{g}$, with α varies in the positive cone Q_+ of the root lattice. They have been introduced by Khovanov-Lauda and independently Rouquier to categorify the negative part of the quantum group. For each dominant weight Λ of $\mathfrak{g}$, the highest weight $\mathfrak{g}$-module $V(\Lambda)$ is categorified by the direct sum (over α) of the category of projective modules over a certain quotient $R^\Lambda(\alpha)$ of $R(\alpha)$, called the cyclotomic quiver Hecke algebras. We are interested in the structure of the center $Z(R^\Lambda(\alpha))$ and the cocenter $\mathrm{Tr}(R^\Lambda(\alpha))$ of $R^\Lambda(\alpha)$.

In [2] we construct $\mathbb{Z}$-graded representations of another Lie algebra $L\mathfrak{g}$ on

$$\mathrm{Tr}^\Lambda = \bigoplus_{\alpha\in Q_+} \mathrm{Tr}(R^\Lambda(\alpha)), \quad Z^\Lambda = \bigoplus_{\alpha\in Q_+} Z(R^\Lambda(\alpha)).$$

The algebra $L\mathfrak{g}$, defined in terms of generators and relations, is a variation of the current algebra. It coincides with $U(\mathfrak{g}[t])$ in type ADE. The $L\mathfrak{g}$-module Tr^Λ is cyclic. An induction argument allows to control the bounds on the grading on $\mathrm{Tr}(R^\Lambda(\alpha))$. Since the grading on $\mathrm{Tr}(R^\Lambda(\alpha))$ and on $Z(R^\Lambda(\alpha))$ are related by a homogenous symmetric form, we deduce that $Z(R^\Lambda(\alpha))$ is positively graded.

By work of Varagnolo [4], the same algebra $L\mathfrak{g}$ also acts on the cohomology of quiver varieites $\mathfrak{M}(\alpha,\Lambda)$ associated with $\mathfrak{g}$ and on the Borel-Moore homology of a Lagrangian subvariety $\mathfrak{L}(\alpha,\Lambda)$ of $\mathfrak{M}(\alpha,\Lambda)$.

Theorem 1. *If $\mathfrak{g}$ is of type ADE, there are isomorphisms of $\mathbb{Z}$-graded $L\mathfrak{g}$-modules*

$$\mathrm{Tr}^\Lambda \simeq \bigoplus_{\alpha\in Q_+} H^{\mathrm{BM}}_{[*]}(\mathfrak{L}(\alpha,\Lambda)), \quad Z^\Lambda \simeq \bigoplus_{\alpha\in Q_+} H^*(\mathfrak{M}(\alpha,\Lambda)).$$

The first module is called the local Weyl module, and the second one the local dual Weyl module. There is also a global version of the theorem, where we consider the equivariant (co)homology with respect to the action of a group G_Λ on $\mathfrak{L}(\alpha,\Lambda)$ and $\mathfrak{M}(\alpha,\Lambda)$. On the left hand side, the algebra $R^\Lambda(\alpha)$ is defined over the ring $H^*_{G_\Lambda}(\mathrm{pt})$ with a particular choice of the cyclotomic polynomial.

It also makes sense to consider the construction above for more general $\mathfrak{g}$. We studied the situation where $\mathfrak{g}$ is a Heisenberg algebra. In this case, the quiver varieties $\mathfrak{M}(r,n)$ are the Gieseker spaces. In the particular case $r=1$, it is the Hilbert scheme of n points on $\mathbb{C}^2$. We consider the equivariant cohomology of $\mathfrak{M}(r,n)$ with respect to a symplectic action of $G=\mathrm{GL}_r\times\mathbb{C}^\times$. Schiffmann-Vasserot[3] and Maulik-Okounkov[1] constructed an action of $W_{1+\infty}$ on (a localization of)

$\bigoplus_n H^*_G(\mathfrak{M}(r,n))$. The algebra $W_{1+\infty}$ is the homologue of $L\mathfrak{g}$ in this case. On the algebra side, we replace $R^\Lambda(\alpha)$ by a level r quotient $R^r(n)$ of the graded affine Hecke algebra associated with the symmetric group $\mathfrak{S}_n$ defined over the ring $H_G(\mathrm{pt}) = H_{\mathrm{GL}_r}(\mathrm{pt})[\hbar]$. Let $R^r(n)_1$ be its specialization at $\hbar = 1$. The center of $R^r(n)_1$ has a natural filtration defined in terms of Jucy-Murphy elements. Let $\mathrm{Rees}(Z(R^r(n)_1))$ be the corresponding Rees algebra.

Theorem 2. *(a) There is a level r representation of $W_{1+\infty}$ on $\bigoplus_n Z(R^r(n))$, which is isomorphic to the representation $\bigoplus_n H^*_G(\mathfrak{M}(r,n))$ after localization.*

(b) There is a $\mathbb{Z}$-graded algebra isomorphism $\mathrm{Rees}(Z(R^r(n)_1)) \simeq H^*_G(\mathfrak{M}(r,n))$.

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Modular perverse sheaves on flag varieties

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(joint work with Simon Riche)

Gradings and mixed sheaves. Let $\mathcal{A}$ be an abelian category. A *grading on* $\mathcal{A}$, or a *graded version of* $\mathcal{A}$, consists of a new abelian category $\mathbb{A}$ equipped with an autoequivalence $\langle 1\rangle : \mathbb{A} \to \mathbb{A}$ ("shift of grading"), an exact functor $\nu : \mathbb{A} \to \mathbb{A}$ ("forgetting the grading"), and an isomorphism $\nu \circ \langle 1\rangle \xrightarrow{\sim} \nu$, such that the map

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^i_{\mathbb{A}}(X, Y\langle n\rangle) \to \mathrm{Ext}^i_{\mathcal{A}}(\nu(X), \nu(Y))$$

is an isomorphism for all $X, Y \in \mathbb{A}$ and all $i \geq 0$. (In some contexts, one may impose additional conditions, cf. [7, Definition 4.3.1].) The basic example is that in which $\mathbb{A}$ consists of graded modules over a graded ring (with suitable finiteness conditions), and $\mathcal{A}$ consists of *ungraded* modules over the same ring.

Many categories of interest in representation theory turn out to admit rather nonobvious gradings, including category $\mathcal{O}$ for a complex semisimple Lie algebra [7] and the principal block of a quantum group at a root of unity [5]. The existence of graded versions of these categories is intimately related to topics such as Koszul duality, character formulas, Ext-groups calculations, etc.

Where do these "hidden" gradings come from? For categories that can be realized in terms of perverse $\mathbb{C}$-sheaves, the influential work of Beilinson–Ginzburg–Soergel [7] showed how gradings can arise from *mixed geometry*, i.e., mixed $\overline{\mathbb{Q}}_\ell$-sheaves or mixed Hodge modules.

This lecture, based on [1, 2, 3], focuses on *modular* (i.e., positive characteristic) perverse sheaves, where the deep results that make characteristic-0 mixed geometry work are largely unavailable. We propose a new definition of "mixed modular sheaves" that, while far less general and powerful than the classical machinery, nevertheless has favorable properties in key examples, such as on flag varieties.

Sheaves in positive characteristic. Let $\mathbb{F}$ be a field. Let X be a complex algebraic variety equipped with an algebraic stratification

$$X = \bigsqcup_{s \in \mathcal{S}} X_s$$

where each stratum X_s is isomorphic to an affine space: $X_s \cong \mathbb{A}^{\dim X_s}$. Let $D^{\mathrm{b}}_{\mathcal{S}}(X, \mathbb{F})$ be the bounded derived category of complexes of $\mathbb{F}$-sheaves on X that are constructible with respect to the given stratification. The shift functor on this category will be denoted by $\{1\} : D^{\mathrm{b}}_{\mathcal{S}}(X, \mathbb{F}) \to D^{\mathrm{b}}_{\mathcal{S}}(X, \mathbb{F})$, rather than by "[1]."

Let $\mathrm{Parity}_{\mathcal{S}}(X, \mathbb{F}) \subset D^{\mathrm{b}}_{\mathcal{S}}(X, \mathbb{F})$ be the full additive subcategory consisting of *parity sheaves* [9] on X. It inherits an autoequivalence $\{1\} : \mathrm{Parity}_{\mathcal{S}}(X, \mathbb{F}) \to \mathrm{Parity}_{\mathcal{S}}(X, \mathbb{F})$. When $\mathbb{F} = \mathbb{C}$, parity sheaves often turn out to be just direct sums of shifted simple perverse sheaves, and $\mathrm{Parity}_{\mathcal{S}}(X, \mathbb{C})$ is close to the category of *pure complexes* in the sense of [6, §5.4]. For general $\mathbb{F}$, parity sheaves enjoy key properties of characteristic-0 pure complexes, including an analogue of the Decomposition Theorem [9, Proposition 2.34]. These parallels between parity $\mathbb{F}$-sheaves and characteristic-0 pure complexes inspire the following definition.

Definition 1 ([2, 3])**.** The *mixed modular derived category* of X is the category

$$D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F}) := K^{\mathrm{b}}\mathrm{Parity}_{\mathcal{S}}(X, \mathbb{F}).$$

An object $\mathcal{F} \in D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F})$ *has weights* $\leq n$, resp. $\geq n$, if it can be written as a complex in $K^{\mathrm{b}}\mathrm{Parity}_{\mathcal{S}}(X, \mathbb{F})$ that vanishes in degrees $< -n$, resp. $> -n$.

The *Tate twist* $\langle 1 \rangle : D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F}) \to D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F})$ is defined by $\langle 1 \rangle := \{-1\}[1]$.

Thus, $\mathrm{Parity}_{\mathcal{S}}(X, \mathbb{F})$ is identified with the pure objects of weight 0 in $D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F})$.

A nontrivial question is whether the usual sheaf functors make sense on the mixed modular derived category. In some cases, this is easy: if $\phi : D^{\mathrm{b}}_{\mathcal{S}}(X, \mathbb{F}) \to D^{\mathrm{b}}_{\mathcal{T}}(Y, \mathbb{F})$ takes parity sheaves to parity sheaves, then of course it induces a functor $D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F}) \to D^{\mathrm{mix}}_{\mathcal{T}}(Y, \mathbb{F})$. For example, if $i : Z \to X$, resp. $j : U \to X$, is an inclusion of a closed, resp. open, union of strata, then we have functors

$$i_* : D^{\mathrm{mix}}_{\mathcal{S}}(Z, \mathbb{F}) \to D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F}) \qquad \text{and} \qquad j^* : D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F}) \to D^{\mathrm{mix}}_{\mathcal{S}}(U, \mathbb{F}).$$

But for other functors on $D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F})$, additional work is required.

Theorem 2 ([2, 3])**.**
(1) Let $i : Z \to X$, resp. $j : U \to X$, be an inclusion of a closed, resp. open, union of strata. Then i_* and j^* have adjoints on both sides. In other words, the functors i^*, $i^!$, j_*, and $j_!$ make sense.
(2) If f is a proper, smooth, even, stratified morphism, then the functors $f_! = f_*$, f^*, and $f^!$ are defined and obey the usual adjunction properties.

(3) Whenever the functors $f_!$, f_*, f^*, or $f^!$ are defined, they obey the usual inequalities on weights (cf. [6, §5.1.14]).

Part (1) implies the existence of a *perverse t-structure* on $D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F})$. The heart of this t-structure, denoted by $\mathrm{Perv}^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F})$, is stable under the Tate twist.

Although this theory is motivated by a search for hidden gradings, we unfortunately do not yet know the answer to the following question in general.

Question 3. Does there exist a "forgetful functor" $\mathrm{For} : D^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F}) \to D^{\mathrm{b}}_{\mathcal{S}}(X, \mathbb{F})$ that makes $\mathrm{Perv}^{\mathrm{mix}}_{\mathcal{S}}(X, \mathbb{F})$ into a graded version of $\mathrm{Perv}_{\mathcal{S}}(X, \mathbb{F})$?

Applications. Let G be a connected complex reductive group, with Borel subgroup $B \subset G$, and Langlands duals $\check{B} \subset \check{G}$. We write $D^{\mathrm{mix}}_{(B)}(G/B, \mathbb{F})$ for the mixed modular derived category of G/B with respect to the stratification by B-orbits. From now on, assume that the characteristic of $\mathbb{F}$ is good for G and $\check{G}$.

Theorem 4 ([2])**.** There is an equivalence of categories

$$\kappa : D^{\mathrm{mix}}_{(B)}(G/B, \mathbb{F}) \xrightarrow{\sim} D^{\mathrm{mix}}_{(\check{B})}(\check{G}/\check{B}, \mathbb{F})$$

that of "Koszul type"; i.e., $\kappa(\mathcal{F}\langle n\rangle) \cong \kappa(\mathcal{F})\langle -n\rangle[n]$. Moreover, κ swaps parity sheaves with mixed tilting perverse sheaves.

For $\mathbb{F} = \mathbb{C}$, this result is due to Bezrukavnikov–Yun [8]. A precursor of this theorem, relating $\mathrm{Parity}_{(B)}(G/B, \mathbb{F})$ to (ordinary, not mixed) tilting perverse sheaves on $\check{G}/\check{B}$, appeared in [1]. As a consequence of Theorem 4, we are able to give an affirmative answer to Question 3 for flag varieties.

Theorem 5 ([2])**.** (1) There is a functor $D^{\mathrm{mix}}_{(B)}(G/B, \mathbb{F}) \to D^{\mathrm{b}}_{(B)}(G/B, \mathbb{F})$ that makes $\mathrm{Perv}^{\mathrm{mix}}_{(B)}(G/B, \mathbb{F})$ into a graded version of $\mathrm{Perv}_{(B)}(G/B, \mathbb{F})$.

(2) Let $\mathcal{E} \in \mathrm{Parity}_{(B)}(G/B, \mathbb{F})$ be an object whose direct summands generate $D^{\mathrm{b}}_{(B)}(G/B, \mathbb{F})$, and let $A = \mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E})$. Then $D^{\mathrm{b}}_{(B)}(G/B, \mathbb{F})$ is equivalent to the derived category of finitely generated dg-A-modules.

Part (2) was previously obtained under more restrictive hypotheses on the characteristic of $\mathbb{F}$ by Riche–Soergel–Williamson [11]. Theorems 4 and 5 lead to a new description of Soergel's modular category $\mathcal{O}$ [12] in terms of $\mathrm{Perv}_{(B)}(G/B, \mathbb{F})$.

The mixed modular derived category has also been used to study the affine Grassmannian in [4, 10]. However, Question 3 remains open for (partial) affine flag varieties, including the affine Grassmannian.

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Categorical actions and KLR algebras

Ruslan Maksimau

We study the parabolic category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_n$. By [3], the Koszul dual category of a parabolic category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_n$ at a negative level is a parabolic category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_n$ at a positive level. Moreover, the paper [2] constructs endofunctors E and F of $\mathcal{O}(\widehat{\mathfrak{gl}}_n)$ at a negative level, that yield a categorical action structure. Assume that the level of the affine category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_n$ is $-e-n$. Then by [2] the functors E, F allow decompositions $E = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} E_i$, $F = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} F_i$. The obtained functors E_i, F_i yield an action of the Lie algebra $\widehat{\mathfrak{sl}}_e$ on the parabolic category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_n$. (The Lie algebra $\widehat{\mathfrak{sl}}_e$ is generated by some elements e_i, f_i for $i \in \mathbb{Z}/e\mathbb{Z}$. The functors E_i, F_i act on the Grothendieck group of $\mathcal{O}(\widehat{\mathfrak{gl}}_n)$ by endomorphisms that satisfy the relations in $\widehat{\mathfrak{sl}}_e$ between the elements e_i, f_i for $i \in \mathbb{Z}/e\mathbb{Z}$.)

My goal is to calculate the Koszul dual functors to E and F. This question is motivated by a construction of a categorification of the Fock space via some subcategories of the parabolic category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_n$. The Fock space of level l is an $\widehat{\mathfrak{sl}}_e$-module that also has a structure of an $\widehat{\mathfrak{sl}}_l$-module. The functors E_i, F_i categorify the $\widehat{\mathfrak{sl}}_e$-action. The identification of their Koszul duals is necessary to understand the categorification of the $\widehat{\mathfrak{sl}}_l$-action.

A similar study of the same problem for $\mathcal{O}(\mathfrak{gl}_n)$ (i.e., in finite type) in [1] suggests the conjecture that the answer is given by Zuckerman functors. The affine case is much more delicate than the finite type one, mainly because there is no satisfactory theory of projective functors for $\mathcal{O}(\widehat{\mathfrak{gl}}_n)$.

The Zuckerman functors are defined as compositions of two "smaller" functors. Therefore, we must first decompose the functors E, F in "smaller" functors either. This can be done using the following inclusion of Lie algebras $\widehat{\mathfrak{sl}}_e \subset \widehat{\mathfrak{sl}}_{e+1}$

$$e_r \mapsto \begin{cases} e_r & \text{if } r \in [0, k-1], \\ [e_k, e_{k+1}] & \text{if } r = k, \\ e_{r+1} & \text{if } r \in [k+1, e-1], \end{cases} \qquad f_r \mapsto \begin{cases} f_r & \text{if } r \in [0, k-1], \\ [f_{k+1}, f_k] & \text{if } r = k, \\ f_{r+1} & \text{if } r \in [k+1, e-1] \end{cases}$$

for some $k \in [0, e-1]$. The main idea is to decompose the functors E_k and F_k using the fact that the elements $e_k, f_k \in \widehat{\mathfrak{sl}}_e$ go to $[e_k, e_{k+1}], [f_{k+1}, f_k] \in \widehat{\mathfrak{sl}}_{e+1}$ respectively.

To realize this approach, we fould a categorical version of the inclusion $\widehat{\mathfrak{sl}}_e \subset \widehat{\mathfrak{sl}}_{e+1}$. We proved that for an abelian category $\mathcal{C}$ with an $\widehat{\mathfrak{sl}}_{e+1}$-action, we can find an abelian subcategory $\mathcal{C}' \subset \mathcal{C}$ that inherits an $\widehat{\mathfrak{sl}}_e$-action from the $\widehat{\mathfrak{sl}}_{e+1}$-action on $\mathcal{C}$. The main point that makes this work is that we constructed an isomorphism between the KLR algebra of type $A^{(1)}_{e-1}$ and a subquotient of the KLR algebra of type $A^{(1)}_e$. This construction can be generalized to a rather large class of KLR algebras. In particular it yields alternative proofs of already known results on the decomposition of the functors E, F for $\mathcal{O}(\mathfrak{gl}_n)$.

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The multiplicative eigenvalue problem and deformed quantum cohomology (An abstract)

Shrawan Kumar

(joint work with Prakash Belkale)

We construct deformations of the small quantum cohomology rings of homogeneous spaces G/P, and obtain an irredundant set of inequalities determining the multiplicative eigenvalue problem for the compact form K of G.

Let G be a simple, connected, simply-connected complex algebraic group. We choose a Borel subgroup B and a maximal torus $H \subset B$. We denote their Lie algebras by $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}$ respectively. Let $R = R_{\mathfrak{g}} \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ and let R^+ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$). Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset R^+$ be the set of simple roots.

Consider the *fundamental alcove* $\mathscr{A} \subset \mathfrak{h}$ defined by

$$\mathscr{A} = \{\mu \in \mathfrak{h} : \alpha_i(\mu) \geq 0 \text{ for all simple roots } \alpha_i \text{ and } \theta_o(\mu) \leq 1\},$$

where θ_o is the highest root of $\mathfrak{g}$. Then, $\mathscr{A}$ parameterizes the K-conjugacy classes of K under the map $C : \mathscr{A} \to K/\operatorname{Ad} K$,

$$\mu \mapsto c(\operatorname{Exp}(2\pi i\mu)),$$

where K is a maximal compact subgroup of G and $c(\operatorname{Exp}(2\pi i\mu))$ denotes the K-conjugacy class of $\operatorname{Exp}(2\pi i\mu)$. Fix a positive integer $n \geq 3$ and define the *multiplicative polytope*

$$\mathscr{C}_n := \{(\mu_1, \dots, \mu_n) \in \mathscr{A}^n : 1 \in C(\mu_1) \dots C(\mu_n)\}.$$

Then, $\mathscr{C}_n$ is a rational convex polytope with nonempty interior in $\mathfrak{h}^n$. Our aim is to describe the facets (i.e., the codimension one faces) of $\mathscr{C}_n$ which meet the interior of $\mathscr{A}^n$.

We need to introduce some more notation before we can state our results. Let P be a standard parabolic subgroup (i.e., $P \supset B$) and let $L \subset P$ be its Levi subgroup containing H. Then, $B_L := B \cap L$ is a Borel subgroup of L. We denote the Lie algebras of P, L, B_L by the corresponding Gothic characters: $\mathfrak{p}, \mathfrak{l}, \mathfrak{b}_L$ respectively. Let $R_\mathfrak{l}$ be the set of roots of $\mathfrak{l}$ and $R_\mathfrak{l}^+$ be the set of roots of $\mathfrak{b}_L$. We denote by Δ_P the set of simple roots contained in $R_\mathfrak{l}$ and we set

$$S_P := \Delta \setminus \Delta_P.$$

For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}$ by

$$\alpha_i(x_j) = \delta_{i,j}, \ \forall\ 1 \leq i \leq \ell.$$

Let W be the Weyl group of G and let W^P be the set of the minimal length representatives in the cosets of W/W_P, where W_P is the Weyl group of P. For any $w \in W^P$, let $X_w^P := \overline{BwP/P} \subset G/P$ be the corresponding Schubert variety and let $\{\sigma_w^P\}_{w \in W^P}$ be the Poincaré dual (dual to the fundamental class of X_w^P) basis of $H^*(G/P, \mathbb{Z})$.

We begin with the following theorem. It was proved by Biswas in the case $G = \mathrm{SL}_2$; by Belkale for $G = \mathrm{SL}_m$ (and in this case a slightly weaker result by Agnihotri-Woodward where the inequalities were parameterized by $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d \neq 0$); and by Teleman-Woodward for general G. It may be recalled that the precursor to these results was the result due to Klyachko determining the additive eigencone for SL_m.

Theorem 1. *Let $(\mu_1, \dots, \mu_n) \in \mathscr{A}^n$. Then, the following are equivalent:*

(a) $(\mu_1, \dots, \mu_n) \in \mathscr{C}_n$,

(b) For any standard maximal parabolic subgroup P of G, any $u_1, \dots, u_n \in W^P$, and any $d \geq 0$ such that the Gromov-Witten invariant

$$\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d = 1,$$

the following inequality is satisfied:

$$\mathscr{I}^P_{(u_1,\dots,u_n;d)} : \qquad \sum_{k=1}^n \omega_P(u_k^{-1}\mu_k) \leq d,$$

where ω_P is the fundamental weight ω_{i_P} such that α_{i_P} is the unique simple root in S_P.

Even though this result describes the inequalities determining the polytope $\mathscr{C}_n$, however for groups other than G of type A_ℓ, the above system of inequalities has redundancies. The aim of our work is to give an irredundant subsystem of inequalities determining the polytope $\mathscr{C}_n$.

To achieve this, similar to the notion of Levi-movability of Schubert varieties in $X = G/P$ (for any parabolic P) introduced earlier by Belkale-Kumar which gives rise to a deformed product in the cohomology $H^*(X)$, we have introduced here the notion of *quantum Levi-movability* resulting into a deformed product in the quantum cohomology $QH^*(X)$ parameterized by $\{\tau_i\}_{\alpha_i \in S_P}$ as follows. As a $\mathbb{Z}[q,\tau]$-module, it is the same as $H^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q,\tau]$, where q (resp. τ) stands for multi variables $\{q_i\}_{\alpha_i \in S_P}$ (resp. $\{\tau_i\}_{\alpha_i \in S_P}$). For $u, v \in W^P$, define the $\mathbb{Z}[q,\tau]$-linear *quantum deformed product* by

$$\text{(1)} \quad \sigma_u^P \circledast \sigma_v^P = \sum_{d \geq 0 \in H_2(X,\mathbb{Z}); w \in W^P} \Big(\prod_{\alpha_i \in S_P} \tau_i^{A_i(u,v,w,d)} \Big) q^d \langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d \sigma^P_{w_o w w_o^P},$$

where w_o (resp. w_o^P) is the longest element of W (resp. W_P),

$$A_i(u,v,w,d) = (\chi_e - \chi_u - \chi_v - \chi_w)(x_i) + \frac{2a_i g^*}{\langle \alpha_i, \alpha_i \rangle},$$

$\chi_w = \sum_{\beta \in (R^+ \setminus R^+_{\mathfrak{l}}) \cap w^{-1} R^+} \beta$, g^* is the dual Coxeter number of $\mathfrak{g}$ and a_i is defined by the following identity:

$$\text{(2)} \qquad d = \sum_{\alpha_i \in S_P} a_i \mu(X^P_{s_i}) \in H_2(X,\mathbb{Z}),$$

It is shown that, for a cominuscule maximal parabolic subgroup P, the deformed product coincides with the original product in the quantum cohomology of X.

W have the following result obtained by crucially using deformation theory.

Theorem 2. *Let $u_1, \ldots, u_n \in W^P$ and $d = (a_i)_{\alpha_i \in S_P} \in H_2(X,\mathbb{Z})$ be such that $\langle \sigma^P_{u_1}, \sigma^P_{u_2}, \ldots, \sigma^P_{u_n} \rangle_d \neq 0$. Then, for any $\alpha_i \in S_P$,*

$$(\chi_e - \sum_{k=1}^n \chi_{u_k})(x_i) + 2g^* \langle x_i, \tilde{d} \rangle \geq 0,$$

where $\tilde{d} = \sum_{\alpha_j \in S_P} a_j \alpha_j^\vee$.

Evaluating each $\tau_i = 0$ in the identity (1) (which is well defined because of the above theorem), we get

$$\sigma_u^P \circledast_0 \sigma_v^P = \sum_{d,w}{}' q^d \langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d \sigma^P_{w_o w w_o^P},$$

where the sum is restricted over those $d \geq 0 \in H_2(X,\mathbb{Z})$ and $w \in W^P$ so that $A_i(u,v,w,d) = 0$ for all $\alpha_i \in S_P$. We shall denote the coefficient of $q^d \sigma^P_{w_o w w_o^P}$ in

$\sigma_u^P \circledast_0 \sigma_v^P$ by $\langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d^{\circledast_0}$. Similarly, we shall denote the coefficient of $q^d \sigma^P_{w_o u_n w_o^P}$ in $\sigma_{u_1}^P \circledast_0 \dots \circledast_0 \sigma_{u_{n-1}}^P$ by $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\circledast_0}$. Now our first main theorem on the multiplicative eigen problem is the following:

Theorem 3. *Let $(\mu_1, \dots, \mu_n) \in \mathscr{A}^n$. Then, the following are equivalent:*

(a) $(\mu_1, \dots, \mu_n) \in \mathscr{C}_n$,

(b) For any standard maximal parabolic subgroup P of G, any $u_1, \dots, u_n \in W^P$, and any $d \geq 0$ such that

$$\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\circledast_0} = 1,$$

the following inequality is satisfied:

$$\mathscr{I}^P_{(u_1,\dots,u_n;d)}: \qquad \sum_{k=1}^{n} \omega_P(u_k^{-1}\mu_k) \leq d.$$

The role of the flag varieties $(G/B)^n$ is replaced here by the quasi-parabolic moduli stack Parbun_G of principal G-bundles on $\mathbb{P}^1$ with parabolic structure at the marked points $b_1, \dots, b_n \in \mathbb{P}^1$. The proof makes crucial use of the canonical reduction of parabolic G-bundles and a certain *Levification process* of principal P-bundles, which allows degeneration of a principal P-bundle to a L-bundle (a process familiar in the theory of vector bundles as reducing the structure group to a Levi subgroup of P).

Our second main theorem on the multiplicative eigen problem asserts that the inequalities given by the (b)-part of the above theorem provide an irredundant system of inequalities defining the polytope $\mathscr{C}_n$. Specifically, we have the following result. This result for $G = \mathrm{SL}_m$ was proved by Belkale. It is the multiplicative analogue of Ressayre's result. Our proof is a certain adaptation of Ressayre's proof. There are additional technical subtleties involving essential use of the moduli *stack* of G-bundles and its smoothness, conformal field theory, affine flag varieties and the classification of line bundles on them.

Theorem 4. *Let $n \geq 2$. The inequalities*

$$\mathscr{I}^P_{(u_1,\dots,u_n;d)}: \qquad \sum_{k=1}^{n} \omega_P(u_k^{-1}\mu_k) \leq d,$$

given by part (b) of the above theorem (as we run through the standard maximal parabolic subgroups P, n-tuples $(u_1, \dots, u_n) \in (W^P)^n$ and non-negative integers d such that $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\circledast_0} = 1$) are pairwise distinct (even up to scalar multiples) and form an irredundant system of inequalities defining the eigen polytope $\mathscr{C}_n$ inside $\mathscr{A}^n$, i.e., the hyperplanes given by the equality in $\mathscr{I}^P_{(u_1,\dots,u_n;d)}$ are precisely the (codimension one) facets of the polytope $\mathscr{C}_n$ which intersect the interior of $\mathscr{A}^n$.

To show that the inequality $\mathscr{I}^P_{(u_1,\dots,u_n;d)}$ can not be dropped, we produce (following Ressayre's general strategy) a collection of points of $\mathscr{C}_n$ for which the above inequality is an equality, and such that their convex span has the dimension of a facet (i.e., $-1+n \dim \mathfrak{h}$). This is achieved by the parabolic analogue of Narasimhan-Seshadri theorem for the Levi subgroup L resulting in a description of $\mathscr{C}_n$ for L in terms of the non-vanishing of the space of global sections of certain line bundles on the moduli stack $\mathrm{Parbun}_L(d)$ of quasi-parabolic L-bundles of degree d applied to the semisimple part of L. To be able to use the parabolic analogue of Narasimhan-Seshadri theorem, we need a certain *Levi twisting*, which produces an isomorphism of $\mathrm{Parbun}_L(d)$ with $\mathrm{Parbun}_L(d \pm 1)$.

It may be remarked that our work completes the multiplicative eigenvalue problem for compact simply-connected groups in the sense that we determine the multiplicative eigen polytope $\mathscr{C}_n$ by giving an irredundant system of inequalities defining it. The problem of a recursive description of $\mathscr{C}_n$ in terms of eigen polytopes of "smaller groups" remains open for general G (for $G = \mathrm{SL}(n)$ this has been carried out by Belkale).

W-algebra and Whittaker coinvariants for $\mathrm{GL}(m|n)$

Simon M. Goodwin

(joint work with Jonathan Brown and Jonathan Brundan)

In [2] we began the study of the *principal W-algebra* $W = W_{m|n}$ associated to the general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}_{m|n}(\mathbb{C})$. This associative superalgebra is a quantization of the Slodowy slice to the principal nilpotent orbit in $\mathfrak{g}$; see for example [7] for a survey of the theory finite W-algebras in the purely even case.

We briefly recall the definition of $W = W_{m|n}$. Without loss of generality we assume that $m \leq n$ and consider the two row left justified *pyramid* π with m boxes in the top row and n boxes in the second row: for example, for $m = 2$ and $n = 5$ the pyramid π is

1	2			
3	4	5	6	7

We define $e \in \mathfrak{g}_{\bar{0}}$ to be the principal nilpotent element

$$e = e_{1,2} + e_{2,3} + \cdots + e_{m-1,m} + e_{m+1,m+2} + e_{m+2,m+3} + \cdots + e_{m+n-1,m+n},$$

and let $\chi \in \mathfrak{g}^*$ be dual to e via the supertrace form on $\mathfrak{g}$. We define

$$\mathfrak{p} = \langle e_{ij} \mid \mathrm{col}(i) \leq \mathrm{col}(j) \rangle \quad \text{and} \quad \mathfrak{m} = \langle e_{ij} \mid \mathrm{col}(i) > \mathrm{col}(j) \rangle,$$

where $\mathrm{col}(i)$ denotes the column containing i in π and we number columns from left to right. We also define $\mathfrak{m}_\chi = \{x - \chi(x) \mid x \in \mathfrak{m}\} \subseteq U(\mathfrak{m})$. Then we have by definition that

$$W = \{u \in U(\mathfrak{p}) \mid u\, \mathfrak{m}_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g})\}.$$

(We note that the W-algebra can be defined for more general pyramids, but we choose not to go in to this here. Also we remark that the assumption $m \leq n$ is necessary for the results below to be true as stated.)

In [2] we obtained a presentation for W by generators and relations, showing that it is a certain truncated shifted version of the Yangian $Y(\mathfrak{gl}_{1|1})$. In particular, it is quite close to being supercommutative and we determined a PBW basis for

W. We also developed a highest weight theory for W, which we note replicates much of that given in the purely even case for any nilpotent orbit in [5].

In order to explain the consequences of this highest theory we introduce some notation. We denote by Tab_π the set of all fillings of π by complex numbers. Then for any $A \in \mathrm{Tab}_\pi$ we can define a Verma module $\overline{M}(A)$ for W, which has simple head $\overline{L}(A)$ as in [2]. The following theorem summarizes the results on the highest weight theory for W obtained in [2].

Theorem 1. *Every irreducible W-module is finite dimensional and is isomorphic to one of the modules $\overline{L}(A)$ for some $A \in \mathrm{Tab}_\pi$. Further $\overline{L}(A) \cong \overline{L}(B)$ if and only if A is row equivalent to B.*

Moreover, there is another more explicit construction of the irreducible representations, implying that they have dimension 2^{m-a} for some *atypicality* $0 \leq a \leq m$, see [2, Thm. 8.4].

We move on to discuss the Whittaker coinvariants functor, which is studied in [3]. For any left $\mathfrak{g}$-module M, it is clear from the definition of W that the space

$$H_0(M) = M/\mathfrak{m}_\chi M$$

of *Whittaker coinvariants* is a W-module. This defines a functor H_0 from the analog of the BGG category $\mathcal{O}$ for $\mathfrak{g}$ (defined with respect to the standard maximal toral subalgebra $\mathfrak{t}$ and Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{b}_{\bar{0}} = \mathfrak{p}_{\bar{0}}$) to the category of finite dimensional W-modules.

After showing that H_0 is exact, we move on to compute the effect of H_0 on various natural families of modules in $\mathcal{O}$. To state our main results in this direction we require some further notation. We let $\{\delta_i\}_{1 \leq i \leq m+n}$ be the basis for $\mathfrak{t}^*$ dual to the basis $\{e_{i,i}\}_{1 \leq i \leq m+n}$ of $\mathfrak{t}$, and we write $(\cdot\,,\cdot)$ for the form on $\mathfrak{t}^*$ induced from the supertrace form on $\mathfrak{g}$. For $A = {}^{a_1 \cdots a_m}_{b_1 \cdots b_n} \in \mathrm{Tab}_\pi$, we define $\lambda_A \in \mathfrak{t}^*$ so that (λ_A, δ_i) is the entry in the ith box of A. We say that $A = {}^{a_1 \cdots a_m}_{b_1 \cdots b_n} \in \mathrm{Tab}_\pi$ is *antidominant* if $a_i - a_j \notin \mathbb{Z}_{<0}$ for $i < j$ and $b_i - b_j \notin \mathbb{Z}_{>0}$ for $i < j$. We define $\rho \in \mathfrak{t}^*$ by

$$\rho = -\delta_2 - 2\delta_3 - \cdots - (m-1)\delta_m + (m-1)\delta_{m+1} + (m-2)\delta_{m+2} + \cdots + (m-n)\delta_{m+n}.$$

Then we define $M(A)$ to be the Verma module for $U(\mathfrak{g})$ with highest weight $\lambda_A - \rho$, and let $L(A)$ be the irreducible head of $M(A)$.

The action of H_0 on the Verma modules and irreducible modules in $\mathcal{O}$ is described in the following theorem.

Theorem 2. *Let $A \in \mathrm{Tab}_\pi$. Then*

(i) $H_0(M(A)) \cong \overline{M}(A)$.

(ii) $H_0(L(A)) \cong \begin{cases} \overline{L}(A) & \textit{if } A \textit{ is antidominant}, \\ 0 & \textit{otherwise.} \end{cases}$

We also compute the composition multiplicities of Verma modules for W, showing that they are multiplicity-free of composition length 2^a where a is the atypicality mentioned earlier.

We move on to discuss how we can relate H_0 to the analogue for $\mathfrak{g}$ of Soergel's functor $\mathbb{V}$ for a semisimple Lie algebra. Let $\mathcal{O}_\mathbb{Z}$ be the sum of all blocks of $\mathcal{O}$ with integral central character. The isomorphism classes of irreducible modules in $\mathcal{O}_{\mathbb{Z}}$ of maximal Gelfand-Kirillov dimension are given by the $L(A)$ for A antidominant and with entries in $\mathbb{Z}$. Their projective covers, denoted $P(A)$, are also injective, and give representatives for the indecomposable projective-injective objects of $\mathcal{O}_{\mathbb{Z}}$. Let P be the direct sum of all these $P(A)$ for A antidominant. Then consider the locally finite endomorphism algebra

$$C = \mathrm{End}_{\mathfrak{g}}^{\mathrm{fin}}(P),$$

so C consists of endomorphisms that are zero on all but finitely many summands of P. This is a locally unital algebra, equipped with a system of local units arising from the projections onto the indecomposable summands of P. Let mof-C be the category of locally unital finite dimensional right C-modules. The familiar quotient functor

$$\mathbb{V} = \mathrm{Hom}_{\mathfrak{g}}(P, -) : \mathcal{O}_{\mathbb{Z}} \to \text{mof-}C$$

is the counterpart for $\mathfrak{g}$ of Soergel's functor from [9] for a semisimple Lie algebra. In particular, as established by Brundan, Losev and Webster, in [6], this functor is fully faithful on projectives; this is the analog of Soergel's *Struktursatz.*

We connect the functor $\mathbb{V}$ just defined to the Whittaker coinvariants functor. Let $\mathcal{R}$ be the essential image of the restriction of H_0 to $\mathcal{O}_{\mathbb{Z}}$, i.e. it is the full subcategory of the category of W-modules consisting of all modules isomorphic to $H_0(M)$ for $M \in \mathcal{O}_{\mathbb{Z}}$. We show that $\mathcal{R}$ is closed under taking submodules, quotients and direct sums; in particular it is abelian. Then our main result is the following theorem, which identifies H_0 with $\mathbb{V}$.

Theorem 3. *There is an equivalence of categories $\mathbb{J}$ making the following diagram of functors commute up to isomorphism:*

$$\begin{tikzcd} & \mathcal{O}_{\mathbb{Z}} \arrow[dl, "\mathbb{V}"'] \arrow[dr, "H_0"] & \\ \textit{mof-}C \arrow[rr, "\sim", "\mathbb{J}"'] & & \mathcal{R} \end{tikzcd}$$

In particular, this implies that H_0 is fully faithful on projective modules, and satisfies the universal property of the Serre quotient of $\mathcal{O}_{\mathbb{Z}}$ by the subcategory consisting of its modules of strictly less than maximal Gelfand-Kirillov dimension.

This is similar to a result of Backelin [1], who established a similar relationship between H_0 and Soergel's functor $\mathbb{V}$ in the purely even setting.

We mention that a fuller survey of the context of this research is given in [4].

Finally we draw attention to some remarkable recent work of Losev [8]. This paper includes a study of Whittaker coinvariant functors associated to arbitrary nilpotent orbits in semisimple Lie algebras. In §6.3 of this paper Losev mentions that the methods are applicable to give an alternative approach in the case of Lie superalgebras.

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Ringel duality for perverse sheaves on hypertoric varieties

Tom Braden

(joint work with Carl Mautner)

In recent years many features of "classical" geometric representation theory associated to the Springer resolution $T^*(G/B) \to \mathcal{N}$ of the nilcone have been generalized to other conical symplectic resolutions $\widetilde{\mathfrak{M}} \to \mathfrak{M}$. In particular, (most) symplectic resolutions admit analogs of enveloping algebras, category $\mathcal{O}$, Kazhdan-Lusztig cells, Springer theory, etc. The result given in this talk gives evidence that at least some features of modular (generalized) Springer theory may generalize to a broader class of conical symplectic resolutions.

A conical symplectic resolution is a resolution of singularities $\pi\colon \widetilde{\mathfrak{M}} \to \mathfrak{M}$, where $\widetilde{\mathfrak{M}}$ is a smooth variety over $\mathbb{C}$ admitting an algebraic symplectic form ω, $\mathfrak{M} = \operatorname{Spec}(\widetilde{\mathfrak{M}})$ is the affinization of $\widetilde{\mathfrak{M}}$, and $\widetilde{\mathfrak{M}}$, $\mathfrak{M}$ admit an action of $\mathbb{C}^*$ contracting $\mathfrak{M}$ to a point and scaling ω by a positive weight. In addition, for many purposes (such as constructing category $\mathcal{O}$), we must assume that $\widetilde{\mathfrak{M}}$ has a hamiltonian action of a torus T with finite fixed point set $\widetilde{\mathfrak{M}}^T$.

In addition, the base $\mathfrak{M}$ of each symplectic resolution is expected to have a partner $\mathfrak{M}^!$ which is related by "symplectic duality". This duality is known to physicists in the form of a mirror duality for three-dimensional gauge theories. Some representation-theoretic consequences of this duality are described in [2], and a possible definition of the dual variety has been recently given by Braverman, Finkelberg and Nakajima [8]. For this talk, only two very coarse features of symplectic duality are relevant: $\mathfrak{M}$ and $\mathfrak{M}^!$ should admit an order-reversing bijection between certain subsets of Poisson leaves, called "special" leaves, and the torus-fixed points of the symplectic resolutions $\widetilde{\mathfrak{M}}$ and $\widetilde{\mathfrak{M}}^!$ should be in natural bijection. Examples of symplectic dual pairs are nilpotent cones $\mathcal{N}_G$, $\mathcal{N}_{G^\vee}$ of Langlands dual groups, and Gale dual hypertoric varieties.

In work adapted from his thesis, Mautner [7] showed the following. Let $\mathcal{N} = \mathcal{N}_{GL_n}$ be the nilpotent cone of $G = GL_n$, and let k be a field of arbitrary characteristic. Then the category $\mathrm{Perv}_G(\mathcal{N}; k)$ of G-equivariant perverse sheaves on $\mathcal{N}$ with coefficients in k is equivalent to the category of rational polynomial representations of $GL_n(k)$ of degree n. This category is in turn isomorphic to modules over the Schur algebra $S_k(n, n)$. It is highest weight, and by a result of Donkin [5] it is self-Ringel dual: if T_λ, P_λ are the tilting and projective objects corresponding to the simple supported on closure of the nilpotent orbit $\mathcal{O}_\lambda \subset \mathcal{N}$, where λ is a partition of n, then

$$\mathrm{End}(\oplus_\lambda P_\lambda) \cong \mathrm{End}(\oplus_\lambda T_\lambda),$$

and under this isomorphism P_λ corresponds to T_{λ^t}. Achar and Mautner [1] gave a geometric proof of Donkin's result by showing that Fourier transform followed by restriction gives an auto-equivalence of $D^b_G(\mathcal{N}; k)$ which sends T_{λ^t} to P_λ. In addition, they showed that the tilting sheaves have a nice construction: T_λ is isomorphic to the pushforward of the constant sheaf along the projection map $T^*(G/P_{\lambda^t}) \to \mathcal{N}$. In particular, T_λ is a parity complex, as defined by Juteau, Mautner and Williamson [6].

Our result says that the same phenomenon holds for Gale dual pairs of hypertoric varieties. Let $T^n = (\mathbb{C}^*)^n$, with action on $T^*\mathbb{C}^n$ induced from the natural ccoordinate action on $\mathbb{C}^n$. Given a rank d sublattice V of the character lattice $X(T^n) = \mathbb{Z}^n$, one has an affine hypertoric variety $\mathfrak{M} := \mu_K^{-1}(0)//K$, where $K \subset T^n$ is the simultaneous kernel of all characters in V, and $\pi_V : T^*\mathbb{C}^n \to (\mathrm{Lie}\, K)^*$ is the algebraic moment map of the K-action. It admits a symplectic (orbifold) resolution $\widetilde{\mathfrak{M}} \to \mathfrak{M}$, obtained by replacing the categorical quotient by a GIT quotient for a generic character of K (different choices may lead to non-isomorphic resolutions). The d-dimensional torus $T = T^n/K$ acts on $\widetilde{\mathfrak{M}}$ and $\mathfrak{M}$. The Gale dual hypertoric variety $\mathfrak{M}^!$ is constructed in the same way, using the lattice $V^! := V^\perp \subset (\mathbb{Z}^n)^* \cong \mathbb{Z}^n$. We have $\dim \mathfrak{M} = 2d$, $\dim \mathfrak{M}^! = 2n - 2d$.

The n linear forms x_i restrict to linear forms on the vector space $V_\mathbb{R}$, giving a hyperplane arrangment which governs the geometry of $\mathfrak{M}$. In particular, there is a stratification $\mathcal{S}$ with (orbifold) strata indexed by flats (intersections of hyperplanes) which are coloop-free; the closure of the stratum S_F is a smaller hypertoric variety $\mathfrak{M}^F$. Taking the complement of the set of hyperplanes of a coloop-free flat gives a coloop-free flat of the arrangment in $V^!_\mathbb{R}$, so we have an order-reversing bijection of strata of $\mathfrak{M}$ and $\mathfrak{M}^!$, as required.

Theorem 1. *Let k be good for V. Then the category $\mathrm{Perv}_{\mathcal{S},T}(\mathfrak{M}; k)$ is highest weight and Ringel dual to $\mathrm{Perv}_{\mathcal{S}^!,T^!}(\mathfrak{M}^!; k)$. A (not necessarily indecomposable) tilting object supported on $\overline{S_F} = \mathfrak{M}^F$ is given by the pushforward of the constant k-sheaf along $\widetilde{\mathfrak{M}}^F \to \mathfrak{M}^F$; it is a parity complex.*

The proof comes from a purely combinatorial result about matroids: if M is a matroid, representable or not, we define a finite-dimensional algebra which we call a *matroidal Schur algebra*. This algebra is quasi-hereditary, so its category of modules is highest weight, and the algebras for dual matroids are Ringel dual. In the case that the matroid comes from a hyperplane arrangement and k is good for the associated hypertoric variety $\mathfrak{M}$, the matroidal Schur algebra is isomorphic to the endomorphisms of a projective generator of $\mathrm{Perv}_{\mathcal{S},T}(\mathfrak{M}; k)$.

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On Deligne's conjecture about tensor categories

Vera Serganova

(joint work with Inna Entova-Aizenbud, Vladimir Hinich)

Let $\mathbb{F}$ denote a field of characteristic zero. In [D] P. Deligne constructed a rigid symmetric Karoubian $\mathbb{F}$-linear tensor category $\mathcal{D}_t$ generated by an object X_t of dimension t. This category satsfies the following properties:

- Indecomposable objects of $\mathcal{D}_t$ are parametrized by pairs of partitions (λ, μ);
- If t is not integer, then the category $\mathcal{D}_t$ is semisimple, the dimension of the indecomposable object $Y(\lambda, \mu)$ is a polynomial of t;
- If t is an integer, then the quotient of the category $\mathcal{D}_t$ by the ideal generated by all negligible morphisms is equivalent to the category of representations of the algebraic group $GL(t)$ for $t > 0$ and the category of representations of the supergroup $GL(0|t)$ such that -1 acts as the grading operator for $t < 0$.

Furthermore, the category $\mathcal{D}_t$ satisfies the following universal property.

Theorem 1. *Let $\mathcal{A}$ be an abelian rigid symmetric $\mathbb{F}$-linear tensor category. Then the category of all objects in $\mathcal{A}$ of dimension t (with morphisms given by isomorphisms) and the category of tensor functors $F : \mathcal{D}_t \to \mathcal{A}$ are equivalent. In particular, given an object X in $\mathcal{A}$ of dimension t, there exists a unique up to isomorphism tensor functor $F_X : \mathcal{D}_t \to \mathcal{A}$ such that $F_X(X_t) = X$.*

The goal of the present talk is to construct an *abelian* tensor category $\mathcal{V}_t$ which satisfies similar universal property. The construction uses tensor categories $\operatorname{Rep} GL(m|n)$ of representations of the supergroups $GL(m|n)$ for all $m, n \in \mathbb{N}$ such that $m - n = t$. Using [DS] we define a functor $H_{m,n} : \operatorname{Rep} GL(m|n) \to \operatorname{Rep} GL(m-1|n-1)$ as follows. Let x be an odd element in the Lie superalgebra $\mathfrak{gl}(m|n)$ such that $x^2 = 0$ and the rank of x is 1. For any $GL(m|n)$-module M we set

$$H_{m,n}(M) := \operatorname{Ker} x / \operatorname{Im} x.$$

Lemma 2. *The functor* $H_{m,n} : \operatorname{Rep} GL(m|n) \to \operatorname{Rep} GL(m-1|n-1)$ *is tensor. It sends the standard representation* $\mathbb{F}^{m|n}$ *to the standard representation* $\mathbb{F}^{m-1|n-1}$.

Consider the filtration of $\operatorname{Rep} GL(m|n)$

$$\operatorname{Rep}^1 GL(m|n) \subset \operatorname{Rep}^2 GL(m|n) \subset \cdots \subset \operatorname{Rep}^k GL(m|n) \subset \dots,$$

where $\operatorname{Rep}^k GL(m|n)$ is the abelian subcategory of $\operatorname{Rep} GL(m|n)$ generated by tensor powers of the standard and costandard representations of total degree not greater than k.

Lemma 3. *If* $4k < min(m, n)$, *then the restriction* $H_{m,n} : \operatorname{Rep}^k GL(m|n) \to \operatorname{Rep}^k GL(m-1|n-1)$ *is an equivalence of abelain categories.*

We define the category $\mathcal{V}_t^k$ as the inverse limit of $\operatorname{Rep}^k GL(m|n)$ for $m - n = t$ and the category $\mathcal{V}_t$ as the direct limit $\varinjlim \mathcal{V}_t^k$.

Proposition 4. *(1) The category* $\mathcal{V}_t$ *is an abelian rigid symmetric tensor category.*

(2) There exists a fully faithful tensor functor $I : \mathcal{D}_t \to \mathcal{V}_t$ *that sends* X_t *to the inverse limit of the standard representations* $\mathbb{F}^{m|n}$.

(3) Furthermore, $\mathcal{V}_t$ *is a highest weight category with indecomposable tilting objects isomorphic to* $I(Y(\lambda, \mu))$ *for all pairs of partitions* (λ, μ).

(4) Any object in $\mathcal{V}_t$ *is isomorphic to the image of a morphism* $I(f) : I(D_1) \to I(D_2)$ *for some objects* D_1 *and* D_2 *in* $\mathcal{D}_t$.

(5) Let $0 \to A \to B \to C \to 0$ *be a short exact sequence in* $\mathcal{V}_t$. *There exists an object* D *in* $\mathcal{D}_t$ *such that the exact sequence* $0 \to A \otimes I(D) \to B \otimes I(D) \to C \otimes I(D) \to 0$ *splits.*

Recall, [D1], that if G is a group scheme in a tensor category $\mathcal{A}$ and $\pi(\mathcal{A})$ is the fundamental group of $\mathcal{A}$, then for any homomorphsim $\varepsilon : \pi(\mathcal{A}) \to G$ one defines the category $\operatorname{Rep}(G, \varepsilon)$ as the full subcategory of $\operatorname{Rep}(G)$ consisting of M such that the image of $\pi(\mathcal{A})$ in $GL(M)$ factors through ε. In particular, for any object $X \in \mathcal{A}$ one defines the tensor category $\operatorname{Rep}(GL(X), \varepsilon)$, where ε is the tautological homomorphism $\pi(\mathcal{A}) \to GL(X)$. The universality property of $\mathcal{V}_t$ is summarized in the following theorem, which gives an affirmative answer to the question 10.18 in [D]. Similar result for the category $\mathcal{S}_t$ is proven in [CO].

Theorem 5. *Let* $\mathcal{A}$ *be a rigid symmetric* $\mathbb{F}$*-linear tensor category and* X *be an object in* $\mathcal{A}$ *of dimension* t.

(1) *If X is not annihilated by any Schur functor then F_X uniquely factors through the embedding $I : \mathcal{D}_t \to \mathcal{V}_t$ and gives rise to an equivalence of tensor categories*
$$\mathcal{V}_t \longrightarrow \mathrm{Rep}(GL(X), \varepsilon)$$
sending $I(X_t)$ to X.

(2) *If X is annihilated by some Schur functor then there exists a unique pair $m, n \in \mathbb{Z}_+$, $m - n = t$, such that F_X factors through the tensor functor $\mathcal{D}_t \longrightarrow \mathrm{Rep}\, GL(m|n)$ and gives rise to an equivalence of tensor categories*
$$\mathrm{Rep}\, GL(m|n) \longrightarrow \mathrm{Rep}(GL(X), \varepsilon)$$
sending the standard representation $\mathbb{F}^{m|n}$ to X.

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