QUOTIENTS OF THE ORBIFOLD FUNDAMENTAL GROUP OF STRATA OF ABELIAN DIFFERENTIALS

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Abstract. Let S be a closed oriented surface of genus $g \geq 2$. The orbifold fundamental group of a component $\mathcal{Q}$ of a stratum of abelian differentials maps into the mapping class group $\mathrm{Mod}(S_{g,m})$ of S with $m \geq 1$ marked points at the zeros of the differentials in $\mathcal{Q}$. We give an explicit description of the image of this homomorphism.

1. Introduction

The moduli space $\mathcal{H}$ of *abelian differentials* on a closed oriented surface S of genus $g \geq 2$ (also called the *Hodge bundle*) decomposes into *strata*. Each such stratum is determined by a partition $2g-2=\sum_{i=1}^{m} k_i$ for some numbers $k_i \geq 1$, and it consists of all abelian differentials with the same number $m \geq 1$ of zeros of the same multiplicities k_i. Such a stratum, denoted by $\mathcal{H}(k_1,\dots,k_m)$, is a complex orbifold of complex dimension $2g-1+m$. Strata are not necessarily connected, but their connected components were classified by Kontsevich and Zorich [KZ03].

Up to date, not much is known about the orbifold fundamental group of components of strata except for some strata in small genus. The case $g=2$ is exceptional as all surfaces of genus 2 are hyperelliptic. In this case there are only two strata. These strata are connected, and they are classifying spaces for groups commensurable to braid groups [FM12]. Looijenga and Mondello [LM14] found that several components of strata in genus $g=3$ are classifying spaces for *finite type Artin groups.*

For every component $\mathcal{Q}$ of a stratum of abelian differentials with $m \geq 1$ zeros there is a natural forgetful map of $\mathcal{Q}$ into the moduli space $\mathcal{M}_{g,m}$ of complex structures on S with m marked points. This map associates to an abelian differential $q \in \mathcal{Q}$ the underlying complex structure and the zeros of q, viewed as marked points. It induces a homomorphism P from the orbifold fundamental group $\pi_1(\mathcal{Q})$ of $\mathcal{Q}$ into the mapping class group $\mathrm{Mod}(S_{g,m})$ of a surface $S_{g,m}$ of genus g with m marked points, well defined up to conjugation.

The homomorphism P is also defined for components of strata of quadratic differentials. Walker [W09, W10] investigated its image. To describe her result,

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recall that the marked point forgetful map $S_{g,m} \to S = S_{g,0}$ induces a Birman exact sequence

$$0 \to \Gamma_{g,m} \to \mathrm{Mod}(S_{g,m}) \xrightarrow{\Pi} \mathrm{Mod}(S) \to 0.$$

Walker found that for strata $\mathcal{Q}$ of quadratic differentials with at least g simple zeros, the subgroup $P\pi_1(\mathcal{Q})$ of $\mathrm{Mod}(S_{g,m})$ maps onto $\mathrm{Mod}(S_{g,0})$ [W09]. She also observed that the intersection of $P\pi_1(\mathcal{Q})$ with the group $\Gamma_{g,m}$ is contained in the kernel of a version of an Abel Jacobi map and, furthermore, that this intersection equals the kernel of the Abel Jacobi map provided that the stratum consists of quadratic differentials with sufficiently many simple zeros [W10].

The goal of this article is to give an explicit description of the image of the homomorphism P for all components of strata of abelian differentials in every genus. For the formulation of our result, we use the following definition.

Definition 1. An abelian differential $q \in \mathcal{Q}$ is called *completely periodic admissible* for the component $\mathcal{Q}$ if the following conditions are satisfied.

(1) The differential q is horizontally and vertically periodic. Equivalently, every regular leaf of the horizontal or the vertical foliation is closed.
(2) The collection $\mathcal{C}$ of core curves of the horizontal and vertical cylinders decompose S into m disks where $m \geq 1$ is the number of zeros of a differential $z \in \mathcal{Q}$.
(3) Any two curves from the collection $\mathcal{C}$ of simple closed curves on S intersect in at most one point. Furthermore, the graph whose vertices are the curves from $\mathcal{C}$ and where two such vertices are connected by an edge if they intersect is a tree.

In Section 2, we construct explicitly a completely periodic admissible differential q for every component of any stratum of abelian differentials on any surface of genus $g \geq 2$. These differentials are all square-tiled, i.e. they are pullbacks of a holomorphic one-form on a square torus T^2 by a covering branched over a single point of T^2.

If $q \in \mathcal{Q}$ is such a differential with its collection $\mathcal{C}$ of core curves of horizontal and vertical cylinders and if we mark a point in each component of $S - \mathcal{C}$, then the Dehn twists about the curves from $\mathcal{C}$ can be viewed as elements of the mapping class group $\mathrm{Mod}(S_{g,m})$. Our main result is the following

Theorem 2. *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials with $m \geq 1$ zeros. Let $q \in \mathcal{Q}$ be a completely periodic differential which is admissible for $\mathcal{Q}$; then up to conjugation, the image $P\pi_1(\mathcal{Q})$ in $\mathrm{Mod}(S_{g,m})$ of the orbifold fundamental group $\pi_1(\mathcal{Q})$ of $\mathcal{Q}$ is generated by the Dehn twists about the core curves of the horizontal and vertical cylinders of q.*

Theorem 2 is also valid for the two-torus T^2. In this case it is equivalent to the well known fact that the mapping class group $SL(2,\mathbb{Z})$ of the torus T^2 (with or without a marked point) is generated by the elements $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

A completely periodic abelian differential q which is admissible for $\mathcal{Q}$ is by no means unique. It is not hard to see that for each such differential q, the subgroup Γ of $\mathrm{Mod}(S_{g,m})$ generated by the Dehn twists about the core curves of the horizontal and vertical cylinders of q is contained in $P\pi_1(\mathcal{Q})$ (see Corollary 3.8, and compare with [H14] for an earlier explicit statement). An analogous result is also true for components of strata of quadratic differentials. The major part of this article is devoted to showing that $P\pi_1(\mathcal{Q}) = \Gamma$. In particular, the group Γ does not depend on the choice of the completely periodic differential which is admissible for $\mathcal{Q}$.

The description of the groups $P\pi_1(\mathcal{Q})$ in Theorem 2, although explicit, does not provide an understanding of these groups. As they intersect the group $\Gamma_{g,m}$ in the kernel of a version of an Abel Jacobi map [W10], they are of infinite index in $\mathrm{Mod}(S_{g,m})$. But one of the natural questions one might ask is whether or not they project to a subgroup of finite index in the mapping class group $\mathrm{Mod}(S)$ of S. As the mapping class group of S acts by pull-back on the Teichmüller space $\tilde{\mathcal{H}}$ of marked abelian differentials on S, this question is equivalent to asking whether the number of components of the preimage of the stratum in the Teichmüller space of abelian differentials is finite.

The easier part of Theorem 2 together with some deep results of Salter [Sa17] yields some information to this end. A much stronger version of its second and third part is due to Calderon [Cal18] in independent work. His interesting approach rests on an analysis of higher spin structures as suggested in [Sa17]. Here for a number $k \geq 2$ which divides $2g-2$, a $\mathbb{Z}/k\mathbb{Z}$- spin structure is a k-th root of the canonical bundle of S.

Theorem 3. *Let $\mathcal{Q}$ be a non-hyperelliptic component of a stratum of abelian differentials.*

1. *If $q \in \mathcal{Q}$ has at least one simple zero then the orbifold fundamental group of $\mathcal{Q}$ surjects onto* $\mathrm{Mod}(S)$.
2. *If $g \geq 5$ and if $q \in \mathcal{Q}$ has at least one zero of odd order $k < g-1$ which divides $g-1$ then the orbifold fundamental group of $\mathcal{Q}$ surjects onto the finite index subgroup of* $\mathrm{Mod}(S)$ *which preserves a $\mathbb{Z}/k\mathbb{Z}$-spin structure.*
3. *If $\mathcal{Q} \subset \mathcal{H}(k_1, \dots, k_m)$ with all k_i even and if $g \geq \max\{2k_i + 1, 21\}$ for some i such that k_i divides $2g-2$, then the orbifold fundamental group of $\mathcal{Q}$ surjects onto a finite index subgroup of* $\mathrm{Mod}(S)$.
4. *If $\mathcal{Q}$ is the non-hyperelliptic component of $\mathcal{H}(4)$ then the orbifold fundamental group of $\mathcal{Q}$ surjects onto the stabilizer of a $\mathbb{Z}/4\mathbb{Z}$-spin structure in* $\mathrm{Mod}(S)$.

As a consequence of Theorem 2 and Theorem 3, in particular its extension by Calderon [Cal18], we find new generating sets of $\mathrm{Mod}(S)$ consisting of $2g+1$ Dehn twists about non-separating simple closed curves. Namely, take any collection $\mathcal{C}$ of $2g+1$ simple closed curves on S with properties (2) and (3) in Definition 1 which decompose S into a two disks whose boundaries are polygons with $4r+2$ and $4g-4r+2$ sides, respectively, where r is odd and prime to $g-1$. Then the collection of Dehn twists about these curves generate $\mathrm{Mod}(S)$. We refer to Section 2 for more details.

In view of the work of Wright [Wr15], we expect that there also is a version of Theorem 2 for affine invariant manifolds and, in particular, for strata of quadratic differentials. We refer to [H14] for some partial result in this direction.

In general, the homomorphism $P : \pi_1(\mathcal{Q}) \to \mathrm{Mod}(S_{g,m})$ is not injective. In particular, a connected component of the preimage of $\mathcal{Q}$ in the Teichmüller space of marked abelian differentials may not be simply connected. As an example, Looijenga and Mondello [LM14] showed that the orbifold fundamental group of the non-hyperelliptic component $\mathcal{Q}$ of the stratum $\mathcal{H}(4)$ is the quotient of the Artin group $\mathcal{A}$ of finite type E_6 by its center. There is a natural so-called *geometric* homomorphism of $\mathcal{A}$ into $\mathrm{Mod}(S_{3,1})$ which maps the standard generators of $\mathcal{A}$ to Dehn twists about the core curves of the horizontal and vertical cylinders of a completely periodic admissible differential $q \in \mathcal{Q}$. Theorem 2 implies that the subgroup $P(\mathcal{A})$ of $\mathrm{Mod}(S_{3,1})$ is the image of $\mathcal{A}$ under a geometric homomorphism. However, by a result of Waijnryb [Wj99] (see also [Ma00]), the kernel of any geometric homomorphism $\mathcal{A} \to \mathrm{Mod}(S_{3,1})$ is not contained in the center of $\mathcal{A}$.

To summarize, Theorem 2 describes an explicit quotient of $\pi_1(\mathcal{Q})$ which however is non-trivial in general. In forthcoming work, we use Theorem 2 to compute the orbifold fundamental group of components of strata with a single zero. This however requires different tools.

As an application of the methods used for the proof of Theorem 2, we provide a fairly easy topological computation of the image of the orbifold fundamental group of a component of a stratum in the symplectic group $Sp(2g,\mathbb{Z})$.

The Hodge bundle $\mathcal{H}$ over the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g is equipped with a natural flat connection, the so-called *Gauss-Manin connection.* This connection defines a trivialization of $\mathcal{H}$ over any contractible subset of moduli space not containing any singular point, and this trivialization is unique up to conjugation. If U is a contractible subset of a component $\mathcal{Q}$ of a stratum, then there is a natural trivialization of the pullback $\Pi^*\mathcal{H}$ of the Hodge bundle $\mathcal{H}$ over U, defined by the pullback connection. If we fix such a trivialization, then for any fixed basepoint $x \in U$, we can study the monodromy of the pullback connection along loops based at x, and we can relate it to the monodromy along periodic orbits of Φ^t passing through U. Equivalently (see [H14] for a detailed discussion), we can study the subsemigroup of $Sp(2g,\mathbb{Z})$ generated by the return maps of periodic orbits γ for Φ^t passing through U.

Definition 4. The *local monodromy group* of the component $\mathcal{Q}$ of a stratum is the following subgroup G of $Sp(2g,\mathbb{Z})$. For $q \in \mathcal{Q}$ and a neighborhood U of q in $\mathcal{Q}$, let $G(U)$ be the subgroup of $Sp(2g,\mathbb{Z})$ generated by the monodromy maps of parametrized periodic orbits for Φ^t beginning at a point in U, and define $G = \cap_{U\ni q} G(U)$.

It is shown in [H14] that the group G does not depend on the point $q \in \mathcal{Q}$.

Components of strata of abelian differentials with all zeros of even order which are not hyperelliptic are distinguished by the *parity of their spin structure* [KZ03].

Such a spin structure is a square root of the canonical bundle of S, and it determines a *quadratic form* $\mathfrak{q}$ on $H^1(S,\mathbb{Z}/2\mathbb{Z})$. The parity of the quadratic form is its *Arf invariant*, an element of $\mathbb{Z}/2\mathbb{Z})$. Any two quadratic forms with the same Arf invariant are conjugate under the action of the symplectic group $Sp(2g,\mathbb{Z}/2\mathbb{Z})$.

For a quadratic form $\mathfrak{q}$ on $H_1(S,\mathbb{Z}/2\mathbb{Z})$ denote by $G(\mathfrak{q})\subset Sp(2g,\mathbb{Z})$ the finite index subgroup which maps under mod 2 reduction of coefficients to the stabilizer of $\mathfrak{q}$ in $Sp(2g,\mathbb{Z}/2\mathbb{Z})$. The first part of the following result is due to Avila, Matheus and Yoccoz [AMY18], and the second and third part are due to Gutierrez-Romo [GR17]. Their proofs use Rauzy Veech induction as their main tool while our approach is purely topological.

Theorem 5 (Avila-Matheus-Yoccoz [AMY18], Gutierrez-Romo [GR17])**.** *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials.*

(1) *If Q is hyperelliptic, then the local monodromy group is an extension of the level two congruence subgroup of $Sp(2g,\mathbb{Z})$ by the symmetric group in either $2g+1$ variables (for the component $\mathcal{H}^{\rm hyp}(2g-2)$) or $2g+2$ variables (for the component $\mathcal{H}^{\rm hyp}(g-1,g-1)$).*
(2) *If $\mathcal{Q}$ is a non-hyperelliptic component of a stratum of abelian differentials with only zeros of even order and quadratic form $\mathfrak{q}$, then the local monodromy group of $\mathcal{Q}$ equals the group $G(\mathfrak{q})$.*
(3) *If $\mathcal{Q}$ is a non-hyperelliptic component of a stratum of abelian differentials with at least one zero of odd order, then the local monodromy group of $\mathcal{Q}$ equals the group $Sp(2g,\mathbb{Z})$.*

For hyperelliptic components, Theorem 5 was established in [AMY18] using the language of Rauzy Veech induction, and non-hyperelliptic components are treated in [GR17], see also [H14].

Plan of the paper and strategy of the proof: In Section 2 we introduce admissible curve systems and relate them to completely periodic abelian differentials. In Section 3 we introduce the Arf invariant of a simple admissible curve system and compute it for a specific collection of curve systems which are chosen in such a way that they determine completely periodic differentials which are admissible for all components strata with a single zero on a surface of genus $g\geq 2$. We also prove Theorem 3. In Section 4 we use the Dehn twists about the core curves of the cylinders to give a fairly easy topological proof of a global version of Theorem 5.

In Sections 5 and 6 we associate to a component $\mathcal{Q}$ of a stratum of abelian differentials a family of train tracks on S. In Section 7 we construct from a completely periodic abelian differential q which is admissible for $\mathcal{Q}$ a particular such train track. In Section 8 we show that these train tracks can be used to navigate in the component $\mathcal{Q}$. This is then used to show that the subgroup of $P\pi_1(\mathcal{Q})$ of $\mathrm{Mod}(S_{g,m})$ defined by a completely periodic admissible differential is independent of the differential.

Section 9 is mainly technical and establishes additional information on subgroups of punctured mapping class groups generated by Dehn twists about the curves of admissible curve systems. Section 10.1 introduces higher spin structures and gives some first relation to degenerations of abelian differentials to abelian differentials on surfaces with nodes.

Section 11 contains the proof of part (4) of Theorem 3. This result is used as the base for an argument by induction which leads to the proof of Theorem 2 in Section 12.

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2. Curve diagrams and components of strata

Consider a closed surface S of genus $g \geq 2$. The goal of this section is to introduce admissible curve systems and show that they correspond to completely periodic admissible abelian differentials for components of strata of abelian differentials on S in the sense of the introduction.

We begin with reviewing the classification of components of strata of abelian differentials due to Kontsevich and Zorich [KZ03]. A *hyperelliptic component* of a stratum is the pull-back of a stratum of meromorphic quadratic differentials on $\mathbb{C}P^1$ under a two-sheeted branched cover $S \to \mathbb{C}P^1$ obtained by quotienting S by a hyperelliptic involution.

Kontsevich and Zorich [KZ03] found that with the exception of the stratum $\mathcal{H}(2\ell-1, 2\ell-1)$ ($\ell \geq 2$), all strata which contain a zero of odd order are connected. The stratum $\mathcal{H}(2\ell-1, 2\ell-1)$ consists of two components. One component $\mathcal{H}^{\rm hyp}(2\ell-1, 2\ell-1)$ is hyperelliptic, the other is not. The description of the components of a stratum with all zeros of even order is as follows.

The zeros of an abelian differential on S, counted with multiplicities, define a divisor on S whose dual line bundle is the canonical bundle for the complex structure on S underlying the differential. If all orders of the zeros are even, then the square root of the divisor is defined. Its dual line bundle is a square root of the canonical bundle.

Let $Z \to S$ be the circle bundle of directions of non-zero tangent vectors on S. The square roots of the canonical bundle correspond precisely to spin structures on S, and these are classified by a coset of $H^1(S, \mathbb{Z}/2\mathbb{Z})$ in $H^1(Z, \mathbb{Z}/2\mathbb{Z})$. We refer to Section 3 of [KZ03] for details about these facts.

A spin structure determines a quadratic form on the symplectic vector space $H_1(S, \mathbb{Z}/2\mathbb{Z})$, equipped with the mod 2 intersection form ι. By definition, a quadratic form is a function $\mathfrak{q} : H_1(S, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ which satisfies

$$\mathfrak{q}(a+b) = \mathfrak{q}(a) + \mathfrak{q}(b) + \iota(a, b).$$

The *Arf invariant* $\mathrm{Arf}(\mathfrak{q})$ of such a quadratic form $\mathfrak{q}$ is defined as follows. Choose any symplectic basis $a_1, b_1, \ldots, a_g, b_g$ of $H_1(S, \mathbb{Z}/2\mathbb{Z})$ and write

$$\mathrm{Arf}(\mathfrak{q}) = \sum_i \mathfrak{q}(a_i)\mathfrak{q}(b_i).$$

The *parity* of the spin structure is defined to be the Arf invariant of its quadratic form. Thus each abelian differential with all zeros of even order determines a parity, and the parity is constant on components of strata with all zeros of even order. We refer to [KZ03] for a detailed discussion and for references.

The classification of components of strata with all zeros of even order is then as follows [KZ03].

Theorem 2.1 (Kontsevich-Zorich)**.** (1) *For $g \geq 4$, any stratum with all zeros of even order contains two non-hyperelliptic connected components, distinguished by the parity of their spin structure.*

(2) *$\mathcal{H}(4)$ and $\mathcal{H}(2,2)$ contain a single non-hyperelliptic component with odd spin structure.*

(3) *For $g = 2\ell \geq 4$ even, the stratum $\mathcal{H}(4\ell-2)$ contains a hyperelliptic component $\mathcal{H}^{\text{hyp}}(4\ell-2)$. The only other hyperelliptic component for this genus is the component $\mathcal{H}^{\text{hyp}}(2\ell-1, 2\ell-1)$ which contains zeros of odd order. The parity of the spin structure of $\mathcal{H}^{\text{hyp}}(4\ell-2)$ is even if and only if $g \equiv 0$ mod 4.*

(4) *For $g = 2\ell+1 \geq 3$ odd, the hyperelliptic components are $\mathcal{H}^{\text{hyp}}(2\ell, 2\ell)$ and $\mathcal{H}^{\text{hyp}}(2\ell)$. The parity of the spin structure of each of these components is even if and only if $g \equiv 3$ mod 4.*

For the construction of a completely periodic abelian differential q which is admissible for a component $\mathcal{Q}$ of $\mathcal{H}(2g-2)$ we have to compute the parity of the spin structure defined by a completely periodic differential q. We first introduce some terminology.

Definition 2.2. A *curve system* on S is a finite collection of simple closed smoothly embedded non-contractible mutually not freely homotopic curves on S such that any two curves from this collection intersect transversely in at most one point.

Note that the mapping class group of S naturally acts on the family of all curve systems.

To each curve system is associated its *curve diagram*. This diagram is a finite graph whose vertices are the curves from the system and where two vertices are connected by an edge if and only if the curves representing these vertices intersect.

Remark 2.3. In [Lei04], in a slightly different context, a curve diagram as defined above is called a *configuration graph*.

Example 2.4. Define a *Humphries system* for a surface of genus g to be a curve system whose diagram consists of a line segment of length $2g$ with a single edge attached to vertex 4. By Proposition 2.1 of [Lei04], any two Humphries systems are equivalent under the action of the mapping class group. The mapping class group of S is generated by the Dehn twists about the curves of a Humphries system (Section 4.4.3 of [FM12]).

Definition 2.5. A curve system is *admissible* if it decomposes S into simply connected components and if its curve diagram is a tree. An admissible curve system is called *simple* if it consists of precisely $2g$ curves.

With this terminology, a Humphries system is admissible, but it is not simple.

Since the curve diagram of an admissible curve system $\mathcal{C}$ is connected, each curve $c \in \mathcal{C}$ intersects at least one other simple closed curve on S transversely in a single point and hence it is non-separating. If $\mathcal{C}$ is a simple curve system then its components, equipped with an arbitrary orientation, define a basis for the first integral homology group $H_1(S, \mathbb{Z})$ of S.

We next discuss the significance of curves systems for components of strata. The following notion will be useful.

Definition 2.6. A *consistent orientation* of a curve system $\mathcal{C} = \{c_1, \dots, c_k\}$ is an orientation of each of the curves c_i with the following properties. Let $p \in c_i \cap c_j$ and let us assume that the oriented basis (c_i', c_j') of T_pS is positive (or negative) where the tangent c_i', c_j' is determined by the orientation of c_i, c_j. Then for any $s \neq i, j$ with $c_i \cap c_s = q \neq \emptyset$, the oriented basis (c_i', c_s') of T_qS is positive (or negative).

It is immediate from the definition that a consistent orientation partitions the admissible curve system $\mathcal{C}$ into two disjoint sets. The *positive* elements are those oriented curves c for which the orientation of T_pS given by the ordered basis (c', d') is positive for any curve $d \in \mathcal{C}$ which intersects c in some point p, and the negative elements are the remaining curves. The positive curves from the curve system $\mathcal{C}$ are pairwise disjoint, and the same holds true for the negative curves. Since the curve diagram of an admissible curve system is connected, up to exchanging the positive and negative curves, such a decomposition is unique, and it corresponds to a realization of the curve diagram as a bipartite graph.

Recall that trees are bipartite graphs. We obtain

Lemma 2.7. *A curve system whose curve diagram is a tree admits a consistent orientation.*

Proof. We proceed by induction on the number of vertices in the curve diagram. If this diagram consists of two vertices and one edge, then any choice of orientation for the two curves in the system will do.

Now assume that the lemma holds true for all curve systems whose curve diagram is a tree with at most $k-1$ vertices. Let $\mathcal{C}$ be a curve system whose curve diagram T is a tree with k vertices. Let T' be obtained from T by removing one leaf a. Then T' is the curve diagram of a curve system $\mathcal{C}' \subset \mathcal{C}$ which is obtained from $\mathcal{C}$ by removing the curve c corresponding to the endpoint of the leaf a. By induction hypothesis, there exists a consistent orientation for $\mathcal{C}'$. Let c' be the unique curve from $\mathcal{C}'$ which intersects c. Define the orientation of c so that c is positive if c' is negative, and define c to be negative if c' is positive. This clearly is a consistent orientation for the curves of $\mathcal{C}$. The induction step follows. □

Remark 2.8. If the curve system $\mathcal{C}$ is simple, then the curves from $\mathcal{C}$ define a basis of the first homology group of S. Thus there are at most g vertices which are pairwise not connected by an edge and hence the number of vertices of each type in the curve diagram of a simple curve system is exactly g. The same argument also shows that for any admissible curve system, the number of vertices of each type in the curve diagram is at least g.

As in the introduction, call an abelian differential completely periodic if its horizontal and vertical measured foliation, respectively, decomposes S into a union of foliated cylinders (or, equivalently, if every non-singular leaf of the horizontal or vertical foliation is closed). The following construction is well known and goes back to Thurston [T88] and Veech [V89], see also Section 5 of [Lei04] for a nice account.

Lemma 2.9. *Let $\mathcal{C}$ be an admissible curve system. Then there exists a completely periodic abelian differential $q(\mathcal{C})$ whose cylinders are homotopic to the curves from $\mathcal{C}$. The number of zeros of $q(\mathcal{C})$ equals the number m of components of $S - \mathcal{C}$.*

Proof. We only outline this well known construction.

View $\mathcal{C}$ as a graph on S. Let $\Lambda \subset S$ be the dual graph to $\mathcal{C}$ embedded in S. Then Λ has m vertices $x_1, \dots, x_m$, one for each component of $S - \mathcal{C}$. Furthermore, Λ defines a cell decomposition of S whose two-cells are rectangles.

Declaring each of these rectangles to be an euclidean square defines an euclidean metric on S with m cone points. Note that by the requirement that $\mathcal{C}$ is admissible and hence its curve diagram is a tree, each complementary component of $\mathcal{C}$, viewed as a polygon, has at least six sides and hence each of the points x_i is indeed a cone point of cone angle bigger than 2π.

If we write $\mathcal{C} = A \cup B$ where A are the positive and B the negative curves for a consistent orientation of $\mathcal{C}$, then for each of these squares, one pair of opposite sides is disjoint from A, and the second pair is disjoint from B. Defining the sides disjoint from A to be horizontal and the sides disjoint from B to be vertical is consistent with the gluing and determines a collection of $\mathbb{C}$-valued charts on $S - \{x_1, \dots, x_m\}$ whose chart transitions are translations. Thus this construction yields a square tiled translation surface with precisely m singular points. It is determined by an abelian differential $q(\mathcal{C})$ with the properties stated in the lemma. ☐

We call the abelian differential $q(\mathcal{C})$ constructed in Lemma 2.9 a *realization* of $\mathcal{C}$ by an abelian differential. Recall that $q(\mathcal{C})$ may be chosen to be square tiled, i.e. it is the pull-back of a holomorphic one-form on the two-torus T^2 by a cover branched over a single point of T^2.

Remark 2.10. The proof of Lemma 2.9 also shows that given an admissible curve system $\mathcal{C}$, the order of the zeros of the differential $q(\mathcal{C})$ are easily computable from the curve system $\mathcal{C}$. Namely, $\mathcal{C}$ determines the cone angle at the singular points for the singular euclidean metric on S resulting from the construction.

3. The Arf invariant of a simple admissible curve system

A simple admissible curve system $\mathcal{C} = \{v_1, \dots, v_{2g}\}$ on S determines a quadratic form $\mathfrak{q} = \mathfrak{q}(\mathcal{C})$ on $H_1(S, \mathbb{Z}/2\mathbb{Z})$ as follows.

Let D be the curve diagram of the curve system $\mathcal{C}$. Let $[v_i] \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ be the mod 2 homology class defined by the curve v_i; it does not depend on a choice of an orientation for v_i. A homology class $b \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ can uniquely be written in the form $b = \sum a_i[v_i]$ with $a_i \in \mathbb{Z}/2\mathbb{Z}$. Let $i_1, \dots, i_s$ $(s \leq 2g)$ be those numbers so that $a_{i_j} \neq 0$ and define $\mathfrak{q}(b)$ as the Euler characteristic mod 2 of the full subgraph $D(b)$ of D whose vertices are the elements v_{i_j} (see p.176 of [FM12]).

Lemma 3.1. $\mathfrak{q}$ *is a quadratic form on* $H_1(S, \mathbb{Z}/2\mathbb{Z})$ *which is invariant under the subgroup of* $\mathrm{Mod}(S)$ *generated by the Dehn twists about the curves in* $\mathcal{C}$*.*

Proof. For $b \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ let as above $D(b)$ be the full subgraph of the curve diagram D whose vertices are the elements with non-zero coefficients for the representation of b. Let $V(b), E(b)$ be the set of vertices and the set of edges of $D(b)$, respectively.

Let as before ι be the symplectic form mod 2 on $H_1(S, \mathbb{Z}/2\mathbb{Z})$. To show that $\mathfrak{q}$ is a quadratic form we have to show that $\mathfrak{q}(a + b) = \mathfrak{q}(a) + \mathfrak{q}(b) + \iota(a, b)$ for all $a, b \in H_1(S, \mathbb{Z}/2\mathbb{Z})$. To this end note that for every class $a \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ the value $\mathfrak{q}(a)$ equals $(|V(a)| - |E(a)|)$ mod 2, or, equivalently, it equals the number mod 2 of connected components of the graph $D(a)$ (recall that D is a tree by assumption).

Write $b_1 = \sum_{v \in V(a) \cap V(b)} [v]$; then

$$|V(a + b)| = |V(a)| + |V(b)| - 2|V(b_1)|$$

and hence we have $|V(a + b)|$ mod $2 = |V(a)| + |V(b)|$ mod 2.

To compute the cardinality mod 2 of the set $E(a + b)$ of edges of $D(a + b)$ we partition the set $\mathcal{E} = E(a) \cup E(b) \cup E(a + b)$ into three subsets.

$\mathcal{E}_1$ is the set of all elements of $\mathcal{E}$ which either have both endpoints in $V(a) - V(b_1)$ or in $V(b) - V(b_1)$. We have $\mathcal{E}_1 \subset E(a + b) \cap (E(a) \cup E(b))$.

The set $\mathcal{E}_2$ contains those elements of $\mathcal{E}$ with one endpoint in $V(b_1)$ and the second endpoint in $(V(a) \cup V(b)) - V(b_1)$. We have $\mathcal{E}_2 \subset (E(a) \cup E(b)) - E(a + b)$.

The set $\mathcal{E}_3$ contains all elements of $\mathcal{E}$ with one endpoint in $V(b) - V(b_1)$ and the second endpoint in $V(a) - V(b_1)$. We have $\mathcal{E}_3 \subset E(a + b) - (E(a) \cup E(b))$.

To summarize, we have $p = |E(a)| + |E(b)|$ mod $2 = |\mathcal{E}_1| + |\mathcal{E}_2|$ mod 2. Furthermore, $r = |E(a + b)|$ mod $2 = (|\mathcal{E}_1| + |\mathcal{E}_3|)$ mod 2. For the proof of the first part of the lemma, it now suffices to show that $r - p = \iota(a, b)$, and this is equivalent to stating that $|\mathcal{E}_3| - |\mathcal{E}_2|$ mod $2 = \iota(a, b)$.

To see that this is indeed the case observe that as $V(b_1) = V(a) \cap V(b)$, each edge in $\mathcal{E}_2$ contributes one to $\iota(a, b)$, and the same holds true for each edge in $\mathcal{E}_3$.

Furthermore, the edges in $\mathcal{E}_1$ do not contribute to $\iota(a,b)$. But this just means that $\iota(a,b) = |\mathcal{E}_2 \cup \mathcal{E}_3| \bmod 2$ which is what we wanted to show.

We are left with showing that the quadratic form $\mathfrak{q}$ is invariant under the subgroup of $\mathrm{Mod}(S)$ generated by the Dehn twists about the curves from $\mathcal{C}$. To this end note that a Dehn twist about the curve v_i acts on $H_1(S,\mathbb{Z}/2\mathbb{Z})$ as a *transvection* by the element $[v_i]$, i.e. this action is just the map

$$\tau_i(b) = b + \iota(b,[v_i])[v_i]$$

(Proposition 6.3 of [FM12]).

That this map preserves $\mathfrak{q}$ is worked out on p.175 of [FM12]. For convenience of the reader, we present the short proof.

Namely, if $\iota(b,[v_i]) = 0$ then $\tau_i(b) = b$ and there is nothing to show. On the other hand, if $\iota(b,[v_i]) = 1$ then $\tau_i(b) = b + [v_i]$. But as the curve diagram D is a tree, we have $\iota(b,[v_i]) = 1$ only if one of the following cases is satisfied.

(1) There exists a connected component V of the graph $D(b)$ such that $v_i \in V$, and the number of edges in V which are incident on v_i is odd.
(2) $v_i \notin D(b)$, and there exists an odd number of vertices $v_j \in V(b)$ which are connected to v_i by an edge.

In the first case, removing the vertex v_i from $D(b)$ results in a graph whose number of components differ from the number of components of $D(b)$ by an even number. Then $\mathfrak{q}(b+[v_i]) = \mathfrak{q}(b)$ as desired. In the second case, the graph $D(b+[v_i])$ is obtained from D by merging an odd number of components to a single component. Once again, we have $\mathfrak{q}(b+[v_i]) = \mathfrak{q}(b)$. □

The *Arf invariant* $\mathrm{Arf}(\mathcal{C})$ of the curve system $\mathcal{C}$ is the Arf invariant of the quadratic form $\mathfrak{q}$ defined by $\mathcal{C}$. Call the curve system *even* (or *odd*) if its Arf invariant is even (or odd).

Recall from Lemma 2.9 that a simple admissible curve system $\mathcal{C}$ determines an abelian differential $q(\mathcal{C})$ with a single zero.

Lemma 3.2. *Let $\mathcal{C}$ be a simple admissible curve system. Then $\mathcal{C}$ is even if and only if the abelian differential $q(\mathcal{C})$ is contained in a component of the stratum $\mathcal{H}(2g-2)$ with even spin structure.*

Proof. Any quadratic form $\mathfrak{q}$ on $H_1(S,\mathbb{Z}/2\mathbb{Z})$ is determined by its values on a basis of $H_1(S,\mathbb{Z}/2\mathbb{Z})$.

By definition of the quadratic form $\mathfrak{q}$ for $\mathcal{C}$, we have $\mathfrak{q}([c]) = 1$ for all $c \in \mathcal{C}$ where $[c]$ denotes the mod two homology class of c. On the other hand, each $c \in \mathcal{C}$ can be represented by a smooth curve which is transverse to either the horizontal or the vertical foliation of $q(\mathcal{C})$. Thus if $\mathfrak{q}'$ denotes the quadratic form on $H_1(S,\mathbb{Z}/2\mathbb{Z})$ determined by the spin structure of $q(\mathcal{C})$ then by Lemma 2 of [KZ03], we have $\mathfrak{q}'([c]) = 1$ for all $c \in \mathcal{C}$. As the homology classes $[c]$ $(c \in \mathcal{C})$ span $H_1(S,\mathbb{Z}/2\mathbb{Z})$, the lemma follows. □

For $g \geq 2$ we introduce now three curve systems by the following curve diagrams.

(1) *Type* A_{2g}*:* For $g \geq 2$ a line with $2g$ vertices. This is the Dynkin diagram A_{2g}.
(2) *Type* U_{2g}*:* For $g \geq 3$ a line with $2g-2$ vertices and a segment of length two coming out of vertex 3. Note that U_6 is the Dynkin diagram E_6, and U_8 is the diagram of the hyperbolic analog Eh_8 of the spherical Coxeter group E_6.
(3) *Type* V_{2g}*:* For $g \geq 3$ a line with $2g-1$ vertices and an edge coming out of vertex 5. We have $V_6 = A_6$, V_8 is the diagram E_8, and V_{10} is the diagram of the hyperbolic analog Eh_{10} of the spherical Coxeter group E_8.

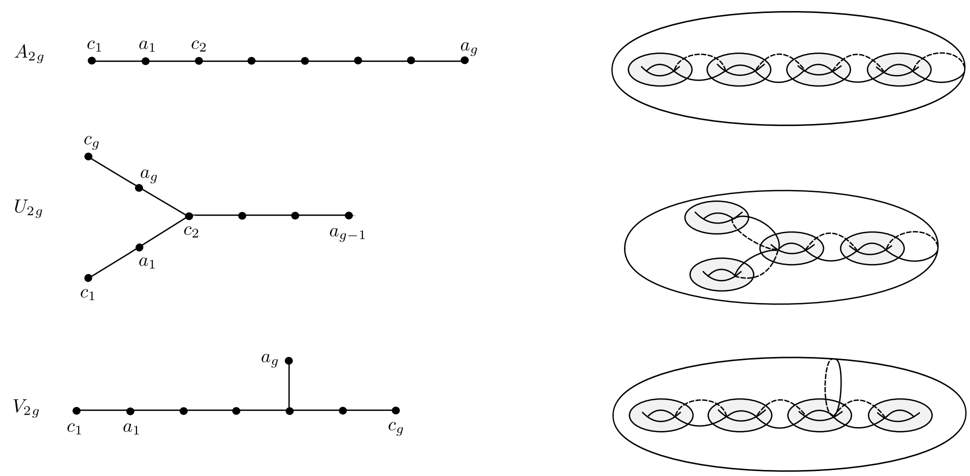


Figure A

Example 3.3. Consider a curve system $\mathcal{C}$ of type A_{2g} as in Figure A. It is invariant under a hyperelliptic involution ν which preserves each of the curves c_i and reverses its orientation. This involution has precisely $2g+2$ fixed points, and $2g+1$ of these fixed points are contained in the curve system $\mathcal{C}$.

Let S_0 be the surface obtained from S by removing the interior of a small ν-invariant disk about the fixed point of ν not contained in $\mathcal{C}$. The involution ν restricts to an involution of S_0. The quotient of S_0 under ν is a disk with $2g+1$ marked points. The Dehn twists about the curves from $\mathcal{C}$ descend to the standard generators of the Artin braid group with $2g+1$ strands (see p.254 and p.255 of [FM12]). As a consequence, the Dehn twists about the curves from $\mathcal{C}$ generate the *symmetric mapping class group* of S_0 (Theorem 9.2 of [FM12]), i.e. the centralizer of ν in $\mathrm{Mod}(S_0)$.

In view of Lemma 3.2, part (1) of the following lemma is equivalent to Corollary 5 of Appendix B in [KZ03].

Lemma 3.4. (1) *If* $\mathcal{C} \subset S$ *is a simple admissible curve system of type* A_{2g} *then* $\mathrm{Arf}(\mathcal{C}) = 0$ *if and only if* $g \equiv 0, 3$ *mod* 4.

(2) *If* $\mathcal{C} \subset S$ *is a simple admissible curve system of type* U_{2g} *for* $g \geq 3$ *then* $\mathrm{Arf}(\mathcal{C}) = 0$ *if and only if* $g \equiv 1, 2$ *mod* 4.
(3) *If* $\mathcal{C} \subset S$ *is a simple admissible curve system of type* V_{2g} *for* $g \geq 3$ *then* $\mathrm{Arf}(\mathcal{C}) = 0$ *if and only of* $g \equiv 0, 3$ *mod* 4.

Proof. We begin with showing part (1) of the lemma. Let $\mathfrak{q}_g$ be the quadratic form defined by the curve diagram A_{2g}. Label the vertices in the diagram in consecutive order with the labels $c_1, a_1, \ldots, c_g, a_g$. The simple closed curves corresponding to the vertices a_i or to the vertices c_j $(i, j = 1, \ldots, g)$ are pairwise disjoint and hence their homology classes $[a_i]$ or $[c_j]$ mod 2 span a Lagrangian subspace of $H_1(S, \mathbb{Z}/2\mathbb{Z})$. If we denote by ι the symplectic form mod 2 in $H_1(S, \mathbb{Z}/2\mathbb{Z})$ then we have $\iota([a_i], [c_i]) = \iota([a_i], [c_{i+1}]) = 1$ for $1 \leq i \leq g-1$, and $\iota([a_i], [c_j]) = 0$ for $j \neq i, i+1$.

If we write $[b_i] = \sum_{j \leq i}[c_j]$ then $[a_1], [b_1], \ldots, [a_g], [b_g]$ is a symplectic basis of $H_1(S, \mathbb{Z}/2\mathbb{Z})$. We have $\mathfrak{q}_g([a_i]) = 1$ for all i, and $\mathfrak{q}_g([b_i]) \equiv i$ mod 2. Thus

$$\mathrm{Arf}(\mathfrak{q}_g) = \sum_{i=1}^{g} i \bmod 2 = \frac{g}{2}(g+1) \bmod 2.$$

This implies $\mathrm{Arf}(\mathfrak{q}_g) = \mathrm{Arf}(\mathfrak{q}_{g+4})$ for all g, and $\mathrm{Arf}(\mathfrak{q}_g) = 0$ if and only if $g \equiv 0, 3$ mod 4.

To show (2) denote again by $\mathfrak{q}_g$ the quadratic form defined by the curve diagram U_{2g}. Number the vertices of the long line of the diagram in consecutive order $c_1, a_1, \ldots, c_{g-1}, a_{g-1}$. Denote moreover by a_g, c_g, respectively, the midpoint and the endpoint of the segment of length two issuing from c_2. Define $b_1 = c_1, b_j = \sum_{i \leq j} c_j + c_g$ for $2 \leq j \leq g-1$ and $b_g = c_g$. It follows as before that a_i, b_i is a symplectic basis of $H_1(S, \mathbb{Z}/2\mathbb{Z})$. Now $\mathfrak{q}_g(a_i) = 1$ mod 2 for all i, $\mathfrak{q}_g(b_j) = j+1$ mod 2 for $2 \leq j \leq g-1$, and $\mathfrak{q}_g(b_1) = 1 = \mathfrak{q}_g(b_g) = 1$ mod 2. Therefore

$$\mathrm{Arf}(\mathfrak{q}_g) = \frac{g}{2}(g+1) - 3 + 2 \bmod 2.$$

But this just means that $\mathrm{Arf}(\mathfrak{q}_g) = 0$ if and only if $g \equiv 1, 2$ mod 2 which is what we wanted to show.

Finally consider the curve system V_{2g}. Number the vertices on the long line in the order $c_1, a_1, \ldots, c_g$ and let a_g be the vertex at the endpoint of the leaf of the tree coming out of the vertex 5. Using the same notations as before, we have

$$\mathrm{Arf}(\mathfrak{q}_g) = 1 + g + \frac{1}{2}(g-3)(g-2) \bmod 2.$$

This implies that $\mathrm{Arf}(\mathfrak{q}_g) \equiv 0$ if and only if $g \equiv 0, 3$ mod 4 and completes the proof of the lemma. □

Remark 3.5. We used the diagrams U_{2g}, V_{2g} to cover all possible Arf invariants in every genus.

Example 3.6. Using the remark 2.8, the curve diagrams of simple curve systems for surfaces of small genus can easily be classified. To this end define a *coloring* of a tree D to be a partition of the vertices of D into two subsets V_1, V_2, such that any edge of D connects a vertex in V_1 to a vertex in V_2.

If $g = 2$ then we look for a tree with 4 vertices and such that the cardinality of V_i equals two. The only possibility is the diagram A_4.

If $g = 3$ then there are precisely two possibilities, corresponding to the diagrams A_6 and E_6.

If $g = 4$ then there are four possibilities. Three of these possibilities correspond to the diagrams $A_8, U_8, V_8 = E_8$ as well as a diagram D with a single vertex of valence 4, with two line segments of length two and a segment of length one attached. The parity of the diagram is odd. This is consistent with the fact that the Coxeter groups defined by D and U_8 are hyperbolic, while the Coxeter group defined by E_8 is spherical.

An *admissible subsystem* of an admissible curve system $\mathcal{C}$ is an admissible curve system $\mathcal{C}'$ whose components are contained in $\mathcal{C}$.

Corollary 3.7. *Let $\mathcal{Q}$ be any component of a stratum of abelian differentials. Then there exists an admissible curve system $\mathcal{C}$ for $\mathcal{Q}$ which contains a subsystem $\mathcal{C}'$ of type T for $T = A_{2g}$, $T = U_{2g}$ or $T = V_{2g}$. Moreover, if all the zeros of $\mathcal{Q}$ are even and if $\mathcal{Q}$ is not hyperelliptic, then we can choose $\mathcal{C}'$ to be the curve system $T = U_{2g}$ or $T = V_{2g}$ whose parity coincides with the parity of $\mathcal{Q}$.*

Proof. For each $k \geq 1$, an admissible curve system $\mathcal{C}$ for the hyperelliptic component $\mathcal{H}^{\rm hyp}(k,k)$ can be obtained from a system of type A_{2g} ($k = g - 1$) by adding a single curve which intersects the curve c_1 in a single point and is disjoint from any curve $d \neq c_1$. The curves from the system $\mathcal{C}$ are invariant under the hyperelliptic involution.

To construct curve systems for the remaining components, let $g \geq 4$ and let $\mathcal{D}_g$ be the curve system on a surface of genus g whose curve diagram is obtained from the curve diagram of the system of type V_{2g} as follows. Add an edge to the vertex c_1, and an edge to the vertex c_3. Furthermore, if g is even then for each $2 \leq \ell < g/2$ attach two edges to the vertex $c_{2\ell+1}$. If g is odd then for $2\ell < (g-1)/2$ attach two edges to the vertex $c_{2\ell+1}$, and attach one edge to the vertex c_g. The resulting curve system $\mathcal{D}_g$ is admissible, and the zeros of $q(\mathcal{D}_g)$ are all of order two. Furthermore, by Lemma 3.2 and its proof, i.e. as a straightforward consequence of Lemma 2 of [KZ03], the parity of the spin structure of the component of $\mathcal{H}(2,\dots,2)$ containing $q(\mathcal{D}_g)$ equals the parity of the curve system of type V_{2g}. Note that $\mathcal{D}_g$ also contains a subsystem of type A_{2g}. This is consistent with Lemma 3.4 which shows that the parities of the curve systems of type A_{2g} and V_{2g} coincide. We refer to Figure B for an illustration.

A simple admissible curve system for an arbitrary component of a stratum with all zeros of even order and of the same parity as the system of type V_{2g} can be obtained from $\mathcal{D}_g$ by removing some of the curves in $\mathcal{D}_g - \mathcal{C}$.

A similar construction is also valid for a curve system of type U_{2g}. Construct a curve system $\mathcal{E}_g$ whose curve diagram is obtained from the diagram U_{2g} as follows. If g is even, then attach an edge each to the vertices c_1 and c_g. Furthermore, for $2 \leq \ell < g/2$ attach two edges to the curve $c_{2\ell-1}$. If g is odd then attach an edge to

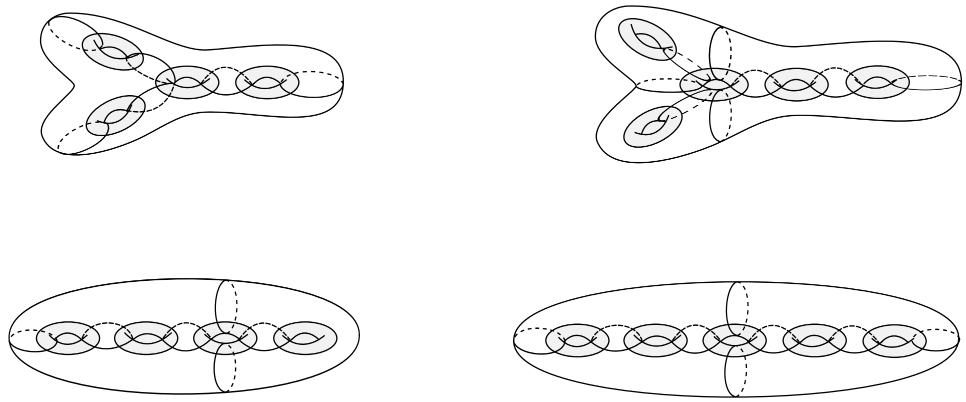


Figure B

the curve c_2, and for $2 \leq \ell \leq (g-1)/2$ attach two edge to the curve $c_{2\ell}$. We refer to Figure B for this construction.

A realization $q(\mathcal{E}_g)$ of a curve system with curve diagram $\mathcal{E}_g$ is contained in the component of $\mathcal{H}(2, \dots, 2)$ whose parity is opposite to the parity of the component containing $q(\mathcal{D}_g)$. Removing some of the curves from $\mathcal{E}_g$ then yields an admissible curve system for each component of a stratum of abelian differentials with only zeros of even order whose parity coincides with the parity of the system U_{2g}.

Finally, for a non-hyperelliptic component with at least one zero of odd order we obtain an admissible curve system by first adding to the curve diagram of $\mathcal{D}_g$ simple closed curves so that we obtain a system whose realization is a differential with only simple zeros. From this system we remove a suitable collection of curves to construct differentials with any prescribed number and order of zeros. We omit the straightforward details of this construction. $\square$

Recall from Lemma 2.9 that an admissible curve system $\mathcal{C}$ determines a completely periodic abelian differential $q(\mathcal{C})$. Let $\mathcal{Q}(\mathcal{C})$ be the component of the stratum of abelian differentials containing $q(\mathcal{C})$ and denote by $\Gamma(\mathcal{C})$ the subgroup of $\mathrm{Mod}(S_{g,m})$ generated by the Dehn twists about the curves from $\mathcal{C}$. Here as before, $m \geq 1$ is the number of zeros of $q(\mathcal{C})$.

Corollary 3.8. *Let $\mathcal{C}$ be any admissible curve system. Then the orbifold fundamental group of $\mathcal{Q}(\mathcal{C})$ contains a subgroup which projects onto a conjugate of $\Gamma(\mathcal{C})$.*

Proof. Let as before $q(\mathcal{C}) \in \mathcal{Q}(\mathcal{C})$ be the completely periodic abelian differential constructed in Lemma 2.9 whose horizontal and vertical cylinders are homotopic to the curves from $\mathcal{C}$.

If C is a horizontal cylinder of $q(\mathcal{C})$, then there is a smooth deformation of $q(\mathcal{C})$ which preserves the restriction of $q(\mathcal{C})$ to the complement of C fixed and shears the foliated cylinder along its horizontal trajectories. We refer to [Wr15] for a comprehensive discussion of this classical construction. For a suitable choice of such a shearing parameter, the endpoint of this transformation is just the Dehn twist about the core curve of the cylinder (see e.g. [H14]). As these shearing transformations do not change the number of zeros of a differential nor its multiplicities, they preserve the component $\mathcal{Q}(\mathcal{C})$. Thus the Dehn twist about the core curve of a horizontal cylinder of $q(\mathcal{C})$ is contained in the projection to $\mathrm{Mod}(S_{g,m})$ of the orbifold fundamental group of $\mathcal{Q}$ based at $q(\mathcal{C})$.

Replacing horizontal cylinders by vertical cylinders of $q(\mathcal{C})$ then shows the corollary. □

As a fairly easy consequence of this observation and deep work of Salter [Sa17], we obtain the first three parts of Theorem 3 from the introduction.

Proposition 3.9. *Let $\mathcal{Q}$ be a non-hyperelliptic component of a stratum of abelian differentials on S.*

(1) *If $q \in \mathcal{Q}$ has at least one simple zero then the orbifold fundamental group of $\mathcal{Q}$ surjects onto* $\mathrm{Mod}(S)$.
(2) *If $g \geq 5$ and if $q \in \mathcal{Q}$ has at least one zero of odd order k which divides $2g-2$ then the orbifold fundamenatal group of $\mathcal{Q}$ surjects onto the finite index subgroup of* $\mathrm{Mod}(S)$ *which preserves a $\mathbb{Z}/k\mathbb{Z}$-spin structure.*
(3) *If $\mathcal{Q} \subset \mathcal{H}(k_1, \dots, k_m)$ with all k_i even, if $k_i < g-1$ divides $2g-2$ for some i and if $g \geq \max\{2k_i+1, 21\}$ then the orbifold fundamental group of $\mathcal{Q}$ surjects onto a finite index subgroup of* $\mathrm{Mod}(S)$.

Proof. To show the first part of the corollary, note that if $\mathcal{Q}$ is a component of abelian differentials with at least one simple zero, then by the above discussion, there exists an admissible curve system for $\mathcal{Q}$ which contains the Humphries system as a subsystem. As the mapping class group of S is generated by the Dehn twists about the curves from the Humphries system, the corollary follows from Corollary 3.8.

Now let $g \geq 5$ and consider a component of a stratum which contains a zero of order $k < g-1$ which divides $g-1$. Let $\mathcal{C}$ be the following curve system on S.

Glue S from a chain of $g-1$ two-holed tori $T_1, \dots, T_{g-1}$ in cyclic order. For each i, the curve c_{2i-1} is embedded in T_i and goes around the hole of T_i. The curves $c_1, \dots, c_{2g-3}$ define a chain of curves, i.e. the curve diagram is a line segment and the numbering of the curves corresponds to the linear order of the vertices in the line segment. The curves a_1, a_2 are embedded in T_1, are disjoint and intersect c_1 in a single point. The curve d is embedded in T_k, intersects c_{2k+1} in a single point and is disjoint from all other curves. The curve a_0 intersects a_1 in a single point, is disjoint from all other curves and is invariant under the obvious cyclic group of diffeomorphisms of S which cyclically permute the tori T_i. Note that a_0 is the unique curve which intersects the common boundary component b of T_{g-1} and T_1.

The curve diagram for $\mathcal{C}$ is the following tree. There is a line segment of length $2g-3$ with linearly numbered vertices $c_1, \dots, c_{2g-3}$. Attached to this line segment are two segments coming out of the vertex c_1, one of length one and the other of length two, and a single edge coming out of the vertex c_{2k+1}.

As the curve diagram of $\mathcal{C}$ is a tree, it can be realized by an abelian differential $q(\mathcal{C})$ with a single zero of order k and a zero of order $2g-2-k > k$. The component $\mathcal{Q}(\mathcal{C})$ of a stratum containing q consists of abelian differentials with two zeros, one of order k and one of order $2g-2-k > k$. In particular, these two zeros define a divisor on S given by points with multiplicities a multiple of k. Thus this divisor determines a k-th root of the canonical bundle and hence a $\mathbb{Z}/k\mathbb{Z}$-valued spin structure on S (see [Sa17] for more information). The Dehn twists about the curves from $\mathcal{C}$ are elements of $P\pi_1(\mathcal{Q})$, and they preserve this spin structure.

Removal of the curve a_0 results in an curve system $\mathcal{C}'$ whose curve diagram is a tree and which fills the subsurface $S - b$ of S. Thus this system fulfills all the conditions in Theorem 9.5 of [Sa17]. Theorem 9.5 of [Sa17] then shows the following.

If k is odd then the stratum $\mathcal{H}(k, 2g-2-k)$ is connected. The group $\Gamma(\mathcal{C})$ surjects onto the stabilizer of a $\mathbb{Z}/k\mathbb{Z}$-spin structure in $\mathrm{Mod}(S)$. As $\Gamma(\mathcal{C}) \subset P\pi_1(\mathcal{Q})$, the second part of the proposition follows.

If k is even then $\mathcal{Q}(\mathcal{C})$ is a non-hyperelliptic component of the stratum $\mathcal{H}(k, 2g-2-k)$. Theorem 9.5 of [Sa17] shows that for $g \geq \max\{2k+1, 21\}$ the group $\Gamma(\mathcal{C})$ is a subgroup of finite index in the stabilizer of the corresponding $\mathbb{Z}/k\mathbb{Z}$-spin structure. This shows the third part of the corollary for one of the two non-hyperelliptic components of $\mathcal{H}(k, 2g-2-k)$.

To cover the second non-hyperelliptic component of $\mathcal{H}(k, 2g-2-k)$, modify the curve diagram by replacing the tail end of length 4 of the line segment with vertices $c_1, \dots, c_{2g-3}$ in the above construction (containing the curves $c_{2g-7}, \dots, c_{2g-3}$) by two segments of length two attached to the vertex c_{2g-7}. This amounts to replacing the circle of $g-1$ tori by a circle of $g-2$ tori and attaching an additional torus T' to the torus T_{g-2}. The curve a_0 is disjoint from T'. This curve system also fulfills the conditions in Theorem 9.5 of [Sa17] and represents the parity opposite to the parity of $\mathcal{C}$.

Finally note that by Corollary 3.7, if $\mathcal{Q}$ is any component of a stratum containing a zero of order $k < g-1$ which divides $2g-2$ then there exists an admissible curve system $\mathcal{D}$ for $\mathcal{Q}$ which can be obtained from one of the above curve systems by adding some curves. Then the group $\Gamma(\mathcal{D})$ contains one of the above groups $\Gamma(\mathcal{C})$ as a subgroup. This completes the proof of the proposition. □

Let us return to the discussion of the hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ as an explicit example.

Example 3.10. Consider again a curve system $\mathcal{C}$ with curve diagram A_{2g}. The group $\Gamma(\mathcal{C})$ equals the Artin braid group in $2g+1$ strands (see Example 3.3 and p.248 of [FM12]). The quotient of this group by its center can be identified with the index $2g+1$ subgroup of the mapping class group $\mathrm{Mod}(S_{0,2g+2})$ of the two-sphere $\mathbb{C}P^1$ with $2g+2$ marked points which fixes one of these marked points.

The hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ of the stratum $\mathcal{H}(2g-2)$ can be obtained as follows [KZ03]. Let $\mathcal{R}$ be the moduli space of meromorphic quadratic differentials on S^2 with $2g+1$ simple poles and one single zero of order $2g-3$.

Taking the double cover of $\mathbb{C}P^1$ branched at each of the singular points defines a biholomorphism of this moduli space with $\mathcal{H}^{\rm hyp}(2g-2)$. As the symmetric mapping class group is isomorphic to the surface braid group $\mathrm{Mod}(S_{0,2g+2})$, we conclude that the fundamental group of the component $\mathcal{H}^{\rm hyp}(2g-2)$ is a subgroup of index $2g+1$ of the symmetric mapping class group.

We complete this section with yet another viewpoint on the groups $\Gamma(\mathcal{C})$ for an admissible curve system $\mathcal{C}$. Namely, let $A(\mathcal{C})$ be the *small type Artin group* defined by the curve diagram corresponding to $\mathcal{C}$. This group has the following presentation.

(1) The vertices of the curve diagram of $\mathcal{C}$ generate $A(\mathcal{C})$.
(2) If two generators are not connected by an edge then they commute.
(3) If a, b are connected by an edge then the braid relation $aba = bab$ holds true.

For a simple closed curve c on S let T_c be the positive Dehn twist about c. If c, d intersect in a single point then the braid relation $T_cT_dT_c = T_dT_cT_d$ is fulfilled. Therefore if we denote by $m \geq 1$ the number of connected components of $S - \mathcal{C}$ then there is a homomorphism

$$\rho : A(\mathcal{C}) \to \mathrm{Mod}(S_{g,m}).$$

This homomorphism maps the generator of $A(\mathcal{C})$ defined by a vertex c of the curve diagram to the positive Dehn twist T_c about c. We call such a homomorphism *geometric*.

The following result is due to Waijnryb [Wj99]. The (positive) result of [PV96] does not apply since a diagram of type D_n can not be a diagram of an admissible curve system on a closed surface.

Theorem 3.11. *Let $\mathcal{C}$ be any admissible curve system. Then the homomorphism ρ is injective if and only if $\mathcal{C}$ is of type A_{2g}.*

Proof. It is not hard to see that any admissible curve diagram $\mathcal{C}$ different from A_{2g} contains a subdiagram consisting of a line segment of length 5 with a single edge attached to the middle vertex. As we will not use the theorem in the sequel, we omit the combinatorial proof. This is the diagram for the finite Coxeter group E_6. Let H be the subgroup of $A(\mathcal{C})$ which is generated by the vertices in this subdiagram. It follows from [vdL83] that H is isomorphic to the finite type Artin group E_6.

The main result of Waijnryb [Wj99] shows that the kernel of any geometric homomorphism of the finite type Artin group E_6 into any mapping class group is not contained in the center of E_6. An application to the group H then yields the theorem. □

The following is a reformulation of Corollary 3.8.

Corollary 3.12. $\rho(\mathcal{A}(\mathcal{C})) \subset P\pi_1(\mathcal{Q}(\mathcal{C})) \subset \mathrm{Mod}(S_{g,m})$ *for any admissible curve system $\mathcal{C}$.*

4. Action on homology

Let as before $\mathcal{Q}$ be a component of a stratum of abelian differentials. Consider the chain of homomorphisms

$$\pi_1(\mathcal{Q}) \xrightarrow{P} \mathrm{Mod}(S_{g,m}) \xrightarrow{\Pi} \mathrm{Mod}(S) \xrightarrow{\Psi} Sp(2g,\mathbb{Z})$$

where $\Pi : \mathrm{Mod}(S_{g,m}) \to \mathrm{Mod}(S)$ is induced by the marked points forgetful map and where $\Psi : \mathrm{Mod}(S) \to Sp(2g,\mathbb{Z})$ defines the natural action of $\mathrm{Mod}(S)$ on the first homology group $H_1(S,\mathbb{Z})$ of S, equipped with the intersection form. The goal of this section is to compute explicitly the group $\Psi\Pi P\pi_1(\mathcal{Q}) \subset Sp(2g,\mathbb{Z})$.

Let $\mathcal{C}$ be an admissible curve system for $\mathcal{Q}$. If we equip $\mathcal{C}$ with a consistent orientation then a curve $c \in \mathcal{C}$ defines a homology class $[c] \in H_1(S,\mathbb{Z})$. The image under the homomorphism Ψ of a positive Dehn twist T_c about c is the *transvection* by $[c]$ in $Sp(2g,\mathbb{Z})$, i.e. the transformation

$$b \to b + \iota(b,[c])[c]$$

where we denote by abuse of notation by ι the symplectic form on $H_1(S,\mathbb{Z})$. Note that this transvection does not depend on the orientation of c.

By Corollary 3.8, the group $\Gamma(\mathcal{C}) \subset P\pi_1(\mathcal{Q})$ generated by the Dehn twists about the curves from $\mathcal{C}$ is contained in $P\pi_1(\mathcal{Q})$ and hence the subgroup $\Psi\Pi(\Gamma(\mathcal{C}))$ of $Sp(2g,\mathbb{Z})$ generated by the transvections by the homology classes of the curves from the system $\mathcal{C}$ is contained in $\Psi\Pi P(\pi_1(\mathcal{Q}))$.

Our first goal is to compute explicitly this group for the simple admissible curve systems introduced in Section 2.

Let $\zeta : Sp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}/2\mathbb{Z})$ be the natural coefficient reduction homomorphism. The kernel $\ker(\zeta)$ of ζ is the level two congruence subgroup of $Sp(2g,\mathbb{Z})$. For a computation of $\Psi\Pi(\Gamma(\mathcal{C}))$ we proceed in the following two steps.

(1) Show that $\Psi\Pi(\Gamma(\mathcal{C}))$ contains the group $\ker(\zeta) \subset Sp(2g,\mathbb{Z})$.
(2) Compute explicitly the finite group $\zeta(\Psi\Pi(\Gamma(\mathcal{C})))$.

We begin with showing property (1) above.

Proposition 4.1. *Let $\mathcal{C}$ be a simple admissible curve system of type $A_{2g}, \mathcal{U}_{2g}, V_{2g}$. Then* $\ker(\zeta) \subset \Psi\Pi(\Gamma(\mathcal{C}))$.

Proof. Recall that the image under Ψ of a positive Dehn twist T_c about a non-separating simple closed curve c is the transvection by the homology class $[c]$ of c. This transvection is independent of the orientation of c. We use the following result of Mumford (appendix to Section 5 of [Mu07], see also Lemma 5 of [Joh85] for a slightly weaker statement). Let (α_i, β_i) be a symplectic basis of $H_1(S,\mathbb{Z})$; then the level two congruence subgroup $\ker(\zeta)$ is generated by the transvections by the elements $2\alpha_i, 2\beta_i, 2(\alpha_i + \beta_i)$.

Let a_i, c_i be the curves from the oriented simple admissible curve system $\mathcal{C}$, where the numbering is as in the proof of Lemma 3.4 (see Figure A), equipped with a consistent orientation. These oriented curves determine homology classes

$[a_i],[c_i]\in H_1(S,\mathbb{Z})$, and these homology classes define a basis of $H_1(S,\mathbb{Z})$. Our goal is to show that there is a symplectic basis of $H_1(S,\mathbb{Z})$ constructed from this homology basis which fulfills the criterion of Mumford.

We discuss each curve system separately. Consider first a curve system $\mathcal{C}$ of type A_{2g}. Using again the conventions from Lemma 3.4 (see Figure A), the homology classes $[a_i]$ $(i\leq g)$ span a Lagrangian subspace of $H_1(S,\mathbb{Z})$. For each i, the transvection by $[a_i]$ is contained in the group $G=\Psi\Pi(\Gamma(\mathcal{C}))$.

A *chain* of simple closed curves on S is a curve system whose curve diagram is a line segment [FM12], equipped with an order induced by the diagram. Its length is the number of curves in the chain. Let $x_1\cdots x_k$ be a chain of simple closed curves on S of odd length k. The oriented boundary of a closed regular neighborhood of the chain consists of two oriented non-separating homologous simple closed curves y_1,y_2. Proposition 4.12 of [FM12] shows that

$$(T_{x_1}\cdots T_{x_k})^{k+1}=T_{y_1}T_{y_2}. \tag{1}$$

As y_1,y_2 are homologous and Ψ is a homomorphism, we have

$$\Psi(T_{x_1}\cdots T_{x_k})^{k+1}=\Psi(T_{y_i})^2.$$

An application to the chain consisting of the curves $a_g,c_g,\dots,c_{g-i+1},a_{g-i}$ as in Figure A yields that for $i\geq 1$ we have

$$(T_{a_g}T_{c_g}\cdots T_{c_{g-i+1}}T_{a_{g-i}})^{2i+2}=T_{d^1_{g-i}}T_{d^2_{g-i}}$$

where d^1_{g-i},d^2_{g-i} are homologous simple closed curves which intersect c_{g-i} in a single point and are disjoint from a_j,c_s for $j\neq g-i$ and for all s. As a consequence, if for $i\geq 1$ we denote by $[b_{g-i}]$ the homology class of d^1_{g-i} and let $[b_g]=[a_g]$, then up to sign, $([c_i],[b_i])$ is a symplectic basis of $H_1(S,\mathbb{Z})$. The chain relation (1) shows that the transvection by $2[b_i]$ is contained in the group G for all i.

Now for each i, the homology class of the simple multicurve $T^{\pm}_{c_i}(2b_i)$ equals $2[b_i]+2[c_i]$ where the choice of $\pm$ depends on the orientation of a_i,b_i. On the other hand, Fact 3.7 in [FM12] shows that $T_{T_{c_i}(2b_i)}=T_{c_i}T^2_{b_i}T^{-1}_{c_i}$ and therefore by the discussions in the last two paragraphs, for each i the transvection by $2[c_i]+2[b_i]$ is contained in the group G. The statement of the proposition for a curve system of type A_{2g} now follows from Mumford's criterion.

We next consider the diagram V_{2g}. We use again the notations from the proof of Lemma 3.4 (see Figure A). For each $i\leq g$ let d_i be a simple closed curve on S which intersects the curve c_i is a single point and does not intersect any other of the curves c_j,a_ℓ. We may assume that $d_3=a_g$ and that for each i, the curves d_i,a_i,d_{i+1} bound a pair of pants. We begin with showing that $\Psi(T_{d_j})^2\in G$ for all j. Recall that T_{d_j} does not depend on an orientation of d_j.

As $d_3=a_g\in\mathcal{C}$, this is clear for $j=3$. The curve d_2 is a boundary component of a tubular neighborhood of the chain $d_3c_3a_2$ of simple closed curves and hence using the version of the chain relation from p.108 of [FM12], we have

$$\Psi(T_{d_2})^2=\Psi(T^2_{d_3}T_{c_3}T_{a_2})^3\in\Psi(G). \tag{2}$$

The same argument also shows that $\Psi(T_{d_3}^2 T_{c_3} T_{a_4})^3 = \Psi(T_{d_4})^2$. Now replace the curve d_3 in the above construction by the curve d_2 or the curve d_4. Using again the fact that Ψ is a homomorphism and $\Psi(T_{d_j}^2) \in G$ $(j = 2, 4)$, we can generate in this way the image under Ψ of the square of the Dehn twist about d_1, d_5. Proceeding inductively, this shows that the image of G under Ψ contains all transvections by the homology classes $2y$ where y runs through the elements of the symplectic basis $[a_1], \cdots, [a_g], [d_1], \cdots, [d_g]$ of $H_1(S, \mathbb{Z})$. Furthermore, as G contains the image under Ψ of the Dehn twists T_{a_i}, we conclude as in the case of the curve system of type A_{2g} that the group G contains the kernel of the homomorphism ζ.

We are left with the curve system of type U_{2g}. Let $a_1, \dots, a_g$ be the maximal system of pairwise disjoint curves in $\mathcal{C}$ which does not contain the curve corresponding to the trivalent vertex of the curve diagram (these are the curves denoted with the same symbols in the proof of Lemma 3.4, see Figure A). The trivalent vertex corresponds to the curve c_2, and the curves on the short line segments disjoint from c_2 are the curves a_g, c_g. Let $b_{g-1} = c_{g-1}$ and for $2 \le i \le g-1$ let b_i be the simple closed curve which intersects a_i in a single point and is disjoint from a_j, c_ℓ for all j, ℓ. Write furthermore $b_1 = c_1$ and $b_g = c_g$. Then $[a_i], [b_i]$ is a symplectic basis of $H_1(S, \mathbb{Z})$. Furthermore, with the reasoning used for the system A_{2g}, V_{2g}, the transvections by the homology classes $2[a_i], 2[b_i]$ are contained in the group G. The argument used for the curve diagram A_{2g} also shows that this is true for the transvections by $2[a_i] + 2[b_i]$. □

Proposition 4.1 implies that for a curve system $\mathcal{C}$ of type A_{2g}, U_{2g}, V_{2g}, the group $\Psi\Pi(\Gamma(\mathcal{C}))$ is a subgroup of $Sp(2g, \mathbb{Z})$ of finite index. To determine this group explicitly we are left with computing the group $\zeta(\Psi\Pi(\Gamma(\mathcal{C})))$.

We begin with the curve system of type A_{2g}. In the following proposition, the subgroup of $Sp(2g, \mathbb{Z}/2\mathbb{Z})$ which is isomorphic to the symmetric group in $2g+1$ variables will be explicitly constructed in its proof. The result is implicitly contained in [AC79] and in [AMY18].

Proposition 4.2. *Let $\mathcal{C}$ be a simple admissible curve system of type A_{2g}. Then the group $\Psi\Pi(\Gamma(\mathcal{C}))$ equals the preimage of the symmetric group in $2g+1$ variables under the reduction of coefficients homomorphism ζ.*

Proof. In Example 3.3, we observed that for a curve system $\mathcal{C}$ with curve diagram A_{2g}, the group $\Gamma(\mathcal{C})$ equals the symmetric mapping class group of the surface S_0 obtained from S by removing the interior of a disk about the unique fixed point of the hyperelliptic involution ν which is not contained in any of the curves from $\mathcal{C}$.

To study the action of $\Gamma(\mathcal{C})$ on $H_1(S, \mathbb{Z}/2\mathbb{Z})$ recall that each generator of $\Gamma(\mathcal{C})$ acts as a transvection on $H_1(S, \mathbb{Z}/2\mathbb{Z})$. Let τ_i be the transvection by the mod 2 homology class of the i-th curve d_i of $\mathcal{C}$ where the order is determined by the curve diagram. These transvections satisfy the following relations.

(1) $\tau_i^2 = \mathrm{Id}$ for all i.
(2) Braid relations: $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ for all i.
(3) $\tau_i \tau_j = \tau_j \tau_i$ for $|i - j| \ge 2$.

These are the defining relations for the Coxeter group with Artin diagram A_{2g}, and this Coxeter group is just the symmetric group in $2g+1$ variables.

Namely, recall that a realization of the curve system $\mathcal{C}$ by an abelian differential is contained in the hyperelliptic component $\mathcal{H}^{\rm hyp}(2g-2)$. The unique zero of a differential in $\mathcal{H}^{\rm hyp}(2g-2)$ is a Weierstrass point. Its image under the quotient map $S\to S/\nu=S_{0,2g+2}$ is a distinguished marked point x_0 in the sphere $S_{0,2g+2}$ with $2g+2$ marked points.

Let $x_1,\dots,x_{2g+1}$ be the remaining marked points. We assume that there is an embedded arc ℓ in $S_{0,2g+2}$ which contains the marked points $x_0,x_1,\dots,x_{2g+1}$ in this order. In other words, ℓ is a union of $2g+2$ arcs, and the i-th segment ℓ_i connects x_{i-1} to x_i.

The preimage of the arc ℓ_i in S is a simple closed curve e_i which defines a homology class $[e_i]\in H_1(S,\mathbb{Z}/2\mathbb{Z})$. The Dehn twist about the curve e_i descends to a standard generator of the Artin braid group A with $2g+1$ strands which acts as a transposition on the marked points, exchanging x_{i-1} and x_i (see [FM12] for details, and compare Example 3.3).

As the symmetric group in $2g+1$ variables is generated by these transpositions and the braid relations are relations in the symmetric group, this Coxeter group indeed equals the symmetric group in $2g+1$ variables. Thus $\zeta(\Psi\Pi(\Gamma(\mathcal{C})))$ is a quotient of the symmetric group in $2g+1$ variables.

On the other hand, as $g\geq 2$, the only nontrivial normal subgroup of the symmetric group is the alternating group which consists of even permutation. Since clearly there is an even permutation of the points $x_1,\dots,x_{2g+2}$ which induces a nontrivial action on $H_2(S,\mathbb{Z}/2\mathbb{Z})$, the kernel of the homomorphism from the symmetric group to $\zeta(\Psi\Pi(\Gamma(\mathcal{C}))$ is trivial. In other words, the group $\zeta(\Psi\Pi(\Gamma(\mathcal{C})))$ equals the symmetric group in $2g+1$ variables. This is what we wanted to show. □

The following immediate consequence will be used later on.

Corollary 4.3. *Let $\mathcal{D}$ be curve system on S which consists of an even number $2k\geq 4$ of curves whose curve diagram is a line segment. Then the group $\Gamma(\mathcal{D})$ generated by the components of $\mathcal{D}$ acts transitively on the curves from $\mathcal{D}$.*

Proof. A tubular neighborhood of the curves from the curve system $\mathcal{D}$ is a surface S_0 of genus k with connected boundary. The group $\Gamma(\mathcal{D})$ preserves S_0 and acts on S_0 as the hyperelliptic mapping class group for an involution of S_0 which preserves $\mathcal{D}$. By Proposition 4.2, via the identification of the hyperelliptic mapping class group with the Artin braid group in $2k+1$ strands, the group acts transitively on its standard generators which is equivalent to the statement of the corollary. □

The following corollary is an easy consequence of Corollary 3.8 and Proposition 4.2.

Corollary 4.4 (A'Campo, Avila-Matheus-Yoccoz)**.** (1) *For each $g \geq 2$, the group $\Psi\Pi P\pi_1(\mathcal{H}^{\rm hyp}(2g-2))$ equals the preimage of the symmetric group in $2g+1$ variables under the reduction of coefficients homomorphism ζ.*
(2) *For each $\ell \geq 1$, the group $\Psi\Pi P\pi_1(\mathcal{H}^{\rm hyp}(\ell,\ell))$ equals the preimage of the symmetric group in $2g+2$ variables under the reduction of coefficients homomorphism ζ.*

Proof. To show the first part of the corollary recall from Example 3.3 that the group $G = P\pi_1(\mathcal{H}^{\rm hyp}(2g-2))$ equals the subgroup of the centralizer of the hyperelliptic involution which fixes the distinguished Weierstrass point at the zero of a differentials $q \in \mathcal{H}^{\rm hyp}(2g-2)$.

Example 3.3 and Example 3.10 show that $P\pi_1(\mathcal{H}^{\rm hyp}(2g-2)) = \Gamma(\mathcal{C})$ where $\mathcal{C}$ is a simple admissible curve system of type A_{2g}. The first part of the corollary thus follows from Proposition 4.2.

The second part of the corollary follows from the same argument. Namely, consider a curve system $\mathcal{C}$ whose curve diagram is a line with $2\ell+3 = 2g+1$ vertices. It is straightforward that $\mathcal{C}$ is admissible for the component $\mathcal{H}^{\rm hyp}(\ell,\ell)$. Note that $\mathcal{C}$ is invariant under a hyperelliptic involution, and that it decomposes S into two connected components.

Following the reasoning in the proof of the first part of the corollary, the group $P\pi_1(\mathcal{H}^{\rm hyp}(\ell,\ell))$ is just the group $\Gamma(\mathcal{C})$ as by equivariance, this group descends to the full spherical braid group on the sphere with $2g+2$ marked points. The image of $\Gamma(\mathcal{C})$ under the homomorphism Ψ can be computed with the argument used in the proof of the first part of the corollary.

Namely, as in the proof of Proposition 4.2, the image of $\Gamma(\mathcal{C})$ under the homomorphism $\zeta\Psi\Pi$ is a quotient of the Coxeter group with Artin diagram A_{2g+1}. The ordering of the elements of $\mathcal{C}$ determined by the curve diagram determines an ordering of the $2g+2$ marked points on the sphere. A standard generator of the Coxeter group acts as a transposition on these marked points, and the braid relations translate into the relations between two transpositions in the symmetric group in $2g+2$ variables. As a consequence, this Coxeter group equals the symmetric group in $2g+2$ variables.

To complete the proof of the corollary, observe that the restriction to the alternating group of the natural homomorphism of this Coxeter group into the group $\zeta\Psi\Pi(\Gamma(\mathcal{C}))$ is non-trivial. As $g \geq 2$, the alternating group is the only non-trivial normal subgroup of the symmetric group in $2g+2$ variables and hence the natural homomorphism is indeed injective. □

We are left with analyzing curve diagrams of type U_{2g} and V_{2g}.

Proposition 4.5. *Let $\mathcal{C}$ be a simple admissible curve system of type U_{2g}, V_{2g} and let $\mathfrak{q}$ be the quadratic form defined by $\mathcal{C}$. Then $\Psi\Pi(\Gamma(\mathcal{C})) \subset Sp(2g,\mathbb{Z})$ is the finite index subgroup which maps under mod 2 reduction to the stabilizer of $\mathfrak{q}$ in $Sp(2g,\mathbb{Z}/2\mathbb{Z})$.*

Proof. Let $\mathfrak{q}$ be the quadratic form defined by the curve diagram. A transvection τ_x by $x \in H_1(S,\mathbb{Z}/2\mathbb{Z})$ preserves $\mathfrak{q}$ if and only if $\mathfrak{q}(x) = 1$. Namely, we have $\tau_x(y) = y + \iota(y,x)x$ and hence

$$\mathfrak{q}(\tau_x y) = \mathfrak{q}(y) + \iota(y,x)\mathfrak{q}(x) + \iota(y,x)$$

which equals $\mathfrak{q}(y)$ if and only if either $\iota(y,x) = 0$ or $\mathfrak{q}(x) = 1$. Since for every x there exists some y with $\iota(y,x) = 1$, the claim follows.

Indeed, by a result of Dieudonné (Proposition 14 of [Die73], see also Theorem 6.3 in [Sa17]), the stabilizer of $\mathfrak{q}$ is generated by transvections by homology classes v with $\mathfrak{q}(v) = 1$. Thus by Proposition 4.1, it suffices to show that the group $G = \zeta(\Psi\Pi(\Gamma(\mathcal{C})))$ contains each transvection by an element $a \in H_1(S,\mathbb{Z}/2\mathbb{Z})$ with $\mathfrak{q}(a) = 1$.

We treat the curve systems for the two curve diagrams U_{2g}, V_{2g} simultaneously. Let D be one of these curve diagrams and recall that D is a tree. We identify each vertex of the tree with the mod 2 homology class it defines. Then the vertices of D define a basis of $H_1(S,\mathbb{Z}/2\mathbb{Z})$. By the definition of $\Gamma(\mathcal{C})$, for each such vertex x, the transvection by x is contained in the group G.

For $x \in H_1(S,\mathbb{Z}/2\mathbb{Z})$ let as before $D(x)$ be the full subgraph of D whose vertices are those basis elements of $H_1(S,\mathbb{Z}/2\mathbb{Z})$ for which the coefficient of the representation of x is non-zero.

Let $y \in H_1(S,\mathbb{Z}/2\mathbb{Z})$ be such that the transvection τ_y by y is contained in G. Assume that the graph $D(y)$ contains a component A which has at least two vertices. Then A is a tree. Let $v \in A$ be a leaf of A; we claim that $\tau_{y+v} \in G$. Namely, we have $\iota(v,y) = 1$ and hence $y + v = \tau_v(y)$ and $\tau_{y+v} = \tau_v\tau_y\tau_v^{-1} \in G$.

Similarly, assume that there exists a component A of $D(y)$ and a vertex $v \in D - D(y)$ such that v is connected to a leaf a of A by an edge and is not connected to any other point in $D(y)$ by an edge; then $\tau_{y+v} \in G$. Again, this follows from the observation that $\iota(v,y) = 1$ and hence $\tau_v(y) = y + v$ which implies $\tau_{y+v} = \tau_v\tau_y\tau_v^{-1} \in G$.

If y, v are as in the previous two paragraphs then we say that we obtain $y+v$ by an *elementary move* from y. Note that elementary moves do not change the number of components of $D(y)$. They consist in adding or subtracting a single vertex to a given diagram $D(y)$. By the consideration in the previous paragraph and induction, we conclude that if z can be obtained from y by a sequence of elementary moves and if $\tau_y \in G$ then also $\tau_z \in G$.

We claim that any two elements x, y so that $D(x), D(y)$ have the same number $k \geq 1$ of components are connected by a sequence of elementary moves. To this end note that $k \leq g$. Consider the element $y_k = \sum_{i=1}^k c_i$ (notations as in the proof of Lemma 3.4, see Figure A). It suffices to show that if $D(x)$ has k components then y can be connected to y_k by a sequence of elementary moves.

We show this by induction on the genus g of S beginning with $g = 3$ and the diagram U_6. Let v be a vertex of $D(y)$ which is closest to c_1 with respect to the simplicial metric on the tree D. If this vertex is unique and if A is the component of $D(y)$ containing v, then no vertex on the unique path ζ connecting c_1 to v is connected to a point on $D(y)$ by an edge. Then attaching this path to $D(y)$ yields an element y' whose diagram contains the element c_1 and which is connected to y by a sequence of elementary moves.

Let A' be the component of $D(y')$ containing c_1. No vertex of A' is connected to any vertex of $D(y') - A'$ by an edge. Thus replacing A' by the single vertex c_1 yields an element y_1 obtained from y by a sequence of elementary moves so that $D(y_1)$ has a component consisting of the single element c_1. Let $y_2 = y_1 - c_1$; then $D(y_2)$ is entirely contained in a subdiagram of D of type A_{2g-2}, and we can use the induction hypothesis to deduce the claim.

If the vertex v is not unique then there are two such vertices v_1, v_2, and these are connected to the unique trivalent vertex w by an edge. Now if D is of type U_{2g} then this means that $D(y)$ contains the vertex a_g but not the vertex c_2. Then we can modify y with a sequence of elementary moves to an element y' with the property that $D(y')$ intersects the component of $D - c_2$ containing a_g in the single vertex c_g. The reasoning in the previous paragraph then shows the induction step.

If D is of type V_{2g} then $D(y)$ contains both a_g and a_3. Now the length of the line segment which connects a_3 to c_g is even and hence by induction hypothesis, we find an element y' obtained from y by a sequence of elementary moves so that $D(y')$ does not contain a_3. The induction step now follows as before.

We are left with showing that for each odd number $k \leq g$, there exists an element $y \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ so that $D(y)$ has k components and such that $\tau_y \in G$. To this end it suffices to show the following. If G contains a transvection τ_y for some y so that $D(y)$ has $k-2$ components, then there also is an element $\tau_z \in G$ for some z so that $D(z)$ has k components.

To this end recall that the curve diagrams U_{2g}, V_{2g} contain each a trivalent vertex v. Let y be such that $D(y)$ has $k-2$ components and contains v and all three vertices connected to v by an edge. Then $\iota(v, y) = 1$ and therefore

$$y + v = \tau_v(y) = y + \iota(v, x + v)v.$$

As before, this implies $\tau_{y+v} \in G$. But $D(y+v) = D(y) - v$ and hence $D(y+v)$ has $k+2$ connected components. This completes the proof of the proposition. $\square$

Corollary 4.6. *Let $\mathcal{Q}$ by a non-hyperelliptic component of a stratum with all zeros of even multiplicity. Then $\Psi\Pi P(\pi_1(\mathcal{Q}))$ equals the stabilizer of a quadratic form on $H_1(S, \mathbb{Z}/2\mathbb{Z})$ which defines a spin structure for $\mathcal{Q}$.*

Proof. By invariance, the group $\Psi\Pi P(\pi_1(\mathcal{Q}))$ is contained in the stabilizer $\mathrm{Stab}(\mathfrak{q})$ of a quadratic form $\mathfrak{q}$ on $H_1(S, \mathbb{Z}/2\mathbb{Z})$, so we have to show that $\Psi\Pi P\pi_1(\mathcal{Q})$ contains $\mathrm{Stab}(\mathfrak{q})$.

By Corollary 3.7, there exists an admissible curve system $\mathcal{C}$ for $\mathcal{Q}$ containing a simple admissible subsystem $\mathcal{C}'$ which determines the parity of the spin structure of $\mathcal{Q}$. But then $\Gamma(\mathcal{C}') \subset P\pi_1(\mathcal{Q})$ and hence $\Psi\Pi(\Gamma(\mathcal{C})') \subset \Psi\Pi P(\pi_1(\mathcal{Q}))$.

Now Proposition 4.5 shows that $\Psi\Pi\Gamma(\mathcal{C}')$ equals the stabilizer of $\mathfrak{q}$ in $Sp(2g,\mathbb{Z})$ and hence $\Psi\Pi P(\pi_1(\mathcal{Q}))$ contains this stabilizer. This is what we wanted to show. ☐

Corollary 4.7. *Let $\mathcal{Q}$ be a non-hyperelliptic component of a stratum with a zero of odd muliplicity; then $\Psi\Pi P\pi_1(\mathcal{Q}) = Sp(2g,\mathbb{Z})$.*

Proof. By Corollary 3.7, there exists an admissible curve system $\mathcal{C}$ for $\mathcal{Q}$ which contains a curve system $\mathcal{C}'$ of type V_{2g} or U_{2g} as a subsystem. By Proposition 4.5, the group $\Psi\Pi\Gamma(\mathcal{C})$ contains the stabilizer of the quadratic form $\mathfrak{q}$ on $H_1(S,\mathbb{Z}/2\mathbb{Z})$ defined as follows.

Let D be the curve diagram of $\mathcal{C}'$. The vertices of D determine a basis of $H_1(S,\mathbb{Z}/2\mathbb{Z})$. For a class x, we have that $\mathfrak{q}(x)$ equals the Euler characteristic of the full subgraph $D(x)$ of D spanned by those vertices whose coefficients in the representation of x is not zero.

As $\mathcal{Q}$ has a zero of odd order, there exists a component c of $\mathcal{C}$ which defines a class $[c]$ in $H_1(S,\mathbb{Z}/2\mathbb{Z})$ with $\mathfrak{q}([c]) = 0$. The graph $D([c])$ has an even number of components. Assume that this number of components equals $2\ell \geq 2$.

By Proposition 4.5 and its proof, $\zeta\Psi\Pi(\Gamma(\mathcal{Q}))$ contains all transvections by elements which define a subgraph of the curve diagram D of $\mathcal{C}'$ with an even number of connected components. But this implies that $\zeta\Psi\Pi(\Gamma(\mathcal{C}))$ contains all transvections. Thus $\zeta\Psi\Pi(\Gamma(\mathcal{C})) = Sp(2g,\mathbb{Z}/2\mathbb{Z})$ and together with Proposition 4.1, this shows the corollary. ☐

5. Train tracks and geodesic laminations

In this section we introduce our main technical tool to promote the results from Section 2 to the Theorem from the introduction, namely train tracks and geodesic laminations. We begin with summarizing some constructions from [PH92, H09] which will be used throughout the paper. We then introduce a class of train tracks which will serve as combinatorial models for components of strata in the later sections, and we discuss some of their properties.

5.1. **Geodesic laminations.** Let S be an oriented surface of genus $g \geq 0$ with $n \geq 0$ marked points (punctures) and where $3g - 3 + n \geq 2$. A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational.* Every geodesic lamination λ consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of λ either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components [CEG87].

A geodesic lamination λ on S is said to *fill up* S if its complementary regions are all topological disks or once punctured monogons or once punctured bigons. Here a once puncture monogon is a once punctured disk with a single cusp at the boundary. A *maximal* geodesic lamination is a geodesic lamination whose complementary regions are all ideal triangles or once punctured monogons.

Definition 5.1. A geodesic lamination λ is called *large* if λ fills up S and if moreover λ can be approximated in the *Hausdorff topology* by simple closed geodesics.

Since every minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed geodesics [CEG87], a minimal geodesic lamination which fills up S is large. However, there are large geodesic laminations with finitely many leaves.

The *topological type* of a large geodesic lamination ν is a tuple

$$(m_1, \ldots, m_\ell; -p_1, p_2) \text{ where } 1 \leq m_1 \leq \cdots \leq m_\ell,\ \sum_i m_i = 4g - 4 + p_1,\ p_1 + p_2 = n.$$

Here $\ell \geq 1$ is the number of complementary regions which are topological disks, and these disks are $m_i + 2$-gons ($i \leq \ell$). There are p_1 once punctured monogons and p_2 once punctured bigons. Let

$$\mathcal{LL}(m_1, \ldots, m_\ell; -p_1, p_2)$$

be the space of all large geodesic laminations of type $(m_1, \ldots, m_\ell; -p_1, p_2)$ equipped with the restriction of the Hausdorff topology for compact subsets of S.

A *measured geodesic lamination* is a geodesic lamination λ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in S with endpoints in the complementary regions of λ which intersects λ nontrivially and transversely. The geodesic lamination λ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. The space $\mathcal{ML}$ of all measured geodesic laminations on S equipped with the weak*-topology is homeomorphic to $S^{6g-7+2n} \times (0, \infty)$. Its projectivization is the space $\mathcal{PML}$ of all *projective measured geodesic laminations.*

The measured geodesic lamination $\mu \in \mathcal{ML}$ *fills up* S if its support fills up S. This support is then necessarily connected and hence minimal. Since a minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed curves [CEG87], there exists a tuple $(m_1, \ldots, m_\ell; -p_1, p_2)$ such that the support of μ defines a point in the set $\mathcal{LL}(m_1, \ldots, m_\ell; -p_1, p_2)$. The projectivization of a measured geodesic lamination which fills up S is also said to fill up S.

There is a continuous symmetric pairing $\iota : \mathcal{ML} \times \mathcal{ML} \to [0, \infty)$, the so-called *intersection form*, which extends the geometric intersection number between simple closed curves.

5.2. **Train tracks.** A *train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. A simple closed curve component of τ contains a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from disks with $0, 1$ or 2 cusps at the boundary and different from annuli and once-punctured disks with no cusps at the boundary. We always identify train tracks which are isotopic. Throughout we use the book [PH92] as the main reference for train tracks. All our train tracks are *marked*, i.e. we think of a train track τ as a (coarsely well defined) point in the marking graph of the subsurface of S filled by τ.

A train track is called *generic* if all switches are at most trivalent. For each switch v of a generic train track τ which is not contained in a simple closed curve component, there is a unique half-branch b of τ which is incident on v and which is *large* at v. This means that every germ of an arc of class C^1 on τ which passes through v also passes through the interior of b. A half-branch which is not large is called *small*. A branch b of τ is called *large* (or *small*) if each of its two half-branches is large (or small). A branch which is neither large nor small is called *mixed*.

Remark 5.2. As in [H09], all train tracks are assumed to be generic. Unfortunately this leads to a small inconsistency of our terminology with the terminology found in the literature.

A *trainpath* on a train track τ is a C^1-immersion $\rho : [k, \ell] \to \tau$ such that for every $i < \ell - k$ the restriction of ρ to $[k+i, k+i+1]$ is a homeomorphism onto a branch of τ. More generally, we call a C^1-immersion $\rho : [a, b] \to \tau$ a *generalized trainpath*. A trainpath $\rho : [k, \ell] \to \tau$ on is *closed* if $\rho(k) = \rho(\ell)$ and if either the image of ρ is a closed curve component of τ or if precisely one of the half-branches $\rho[k, k+1/2], \rho[\ell - 1/2, \ell]$ is large.

A generic train track τ is *orientable* if there is a consistent orientation of the branches of τ such that at any switch s of τ, the orientation of the large half-branch incident on s extends to the orientation of the two small half-branches incident on s. If C is a complementary polygon of an oriented train track then the number of sides of C is even. In particular, a train track which contains a once punctured monogon component is not orientable (see p.31 of [PH92] for a more detailed discussion).

A train track or a geodesic lamination η is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is homotopic to the identity and maps η into τ in such a way that the restriction of the differential of F to the tangent space of η vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of F to η a *carrying map* for η. Write $\eta \prec \tau$ if the train track η is carried by the train track τ. Then every geodesic lamination ν which is carried by η is also carried by τ.

A train track *fills up* S if its complementary components are topological disks or once punctured monogons or once punctured bigons. Note that such a train track τ is connected. Let $\ell \geq 1$ be the number of those complementary components of τ which are topological disks. Each of these disks is an $m_i + 2$-gon for some $m_i \geq 1$ ($i = 1, \dots, \ell$). The *topological type* of τ is defined to be the ordered tuple $(m_1, \dots, m_\ell; -p_1, p_2)$ where $1 \leq m_1 \leq \cdots \leq m_\ell$ and p_1 (or p_2) is the number of once punctured monogons (or once punctured bigons); then $\sum_i m_i = 4g - 4 + p_1$ and $p_1 + p_2 = n$. If τ is orientable then $p_1 = 0$ and m_i is even for all i. A train track of topological type $(1, \dots, 1; -p_1, 0)$ is called *maximal*. The complementary components of a maximal train track are all trigons, i.e. topological disks with three cusps at the boundary, or once punctured monogons.

A *transverse measure* on a generic train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: for every trivalent switch s of τ, the sum of the weights of the two small half-branches incident on s equals the weight of the large half-branch. Particular such transverse measures are the counting measures of simple multicurves c carried by τ. Such a measure associates to a branch b the number of the preimages of an interior point of b under the carrying map. The weight of every branch with respect to this measure is integral. In particular, the ratio of weights of any two branches is rational, and we call a transverse measure with this property *rational*. The set of rational measures is invariant under scaling, and it is dense is the cone of all transverse measures on τ.

A *subtrack* σ of a train track τ is a subset of τ which is itself a train track. Then σ is obtained from τ by removing some of the branches, and we write $\sigma < \tau$. A *vertex cycle* for τ is defined to be an embedded subtrack of τ which either is a simple closed curve or a *dumbbell*, i.e. it consists of two loops with one cusp which are connected by an embedded segment joining the cusps (that this definition is equivalent to the definition defined in other works can for example be found in [Mo03], see also [H06]). An orientable train track does not contain dumbbells. Each vertex cycle supports a single transverse measure up to scale.

The following is well known and will be used several times in the sequel. We refer to [Mo03] for a comprehensive discussion.

Lemma 5.3. *Let $\mathcal{V}(\tau)$ be the space of all transverse measures on τ.*

(1) *$\mathcal{V}(\tau)$ has the structure of a cone over a compact convex polyhedron in a finite dimensional vector space.*
(2) *The vertices of the polyhedron are up to scaling the measures supported on the vertex cycles.*
(3) *There exists a natural homeomorphism of $\mathcal{V}(\tau)$, equipped with the euclidean topology, onto the closed subspace of $\mathcal{ML}$ of all measured geodesic laminations carried by τ.*

The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. We call such a transverse measure μ *positive*, and we write $\mu > 0$ (see [PH92] for more details).

If b is a small branch of τ which is incident on two distinct switches of τ then the graph σ obtained from τ by removing b is a subtrack of τ. We then call τ a *simple extension* of σ. Note that formally to obtain the subtrack σ from $\tau - b$ we may have to delete the switches on which the branch b is incident.

The following lemma is certainly known to the experts. We include it here as it is useful to understand the way train tracks can be used to model of strata of abelian or quadratic differentials and their degenerations.

Lemma 5.4. (1) *A simple extension τ of a recurrent non-orientable connected train track σ is recurrent.*
(2) *An orientable simple extension τ of a recurrent orientable connected train track σ is recurrent.*

Proof. If τ is a simple extension of a connected train track σ then σ can be obtained from τ by the removal of a small branch b which is incident on two distinct switches s_1, s_2. Then s_i is an interior point of a branch b_i of σ $(i = 1, 2)$.

If σ is moreover non-orientable and recurrent then there is a trainpath $\rho_0 : [0, t] \to \tau - b$ which begins at s_1, ends at s_2 and such that the half-branch $\rho_0[0, 1/2]$ is small at $s_1 = \rho_0(0)$ and that the half-branch $\rho_0[t - 1/2, t]$ is small at $s_2 = \rho_0(t)$. Extend ρ_0 to a closed trainpath ρ on $\tau - b$ which begins and ends at s_1. This is possible since σ is non-orientable, connected and recurrent. There is a closed trainpath $\rho' : [0, u] \to \tau$ which can be obtained from ρ by replacing the trainpath ρ_0 by the branch b traveled through from s_1 to s_2. The counting measure of ρ' on τ satisfies the switch condition and hence it defines a transverse measure on τ which is positive on b. On the other hand, every transverse measure on σ defines a transverse measure on τ. Thus since σ is recurrent and since the sum of two transverse measures on τ is again a transverse measure, the train track τ is recurrent as well.

The second part of the lemma follows in exactly the same way, and its proof will be omitted. ☐

Definition 5.5. A train track τ of topological type $(m_1, \dots, m_\ell; -p_1, p_2)$ which carries a minimal large geodesic lamination $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -p_1, p_2)$ is called *fully recurrent*.

Remark 5.6. The above definition is made for convenience to rule out some easy pathological cases. Since the discussion of such cases is not important for what follows, we do not include it here.

Note that by definition, a fully recurrent train track is connected and fills up S. Since a minimal geodesic lamination supports a transverse measure, a fully recurrent train track τ is recurrent.

We pause to relate our discussion to the work of Thurston as documented in [PH92] and the work of Minsky and Weiss [MW14]. To this end let τ be a fully recurrent orientable train track on S of type $(m_1, \dots, m_\ell; 0)$. Mark a point in each complementary component of τ and let Σ be the union of these ℓ marked points.

The set $\mathcal{V}(\tau)$ is the set of non-negative solutions of system of linear equations and hence it has the structure of convex cone in a vector space. We have

Lemma 5.7. *The choice of an orientation for τ determines a linear homeomorphism of $\mathcal{V}(\tau)$ onto a closed cone in $H_1(S-\Sigma;\mathbb{R})$ with dense interior.*

Proof. View τ as an embedded graph in $S-\Sigma$. Thus τ is a one-dimensional chain complex whose one-cells are the branches of τ, and the first cellular homology $H_1(\tau;\mathbb{R})$ of τ is defined. The inclusion $\iota:\tau\to S-\Sigma$ defines a homomorphism $\iota_*:H_1(\tau;\mathbb{R})\to H_1(S-\Sigma;\mathbb{R})$. As each complementary component of τ is a disk in S, and this disk contains precisely one point of Σ, the train track τ is a deformation retract of $S-\Sigma$ and hence the map ι_* is an isomorphism.

Given an orientation for τ and a transverse measure $\mu\in\mathcal{V}(\tau)$, evaluation of μ on the branches of τ can be viewed as an element in the first chain group of τ. By the definition of a consistent orientation, the switch condition is equivalent to stating that this element is a cycle and hence μ defines a class in $H_1(\tau;\mathbb{R})$. As there are no two-cells, the homology group $H_1(\tau;\mathbb{R})$ equals the kernel of the boundary map and hence we obtain an embedding $\mathcal{V}(\tau)\to H_1(S-\Sigma;\mathbb{R})$.

The lemma follows from the observation that as τ is recurrent, the dimension of $\mathcal{V}(\tau)$ coincides with the dimension of $H_1(S-\Sigma,\mathbb{R})$. Namely, adding a sufficiently small solution to the switch conditions to a positive solution result in a positive solution. As the set of positive solutions is non-empty and the space of all solutions is identified with the group $H_1(S-\Sigma,\mathbb{R}$, the lemma follows. ☐

The second statement of the following corollary is immediate from Lemma 5.7, and the first statement follows in the same way by passing to the orientation cover.

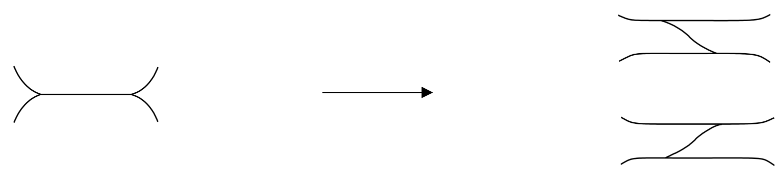
Corollary 5.8.
(1) $\dim\mathcal{V}(\tau)=2g-2+\ell+p_1+p_2$ *for every non-orientable recurrent train track of topological type* $(m_1,\dots,m_\ell;-p_1,p_2)$.
(2) $\dim(\mathcal{V}(\tau)=2g-1+\ell+p_2$ *for every orientable recurrent train track τ of topological type* $(m_1,\dots,m_\ell;0,p_2)$.

Using the identification of $\mathcal{V}(\tau)$ with an open cone in the space of measured laminations on S, there is a geometric interpretation of the map from Lemma 5.7. Namely, the space of measured laminations on S can be identified with the space of measured foliations. If Σ is the singular set of the measured foliation $\mathcal{F}$, then $\mathcal{F}$ defines a relative cohomology class $\zeta\in H^1(S,\Sigma;\mathbb{R})$ by integrating the transverse measure over relative cycles in $H_1(S,\Sigma;\mathbb{R})$. The class in $H_1(S-\Sigma;\mathbb{R})$ defined by the transverse measure μ corresponding to $\mathcal{F}$ is just the Poincaré dual of μ.

There are two simple ways to modify a fully recurrent train track τ to another fully recurrent train track. Namely, if b is a mixed branch of τ then we can *shift* τ along b to a new train track τ'. This new train track carries τ and hence it is fully recurrent since it carries every geodesic lamination which is carried by τ [PH92, H09].

Similarly, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure C. The new small branch in the split track is called the

Figure C



diagonal of the split. A (right or left) split τ' of a train track τ is carried by τ. If τ is of topological type $(m_1,\dots,m_\ell;-p_1,p_2)$, if $\nu\in\mathcal{LL}(m_1,\dots,m_\ell;-p_1,p_2)$ is carried by τ and if e is a large branch of τ, then there is a unique choice of a right or left split of τ at e such that the split track η carries ν. In particular, η is fully recurrent. Note however that there may be a split of τ at e such that the split track is not fully recurrent any more (see Section 2 of [H09] for details).

To each train track τ which fills up S one can associate a *dual bigon track* τ^* (Section 3.4 of [PH92]). There is a bijection between the complementary components of τ and those complementary components of τ^* which are not *bigons*, i.e. disks with two cusps at the boundary. This bijection maps a complementary component C of τ which is an n-gon for some $n\geq 3$ to an n-gon component of τ^* contained in C, and it maps a once punctured monogon or bigon C to a once punctured monogon or bigon contained in C. If τ is orientable then the orientation of S and an orientation of τ induce an orientation on τ^*, i.e. τ^* is orientable.

There is a notion of carrying for bigon tracks which is analogous to the notion of carrying for train tracks. Measured geodesic laminations which are carried by the bigon track τ^* can be described as follows. A *tangential measure* on a train track τ of type $(m_1,\dots,m_\ell;-p_1,p_2)$ assigns to a branch b of τ a weight $\mu(b)\geq 0$ such that for every complementary k-gon of τ or once punctured bigon with consecutive sides $c_1,\dots,c_k$ and total mass $\mu(c_i)$ (counted with multiplicities) the following holds true.

(1) $\mu(c_i)\leq\mu(c_{i-1})+\mu(c_{i+1})$.
(2) $\sum_{i=j}^{k+j-1}(-1)^{i-j}\mu(c_i)\geq 0$, $j=1,\dots,k$.

The complementary once punctured monogons define no constraint on tangential measures. Our definition of tangential measure on τ is stronger than the definition given on p.22 of [PH92] and corresponds to the notion of a *metric* as defined on p.184 of [P88]. We do not use this terminology here since we find it misleading.

The space of all tangential measures on τ has the structure of a convex cone in a finite dimensional real vector space. By Lemma 2.1 of [P88], every tangential measure on τ determines a simplex of measured geodesic laminations which *hit* τ *efficiently*. The supports of these measured geodesic laminations are carried by the bigon track τ^*, and every measured geodesic lamination which is carried by τ^* can be obtained in this way. The dimension of this simplex equals the number of complementary components of τ with an even number of sides. The train track τ

is called *transversely recurrent* if it admits a tangential measure which is positive on every branch.

In general, a measured geodesic lamination ν which hits τ efficiently does not determine uniquely a tangential measure on τ either. Namely, let s be a switch of τ and let a, b, c be the half-branches of τ incident on s and such that the half-branch a is large. If β is a tangential measure on τ and if ν is a measured geodesic lamination in the simplex determined by β then it may be possible to drag the switch s across some of the leaves of ν and modify the tangential measure β on τ to a tangential measure $\mu \neq \beta$. Then $\beta - \mu$ is a multiple of a vector of the form $\delta_a - \delta_b - \delta_c$ where δ_w denotes the function on the branches of τ defined by $\delta_w(w) = 1$ and $\delta_w(a) = 0$ for $a \neq w$.

Definition 5.9. Let τ be a train track of topological type $(m_1, \dots, m_\ell; -p_1, p_2)$.

(1) τ is called *fully transversely recurrent* if its dual bigon track τ^* carries a minimal large geodesic lamination $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -p_1, p_2)$.
(2) τ is called *large* if τ is fully recurrent and fully transversely recurrent.

For a large train track τ let $\mathcal{V}^*(\tau) \subset \mathcal{ML}$ be the set of all measured geodesic laminations whose support is carried by τ^*. Each of these measured geodesic laminations corresponds to a family of tangential measures on τ. With this identification, the pairing

$$(\nu, \mu) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau) \to \sum_b \nu(b)\mu(b) \tag{3}$$

is just the restriction of the intersection form on measured lamination space (Section 3.4 of [PH92]). Moreover, $\mathcal{V}^*(\tau)$ is naturally homeomorphic to a convex cone in a real vector space. The dimension of this cone coincides with the dimension of $\mathcal{V}(\tau)$.

From now on we denote by $\mathcal{LT}(m_1, \dots, m_\ell; -p_1, p_2)$ the set of all isotopy classes of large train tracks on S of type $(m_1, \dots, m_\ell; -p_1, p_2)$.

6. Strata

The goal of this section is to relate large train tracks to components of strata of abelian or quadratic differentials. We begin with introducing some notations for quadratic differentials which were used earlier in some more restricted setting for abelian differentials. The results in this section hold true in both cases.

For a closed oriented surface $S_{g,n}$ of genus $g \geq 0$ with $n \geq 0$ marked points (punctures) let $\tilde{\mathcal{Q}}(S_{g,n})$ be the bundle of marked area one holomorphic quadratic differentials with either a simple pole or a regular point at each of the marked points and no other pole over the *Teichmüller space* $\mathcal{T}(S_{g,n})$ of marked complex structures on $S_{g,n}$.

Fix a complete hyperbolic metric on $S_{g,n}$ of finite area. A quadratic differential $q \in \tilde{\mathcal{Q}}(S_{g,n})$ is determined by a pair (λ^+, λ^-) of measured geodesic laminations which *jointly fill up* S (i.e. we have $\iota(\lambda^+, \mu) + \iota(\lambda^-, \mu) > 0$ for every measured geodesic lamination μ). The *vertical* measured geodesic lamination λ^+ for q corresponds to the equivalence class of the *vertical measured foliation* of q. The *horizontal* measured geodesic lamination λ^- for q corresponds to the equivalence class of the *horizontal measured foliation* of q.

For $p_1 \leq n$, $p_2 = n - p_1$ and $\ell \geq 1$, an ℓ-tuple $(m_1, \dots, m_\ell)$ of positive integers $1 \leq m_1 \leq \dots \leq m_\ell$ with $\sum_i m_i = 4g - 4 + p_1$ defines a *stratum* $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -p_1, p_2)$ in $\tilde{\mathcal{Q}}(S_{g,n})$. This stratum consists of all marked quadratic differentials with p_1 simple poles, p_2 regular marked points and ℓ zeros of order $m_1, \dots, m_\ell$. We require that these differentials are not squares of holomorphic one-forms. The stratum is a complex manifold of dimension

$$h = 2g - 2 + \ell + p_1 + p_2. \tag{4}$$

The closure in $\tilde{\mathcal{Q}}(S_{g,n})$ of a stratum is a union of components of strata. Strata are invariant under the action of the mapping class group $\mathrm{Mod}(S_{g,n})$ of $S_{g,n}$ and hence they project to strata in the moduli space $\mathcal{Q}(S_{g,n}) = \tilde{\mathcal{Q}}(S_{g,n})/\mathrm{Mod}(S_{g,n})$ of quadratic differentials on $S_{g,m}$. Denote by $\mathcal{Q}(m_1, \dots, m_\ell; -p_1, p_2)$ the projection of the stratum $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -p_1, p_2)$. The strata in moduli space need not be connected, but their connected components have been identified by Lanneau [La08]. A stratum in $\mathcal{Q}(S_{g,n})$ has at most two connected components. The number of components of the stratum $\mathcal{Q}(m_1, \dots, m_\ell; -p_1, p_2)$ equals the number of components of $\mathcal{Q}(m_1, \dots, m_\ell; -p_1, 0)$.

Similarly, let $\tilde{\mathcal{H}}(S_{g,n})$ be the bundle of marked holomorphic one-forms over Teichmüller space $\mathcal{T}(S_{g,n})$ of $S_{g,n}$. Each of the marked points of $S_{g,n}$ is required to be a regular point for the differential. In particular, the bundle is non-empty only if $g \geq 1$. For an ℓ-tuple $k_1 \leq \dots \leq k_\ell$ of positive integers with $\sum_i k_i = 2g - 2$, the stratum $\tilde{\mathcal{H}}(k_1, \dots, k_\ell; n)$ of marked holomorphic one-forms on S with ℓ zeros of order k_i $(i = 1, \dots, \ell)$ and n regular marked points is a complex manifold of dimension

$$h = 2g - 1 + \ell + n. \tag{5}$$

It projects to a stratum $\mathcal{H}(k_1, \dots, k_\ell; n)$ in the moduli space $\mathcal{H}(S_{g,n})$ of area one holomorphic one-forms on $S_{g,n}$. Strata of holomorphic one-forms in moduli space need not be connected, but the number of connected components of a stratum is at most three [KZ03].

We continue to use the assumptions and notations from Section 5. For a marked large train track $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -p_1, p_2)$ let

$$\mathcal{Q}(\tau) \subset \tilde{\mathcal{Q}}(S_{g,n})$$

be the set of all marked quadratic differentials whose vertical measured geodesic lamination is contained in $\mathcal{V}(\tau)$ and whose horizontal measured geodesic lamination is carried by the dual bigon track τ^* of τ. Since τ and τ^* both carry a minimal large geodesic lamination, and such a lamination supports a transverse measure and fills $S = S_{g,n}$, for a large train track τ on $S = S_{g,n}$ the set $\mathcal{Q}(\tau)$ is not empty. Recall that no geodesic lamination can be carried by both τ and τ^*.

The next observation relates $\mathcal{Q}(\tau)$ to components of strata. It is related to Theorem 1.2 of [MW14] which is used in its proof. The lemma can also be shown in a more elementary way, however we felt that the proof we provide is better suited to connect to the results in [MW14].

Lemma 6.1. (1) *Let $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -p_1, p_2)$ be non-orientable and let $q \in \mathcal{Q}(\tau)$. If the support of the horizontal measured geodesic lamination of q is contained in $\mathcal{LL}(m_1, \dots, m_\ell; -p_1, p_2)$ then $q \in \tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -p_1, p_2)$.*
(2) *Let $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; 0, p_2)$ be orientable and let $q \in \mathcal{Q}(\tau)$. If the support of the horizontal measured geodesic lamination of q is contained in $\mathcal{LL}(m_1, \dots, m_\ell; 0, p_2)$ then $q \in \tilde{\mathcal{H}}(m_1/2, \dots, m_\ell/2; p_2)$.*

Proof. A marked abelian or quadratic differential $z \in \tilde{\mathcal{Q}}(S_{g,n})$ defines a singular euclidean metric on $S_{g,n}$. A singular point for z is a zero or a pole or a marked regular point. A *saddle connection* for z is a geodesic segment for this singular euclidean metric which connects two singular points and does not contain a singular point in its interior. A *separatrix* is a maximal geodesic segment or ray which begins at a singular point and does not contain a singular point in its interior.

A complex structure on $S_{g,n}$ determines a complete finite area hyperbolic metric h on $S_{g,n}$ with cusps at the p_1 marked points appearing in the definition. Let ξ be the support of the horizontal measured geodesic lamination of the quadratic differential z, realized in the hyperbolic structure defined by z. By [Lev83], the geodesic lamination ξ can be obtained from the horizontal foliation of z by cutting $S_{g,m}$ open along each horizontal separatrix and straightening the remaining leaves so that they become geodesics for h. In particular, up to homotopy, a horizontal saddle connection s of z is contained in the interior of a complementary component C of ξ which is uniquely determined by s.

As marked regular points play no role for the conclusion of the lemma let us consider an orientable large train track $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; 0, 0)$. Let $q \in \mathcal{Q}(\tau)$, with horizontal measured geodesic lamination $\mu \in \mathcal{V}(\tau)$ whose support $\mathrm{supp}(\mu)$ is contained in $\mathcal{LL}(m_1, \dots, m_\ell; -p_1, p_2)$. Mark a point in each complementary component of τ and let Σ be the union of these marked points. By Lemma 5.7, μ defines a homology class in $H_1(S - \Sigma; \mathbb{R})$, and this class is the Poincaré dual of the measured foliation which corresponds to μ, viewed as a relative cohomology class in $H^1(S, \Sigma; \mathbb{R})$. Note that as μ is minimal, we can choose this foliation in such a way that its singular set is precisely Σ. We refer to [MW14] and the discussion after Lemma 5.7 for more information.

Now the dual bigon track τ^* is orientable, and for a given orientation of τ there exists an orientation of τ^* so that each intersection between τ and τ^* is an interior point of an oriented branch b, b^* of τ, τ^* so that the ordered pair (b, b^*) defines the orientation of S. But this just means that for any measured lamination ν carried by τ^*, the intersection between the homology class in $H_1(S - \Sigma; \mathbb{R})$ defined by ν and the homology class defined by any measured lamination whose support equals the support of μ is positive.

By Theorem 1.2 of [MW14], the pair (μ,ν) defines an abelian differential $q \in \mathcal{H}(m_1,\dots,m_\ell)$. As an abelian differential is uniquely determined by its horizontal and vertical measured laminations, this is what we wanted to show.

In the case that τ is non-orientable the same holds true for $\mathrm{supp}(\mu)$ since an orientation of $\mathrm{supp}(\mu)$ would induce an orientation on τ. The claim now follows in the same way by passing to the orientation cover of τ. □

We use Lemma 6.1 to show

Proposition 6.2. (1) *Let $\tau \in \mathcal{LT}(m_1,\dots,m_\ell;-p_1,p_2)$ be a large non-orientable train track. Then there is a component $\tilde{\mathcal{Q}}$ of $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$ such that $\mathcal{Q}(\tau)$ is the closure in $\tilde{\mathcal{Q}}(S_{g,n})$ of an open path connected subset of $\tilde{\mathcal{Q}}$.*
(2) *For every large orientable train track $\tau \in \mathcal{LT}(m_1,\dots,m_\ell;0,n)$ there is a component $\tilde{\mathcal{Q}}$ of $\tilde{\mathcal{H}}(m_1/2,\dots,m_\ell/2,n)$ such that $\mathcal{Q}(\tau)$ is the closure in $\tilde{\mathcal{H}}(S_{g,n})$ of an open path connected subset of $\tilde{\mathcal{Q}}$.*

Proof. In the proof of the proposition, we do not distinguish between the orientable and the non-orientable case.

Let $\tau \in \mathcal{LL}(m_1,\dots,m_\ell;-p_1,p_2)$ and let $\mu \in \mathcal{V}(\tau)$ be such that the support $\mathrm{supp}(\mu)$ of μ is contained in $\mathcal{LL}(m_1,\dots,m_\ell;-p_1,p_2)$. Then μ defines a positive transverse measure on τ. If $\beta \in \mathcal{V}^*(\tau)$ is arbitrary then the measured geodesic laminations μ,β jointly fill up S (since the support of β is different from the support of μ and $\mathrm{supp}(\mu)$ fills up S) and hence the pair (μ,β) defines a point $q \in \mathcal{Q}(\tau)$. By Lemma 6.1, we have $q(\mu,\beta) \in \tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$.

Recall that $\mathcal{V}^*(\tau)$ is homeomorphic to a cone over a closed cell whose dimension equals half of the dimension of $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$. Let V be the interior of this cell. By continuity and invariance of domain, we conclude that the set $\{q(\mu,\beta) \mid \beta \in V\}$ is a neighborhood of $q(\mu,\nu)$ in the *strong stable manifold* of $q(\mu,\nu)$ in $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$ which in *period coordinates* consists of all quadratic differentials in $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$ with the same real part.

As measured laminations which are minimal and of the same topological type as τ are dense in the set of all measured laminations in such a strong stable manifold (see [KMS86] for a comprehensive discussion of this fact), we conclude that such measured laminations are dense in $\mathcal{V}^*(\tau)$.

Now assume that the support of the lamination $\nu \in \mathcal{V}^*(\tau)$ is contained in $\mathcal{LL}(m_1,\dots,m_\ell;-p_1,p_2)$. Using exactly the same reasoning as above, we deduce that for each $\alpha \in \mathcal{V}(\tau)$, the pair (α,ν) defines a quadratic differential $q(\alpha,\nu) \in \tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$. Furthermore, the measured laminations whose support is minimal and of the same topological type as τ are dense in $\mathcal{V}(\tau)$.

To summarize, there exists an open dense subset of $Q(\tau)$ which is contained in $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$. To complete the proof of the proposition, it suffices to show that the set of all pairs $(\alpha,\beta) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau)$ which gives rise to a differential $q(\alpha,\beta) \in \tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$ is path connected.

Thus let $(\alpha,\beta),(\alpha',\beta')$ be two pairs with this property. As jointly filling up S is an open condition for pairs of measured laminations and as the set of all measured laminations whose support is of type $\mathcal{LL}(m_1,\dots,m_\ell;-p_1,p_2)$ is dense in $\mathcal{V}(\tau)$, there exist laminations $\mu,\mu'\in\mathcal{V}(\tau)$ with the following property. The support of μ,μ' is of type $\mathcal{LL}(m_1,\dots,m_\ell;-p_1,p_2)$, and there are paths $c,c':[0,1]\to\mathcal{V}(\tau)$ connecting α to μ, α' to μ' so that for every $t\in[0,1]$, the pair $(c(t),\beta)$ and $(c'(t),\beta')$ determines a quadratic differential $q(c(t),\beta),q(c'(t),\beta')$, and the resulting paths $t\to q(c(t),\beta),t\to q(c'(t),\beta')$ are contained in $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$.

By Lemma 6.1, for every measured lamination $\xi\in\mathcal{V}^*(\tau)$, the pair (μ,ξ) defines a quadratic differential $q(\mu,\xi)$ in $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$, and the same holds true for the pair (μ',ξ). Now choose a measured lamination $\nu\in\mathcal{V}^*(\tau)$ whose support is contained in $\mathcal{LL}(m_1,\dots,m_\ell;-p_1,p_2)$. Using the reasoning in the previous paragraph, the differential $q(\mu,\beta)$ can be connected to $q(\mu,\nu)$ by a path in $Q(\tau)$ which is contained in $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$, and $q(\mu',\beta')$ can be connected to $q(\mu',\nu)$ by a path with the same properties. Using this argument once more we conclude that the differential $q(\mu',\nu)$ can be connected to $q(\mu,\nu)$ by a path in $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)\cap\mathcal{Q}(\tau)$.

Together we find that the differentials $q(\alpha,\beta)$ and $q(\alpha',\beta')$ can both be connected to $q(\mu,\nu)$ by a path in $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)\cap Q(\tau)$. As the pairs (α,β) and (α',β') were arbitrarily chosen with the property that they determine quadratic differentials in $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$, the proposition follows. □

The next proposition is a converse to Proposition 6.2 and shows that train tracks can be used to define coordinates on strata. The main point here is to keep track of the singularities of the differential in a given stratum. This issue also arises in [MW14] and is taken care of there with a different construction which however does not serve our needs.

Proposition 6.3. (1) *For every $q\in\tilde{\mathcal{H}}(k_1,\dots,k_s;n)$ there is an orientable train track $\tau\in\mathcal{LT}(2k_1,\dots,2k_s;0,n)$ so that q is an interior point of $\mathcal{Q}(\tau)$.*

(2) *For every $q\in\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$ there is a non-orientable train track $\tau\in\mathcal{LT}(m_1,\dots,m_\ell;-p_1,p_2)$ so that q is an interior point of $\mathcal{Q}(\tau)$.*

Furthermore, if q contains a horizontal cylinder then the core curve of this cylinder is embedded in τ.

Proof. Let $q\in\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-p_1,p_2)$ and let $\Sigma=\{u_1,\dots,u_s\}$ $(s=\ell+m+p)$ be the singular set of q, i.e. the union of the zeros and poles and marked regular points.

Recall that q defines a singular euclidean metric on $S_{g,n}$ as well as two measured foliations, the horizontal and the vertical measured foliation. If x is a singular point of this metric, then x is a cone point of cone angle $k\pi$ for some $k\geq 1$. There are precisely k horizontal and precisely k vertical separatrices which begin at x. The zeros of the differential correspond to the cone points with cone angle $k\pi$ for some $k\geq 3$.

Choose a number $\epsilon > 0$ which is smaller than one eighth of the distance in the flat metric between any two singular points. Let $u_i \in \Sigma$ be a singular point of cone angle $k\pi$ for some $k \geq 1$. There exists a closed neighborhood V_i of u_i with the following properties. The boundary ∂V_i of V_i is a polygon with $2k$ sides. The sides are alternating between vertical arcs of fixed length $\sigma < \epsilon/10$ and horizontal arcs. The midpoint of a vertical arc is a point of distance ϵ on a horizontal separatrix through u_i. Note that the polygon is uniquely determined by these requirements.

Out of the polygons V_i $(i \leq s)$ we construct a train track η_i with stops whose switches are the midpoints of the vertical sides of the polygon ∂V_i. Thus each switch is a point of distance ϵ to the singular point u_i on a horizontal separatrix ζ_i.

Two switches on separatrices ζ_i^1, ζ_i^2 are connected by a branch in η_i if the angle at x_i between ζ_i^1, ζ_i^2 equals π, or, equivalently, if there is a path in ∂V_i connecting ζ_i^1, ζ_i^2 which travels through precisely one horizontal side of ∂V_i. This branch is constructed in such a way that all the vertical sides of the polygons ∂V_i are replaced by a cusp. Furthermore, we require that all branches are contained in V_i and do not intersect u_i. Figure D shows this construction.

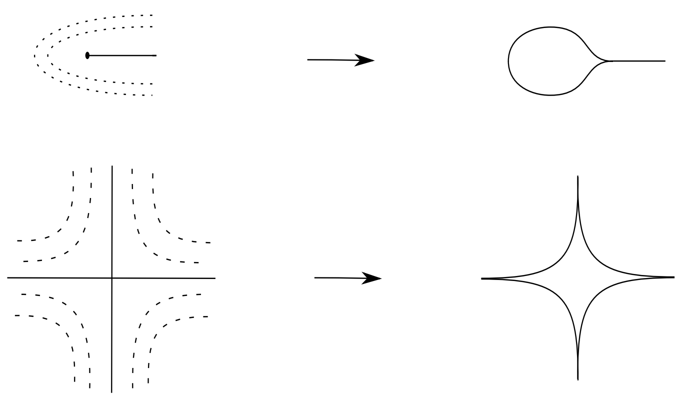


Figure D

The construction can be done in such a way that η_i is transverse to the vertical measured foliation of q. More precisely, by adjusting the constant σ we can assume that the tangent of η_i is arbitrarily close to the horizontal direction and that η_i is an arc of arbitrarily small geodesic curvature for the euclidean metric defined by q. There is a complementary component C_i of $V_i - \eta_i$ which is a polygon with $2k$ cusps. Its closure is contained in V_i and meets ∂V_i only at the cusps. It contains the singular point u_i. The cusps of the component are the vertices of η_i. The component C_i is a once punctured monogon if u_i is a pole, a once punctured bigon if u_i is a regular marked point, or an $m_i + 2$-gon if u_i is a zero of order m_i.

Let $\hat\eta$ be the union of the train tracks with stops η_i; this union consists of $\ell+p_1+p_2$ connected components, and it has $\sum_i(m_i + 2) + p_1 + 2p_2$ vertices. The graph $\hat\eta$ is transverse to the vertical foliation of q. By construction, $\hat\eta$ also is transverse to the straight line foliation on S defined by any direction for the singular euclidean metric on S which is sufficiently close to the vertical direction.

A *generalized bigon track* is a graph with all properties of a train track except that we allow the existence of complementary bigons, and we allow complementary annuli. Out of the train track with stops $\hat{\eta}$ we construct a generalized bigon track η on S by inductively replacing a stop by a switch and adding additional branches as follows.

For $i \leq s$ let β be a vertical side of the polygon ∂V_i. Then for small enough $h > 0$, β is a side of an euclidean (right angled) rectangle R_h in S of width h which intersects the polygonal disk V_i precisely in β. We require that the interior of R_h is disjoint from the disks V_j. The area of R_h equals σh. Thus by consideration of area, there exists a smallest number $h_0 > 0$ such that the vertical side β' of R_{h_0} distinct from β intersects one of the polygonal disks V_j. Then $\beta' \cap \partial V_j$ is a (possibly degenerate) subarc of a vertical side ξ of ∂V_j.

There are two possibilities. In the first case, $\beta' = \xi$. Then $V_i \cup R_{h_0} \cup V_j$ contains a horizontal saddle connection joining u_i to u_j. Connect the cusp of the component η_i of $\hat{\eta}$ contained in V_i to the cusp of the component of η_j contained in V_j by the subsegment of the saddle connection which is contained in R_{h_0}. This construction yields a new generalized bigon track $\hat{\eta}'$ with stops, and the number of stops of $\hat{\eta}'$ equals the number of stops of $\hat{\eta}$ minus two.

The second possibility is that $\beta' \cap \xi$ is a proper subarc of ξ (perhaps degenerate to a single point). Then precisely one of the endpoints of ξ is contained in β'; denote this point by z. The point z is contained in a horizontal side of the polygon V_j and hence it determines a branch b of the train track with stops $\hat{\eta}$. Connect the midpoint y of the side β of R_{h_0} (which is a stop of η_i) to an interior point of the branch b with an arc ν in such a way that the union $\eta_i \cup \nu \cup \eta_j$ is a generalized bigon track with stops, and that there is an arc of class C^1 contained in this generalized bigon track connecting the stop y of η_i to the endpoint of the branch b which is distinct from z. In the resulting generalized bigon track $\hat{\eta}'$ with stops, the midpoint y of the vertical side β of V_i (which was a stop in η_i) is a trivalent switch. The number of stops of $\hat{\eta}'$ equals the number of stops of $\hat{\eta}$ minus one.

Doing this construction with each of the stops of $\hat{\eta}$ replaces $\hat{\eta}$ by a generalized bigon track η. This can be done in such a way that each branch of η is a smooth arc whose tangent is everywhere close to the horizontal subbundle of the tangent bundle of $S - \Sigma$.

We show next that a complementary component of η which does not contain a singular point of q either is a bigon or an annulus. For this it suffices to show that the Euler characteristic of each complementary region which does not contain any marked point vanishes. Namely, by construction, the sum of the Euler characteristics of the complementary regions of η containing marked points equals the Euler characteristic of $S_{g,n}$. Furthermore, there are no complementary monogons as those would have to encircle a cusp. As the Euler characteristic of any complementary component different from a monogon is non-positive, the Euler characteristic of every complementary component not containing a marked point has to vanish. Hence each such component either is a bigon or an annulus. An annulus component corresponds to a horizontal cylinder of q.

To construct a train track out of η we begin with collapsing successively the complementary bigons of η as follows. The set of all directions for the flat metric defined by q which are tangent to some saddle connection is countable and hence we can find arbitrarily near the vertical direction a direction which is not tangent to any saddle connection. By construction of η, we may assume that this direction is transverse to η. For simplicity of exposition we will call this direction vertical in the sequel. We use the vertical flow to collapse the complementary bigons of η as follows.

Let B be a complementary bigon of η. The boundary ∂B of B consists of two arcs a_1, a_2 which are nearly horizontal and which meet tangentially at their endpoints. The vertical foliation is transverse to these sides, and non-singular in the bigon.

By transversality and compactness, a point x in the interior of the side a_1 of B is the starting point of a vertical arc γ whose interior is contained in the interior of B and whose second endpoint y is contained in a side of ∂B. If y is contained in the same side a_1 of ∂B as x then y bounds together with the subarc of a_1 connecting x to y an euclidean disk whose boundary consists of two smooth arcs with small curvature which meet at the endpoints with an angle close to $\pi/2$. However, this violates the Gauss Bonnet theorem. This implies that B is foliated by vertical arcs with one endpoint on a_1 and the second endpoint on a_2.

Now although the boundary of B may not be embedded in S (we only know that the interior of B is embedded), the two endpoints of any vertical arc as above are distinct since there is no vertical closed geodesic by assumption. This means that we can collapse these vertical arcs to points and collapse in this way the bigon B to a single arc. Let θ be the generalized bigon track obtained in this way.

There is a collapsing map $F' : S \to S$ of class C^1 which is homotopic to the identity, which equals the identity outside of a small neighborhood of the bigon B and which maps η to θ by collapsing the vertical arcs crossing through B. As the sides of B are nearly horizontal, we may assume that the differential of the restriction of the collapsing map F' to each horizontal arc for q vanishes nowhere.

Using once more the fact that vertical trajectories do not contain loops, we can repeat this process with any other bigon. In finitely many such steps we construct a generalized bigon track $\hat{\tau}$ and a collapsing map $F : \eta \to \hat{\tau}$ with the following properties.

(1) $\hat{\tau}$ does not have any complementary bigon components.
(2) F is homotopic to the identity and of class C^1.
(3) The differential of the restriction of F to the horizontal foliation of q vanishes nowhere, and it maps the intersection of the horizontal foliation of q with the bigon complementary components of η to smoothly immersed arcs in $\hat{\tau}$.

The bigon track $\hat{\tau}$ may not be a train track as it may have complementary components which are annuli. However, the above construction can also be used to collapse annuli to circles. To this end let A be a complementary annulus of $\hat{\tau}$. By construction of $\hat{\tau}$, A is contained in a horizontal cylinder C for q, and its closure does not contain a singular point of q. Furthermore, its boundary curves are transverse to the vertical foliation, and they intersect the interior of the horizontal cylinder C in m embedded arcs where m is the number of singular points on the boundary of C.

Let a_1, a_2 be the two boundary curves of A. For a point $x \in a_1$, there is a unique subarc $v(x)$ of a vertical trajectory starting at x which is entirely contained in A and connects x to a point $\psi(x)$ contained in the boundary of A. Using once more the Gauss Bonnet theorem, we conclude that in fact $\psi(x) \in a_2$. As there are no vertical cylinders and the closure of A does not contain singular points, we have $\psi(x) \neq x$. Furthermore, the arc $v(x)$ depends smoothly on x.

By the discussion in the previous paragraph, for each $x \in a_1$ we can collapse the arc $v(x)$ to a point. The result is a new generalized bigon track τ', and there is a collapsing $\hat{\tau} \to \tau'$ of class C^1 which is homotopic to the identity and whose differential restricted to any horizontal arc vanishes nowhere. The number of complementary components which are annuli is strictly smaller than the number of annuli components of $\hat{\tau}$. Repeating this construction with all the finitely many annuli components of $S - \hat{\tau}$, we construct in this way from $\hat{\tau}$ a train track τ on S which carries the horizontal measured geodesic lamination of q. Furthermore, the vertical measured geodesic lamination of q hits τ efficiently (see [PH92]) and hence it is carried by the dual bigon track τ^* of τ. Each complementary component of τ contains precisely one singular point of q, and the component is a $k+2$-gon if and only if the singular point is a zero of order k. This yields that τ is of topological type $(m_1, \dots, m_\ell; -p_1, p_2)$.

We are left with showing that τ is large. Now by construction, τ carries the horizontal lamination of $e^{is}q$ provided s is sufficiently close to 0. But the set of directions for the singular euclidean metric defined by q so that the horizontal foliation in this direction is minimal and of the type predicted by the number and multiplicities of the zeros of q is dense [KMS86]. This implies that τ carries a lamination which is minimal, large and of the same topological type as τ. Similarly, for s sufficiently close to zero, the vertical measured geodesic lamination of $e^{is}q$ hits τ sufficiently. Thus as before, $\mathcal{V}^*(\tau)$ carries a minimal large geodesic lamination of the same topological type as τ. In other words, τ has all the properties required in the proposition.

Now if q is an abelian differential then the horizontal and vertical foliations of q are orientable. As $\hat{\eta}$ is constructed from the horizontal foliation of q, it inherits an orientation from the orientation of the horizontal foliation of q. The collapsing construction uses the orientable vertical foliation, and it is straightforward that this construction respects orientations as well. Then the resulting train track τ is orientable. ☐

We summarize the discussion in this section as follows.

Let $\mathcal{Q}$ be a component of the stratum $\mathcal{Q}(m_1, \dots, m_\ell; -p_1, p_2)$ of $\mathcal{Q}(S_{g,n})$ (or of the stratum $\mathcal{H}(m_1/2, \dots, m_\ell/2; p)$ of $\mathcal{H}(S_{g,n})$) and let $\tilde{\mathcal{Q}}$ be the preimage of $\mathcal{Q}$ in

$\tilde{\mathcal{Q}}(S_{g,n})$ (or in $\tilde{\mathcal{H}}(S_{g,n})$). Then there is a collection

$$\mathcal{LT}(\tilde{\mathcal{Q}}) \subset \mathcal{LT}(m_1, \dots, m_\ell; -p_1, p_2)$$

of large marked train tracks τ of the same topological type as $\mathcal{Q}$ such that for every $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$, the set $\mathcal{Q}(\tau)$ contains an open path connected subset of $\tilde{\mathcal{Q}}$.

The set $\mathcal{LT}(\tilde{\mathcal{Q}})$ is invariant under the action of the mapping class group. Its quotient $\mathcal{LT}(\mathcal{Q})$ under this action is finite and will be called the *set of combinatorial models for* $\mathcal{Q}$. The subset

$$\cup_{\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})} \mathcal{Q}(\tau)$$

of the Teichmüller space of abelian or quadratic differentials is closed, Mod(S)-invariant and contains $\tilde{\mathcal{Q}}$ as an open dense subset, i.e. it coincides with the closure of $\tilde{\mathcal{Q}}$ in $\tilde{\mathcal{Q}}(S_{g,n})$.

Lemma 6.4. *Let $\mathcal{Q}$ be a component of a stratum, with preimage $\tilde{\mathcal{Q}}$ in $\tilde{\mathcal{Q}}(S_{g,n})$, let $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ and let η be a large train track of the same topological type as τ which is carried by τ. Then $\eta \in \mathcal{LT}(\tilde{\mathcal{Q}})$.*

Proof. A point in $\mathcal{Q}(\tau)$ is defined by a pair (λ, ν) where $\lambda \in \mathcal{V}(\tau)$ and where $\nu \in \mathcal{V}^*(\tau)$. If we choose λ in such a way that its support supp(λ) is of the same topological type as τ and such that λ is carried by the train track η, then (λ, ν) defines a differential in $\mathcal{Q}(\eta)$. □

As a fairly immediate consequence of the above discussion and Section 3 of [H09], we obtain a method to construct large train tracks of a given topological type. Namely, for a fixed choice of a complete hyperbolic metric on S of finite volume and numbers $a > 0, \epsilon > 0$, there is a notion of *a-long* train track which *ϵ-follows* a large geodesic lamination λ. By definition, this means the following. Fix a hyperbolic metric on S. The *straightening* of a train track τ is obtained from τ by replacing each branch b by a geodesic segment which is homotopic with fixed endpoints to b. We require that the length of each of the straightened edges is at least a, that their tangent lines are contained in the ϵ-neighborhood of the projectivized tangent bundle of λ and that moreover the straightening of every trainpath on τ is a piecewise geodesic whose exterior angles at the breakpoints are not bigger than ϵ.

Lemma 3.2 of [H09] shows that for every geodesic lamination λ on S and every $\epsilon > 0$ there is an a-long generic transversely recurrent train track τ which carries λ and ϵ-follows λ.

Corollary 6.5. *Let $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ and let λ be a minimal large geodesic lamination of the same topological type as τ which is carried by τ. Then for sufficiently small $\epsilon > 0$, an a-long train track η which ϵ-follows λ is contained in $\mathcal{LT}(\tilde{\mathcal{Q}})$.*

Proof. By construction, if λ is large, then for sufficiently small ϵ and sufficiently large $a > 0$, an a-long train track η which ϵ-follows λ is of the same topological type as λ. Furthermore, η carries a minimal large geodesic lamination of the same topological type as τ and hence η is fully recurrent and transversely recurrent.

If λ is carried by a large train track τ then for sufficiently small $\epsilon > 0$ and sufficiently large $a > 0$, η is carried by τ (see Section 3 of [H09]). Then a large geodesic lamination which is carried by τ^* is carried by η^* and hence η is large as claimed. □

7. Realizations of admissible curve systems

In this section we construct from a curve system which is admissible for a component $\mathcal{Q}$ of a stratum of abelian differentials on a closed oriented surface S a large train track τ which is contained in $\mathcal{LT}(\tilde{\mathcal{Q}})$. We use this train track to obtain a dynamical version of Corollary 4.4 and Proposition 4.5 in the spirit of a conjecture of Zorich [Z99] (see [AMY18, GR17]).

In Section 8 we will use the results in this section to navigate between subgroups of $\mathrm{Mod}(S_{g,m})$ which are generated by the Dehn twists about the components of curve systems which are admissible for $\mathcal{Q}$.

We assume throughout that all curves from a curve system $\mathcal{C}$ intersect transversely in the minimal number of points. An admissible curve system $\mathcal{C}$ on S decomposes S into $m \geq 1$ topological disks. By Lemma 2.7, there is a consistent orientation for $\mathcal{C}$. Recall also that a vertex cycle of an oriented train track τ is an embedded simple closed curve in τ.

Lemma 7.1. *Let $\mathcal{C}$ be an admissible curve system. Then a consistent orientation of $\mathcal{C}$ defines a recurrent orientable train track τ on S which contains each of the curves from $\mathcal{C}$ as a vertex cycle.*

Proof. Equip the admissible curve system with a consistent orientation. Construct from the oriented curve system $\mathcal{C}$ a train track τ by replacing each intersection between oriented curves by a large branch as shown in Figure E. Informally, this

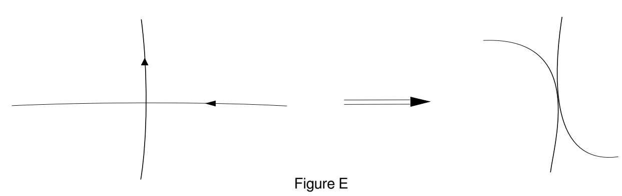


Figure E

amounts to passing through the intersecting curves $a, c \in \mathcal{C}$ in the direction prescribed by their orientation, and if we follow the curve a in an orientation preserving fashion, then we allow to turn into the curve c in an orientation preserving fashion. This does not depend on the order of the curves a, c. There exists an orientation on the train track τ which induces on each of the curves $c \in \mathcal{C}$ the orientation from the consistent orientation used in its construction. Namely, we only have to check consistency of the orientation at the switches, which is immediate from the definition of a consistent orientation.

By construction, each of the curves from the system $\mathcal{C}$ is embedded in τ.

Thus we are left with showing that the train track τ is recurrent. To this end we just have to check that we can transit from any curve defining a leaf in the tree of the curve diagram to any other leaf with an oriented trainpath. However, by construction, we can transit from any curve c in the system to any curve which intersects c and hence recurrence follows by a straightforward induction. $\square$

We call the train track τ constructed in Lemma 7.1 from a curve system $\mathcal{C}$ a *train track realization* of $\mathcal{C}$. All switches of such a train track are trivalent.

The following observation follows easily from the definition of a consistent orientation. For its formulation, let c be an embedded simple closed curve in an oriented train track τ on S. Equip the curve c with the orientation inherited from τ. If v is a switch contained in c then there exists a unique half-branch incident on v which is *not* contained in c (recall that we require all switches to be trivalent). For the orientation of c and the given orientation of S, the switch v is called a *right* (or *left*) switch if locally in a small neighborhood of v, the half-branch incident on v and not contained in c is to the right of c (or to the left of c). Furthermore, the switch v is called *incoming* if the orientation of the branch b incident on v and not contained in c is such that the oriented subarc of b which begins at the midpoint of b ends at v, and it is called *outgoing* otherwise.

Recall that a choice of a consistent orientation for the curve system $\mathcal{C}$ defines a partition $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$ into the *positive* curves $\mathcal{C}^+$ and the *negative* curves $\mathcal{C}^-$. We have

Lemma 7.2. *Let $\mathcal{C}$ be an admissible curve system and let τ be a train track which realizes $\mathcal{C}$, defined by a choice of a consistent orientation for $\mathcal{C}$. Then for each component c of $\mathcal{C}^+$ (or c of $\mathcal{C}^-$), the train track τ can be split to its image under a positive (or negative) Dehn twist about c.*

Proof. By the definition of τ and the definition of a consistent orientation of $\mathcal{C}$, if $c \in \mathcal{C}^+$ then all right switches are incoming and all left switches are outgoing, and if $c \in \mathcal{C}^-$ then all left switches are incoming and all right switches are outgoing (see Figure F for an illustration).

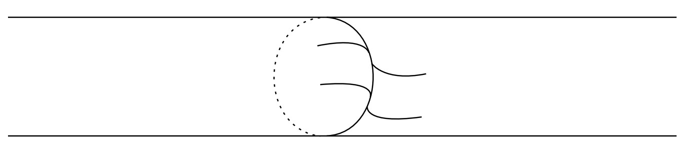


Figure F

Now let us assume that $c \in \mathcal{C}^+$, the case that $c \in \mathcal{C}^-$ is equivalent and will be omitted. As τ is recurrent, there is a large branch e of τ contained in c. This large branch is incident on two distinct switches. By construction of τ, one switch is a right and the other a left switch, and one is incoming and the other outgoing. In particular, a split τ' of τ at e which carries c is unique, and it slides the right incoming branch across the outgoing one. As c crosses through the diagonal of the split, the split track contains c as an embedded simple closed curve.

Now if c consists of only two branches, then the branch contained in c which is distinct from e is a small branch, and the split track τ' is the image of τ by a positive Dehn twist about c. Namely, this Dehn twist maps the homology class $[d]$ defined by the oriented curve $d \in \mathcal{C}^-$ which intersects c in a point corresponding to e to the class $[d]+[c]$ where $[c]$ is the class defined by the orientation of c, and if ι is the intersection form on $H_1(S,\mathbb{Z})$ then $\iota([c],[d]) = 1$.

Otherwise we repeat this construction with the train track τ'. In finitely many such steps we obtain a train track η which on the one hand is obtained from τ by a splitting sequence, on the other hand it is the image of τ by a positive Dehn twist about c. □

Recall from Section 2 that each admissible curve system $\mathcal{C}$ determines a component $\mathcal{Q}(\mathcal{C})$ of a stratum of abelian differentials. We denote by $\tilde{\mathcal{Q}}(\mathcal{C})$ its preimage in the Teichmüller space of marked abelian differentials.

Corollary 7.3. *Let τ be a train track realization of an admissible curve system $\mathcal{C}$ on S; then $\tau \in \mathcal{LT}(\mathcal{Q}(\mathcal{C}))$.*

Proof. By construction, the topological type of τ coincides with the topological type of a train track in $\mathcal{LT}(\tilde{\mathcal{Q}}(\mathcal{C}))$. We have to show that τ is large and determines the component $\mathcal{Q}(\mathcal{C})$.

To this end decompose once again $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$. The Thurston Penner construction (see Theorem 3.1 of [P88]), applied to the curves from $\mathcal{C}$, shows the following. Let φ_1 be the concatenation of the positive Dehn twists about the curves from $\mathcal{C}^+$ and let φ_2 be the concatenation of the negative Dehn twists about the curves from $\mathcal{C}^-$. As the curves in $\mathcal{C}^\pm$ are pairwise disjoint and hence the Dehn twists about its components commute, the maps φ_i $(i = 1,2)$ do not depend on choices made. Furthermore, $\varphi = \varphi_1\varphi_2$ is a pseudo-Anosov mapping class.

Each pseudo-Anosov mapping class preserves a flow line of the Teichmüller flow on the Teichmüller space of quadratic differentials or abelian differentials, and it acts on this flow line by translation. We claim that the invariant flow line of φ projects to a periodic flow line of the Teichmüller flow in the component $\mathcal{Q}(\mathcal{C})$.

Namely, by the definition of a realization $q(\mathcal{C})$ of $\mathcal{C}$, the abelian differential $q(\mathcal{C})$ is square tiled, with completely periodic horizontal and vertical measured foliation. In general, $q(\mathcal{C})$ is not contained in the invariant flow line for φ, but it can deformed to a differential in this flow line as follows (see Section 5 of [Lei04]).

Number the curves in $\mathcal{C}^\pm$ in an arbitrary way. For simplicity of notation, we denote the curves in $\mathcal{C}^+$ by a_i, and the curves in $\mathcal{C}^-$ by b_j. Define $N = N_{A,B}$ to be the (n,m)-matrix whose (i,j)-entry is $\iota(a_i,b_j)$. The connectivity of $\mathcal{C}$ guarantees that NN^t is irreducible. Let v be an Perron Frobenius eigenvector for NN^t where $\mu > 0$ is the Perron Frobenius eigenvalue.

If we put $v' = \mu^{-1/2}N^t v$ then

$$N^t N v' = \mu v'$$

and also $v = \mu^{-1/2}N$ (Section 5 of [Lei04]). Deform the euclidean squares which compose $q(\mathcal{C})$ to recangles for which the sides transverse to a_i have length v_i and the sides transverse to b_j have length v'_j. Such a deformation can be achieved within the stratum containing $q(\mathcal{C})$. Let z be the image of this deformation.

By the results in Section 5 of [Lei04], each of the mapping classes contained in the subgroup of $\mathrm{Mod}(S)$ generated by φ_1, φ_2 can be represented by an affine transformation for the translation surface z. But this just means that φ is represented by an affine transformation for z and hence the invariant flow line for φ passes through z. This is what we wanted to show.

On the other hand, it follows from Lemma 7.2 that τ is splittable to $\varphi(\tau)$. Let $P\mathcal{V}(\tau) \subset \mathcal{PML}$ be the projectivization of $\mathcal{V}(\tau)$. Then $P\mathcal{V}(\tau)$ is a non-empty compact subset of $\mathcal{PML}$, and $\varphi(P\mathcal{V}(\tau)) \subset P\mathcal{V}(\tau)$. Then $\cap_k \varphi^k P\mathcal{V}(\tau)$ is a non-empty compact φ-invariant subset of $\mathcal{V}(\tau)$.

Since φ acts with north-south dynamics on $\mathcal{PML}$, the set $\cap_k \varphi^k P\mathcal{V}(\tau)$ contains the attracting fixed point for the action of φ. Thus τ carries this attracting fixed point. If λ is the support of a measured geodesic lamination whose projective class is this fixed point, then λ is minimal and of the same topological type as τ. Thus τ is fully recurrent.

The same reasoning, applied to the inverse of φ, shows that τ^* carries the repelling fixed point of φ which is minimal and large. This implies that indeed, $\tau \in \tilde{\mathcal{Q}}(\mathcal{C})$. ☐

Let now q be a point in the component $\mathcal{Q} = \mathcal{Q}(\mathcal{C})$ and let U be a contractible neighborhood of q in $\mathcal{Q}$. Since there exists a probability measure on $\mathcal{Q}$ of full support which is invariant and ergodic under the *Teichmüller flow* Φ^t, the set U contains forward recurrent points for Φ^t. Let Γ be the set of periodic orbits for the Teichmüller flow Φ^t passing through U. Choose a component $\tilde{U}$ of the preimage of U in the Teichmüller space of abelian differentials. A periodic orbit of Φ^t beginning at a point in U then lifts to an orbit of Φ^t in the Teichmüller space of abelian differentials which begins in $\tilde{U}$ and is invariant under a pseudo-Anosov element $\varphi \in \mathrm{Mod}(S_{g,m})$. Define $\Omega(\Gamma)$ to be the subgroup of $\mathrm{Mod}(S_{g,m})$ generated by such pseudo-Anosov elements φ. Observe that as U is contractible, the choice of $\tilde{U}$ determines an element of $\mathrm{Mod}(S_{g,m})$ rather than a conjugacy class in $\mathrm{Mod}(S_{g,m})$; we refer to [H14] for a comprehensive discussion. The group $\Omega(\Gamma)$ depends on the choice of $\tilde{U}$, but different choices give rise to conjugate groups.

We next observe that if q lifts to a differential in $\mathcal{Q}(\tau)$ for a train track realization τ of $\mathcal{C}$ then the group $\Omega(\Gamma)$ contains the group $\Gamma(\mathcal{C})$.

Lemma 7.4. *Let τ be a train track realization of an admissible curve system $\mathcal{C}$ for $\mathcal{Q}$. Let $\tilde{q} \in \mathcal{Q}(\tau)$ be the preimage of a differential $q \in \mathcal{Q}(\mathcal{C})$ which is contained in a periodic orbit for Φ^t, and let $\tilde{U}$ be any neighborhood of q in the component $\tilde{\mathcal{Q}}$ of* *the preimage of $\mathcal{Q}$ which contains $\tilde{q}$. Then for each $c \in \mathcal{C}$, the Dehn twist about c is contained in $\Omega(\Gamma)$.*

Proof. Recall from Lemma 7.2 that for each $c \in \mathcal{C}$ there exists a splitting sequence connecting τ to the image $T_c^{\pm}\tau$ of τ by a (positive or negative) Dehn twist $T_c^{\pm}$ about c.

As periodic points for the Teichmüller flow are dense in any component of a stratum, there exists a periodic point $q \in \mathcal{Q}(\tau)$, and this point is contained in the projection of the cotangent line of the axis of a pseudo-Anosov mapping class φ. Furthermore, we may assume that the mapping class φ admits τ as a *train track expansion*. This means that $\varphi(\tau)$ is carried by τ.

A pseudo-Anosov mapping class acts with north-south dynamics on the space of projective measured laminations, and this property characterizes pseudo-Anosov mapping classes. A standard fixed point argument (this is explained in detail in [H14]) then yields that for large enough k, the mapping class $T_c^{\pm} \circ \varphi^k$ is pseudo-Anosov, and it admits τ as a train track expansion. It then follows that there is some $m > 1$ so that the axis of $\varphi^m T_c \varphi^k$ passes through an arbitrarily chosen neighborhood U of q. We refer to Section 4 of [H14] where this is explained in detail.

As $c \in \mathcal{C}$ was arbitrary, this shows that $\Gamma(\mathcal{C}) \subset \Omega(\Gamma)$. □

Recall from the introduction the definition of the local monodromy group of a component $\mathcal{Q}$ of a stratum of abelian differentials. As an immediate corollary of Lemma 7.4 and Lemma 4.7 of [H14] we obtain

Corollary 7.5. *Let $\mathcal{C}$ be an admissible curve system for a component $\mathcal{Q}$ of a stratum of abelian differentials. Then the local monodromy group of $\mathcal{Q}$ contains the image of the group $\Gamma(\mathcal{C})$ generated by the Dehn twists about the components of $\mathcal{C}$ under the homomorphism $\Psi\Pi$.*

Together with Corollary 4.4 and Corollary 4.6, Corollary 7.5 yields Theorem 5 from the introduction.

8. Navigating within the punctured mapping class group

A component Q of a stratum of abelian differentials is a complex orbifold. A zero of a differential $q \in \mathcal{Q}$ can be viewed as a marked point on the surface S and hence if q has $m \geq 1$ zeros then there is a homomorphism

$$P : \pi_1(\mathcal{Q}) \to \mathrm{Mod}(S_{g,m}),$$

well defined up to conjugation, where $\pi_1(\mathcal{Q})$ denotes the orbifold fundamental group of $\mathcal{Q}$. The main result of this section is the following. Let $\mathcal{C}$ be any admissible curve system for $\mathcal{Q}$; then $\Gamma(\mathcal{C})$ is a normal subgroup of $P\pi_1(\mathcal{Q})$ which does not depend on the choice of $\mathcal{C}$.

Recall that a *vertex cycle* of an oriented train track τ is a simple closed curve embedded in τ. We begin with determining the vertex cycles of a train track realization of an admissible curve system $\mathcal{C}$.

Lemma 8.1. *Let τ be the train track realization of an admissible curve system $\mathcal{C}$. Then the vertex cycles of τ are precisely the curves of $\mathcal{C}$.*

Proof. Any embedded simple closed curve in τ defines a subtrack of τ which consists of at least two branches. Furthermore, since τ is recurrent, this subtrack contains a branch which is large in τ [PH92].

Let $c \subset \tau$ be a vertex cycle and let e be a large branch of τ contained in c. By construction of a train track realization of $\mathcal{C}$, such a large branch corresponds to the intersection of two curves $a_1, a_2 \in \mathcal{C}$. In other words, it corresponds to an edge χ in the curve diagram D of $\mathcal{C}$. Furthermore, both a_1, a_2 are embedded in τ, and $e = a_1 \cap a_2$.

For the orientation of τ induced by the consistent orientation of $\mathcal{C}$ which is used in the construction of τ, let b be the branch of τ contained in c which is forward adjacent to the branch e. By construction of τ, the branch b is small and it either is contained in the curve a_1 or in the curve a_2, but these two possibilities are exclusive. Assume that b is contained in a_1. Our goal is to show that $a_1 = c$.

Let $\mathcal{C}' \subset \mathcal{C}$ be the subsystem of the curve system $\mathcal{C}$ whose elements correspond to the vertices of the component D' of $D - \chi$ containing a_1 (recall that D is a tree). There is a subtrack η of τ which is a train track realization of the curve system $\mathcal{C}'$. The number of branches of η is strictly smaller than the number of branches of τ. We claim that η contains the vertex cycle c.

Namely, as the curve diagram D of $\mathcal{C}$ is a tree, there are precisely two half-branches of $\tau - \eta$ which are incident on vertices in η. These half-branches are incident on the two distinct endpoints of e. After removal of $\tau - \eta$, the endpoints of e are contained in the interior of some edge of η.

As τ is oriented, any trainpath ρ on τ which begins at the switch v of e on which the branch b is incident, which passes first through b and which eventually intersects $\tau - \eta$ has to pass through e before turning into $\tau - \eta$ (see Figure E). If this path parametrizes the embedded simple closed curve c, then it has to end at v. Consequently we have $c \subset \eta$ as claimed.

Now c is embedded in the orientable train track η and hence c is a vertex cycle of η. As η is a train track realization of a proper subsystem $\mathcal{C}'$ of $\mathcal{C}$, if $c \subset \eta$ only consists of two branches then clearly $c = a_1 \in \mathcal{C}$ and we are done. Otherwise we can apply the above discussion to c viewed as a vertex cycle of η. After finitely many steps, each of which strictly decreases the number of edges of the train track containing c as a vertex cycle, we find that indeed, $c = a_1 \in \mathcal{C}$. □

For an orientable large train track $\eta \in \mathcal{LT}(\tilde{\mathcal{Q}})$ define $\Gamma(\eta)$ to be the subgroup of $\mathrm{Mod}(S_{g,m})$ generated by the Dehn twists about the vertex cycles of η. Here as before, m is the number of zeros of a differential in $\mathcal{Q}$.

Corollary 8.2. *Let τ be a train track realization of an admissible curve system $\mathcal{C}$; then $\Gamma(\tau) = \Gamma(\mathcal{C})$.*

Proof. As by Lemma 8.1 the vertex cycles of τ are precisely the curves from the curve system $\mathcal{C}$, the corollary is immediate from the definitions. □

The following technical observations are recordered here for later use.

Lemma 8.3. *Let τ be a large orientable train track on S and let c be a vertex cycle of τ.*

(1) *The numbers of large and small branches of τ contained in c coincide and are non-zero.*
(2) *If d is another vertex cycle of τ then every component of $c \cap d$ contains a large branch.*
(3) *Let e be a large branch of τ contained in c. Then τ can be split at large branches contained in c and distinct from e to a large train track η which contains c as a vertex cycle and such that e is the unique large branch of η contained in c.*

Proof. Since τ is recurrent, c contains a large branch e of τ. Parametrize c as an oriented periodic trainpath $\rho : [0, k] \to \tau$, beginning at the large branch $e = \rho[0, 1]$. Then ρ determines a function $f : [0, k-1] \to \{0, 1\}$ as follows. Define $f(j) = 0$ if the half-branch $\rho[j - 1/2, j]$ is small at $\rho(j)$, and define $f(j) = 1$ otherwise.

We have $f(0) = 0$, and $f(1) = 1$. Furthermore, the branch $\rho[j - 1, j]$ is large if and only if $\rho(j - 1) = 0$ and $\rho(j) = 1$, and it is small if and only if $\rho(j - 1) = 1$ and $\rho(j) = 0$. A mixed branch $\rho[j - 1, j]$ is characterized by the property that the function f takes on the same value at its endpoints.

The first part of the lemma is an immediate consequence: c contains the same number of large and small branches, and between any two large branches there is a small branch.

Now if d is another vertex cycle of τ and if $\beta = \rho[i, j]$ is a component of $c \cap d$ then $f(\rho(i)) = 0$ and $f(\rho(j)) = 1$ which immediately implies the second part of the lemma.

We are left with showing the third part of the lemma. To this end let again e be a large branch of τ contained in c. Let $\rho : [0, k] \to \tau$ be a parametrization of c as an oriented trainpath beginning at $\rho[0, 1] = e$. The length k of the path is the *combinatorial length* of c. Define the *oriented combinatorial distance* between e and the branch $\rho[i, i+1]$ to be i. Let $\mu_\tau(c, e)$ be the smallest oriented combintorial distance between e and a small branch of τ contained in c. Note that if $\mu_\tau(c, e) = k - 1$ then c contains a single large branch of τ.

We claim that if c contains at least two large branches then there is a sequence of splits of τ at large branches in c different from e so that the split track η is large and carries c, the combinatorial length of c in η is at most k and that moreover $\mu_\eta(c, e) > \mu_\tau(c, e)$.

To show this claim let $f = \rho[j-1, j]$ be the large branch of τ contained in c of smallest combinatorial distance to e. Split τ at f so that the split track β_1 carries c. Then c defines a trainpath on β_1 of length at most k (the length can be $k-1$ if c is carried by both train tracks arising from τ by a split at f). We may assume that the train track β_1 is large.

We distinguish now two cases. In the first case, the branch $\rho[j-2, j-1]$ of τ is a small branch; then $\mu_\tau(c, e) = j-2$. The branch of β_1 corresponding to $\rho[j-2, j-1]$ is a mixed branch, and as the split does not change the type of the branches $\rho[s-1, s]$ for $1 \leq s \leq j-2$, this implies that $\mu_{\beta_1}(c, e) \geq j-1 = \mu_7 tau(c, e) + 1$. Thus β_1 has the properties we were looking for.

In the second case, the branch $\rho[j-2, j-1]$ of τ is a mixed branch. Then the branch of β_1 corresponding to $\rho[j-2, j-1]$ is large, and the branch corresponding to $\rho[j-1, j]$ is small or mixed. Furthermore, we have $\mu_{\beta_1}(c, e) = \mu_\tau(c, e)$.

Repeat the construction with $\rho[j-2, j-1]$. After finitely many such steps we arrive at a train track β_2 so that the c is carried by β_2, $\mu_{\beta_2}(c, e) = \mu_\tau(c, e) = i$ and that the branch corresponding to $\rho[i+1, i+2]$ in $c \subset \beta_2$ is large. Then β_2 has the properties of the first case above, and we can apply the first case to deduce what we were looking for.

Proceeding inductively, in finitely many such steps we find a splitting sequence beginning at τ with the properties predicted in part (3) of the lemma. □

The proof of the following lemma uses orientability of the train track in an essential way and is not valid for non-orientable train tracks. Recall that we denote by T_c the positive Dehn twist about the simple closed curve c.

Lemma 8.4. *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials and let $\eta \in \mathcal{LT}(\tilde{\mathcal{Q}})$ be obtained from $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ by a single split.*

(1) $\Gamma(\eta) = \Gamma(\tau)$.
(2) *For every vertex cycle c of η, there exist two (not necessarily distinct) vertex cycles c_1, c_2 of τ such that $c = T_{c_1}^{\pm} c_2$.*
(3) *For every vertex cycle d of τ there exists a vertex cycle c of η and an element $\psi \in \Gamma(\eta)$ such that $\psi(c) = d$.*

Proof. Let us assume that η is obtained from τ by a right split at a large branch e. There are two cases. In the first case, the train track η' obtained from τ by a left split is *not* recurrent. Then every measured geodesic lamination which is carried by τ is also carried by η. Now the convex cone of all measured geodesic laminations carried by τ (or η) is the cone over a compact convex polytope whose vertices are up to scaling the vertex cycles of τ (or η) (Proposition 3.11.3 of [Mo03]). Hence any vertex cycle of η is a vertex cycle on τ, and vice versa, every vertex cycle of τ lifts to a vertex cycle on η. Thus there is nothing to show.

Assume now that both η and η' are recurrent. As η is carried by τ, there is a carrying map $F: S \to S$ homotopic to the identity with $F(\eta) = \tau$. The map F can be chosen in such a way that $F^{-1}(e)$ is the union of the diagonal branch

e' of the split and two small half-branches whose second half projects to a branch which is incident on an endpoint of e and which is distinct from e (see Figure C). Furthermore, the restriction of F to $\eta - F^{-1}(e)$ is an embedding.

We first show the second part of the lemma. We only have to consider a vertex cycle c of η which is not a vertex cycle of τ. Then c is an embedded simple closed curve in η, but its image under the projection F to τ is a subtrack of τ which is not a simple closed curve. As the restriction of F to $\eta - F^{-1}(e)$ is an embedding, the image of c under F, viewed as a periodic trainpath $\hat{c}$ on τ, has to pass through the branch e twice. Furthermore, it passes through any of the neighboring branches of e at most once. As there are precisely two half-branches which are incident and small on each endpoint of e, the curve $\hat{c}$ has to pass through each of these half-branches precisely once. This implies that c does not pass through the diagonal branch e' of the split.

Equip c with the orientation inherited from the orientation of η. Let v_1, v_2 be the two (necessarily distinct) switches of τ on which e is incident. Assume that e is oriented from v_1 to v_2. Let b_1, b_2 be the two half-branches of τ which are incident and small at v_1, and let b_3, b_4 be the two half-branches of τ which are incident and small at v_2. As the oriented simple closed curve c does not pass through the branch e', for a suitable numbering of the half-branches b_i there exists a subsegment of $\hat{c}$ which crosses through b_1, e, b_3 in this order, and there exists a subsegment of $\hat{c}$ which crosses through b_2, e, b_4 in this order. We refer to Figure G for an illustration.

As a consequence, there are simple closed embedded curves c_1, c_2 in τ with the following properties. The weight function on τ defined by $c_1 \cup c_2$ (which is just counting the number of preimages of interior points on the branches of τ under a carrying map $c_1 \cup c_2 \to \tau$) equals the weight function of $\hat{c}$. Furthermore, up to homotopy, the simple closed curves c_1, c_2 intersect in a single point, and this point can be chosen to be an interior point of e.

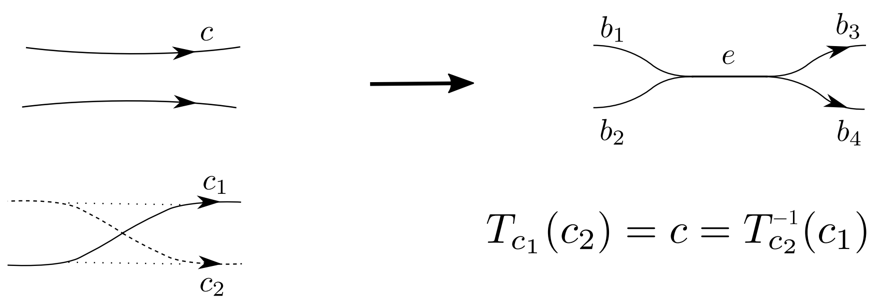


Figure G

Up to exchanging c_1 and c_2, we thus have

$$c = T_{c_1}(c_2) = T_{c_2}^{-1}(c_1).$$

Furthermore,

$$T_c = T_{T_{c_1} c_2} = T_{c_1} T_{c_2} T_{c_1}^{-1} = T_{c_2}^{-1} T_{c_1} T_{c_2} \in \Gamma(\tau)$$

(see p.73 of [FM12]). As c was an arbitrary vertex cycle of η which is not a vertex cycle of τ, this shows the second part of the lemma and also shows that $\Gamma(\eta) \subset \Gamma(\tau)$.

By induction, we then obtain the following. If the large train track ζ can be obtained from τ by a splitting sequence, then $\Gamma(\zeta)\subset\Gamma(\tau)$. Furthermore, for every vertex cycle c of ζ there exists a vertex cycle d of τ and an element $\psi\in\Gamma(\tau)$ such that $c=\psi(d)$.

We are left with showing part 1) and 3) of the lemma. To show that $\Gamma(\tau)\subset\Gamma(\eta)$ observe that using the above notation, one of the curves c_1,c_2 is a vertex cycle of η, and the second curve is not carried by η. But if say c_2 is not carried by η then $c_2=T_{c_1}^{-1}c$ (which settles the third part of the lemma for c_2) and

$$T_{c_2}=T_{T_{c_1}^{-1}c}=T_{c_1}^{-1}T_cT_{c_1}\in\Gamma(\eta).$$

As a consequence, for the completion of the proof of the lemma, it suffices to show the following. If c_2 is any vertex cycle of τ which is not carried by η, then there is a vertex cycle c_1 of η and an element $\psi\in\Gamma(\eta)$ so that $c_2=\psi(c_1)$.

Thus let c_2 be an arbitrary vertex cycle of τ which is not carried by η. Note that this implies that c_2 is carried by the train track η' obtained from τ by a left split at the large branch e.

Inverting the discussion which led to the proof of the second part of the lemma and of $\Gamma(\eta)\subset\Gamma(\tau)$, it suffices to find a large train track ζ with the following properties.

- There exists a train track τ' which contains c_2 as a vertex cycle and which can be obtained from τ by a sequence of splits not involving a split at the large branch e.
- ζ can be obtained from τ' by a single split at e, and η is splittable to ζ.
- There exists a vertex cycle c_1 of ζ whose image in τ under a carrying map $\zeta\to\tau$ intersects c_2 in a single point contained in e.

Namely, for such a vertex cycle c_1 we have $T_{c_1}\in\Gamma(\zeta)\subset\Gamma(\eta)$. The curve $c=T_{c_1}^{\pm}(c_2)$ also is a vertex cycle of ζ, and $c_2=T_{c_1}^{\mp}c$ which shows the third part of the lemma. The relation $T_{c_2}=T_{c_1}^{\pm}T_cT_{c_1}^{\mp}\in\Gamma(\zeta)\subset\Gamma(\eta)$ yields the first part.

To construct a train track ζ and a vertex cycle c_1 for ζ with the required properties, note that as c_2 is embedded in τ, it is a subtrack of τ. By Lemma 8.3, τ can be modified with a sequence of splits at large branches in c_2 different from the large branch e to a large train track τ' which contains c_2 as a vertex cycle and such that e is the only large branch of τ' contained in e. As splits at distinct large branches commute, the train track η is splittable to the train track obtained from τ' by a single right split at e. There exists a vertex cycle d of ζ which crosses through the diagonal of the split connecting τ' to ζ. This vertex cycle also is a vertex cycle of τ', and it intersects c_2 transversely in a single point. The curve d has all the required property. This completes the proof of the lemma. ☐

A *splitting and collapsing sequence* is a sequence (η_i) of train tracks so that for each i, the train track η_{i+1} is obtained from η_i either by a split or a *collapse*, i.e. the inverse of a split.

Corollary 8.5. *Let τ be a train track realization of an admissible curve system $\mathcal{C}$. If the large train track η is obtained from τ by a splitting and collapsing sequence, then $\Gamma(\eta) = \Gamma(\mathcal{C})$ and every vertex cycle of η is the image of a vertex cycle of τ by an element of $\Gamma(\mathcal{C})$.*

Proof. We proceed by induction on the length of the splitting and collapsing sequence. If this length equals one then Lemma 8.1 and Lemma 8.4 show that the corollary holds true, so assume it holds true whenever η can be obtained from τ by a splitting and collapsing sequence of length at most $k-1$ for some $k-1 \geq 1$.

Assume that the large train track η can be obtained from τ by a splitting and collapsing sequence of length k. Let ξ be obtained from τ by a splitting and collapsing sequence of length $k-1$ and assume that ξ is splittable or collapsible to η. By induction hypothesis, we have $\Gamma(\xi) = \Gamma(\mathcal{C})$ and hence $\Gamma(\eta) = \Gamma(\xi) = \Gamma(\mathcal{C})$ by Lemma 8.1 and Lemma 8.4, moreover every vertex cycle of ξ is the image of a vertex cycle of τ by an element of $\Gamma(\mathcal{C}) = \Gamma(\xi)$.

Let now d be a vertex cycle of η. By the second and third part of Lemma 8.4, there exists some vertex cycle c of ξ and an element $\psi \in \Gamma(\xi) = \Gamma(\mathcal{C})$ such that $d = \psi(c)$. By induction assumption, there exists a vertex cycle c' of τ and an element $\psi' \in \Gamma(\mathcal{C})$ such that $c = \psi'(c')$. Then $d = \psi\psi'(c')$ for $\psi'\psi \in \Gamma(\mathcal{C})$ which is what we wanted to show. □

To relate the projection $P\pi_1(\mathcal{Q})$ of the orbifold fundamental group of $\mathcal{Q}$ to $\Gamma(\mathcal{C})$ we use a combinatorial model for $P\pi_1(\mathcal{Q})$ as follows.

Let $\mathcal{G}(\mathcal{Q})$ be the graph whose vertices are the (marked) train tracks in $\mathcal{LT}(\tilde{\mathcal{Q}})$ (i.e. those marked large train tracks which are associated to the component $\mathcal{Q}$). The mapping class group $\mathrm{Mod}(S_{g,m})$ naturally acts properly and faithfully on $\mathcal{G}(\mathcal{Q})$ by precomposition of marking. A $\mathrm{Mod}(S_{g,m})$-orbit of vertices of $\mathcal{G}(\mathcal{Q})$ consists of all large train tracks of the same topological type which only differ by the marking. We connect two marked train tracks $\tau, \eta \in \mathcal{G}(\mathcal{Q})$ by an edge if η is obtained from τ by a split or a collapse. Note that the graph $\mathcal{G}(\mathcal{Q})$ need not be connected, and $\mathrm{Mod}(S_{g,m})$ acts as a group of permutations on the connected components of $\mathcal{G}(\mathcal{Q})$.

If $\mathcal{G}$ is any connected component of $\mathcal{G}(\mathcal{Q})$ then we denote by $\mathrm{Stab}(\mathcal{G})$ the stabilizer of $\mathcal{G}$ in $\mathrm{Mod}(S_{g,m})$. The stabilizer of any other connected components is conjugate to $\mathrm{Stab}(\mathcal{G})$ in $\mathrm{Mod}(S_{g,m})$. We have

Proposition 8.6. $P\pi_1(\mathcal{Q}) = \mathrm{Stab}(\mathcal{G})$ *for some connected component $\mathcal{G}$ of the graph $\mathcal{G}(\mathcal{Q})$. Furthermore,* $\mathrm{Stab}(\mathcal{G})$ *acts properly and cocompactly on $\mathcal{G}$.*

Proof. Let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of abelian differentials. Denote by $\mathrm{Stab}(\tilde{\mathcal{Q}})$ the stabilizer of $\tilde{\mathcal{Q}}$ in $\mathrm{Mod}(S_{g,m})$. We show first that $\mathrm{Stab}(\tilde{\mathcal{Q}})$ preserves a component of $\mathcal{G}(\mathcal{Q})$.

To this end let $q \in \tilde{\mathcal{Q}}$ be a point whose horizontal and vertical measured geodesic lamination, respectively, has large support of the same combinatorial type as $\tilde{\mathcal{Q}}$. Recall that the set of points with this property is dense in $\tilde{\mathcal{Q}}$. Let furthermore

$\psi \in \mathrm{Stab}(\tilde{\mathcal{Q}})$. Then there exists a path $\alpha : [0,1] \to \tilde{\mathcal{Q}}$ which connects $q = \alpha(0)$ to $\psi(q) = \alpha(1)$. Choose once and for all a marked train track $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ so that $q \in \mathcal{Q}(\tau)$. Such a train track exists by Proposition 6.3. Our goal is to show that there exists a path in $\mathcal{G}(\mathcal{Q})$ which connects τ to $\psi(\tau)$.

By Proposition 6.3, every point $z \in \tilde{\mathcal{Q}}$ has an open neighborhood $U_z \subset \tilde{\mathcal{Q}}$ with the following property. There exists a train track $\zeta(U_z) \in \mathcal{LT}(\tilde{\mathcal{Q}}) \subset \mathcal{G}(\mathcal{Q})$ so that U_z is contained in $\mathcal{Q}(\zeta(U_z))$. By Proposition 6.2, we may assume that each of the sets U_z is path connected.

Since $\alpha[0,1] \subset \tilde{\mathcal{Q}}$ is compact it can be covered by finitely many of the sets U_z, say by the sets $U_{\alpha(t_i)}$ for $0 = t_0 < t_1 < \cdots < t_k$. We may assume that $\zeta(U_{\alpha(t_0)}) = \tau$ and that $\zeta(U_{\alpha(t_k)}) = \psi(\tau)$. We may furthermore assume that $U_{\alpha(t_i)} \cap U_{\alpha(t_{i+1})} \neq \emptyset$ for all i. It now suffices to show that for each i, there is a path in $\mathcal{G}(\mathcal{Q})$ which connects $\zeta_i = \zeta(U_{\alpha(t_i)})$ to $\zeta_{i+1} = \zeta(U_{\alpha(t_{i+1})})$.

The set of all points $z \in \tilde{\mathcal{Q}}$ with the property that the support of the horizontal and vertical measured lamination of z is large, of the same combinatorial type as $\tilde{\mathcal{Q}}$, is dense in $\tilde{\mathcal{Q}}$, and the sets $U_{\alpha(t_j)}$ are open. Thus for each i there exists a measured geodesic lamination ν_i whose support is large, of the same topological type as $\tilde{\mathcal{Q}}$, which is carried by both ζ_i and ζ_{i+1}. This measured lamination is the horizontal measured lamination of a point in $U_{\alpha(t_i)} \cap U_{\alpha(t_{i+1})}$.

Fix a hyperbolic metric on S. Then for sufficiently small $\epsilon > 0$, a train track β which carries ν_i and ϵ-follows ν_i is carried by both ζ_i and ζ_{i+1} [H09]. Corollary 6.5 shows that β is large. It now follows from the work of Penner [PH92] that β carries a large train track β' with the property that both ζ_i and ζ_{i+1} are splittable to β'. But this just means that ζ_i can be connected to ζ_{i+1} by a path in $\mathcal{G}(\mathcal{Q})$ which is the concatenation of a splitting path connecting ζ_i to β' and the inverse of a splitting path connecting β' to ζ_{i+1}. This is what we wanted to show.

To summarize, we showed that $\mathrm{Stab}(\tilde{\mathcal{Q}})$ is contained in the stabilizer of the component $\mathcal{G}$ of $\mathcal{G}(\mathcal{Q})$ which contains the fixed marked train track τ. Our next goal is to show that $\mathrm{Stab}(\mathcal{G}) \subset \mathrm{Stab}(\tilde{\mathcal{Q}})$.

To this end let now $\psi \in \mathrm{Stab}(\mathcal{G})$. It suffices to construct for any edge path $\beta : [0,k] \to \mathcal{G}(\mathcal{Q})$ beginning at the fixed marked train track $\tau = \beta(0)$ and ending at $\psi(\tau) = \beta(k)$ and for any point $q \in \tilde{\mathcal{Q}} \cap \mathcal{Q}(\tau)$ a path in $\tilde{\mathcal{Q}}$ connecting q to $\psi(q)$. However, the existence of such an edge path follows in a fairly straightforward way from the definition of the sets $Q(\tau)$ and Proposition 6.2.

Namely, by the definition of $\mathcal{G}(\mathcal{Q})$, $\beta(i+1)$ is obtained from $\beta(i)$ by a split or a collapse for each i. But this implies that $Q(\beta(i)) \cap Q(\beta(i+1))$ contains an open subset of $\tilde{\mathcal{Q}}$. Choose for each i a point $q(i) \in Q(\beta(i)) \cap Q(\beta(i+1)) \cap \tilde{\mathcal{Q}}$. We require that $q(0) = q$ and $q(k) = \psi(q)$. By Proposition 6.2, for each i there exists a path in $Q(\beta(i)) \cap \tilde{\mathcal{Q}}$ which connects $q(i-1)$ to $q(i)$. The concatenation of these paths defines a path in $\tilde{\mathcal{Q}}$ connecting q to $\psi(q)$. In other words, we have $\mathrm{Stab}(\mathcal{G}) \subset \mathrm{Stab}(\tilde{\mathcal{Q}})$ as advertised.

We are left with showing that the action of $\mathrm{Stab}(\mathcal{G})$ on $\mathcal{G}$ is proper and cocompact. To show properness observe that as a large train track $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ decomposes S into disks and once punctured disks, the stabilizer of τ in $\mathrm{Mod}(S_{g,m})$ acts as a group of permutations on the branches of τ, and a mapping class which fixes each of the branches of τ is the identity. Thus the order of the stabilizer of τ is bounded from above by a universal constant. As $\mathrm{Mod}(S_{g.m})$ acts on $\mathcal{G}(\mathcal{Q})$ as a group of simplicial automorphisms, properness of the action follows.

To show cocompactness note that there are only finitely many combinatorial types of large train tracks. Thus if we fix a train track realization τ of the curve diagram $\mathcal{C}$, then uniformly near any train track $\zeta \in \mathcal{G}$ there is a train track in the orbit of τ under the action of $\mathrm{Stab}(\mathcal{G})$. But this just means that $\mathcal{G}$ is the image of a finite subset of $\mathcal{G}$ under the action of $\mathrm{Stab}(\mathcal{G})$. This shows cocompactness of the action of $\mathrm{Stab}(\mathcal{G})$ (compare [H09] for a similar statement). □

The following example shows that the action of $\mathrm{Stab}(\mathcal{G})$ on $\mathcal{G}$ may not be faithful.

Example 8.7. Let $\tilde{\mathcal{H}}^{\mathrm{hyp}}(2g-2)$ be a component of the preimage of the hyperellptic component $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ in the Teichmüller space of abelian differentials. Its stabilizer in $\mathrm{Mod}(S_{g,1})$ contains the hyperelliptic involution which acts by the involution $q \to -q$ on $\tilde{\mathcal{H}}^{\mathrm{hyp}}(2g-2)$. If we denote by $\mathcal{G}$ a component of the graph $\mathcal{G}(\tilde{\mathcal{H}}^{\mathrm{hap}}(2g-2))$ then the hyperelliptic involution acts on $\mathcal{G}$ by fixing each train track and reversing its orientation. Thus the group $P\pi_1(\mathcal{Q})$ may have a non-trivial center, and this center may act trivially on $\mathcal{G}(\mathcal{Q})$ (if we do not distinguish two orientable train tracks which only differ by their orientation).

To summarize, let $\mathcal{Q}$ be a component of a stratum of abelian differentials and let $\tilde{\mathcal{Q}}$ be a component of its preimage in the Teichmüller space of abelian differentials. We identified the stabilizer $\mathrm{Stab}(\tilde{\mathcal{Q}})$ of $\tilde{\mathcal{Q}}$ in $\mathrm{Mod}(S_{g,m})$ with the stabilizer of a component $\mathcal{G}$ in the graph $\mathcal{G}(\mathcal{Q})$. We also know that for every admissible curve system $\mathcal{C}$ for $\mathcal{Q}$, there exists a train track realization $\tau = \tau(\mathcal{C})$ for $\mathcal{C}$. The curve system $\mathcal{C}$ defines a subgroup $\Gamma(\mathcal{C})$ of $P\pi_1(\mathcal{Q})$. By naturality of the action of $\mathrm{Mod}(S_{g,m})$, for every $\psi \in P\pi_1(\mathcal{Q})$ we have

$$\Gamma(\psi(\mathcal{C})) = \psi(\Gamma(\mathcal{C}))\psi^{-1}.$$

On the other hand, Proposition 8.6 shows that $\psi \in \mathrm{Stab}(\mathcal{G})$ for a connected component $\mathcal{G}$ of $\mathcal{G}(\mathcal{Q})$. Thus τ can be connected to $\psi(\tau) = \tau(\psi(\mathcal{C}))$ by an edge-path in $\mathcal{G}$, and such a path is a finite concatenation of splitting and collapsing segments. Hence Corollary 8.5 yields that $\Gamma(\psi(\mathcal{C})) = \Gamma(\mathcal{C})$. From this we conclude

Corollary 8.8. *For every admissible curve system $\mathcal{C}$ for $\mathcal{Q}$, the group $\Gamma(\mathcal{C})$ is a normal subgroup of $P\pi_1(\mathcal{Q})$.*

9. Twist groups

In this section we consider again a curve system $\mathcal{C}$ which is admissible for a component $\mathcal{Q}$ of a stratum of abelian differentials. Our goal is to provide some additional properties of the group $\Gamma(\mathcal{C})$.

We begin with an alternative description of the group $\Gamma(\mathcal{C})$. Although this description is not used in the sequel, we think it helps to understand the structure of the orbifold fundamental group of a component of a stratum of abelian differentials. Furthermore, it relates to the work of Salter [Sa17].

Choose a component $\tilde{\mathcal{Q}}$ of the preimage of $\mathcal{Q}$ in the Teichmüller space of abelian differentials with the property that $\Gamma(\mathcal{C})$ is contained in the stabilizer of $\tilde{\mathcal{Q}}$. Define $\Lambda(\tilde{\mathcal{Q}})$ to be the subgroup of $\mathrm{Mod}(S_{g,m})$ which is generated by the Dehn twists about the core curves of all cylinders for the singular euclidean metrics of all abelian differential in $\tilde{\mathcal{Q}}$. Clearly $\Gamma(\mathcal{C}) \subset \Lambda(\tilde{\mathcal{Q}})$. We have

Proposition 9.1. $\Lambda(\tilde{\mathcal{Q}}) = \Gamma(\mathcal{C})$.

Proof. Let c be the core curve of a cylinder of a differential $q \in \tilde{\mathcal{Q}}$. We have to show that $T_c \in \Gamma(\mathcal{C})$ where as before, T_c is the positive Dehn twist about c.

To this end replace q by $z = e^{i\theta}q$ where $\theta \in \mathbb{R}$ is such that the cylinder with core curve c is horizontal for z.

Using the notations from Section 8, let $\mathcal{G} \subset \mathcal{G}(\mathcal{Q})$ be the component with the property that $\mathrm{Stab}(\mathcal{G}) = \mathrm{Stab}(\tilde{\mathcal{Q}})$. Such a component exists by Proposition 8.6. By Proposition 6.3, there exists a large train track $\eta \in \mathcal{G}$ which contains c as a vertex cycle.

Let τ be a train track realization of the curve system $\mathcal{C}$. Then $\tau \in \mathcal{G}$ and hence there exists a path in $\mathcal{G}$ connecting τ to η. Corollary 8.5 now yields that $T_c \in \Gamma(\mathcal{C})$ which is what we wanted to show. □

Our second goal is to describe the $\Gamma(\mathcal{C})$-orbits of the components of $\mathcal{C}$. The following lemma is true for all components.

Lemma 9.2. *Let $\mathcal{C}$ be an admissible curve system for the component $\mathcal{Q}$ and let $c \in \mathcal{C}$. Then each curve $c' \in \mathcal{C}$ is contained in the $\Gamma(\mathcal{C})$-orbit of c.*

Proof. By Proposition 4.2 and its proof, the lemma holds true for hyperelliptic components. Thus we may assume from now on that $\mathcal{Q}$ is not hyperelliptic. Then by Corollary 3.7 and Lemma 8.4, we only have to show the lemma for a curve system $\mathcal{C}$ which is an extension of a system of type U_{2g} or a system of type V_{2g}.

We begin with a curve system $\mathcal{C}$ of type U_{2g}. Its curve diagram can be described as a line segment L of length $2g-2$ with an arc of length two attached. Or, the diagram can be viewed as two line segments L, L' of length $2g-2$ which are glued together along two subsegments of length $2g-4$.

Let $\mathcal{L} \subset \mathcal{C}$ be the subsystem of curves whose curve diagram is the line L. As $\mathcal{L}$ has an even number of curves, a tubular neighborhood of $\mathcal{L}$ in S is a subsurface S_0 of S of genus $g-1$ with connected boundary (see Section 4.4 of [FM12] for details). This subsurface is invariant under the group $\Gamma(\mathcal{L})$. By Corollary 4.3, applied to $\mathcal{L}$ as a curve system in S_0, the group $\Gamma(\mathcal{L})$ acts transitively on the components of $\mathcal{L}$. In particular, if we denote by c the curve which corresponds to the endpoint of the line L which also is an endpoint of L', then for any $d \in \mathcal{L}$ there exists some $\psi_d \in \Gamma(\mathcal{L}) \subset \Gamma(\mathcal{C})$ with $\psi_d(d) = c$. Similarly, if we denote by $\mathcal{L}'$ the subsystem of $\mathcal{C}$ with curve diagram L', then for all $d' \in \mathcal{L}'$, there exists some $\psi_{d'} \in \Gamma(\mathcal{L}') \subset \Gamma(\mathcal{C})$ with $\psi_{d'}(d') = c$. Thus any curve from $\mathcal{C}$ can be moved to the curve c by an element of $\Gamma(\mathcal{C})$. This yields the statement of the lemma for the system of type U_{2g}.

The argument for the curve system of type V_{2g} is identical and will be omitted.

Now let $\mathcal{C}$ be an extension of a system $\mathcal{D}$ of type U_{2g} or V_{2g}. Then every element $c \in \mathcal{C} - \mathcal{D}$ is contained in a subsystem $\mathcal{E}$ whose diagram is a line with four vertices, three of them contained in $\mathcal{D}$. By the above argument, applied to $\mathcal{E}$, there exists an element $\psi \in \Gamma(\mathcal{E}) \subset \Gamma(\mathcal{C})$ which maps c to a curve in $\mathcal{D}$. By the first part of the proof, this implies that any curve in $\mathcal{C}$ can be mapped to a fixed curve in $\mathcal{D}$ by an element of $\Gamma(\mathcal{C})$. The lemma follows. □

We will use a strengthening of the third part of Lemma 8.4. To this end let $c \neq d$ be vertex cycles of an orientable train track τ. The intersection $c \cap d$ consists of a finite number of embedded segments (which means a finite number of embedded trainpaths). Define such a segment β to be a *crossing segment* if the following holds true. Let $A \subset S$ be an annulus neighborhood of c. Then there is no neighborhood of β in d which intersects A in an arc entirely contained in the closure of a component of $A - c$. In other words, a neighborhood of β in d crosses through c. Otherwise we call the segment *non-crossing*.

Define two vertex cycles c, d on a train track τ to be in *slick position* if $c \cap d$ consists of precisely one crossing segment. Note that if τ is a train track realization of an admissible curve system, then any two of the vertex cycles of τ which are not disjoint are in slick position. Futhermore, if c, d are in slick position then the simple closed curves c, d can be homotoped in such a way that the intersect transversely in a single point.

In the next lemma we denote as before by $\tilde{\mathcal{Q}}$ a component of the preimage of a component $\mathcal{Q}$ of a stratum of abelian differentials on S.

Lemma 9.3. *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials and let $\eta \in \mathcal{LT}(\tilde{\mathcal{Q}})$ be obtained from $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ by a single split. Let c, d be a pair of vertex cycles of η which are in slick position. Then there exists an element $\psi \in \Gamma(\tau)$ and a pair of vertex cycles c', d' of τ in slick position such that $\psi(c') = c, \psi(d') = d$.*

Proof. Using the notations from the lemma, let us first assume that both vertex cycles c, d of η are also vertex cycles of τ. Then the restriction of the carrying map $F : \eta \to \tau$ to $c \cup d$, viewed as a subtrack of η, is an embedding. Thus c, d, viewed as vertex cycles of τ, are in slick position, and we can choose ψ to be the identity.

Assume now that c is not a vertex cycle of τ. Let e be the large branch of τ which is used in the transformation of τ to η, i.e. assume that η is obtained from τ by a right split at e. Equip c with the orientation inherited from the orientation of η. Using the notations from the proof of Lemma 8.4 (see Figure G), we showed that c does not pass through the diagonal branch of the split.

Let $F:\eta\to\tau$ be a carrying map as in the proof of Lemma 8.4. The transverse measure on τ defined by $F(c)$ is a sum of the transverse measures defined by two vertex cycles c_1,c_2 of τ. The vertex cycle c_1 also is a vertex cycle of η, and the vertex cycle c_2 is not carried by η. The curves c_1,c_2 intersect in a single point which can be chosen in the interior of the large branch e. Furthermore, we have $c=T_{c_1}^{\pm}c_2=T_{c_2}^{\mp}c_1$.

By assumption, the vertex cycles $c,d\subset\eta$ are in slick position and hence they intersect in a single crossing arc.

We distinguish four cases.

Case 1: The vertex cycle d of η passes through the diagonal e' of the split.

Using the notations from Figure G, the vertex cycle d is disjoint from the two branches b_1,b_4 of η distinct from e' which are incident on the endpoints of e' and small at these endpoints. Furthermore, d contains the two branches b_2,b_3 of η which are incident on the endpoints of e' and large at these endpoints.

Now by Lemma 8.4 and its proof, as c is not a vertex cycle of τ, the branches b_2,b_3 are contained in c and hence they are contained in $c\cap d$. On the other hand, by assumption $c\cap d$ is connected and hence $c\cap d$ contains the complement of e' in the curve c_1 shown in Figure G. As d is a vertex cycle which contains e', this implies that $d=c_1$. Now $c=T_{c_1}^{\pm}c_2$ where c_2 is a vertex cycle of τ not carried by η and hence in this case we can take $\psi=T_{c_1}^{\pm}$ and $d'=d=c_1$, $c'=c_2$.

Case 2: d is a vertex cycle of τ which does not cross through the diagonal of the split and whose projection to τ does not contain the large branch e.

In this case the restriction of the carrying map $F:\eta\to\tau$ to d is a C^1-diffeomorphism which is disjoint from the large branch e. The restriction of F to $c\cap d$ has the same property and hence $F(c\cap d)\subset c_1$ or $F(c\cap d)\subset c_2$. As a consequence, d is disjoint from either c_2 or from c_1. If d is disjoint from c_2 then we use $c=T_{c_2}^{\pm}c_1$ and $d=T_{c_2}^{\pm}d$ as before. The case that d is disjoint from c_1 is completely analogous.

Case 3: d is a vertex cycle of τ which does not cross through the diagonal e' of the split but which projects onto the large branch e.

Recall that by assumption, $c\cap d$ is a crossing segment. In particular, d crosses through c. Equip c,d with the orientation inherited from the orientation of η. As S is oriented, it makes sense to distinguish between vertex cycles crossing from the right side to the left side of c and vertex cycles crossing from the left side to the right side of c.

Assume that the vertex cycle c_1 of η crosses from the right side to the left side of c, and that the curve c_2 (which is a vertex cycle of τ but which is not carried by η) crosses from the left side to the right side. This means that with a homotopy we can assure that the curve c_1 lies entirely to the right of the curve c except in a small neighborhood of the diagonal branch $e'\subset\eta$ where it crosses through c.

Now let us assume that d crosses from the right to the left through c. Using the notations of Figure G, as d does not pass through the diagonal of the split, it can be homotoped to a curve which intersects c in a unique point contained in the interior of one of the branch b_1 (if it projection onto b_1) or in the interior of the branch b_4 (if it projects onto b_4), and this crossing is from the right side of b_i to the left side. However, this implies that d is disjoint from c_1.

The same argument shows that if d crosses from the left to the right through c then d is disjoint from c_2. As before, we conclude that we can take $\psi = T_{c_1}^{\pm}$ or $\psi = T_{c_2}^{\pm}$.

Case 4: d is not a vertex cycle of τ.

Following Lemma 8.4 and its proof and using the notations from Figure G, d projects onto b_i for each $1 \leq i \leq 4$. As its intersection with c is connected, d projects to both c_1 and c_2 and hence we have $d = c$ which contradicts the assumption on d. □

As a consequence, we obtain

Corollary 9.4. *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials and let $\mathcal{C}$ be an admissible curve system for $\mathcal{Q}$. Assume that $\Gamma(\mathcal{C})$ stabilizes the component $\tilde{\mathcal{Q}}$ of the preimage of $\mathcal{Q}$. Let $\psi \in \mathrm{Stab}(\tilde{\mathcal{Q}})$ and let $c, d \in \mathcal{C}$ a pair of curves which intersect in a single point. Then there exists some $\varphi \in \Gamma(\mathcal{C})$ such that $\varphi(c) = \psi(c), \varphi(d) = \psi(d)$.*

Proof. Let $\mathcal{C}$ be any admissible curve system for $\mathcal{Q}$ and let τ be a train track realization of $\mathcal{C}$. Using the notations from the corollary and from Section 8, let $\mathcal{G}$ be the component of $\mathcal{G}(\mathcal{Q})$ which is stabilized by $\mathrm{Stab}(\tilde{\mathcal{Q}})$.

Let $\eta \in \mathcal{G}$ be a train track which is splittable to τ. Let $c, d \in \mathcal{C}$ be curves which intersect in a single point. We claim that there are vertex cycles c', d' for η in slick position, and there is some $\psi \in \Gamma(\mathcal{C}) = \Gamma(\eta)$ such that $\psi(c') = c, \psi(d') = d$.

We proceed by induction on the length of the splitting sequence transforming η to τ. The case that this length equals one is precisely the statement of Lemma 9.3. Now let us assume that the statement holds true whenever this length is at most $k-1$ for some $k \geq 2$. Let $(\eta_i)_{i \geq 0}$ be a splitting sequence of length k connecting a train track η_0 to $\eta_k = \tau$. By induction hypothesis, applied to the splitting sequence $(\eta_i)_{i \geq 1}$, there exists a pair of vertex cycles $\hat{c}, \hat{d}$ on η_1 in slick position, and there exists some $\varphi \in \Gamma(\eta_1) = \Gamma(\mathcal{C})$ so that $\varphi(\hat{c}) = c, \varphi(\hat{d}) = d$. By Lemma 9.3, we can find vertex cycles c', d' for η_0 in slick position and an element $\theta \in \Gamma(\eta_0) = \Gamma(\mathcal{C})$ such that $\theta(c') = \hat{c}, \theta(d') = \hat{d}$. The claim now follows with $\psi = \theta\varphi$.

A pair of vertex cycles in slick position for τ is a pair of curves $c, d \in \mathcal{C}$ which intersect in a single point. Thus together with Lemma 9.2 and perhaps by replacing ψ by ψ^{-1} we deduce the following. Let us assume that there exists some $\theta \in \mathrm{Stab}(\mathcal{G})$ such that τ is splittable or collapsible to $\theta(\tau)$; then for any pair of curves $c, d \in \mathcal{C}$ which intersect in a single point there exists some $\psi \in \Gamma(\mathcal{C})$ such that $\psi(c) = \theta(c), \psi(d) = \theta(d)$.

We use this to complete the proof of the corollary as follows. Let $\mathrm{Id} \neq \psi \in \mathrm{Stab}(\tilde{\mathcal{Q}})$; then τ can be connected to $\psi(\tau)$ by a splitting and collapsing path. Such an arc consists of $k \geq 1$ subarcs which are either splitting arcs or collapsing arcs. A splitting arc is followed by a collapsing arc.

We proceed by induction on the minimal number k of such arcs in a splitting and collapsing path connecting τ to $\psi(\tau)$. The case $k = 1$ follows from the above discussion, so assume by induction that the claim holds true for some $k - 1 \geq 1$.

Assume that τ can be connected to $\psi(\tau)$ by a splitting and collapsing arc consisting of k segments. Assume first that the first segment is a splitting segment, ending at a train track η.

As $\Gamma(\mathcal{C})$ acts properly and cocompactly on $\mathcal{G}$, for every $\xi \in \mathcal{G}$ there exists some $\varphi_1, \varphi_2 \in \mathrm{Stab}(\mathcal{G})$ such that $\varphi_1(\tau) \prec \xi \prec \varphi_2(\tau)$. This means that we can find some $\varphi \in \mathrm{Stab}(\mathcal{G})$ so that $\varphi(\tau)$ is carried by η. Then we may assume that η is splittable to $\varphi(\tau)$. Replace the splitting and collapsing arc connecting τ to $\varphi(\tau)$ by a splitting and collapsing arc with the same number of segments but where the first segment is a splitting segment connecting τ to $\varphi(\tau)$. We can now apply the induction hypothesis to the splitting and collapsing arc connecting $\varphi(\tau)$ to $\psi(\tau)$ and to the splitting sequence connecting τ to $\varphi(\tau)$.

The case that the first segment is a collapsing arc is completely analogous and will be omitted. ☐

10. Higher spin structures

In this section we introduce $\mathbb{Z}/r\mathbb{Z}$-valued spin structures on a closed surface. We relate such higher spin structures to strata of abelian differential and discuss a first instance of degeneration abelian differentials to abelian differentials on a surface with nodes by controlling the information on the higher spin structures.

10.1. **Higher spin structures and divisors.** Let S be for the moment an arbitrary compact oriented surface. We allow that the boundary of S is non-empty. Let $\mathcal{S}$ be the set of isotopy classes of oriented simple closed curves on S. We do not require that such a curve is essential. The following goes back to Humphries and Johnson [HJ89]; we adopt the viewpoint of Salter [Sa17]. As before, we denote by ι the homological intersection form on $H_1(S, \mathbb{Z})$.

Definition 10.1. A $\mathbb{Z}/r\mathbb{Z}$-valued spin structure on S is a function $\varphi : \mathcal{S} \to \mathbb{Z}/r\mathbb{Z}$ satisfying the following properties.

(1) (Twist-linearity) Let $c, d \in \mathcal{S}$ be arbitrary. Then
$$\varphi(T_c(d)) = \varphi(d) + \iota(d, c)\varphi(c) \pmod r$$
(2) (Normalization) If ζ is the boundary of an embedded disk $D \subset S$, oriented so that D is to the left of ζ, then $\varphi(\zeta) = 1$.

It follows immediately from part (1) of the definition that the Dehn twist about a curve c preserves the spin structure if and only if either $\varphi(c) = 0$ or if c is separating. Furthermore, if φ is a $\mathbb{Z}/r\mathbb{Z}$-spin structure on S then $r \mid (2g-2)$ (Remark 3.6 of [Sa17]).

Namely, let UTS be the unit tangent bundle of S, or, equivalently, the circle bundle over S associated to the tangent bundle. Then $\mathbb{Z}/\mathbb{Z}$-spin structures on S are in correspondence with cyclic r-fold coverings $\widetilde{UTS} \to UTS$ that restrict to connected coverings of the fibre. Equivalently, they correspond to r-th roots of the tangent bundle of S, viewed as an oriented two-dimensional real vector bundle over S.

Any abelian differential q on S with a single zero defines an $\mathbb{Z}/(2g-2)\mathbb{Z}$-spin structure on S. Namely, the unique zero of q, viewed as an divisor on S which defines the cotangent bundle of S, has weight $2g-2$, and the $(2g-2)$-fold tensor product of the line bundle whose divisor equals the zero of q with weight one equals the canonical bundle as holomorphic line bundle.

10.2. **Degeneration to a boundary stratum I.** In this subsection we collect information on a specific type of degeneration of abelian differentials in a component $\mathcal{Q}$ of a stratum of abelian differentials with a single zero on a surface of genus g to a differential on a surface of genus $g-1$ with a single non-separating node.

Let $\mathcal{C}$ be a simple admissible curve system for $\mathcal{Q}$ as defined in Section 2. Let $c \in \mathcal{C}$ be a curve which corresponds to a leaf of the curve diagram of $\mathcal{C}$. Let furthermore $q = q(\mathcal{C}) \in \mathcal{Q}$ be a realization of $\mathcal{C}$ as in Lemma 2.9. We may assume that there is a horizontal cylinder C with core curve c for q. Since c intersects one single curve from $\mathcal{C}$, this cylinder is glued from a single square R by identifying its two vertical sides.

There exists a unique simple closed curve d on S (not a component of $\mathcal{C}$) which intersects c in a single point and does not intersect any other curve from $\mathcal{C}$. Namely, let S_0 be a tubular neighborhood of $\mathcal{C}-c$. As the number of curves in $\mathcal{C}-c$ is odd, S_0 is a surface of genus $g-1$ with two homotopic non-separating boundary components. In particular, $S - S_0$ is a cylinder. Its core curve d is disjoint from $\mathcal{C} - c$, and it is the unique simple closed curve with this property. As up to homotopy, the curve c intersects $S - S_0$ in a connected arc connecting the two boundary components, the curve d has the required properties.

Example 10.2. Consider the curve system $\mathcal{C}$ of type U_{2g}. There are three curves corresponding to leaves in the curve diagram, these curves are the curves with label c_1, c_g, a_{g-1} in Figure A. For each of these curves, say the curve c, there exists a unique simple closed curve d which intersects c in a single point and does not intersect any other curve from $\mathcal{C}$. Adding this curve to $\mathcal{C}$ defines an admissible curve system for a component $\mathcal{Q}'$ of a stratum of abelian differentials. For the curve c_1 and c_g, this is a component of $\mathcal{H}(g-2, g)$, and for the curve a_{g-1} this is a component of $\mathcal{H}(g-1, g-1)$.

Let again R be the square in the construction of $q(\mathcal{C})$ which contains the intersection point of c with the unique curve $d \in \mathcal{C}$ which is not disjoint from c. The two vertical sides of R are identified in $q(\mathcal{C})$. The resulting arc in $q(\mathcal{C})$ connects the singular point x of $q = q(\mathcal{C})$ to itself. In particular, this saddle connection defines a simple closed curve on S which is homotopic to the curve d. We can shrink the height of the cylinder C to zero while keeping the remaining squares fixed. This construction shrinks the flat length of the curve d to zero. The thus defined arc of abelian differentials degenerates to an abelian differential q' on a surface S' of genus $g-1$ with a non-separating node. This node is obtained by shrinking d to a point.

There is more information. The node is the only singular point for the euclidean metric on S' defined by q'. The core curve c of the cylinder degenerates in S' to a horizontal saddle connection connecting the node to itself. We refer to [BCGGM18] for more information.

Let again x be the zero of the differential $q = q(\mathcal{C})$. The horizontal separatrices emanating from x divide a disk neighborhood of x into $4g$ euclidean half-disks of fixed radius $\epsilon > 0$ centered at x. These half-disks are contained in the upper half-plane and contain a horizontal segment of length 2ϵ centered at 0. The point x corresponds to the center of the horizontal boundaries of these half-disks. The left half-segment of a half-disk is identified with the right half-segment of another half-disk.

There are two non-adjacent such half-disks whose interiors are contained in the cylinder C. Removal of the interiors of these half-disks yields two closed sectors which only meet at x. The total angles α_1, α_2 of these sectors are multiples of 2π. A neighborhood of the node for the flat metric defined by the differential q' is obtained by removal of C and by identifying the two boundary arcs of each of the two sectors.

Cutting the surface S' open at the node yields a surface $\hat{S}$ of genus $g-1$ with two marked points x_1, x_2 and an abelian differential $\hat{q}$. The cone angle at these points are α_1, α_2. One of these points may be regular, but the images of the node are the only singular points for $\hat{q}$. In particular, the differential $\hat{q}$ has one or two zeros and no poles. The horizontal cylinder C in q degenerates to a horizontal saddle connection for $\hat{q}$ connecting the marked points x_1, x_2.

Denote by $\mathcal{R}$ the component of a stratum of abelian differentials on a surface of genus $g-1$ which contains $\hat{q}$. The curve system $\mathcal{C}' = \mathcal{C} - \{c\}$ induces an admissible curve system for $\mathcal{R}$.

Example 10.3. The above construction, applied to a realization of the curve system of type U_{2g} and the cylinder whose core curve is the curve labeled with c_1 or c_g yields a differential on a surface of genus $g-1$ with a zero of order $g-3$ and a zero of order $g-1$. In particular, for $g = 3$ we obtain an abelian differential on a surface of genus 2 with a double zero and a regular marked point.

If we apply the above construction to a realization of the curve system of type U_{2g} and the cylinder whose core curve is the curve labeled with a_{g-1} then we obtain an abelian differential on a surface of genus $g-1$ with two zeros of order $g-2$.

Thus in the case $g = 3$ we obtain a differential on a surface of genus 2 with two simple zeros.

The above construction can be reversed as follows; this is called the *figure eight construction* in [EMZ13].

Consider a translation surface $(\hat{S}, \hat{q})$ with two distinguished marked points $x_1 \neq x_2$. We allow that these marked points are regular or singular. Let γ be a saddle connection connecting x_1 to x_2. Assume for simplicity that this saddle connection is horizontal. Cut the surface $\hat{S}$ open along γ and identify the points x_1, x_2. One obtains a surface X with two horizontal geodesic boundary circles of the same length, attached at one point. Glue a flat cylinder C to the two boundary circles. The result is a surface whose genus equals the genus of $\hat{S}$ plus one and which is equipped with an abelian differential q.

There are two degrees of freedom for this construction which can be described by a complex parameter $re^{i\theta}$. Here $r > 0$ is the height of the cylinder C, and $\theta \in [0, 2\pi)$ describes the relative position of the two distinguished points on its two boundary components. Let q be the differential for which the two distinguished points on the boundary of C are connected by a vertical arc whose length equals the height of C.

Given a sufficiently small neighborhood $\hat{U}$ in the moduli space of abelian differentials on $\hat{S}$ with two distinguished marked points x_1, x_2 and a saddle connection connecting x_1 to x_2, the figure eight construction yields an open neighborhood U of q in the moduli space of abelian differentials on S and a holomorphic surjection $\Pi : U \to U'$ whose fibre is a punctured complex disk. Each of the surfaces z in U contains a distinguished cylinder $C(z)$. Going once around a circle in a fibre which generates the fundamental group of the punctured disk results in a Dehn twist about the core curve of the cylinder.

11. The non-hyperelliptic component of $\mathcal{H}(4)$

In this section we use the discussion in Section 10.1 to compute explicitly the image of the orbifold fundamental group $\pi_1(\mathcal{Q})$ in the mapping class group $\mathrm{Mod}(S)$ for the non-hyperelliptic component $\mathcal{Q}$ of the minimal stratum $\mathcal{H}(4)$.

From now on S will always be a surface of genus $g = 3$.

11.1. **Curve graphs.** Let φ by a $\mathbb{Z}/4\mathbb{Z}$-spin structure on S. For the application we have in mind we may assume that the mod 2 reduction of φ is an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure.

Let $\mathcal{CG}$ be the curve graph of S. The vertices of this graph are isotopy classes of simple closed curves on S, and two such curves are connected by an edge if and only if they can be realized disjointly. Denote by $\mathcal{CG}_1$ the complete subgraph of $\mathcal{CG}$ of all non-separating curves c on S with $\varphi(c) = \pm 1$. Note that this condition does not depend on the orientation of c and hence is indeed a condition on the vertices of $\mathcal{CG}$. Furthermore, the condition is equivalent to saying that $\varphi(c)$ is odd for one and hence any of the two possible orientations of c. The finite index subgroup $\mathrm{Mod}(S)[\varphi]$ of $\mathrm{Mod}(S)$ of all elements which preserves the spin structure φ acts on $\mathcal{CG}_1$ as a group of simplicial automorphisms.

The goal of this subsection is to show

Proposition 11.1. *The graph $\mathcal{CG}_1$ is connected.*

Proof. Define a *graph of non-separating pairs* $\mathcal{NS}$ as follows. Vertices of $\mathcal{NS}$ are unordered pairs of simple closed curves (c, d) on S so that $S - (c \cup d)$ is connected. Two such pairs $(c, d), (c', d')$ are connected by an edge of length one if they differ by a single component and can be realized disjointly. The graph $\mathcal{NS}$ is connected (see [H14] for more details and more information on this graph).

We use the graph $\mathcal{NS}$ as an auxiliary structure for the construction of curves c with $\varphi(c) = \pm 1$. Namely, in a non-deterministic way we can associate to a vertex (c, d) in $\mathcal{NS}$ a vertex $\Lambda(c, d)$ of $\mathcal{CG}_1$ as follows.

If at least one of the curves, say the curve c, satisfies $\varphi(c) = \pm 1$ then we choose $\Lambda(c, d) = c$. Otherwise both $\varphi(c), \varphi(d)$ are even. Connect the disjoint curves c, d by an embedded arc ϵ in S whose interior is disjoint from $c \cup d$. A regular neighborhood ν of $c \cup \epsilon \cup d$ is homeomorphic to a three-holed sphere. Two of the boundary components of ν are the curves c, d. We assume that these curves are oriented in such a way that ν lies to the left. The third boundary component $c +_\epsilon d$, oriented in such a way that ν is to its right, satisfies $[c +_\epsilon d] = [c] + [d]$ where as before, $[c]$ denotes the homology class of the oriented curve c.

By Lemma 3.13 of [Sa17], we have $\varphi(c +_\epsilon d) = \varphi(c) + \varphi(d) + 1$, in particular, $\varphi(c +_\epsilon d)$ is odd. Furthermore, as (c, d) is a non-separating pair and $[c +_\epsilon d] = [c] \cup [d]$, the curve $c +_\epsilon d$ is non-separating. We then can define $\Lambda(c, d) = c +_\epsilon d$.

Now let $c = c_0, e$ be two vertices in the graph $\mathcal{CG}_1$. Then c_0, e are non-separating simple closed curves and hence we can find non-separating simple closed curves d_0, f so that (c_0, d_0) and (e, f) are vertices in $\mathcal{NS}$. Connect (c_0, d_0) to (e, f) by an edge path $(c_i, d_i)_{i \leq n}$ in $\mathcal{NS}$. We use this edge path to construct an edge path $a_j \subset \mathcal{CG}_1$ connecting c_0 to e inductively in such a way that it passes through suitable choices for the curves $\Lambda(c_i, d_i)$. In other words, the construction is done in such a way that there is an increasing sequence $j_0 = 0 < j_1 < \cdots < j_m = n$ such that a_{j_i} equals a possible choice for $\Lambda(c_i, d_i)$.

Define $a_0 = c_0$ (we can think of this as being the value of $\Lambda(c_0, d_0)$). Assume by induction that for some $i \geq 0$ we constructed already the path $(a_s)_{s \leq j(i)}$. Our goal will be to construct a path $(a_s)_{j_i \leq s \leq j_{i+1}}$ for some $j_{i+1} \geq j_i + 1$ which connects a_{j_i} to some choice for $\Lambda(c_{i+1}, d_{i+1})$. We consider two cases.

Case 1; One of the values $\varphi(c_i)$ or $\varphi(d_i)$ is odd.

By assumption, in this case we have up to renaming $a_{j_i} = c_i$.

Consider the pair $(c_{i+1}, d_{i+1}) \in \mathcal{NS}$. The curves c_{i+1}, d_{i+1} are disjoint from c_i. If at least one of the value $\varphi(c_{i+1}).\varphi(d_{i+1})$ is odd, say that this holds true for

$\varphi(c_{i+1})$, then define $j_{i+1} = j_i + 1$ and $a_{j_{i+1}} = c_{i+1}$. This is consistent with the requirements for the path (a_j).

Otherwise $\varphi(c_{i+1})$ and $\varphi(d_{i+1})$ are both even. Cut S open along $c_{i+1} \cup d_{i+1}$. The resulting surface is a torus T with four boundary components containing the curve c_i. Denote the two boundary components which are the images of the curve c_{i+1} by C_1, C_2, and denote the two boundary components which are the images of the curve d_{i+1} by D_1, D_2.

If $c_i \subset T$ is non-separating then we can connect the boundary component C_1 of T to the boundary component D_1 by an embedded arc ϵ disjoint from c_i. The lift of this arc to S, again denoted by ϵ, connects c_{i+1} to d_{i+1} and is disjoint from c_i. Define $a_{j_{i+1}} = c_{i+1} +_\epsilon d_{i+1}$. Clearly $a_{j_{i+1}}$ fulfills the requirements for the extension of the path (a_j).

If $c_i \subset T$ is separating then it decomposes T into a torus with one, two or three holes and a sphere with five, four or three holes. We can connect one of the two boundary components C_1, C_2 to one of the two boundary components D_1, D_2 by an embedded arc in $T - C_i$ and proceed as above unless c_i decomposes T into a torus T_0 with three holes and a sphere with three holes where up to renaming, two of the holes are bounded by C_1, C_2.

Choose a non-separating simple closed curve $b \subset T_0$. If $\varphi(b)$ is odd then define $a_{j_i+1} = b$. By construction, there exists an embedded arc ϵ connecting c_{i+1} to d_{i+1} which is disjoint from b, viewed as a simple closed curve in S. Let $j_{i+1} = j_i + 2$ and define $a_{j_{i+1}} = c_{i+1} +_\epsilon d_{i+1}$.

If $\varphi(b)$ is even then choose an embedded arc $\epsilon_1 \subset T_0$ which connects the boundary component D_1 of T_0 to b and define $a_{j_i+1} = d_{i+1} +_{\epsilon_1} b$. Clearly $\varphi(a_{j_i+1}$ is odd and a_{j_i+1} is disjoint from a_{j_i}.

Since $b \subset T_0$ is non-separating, the surface $T_0 - (b \cup \epsilon_1)$ is connected, and it contains the boundary circle D_2 and a boundary component corresponding to c_i. In particular, if we view b and ϵ_1 as a curve and an arc in T then there exists an embedded arc ϵ_2 in $T - (b \cup \epsilon_1)$ which connects the boundary component D_2 to the boundary component C_2. Define $j_{i+1} = j_i + 2$ and $a_{j_{i+1}} = c_{i+1} +_{\epsilon_2} d_{i+1}$. Again this construction fulfills all the requirements and completes the construction in the case that at least one of the numbers $\varphi(c_i), \varphi(d_i)$ is odd.

Case 2: $\varphi(c_i)$ and $\varphi(d_i)$ are both even.

In this case there exists an embedded arc ϵ connecting c_i to d_i such that $a_{j_i} = c_i +_\epsilon d_i$. Assume by renaming that $d_{i+1} = d_i$. The curve c_{i+1} is disjoint from c_i, d_i, but it may not be disjoint from ϵ.

Cut S open along $c_i \cup d_i$. Let T be the resulting four holed torus and let C_1, C_2 and D_1, D_2 be the boundary components of T corresponding to c_i, d_i. For a suitable numbering, the arc ϵ connects the boundary components C_1 and D_1.

We distinguish again two cases. The first case is that $\varphi(c_{i+1})$ is odd and that c_{i+1} does not separate the pair of boundary components C_1, C_2 of T from the pair of boundary components $D_1 \cup D_2$. Let us assume that there is an arc ϵ' in T which is disjoint from c_{j+1} and connects C_1 to D_1.

Consider the graph whose vertices are embedded arcs in T with one endpoint on C_1 and the second endpoint on D_1 and where two arcs are connected by an edge of length one if they can be realized disjointly. This graph is connected (see e.g. [H14]) and hence we can connect the arc ϵ to the arc ϵ' disjoint from c_{j+1} by an edge path, say the path (ϵ_ℓ). We now modify this path as follows.

Let $\epsilon_\ell, \epsilon_{\ell+1}$ be two adjacent arcs. Cut T open along ϵ_ℓ and let $\hat{T}$ be the corresponding three holed torus. Two of the boundary components of $\hat{T}$ are the curves C_2, D_2, and the third component contains the circles C_1, D_1 as subarcs.

There are two cases possible. In the first case, the arc $\epsilon_{\ell+1}$ does not separate C_2 from D_2 in $\hat{T}$. This means that there exists an arc connecting C_2 to D_2 which is disjoint from both ϵ_ℓ and $\epsilon_{\ell+1}$. Define δ_ℓ to be such an arc.

Otherwise the arc $\epsilon_{\ell+1}$ separates C_2 from D_2. As its endpoints lie on the same boundary component of the three-holed torus $T - \epsilon_\ell$, it decomposes the three-holed torus $T - \epsilon_\ell$ into a cylinder with one boundary component, say the component C_2, and a two-holed torus Z with one boundary component D_2. Then there exists an arc ϵ'_ℓ with one endpoint on C_1 and the second endpoint on D_1 which is disjoint from both ϵ_ℓ and $\epsilon_{\ell+1}$ and which cuts Z into a three-holed sphere. The arc ϵ'_i does not separate C_2 from D_2 in $T - \epsilon_\ell$ or $T - \epsilon_{\ell+1}$. Choose an arc δ_ℓ connecting C_2 to D_2 which is disjoint from both ϵ_ℓ and ϵ'_ℓ and an arc δ'_ℓ connecting C_2 to D_2 which is disjoint from ϵ'_ℓ and $\epsilon_{\ell+1}$. Replace the two arcs $\epsilon_\ell, \epsilon_{\ell+1}$ by arcs $\epsilon_\ell, \delta_\ell, \epsilon'_\ell, \delta'_\ell, \epsilon_{\ell+1}$.

Doing this construction for each ℓ yields a sequence β_j of arcs with the following properties.

- $\beta_0 = \epsilon, \beta_{2k} = \epsilon'$ for some $k \geq 1$.
- For each ℓ the arc $\beta_{2\ell+1}$ connects the boundary components C_1 and D_1, and the arc $\beta_{2\ell+1}$ connects C_2 and D_2.
- The arcs $\beta_\ell, \beta_{\ell+1}$ are disjoint.

For each ℓ the simple closed curve $b_\ell = c_i +_{\beta_\ell} d_i$ in S is non-separating curve in S, and we have $\varphi(b_\ell) = 1$ for all ℓ, furthermore the curves b_ℓ and $b_{\ell+1}$ are disjoint. Thus this construction connects b_0 to b_{2k} by a path in $\mathcal{CG}_1$. But b_{2k} is disjoint from c_{j+1} and hence defining $j_{i+1} = j_i + 2k + 1$, this defines a path in $\mathcal{CG}_1$ which connects a_{j_i} to $c_{j+1} = a_{j_{i+1}}$.

The final case to consider is that $\varphi(c_{i+1}), \varphi(d_{i+1})$ are both even. The above construction can be used to connect the curve $a_{j_i} = c_i +_\epsilon d_i$ in $\mathcal{CG}_1$ to $a_u = c_i +'_\epsilon d_i$ where ϵ' is disjoint from c_{j+1}. Furthermore, we may assume that the arc ϵ', viewed as an arc in the four-holed torus T, connects C_1 to D_1.

But then there exists an arc δ connecting C_2 to D_2 which is disjoint from ϵ'. The curve $a_u = c_{i+1} +_\delta d_i$ of a fulfills $\varphi(a_u) = 1$, furthermore there exists as path in $\mathcal{CG}_1$ connecting a_{j_i} to a_u. If s is the length of this path then define $j_{i+1} = j_i + s$. The resulting arc in $\mathcal{CG}$ has the properties we are looking for.

Induction by j yields the statement of the proposition. □

Remark 11.2. There is a simple trick due to Putman [Put08] which can be used to show that a graph G which admits a vertex transitive action of a finitely generated group Γ is connected: The only thing one needs to check is that for a fixed basepoint $x \in G$ and a fixed finite generating set $\{g_i \mid i\}$ of Γ, the vertices $g_i x$ can be connected to x by an edge. We can not use this trick in an obvious way here as we neither know a nice generating set for $\mathrm{Mod}(S)[\varphi]$ (which is a finite index subgroup of the finitely presented group $\mathrm{Mod}(S)$ and hence is finitely generated), neither do we know a priori whether the action of $\mathrm{Mod}(S)$ on $\mathcal{CG}_1$ is vertex transitive.

11.2. **Degeneration and induction.** Consider now a curve system $\mathcal{C}$ of type E_6 on a surface of genus three. It is admissible for the non-hyperelliptic component $\mathcal{Q}$ of the minimal stratum. By the main result of Looijenga and Mondello [LM14], we have $P\pi_1(\mathcal{Q}) = \Gamma(\mathcal{C})$. Furthermore, the following holds true.

Consider a curve c which defines a leaf of the curve diagram of $\mathcal{C}$ so that the curve diagram of $\mathcal{C}' = \mathcal{C} - c$ is the Dynkin diagram D_5. There exists a simple closed non-separating curve $d \notin \mathcal{C}$ which intersects c in a single point and does not intersect any other curve from $\mathcal{C}$. This curve satisfies $\varphi(d) = \pm 1$ which can be seen as follows.

In Section 2 we viewed a $\mathbb{Z}/2\mathbb{Z}$-spin structure on S as a quadratic form on $H_1(S, \mathbb{Z}/2\mathbb{Z})$. The $\mathbb{Z}/4\mathbb{Z}$-spin structure φ on S defines a $\mathbb{Z}/2\mathbb{Z}$-spin structure on S by reduction of coefficients. The corresponding quadratic form ψ is defined by

$$\psi([c]) = (\varphi(c) + 1) \bmod 2$$

where $[c]$ is the mod 2 homology class defined by the oriented curve $c \in \mathcal{S}$. We refer to end of Section 3 of [Sa17] for a comprehensive discussion of this fact. It is immediate from the discussion in Section 3 that $\psi(d) = 0$ and hence $\varphi(d) = \pm 1$.

The subgroup of $\mathrm{Mod}(S_{3,1})$ generated by the Dehn twists about the curves from $\mathcal{C}'$ preserves the subsurface $S - d$. This is a surface of genus two with two boundary components. If we replace the boundary components by cusps then $\mathcal{C}'$ defines a curve system on a surface of genus two with two marked points. For one of these marked points, say the marked point x, two of the curves from the system $\mathcal{C}'$ become homotopic if we remove x.

Denote as before by $\mathrm{Mod}(S)[\varphi]$ the stabilizer of a spin structure φ in the mapping class group of S. Recall the definition of the projection $\Pi : \mathrm{Mod}(S_{3,1}) \to \mathrm{Mod}(S) = \mathrm{Mod}(S_{g,0})$. We use these observations to show

Proposition 11.3. *Let $\mathcal{C}$ be a curve system of type E_6 on a surface of genus 3 and let φ be the $\mathbb{Z}/4\mathbb{Z}$-spin structure invariant under $\Gamma(\mathcal{C})$. Then $\Pi\Gamma(\mathcal{C}) = \mathrm{Mod}(S)[\varphi]$.*

Proof. We use the results of Perron and Vannier [PV96]. Let φ be the spin structure invariant under $\Gamma(\mathcal{C})$. Cutting S open along the simple closed curve d with $\varphi(d) = \pm 1$ which intersects the simple closed curve $c \in \mathcal{C}$ chosen as above in a single point and does not intersect any other curve from the curve system $\mathcal{C}$ yields a surface Σ of genus two with two boundary components. The curve system $\mathcal{C}'$ descends to a curve system on Σ whose curve diagram is the Dynkin diagram of type D_5.

One of the two boundary components of Σ, say the boundary componet v, has the following property. If we cap off v by attaching a disk along v, then two of the curves from $\mathcal{C}'$ become homotopic. Let Σ' be the resulting surface of genus 2 with connected boundary. The curve system $\mathcal{C}'$ descends to a curve system of type A_4 on Σ'.

Let $\Phi : \Sigma \to \Sigma'$ be the map obtained by identifying the boundary component v with a point. It induces a surjection Φ_* of $\Gamma(\mathcal{C}')$ onto the Artin braid group on five strands (which is just the Artin braid group $\mathcal{A}_4$ of type A_4). By Corollary 1 of [PV96], the kernel of the homomorphism Φ_* is the free group with four generators which is identified with the fundamental group of Σ' via the Birman exact sequence

$$0 \to \pi_1(\Sigma') \to \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(\Sigma') \to 0.$$

In other words, if we denote by $\hat{\Sigma}$ the surface obtained from Σ by replacing the boundary component v by a puncture, then this kernel is precisely the kernel of the natural homomorphism $\mathrm{Mod}(\hat{\Sigma}) \to \mathrm{Mod}(\Sigma')$. The spin structure φ on S descends to a $\mathbb{Z}/2\mathbb{Z}$-spin structure ψ on Σ'. This means the following.

Let Σ_0 be the closed surface of genus 2 without marked points or boundary, i.e. Σ_0 is the image of Σ' under the marked point forgetful map. Recall from Lemma 3.4 that the Artin braid group $\mathcal{A}_4$ in five strands, viewed as the subgroup of the mapping class group of Σ_0, preserves an odd spin structure. There are precisely 6 odd spin structures on Σ_0, and these spin structures are permuted by the mapping class group. Each of these spin structures corresponds to one of the six Weierstrass point on Σ. The Artin braid group $\mathcal{A}_4$ equals the stabilizer of such an odd spin structure, and it can be identified with the fundamental group of the stratum $\mathcal{H}(2)$ of abelian differentials with a single zero.

Using again the Birman exact sequence

$$0 \to \pi_1(\Sigma') \to \mathrm{Mod}(\hat{\Sigma}) \to \mathrm{Mod}(\Sigma') \to 0.$$

and the inclusion homomorphism, by Corollary 1 of [PV96] we obtain an exact sequence

$$0 \to \pi_1(\Sigma') \to \Gamma(\mathcal{C}') \to \mathrm{Mod}(\Sigma')[\varphi] \to 0.$$

Thus the group $\Gamma(\mathcal{C})'$ coincides with the fundamental group of the stratum $\mathcal{H}(2;1)$ of abelian differentials on a surface of genus two with a double zero and an additional regular marked point.

Recall that a $\mathbb{Z}/2\mathbb{Z}$-spin structure on Σ_0 corresponds to a quadratic form on $H_1(\Sigma_0, \mathbb{Z}/2\mathbb{Z})$. We conclude that

$$\Gamma(\mathcal{C}') = \mathrm{Mod}(\Sigma')[\varphi]/\mathbb{Z}_2$$

where the quotient is by the center, i.e by the hyperelliptic involution.

Namely, the full mapping class group of Σ_0 is the extension of the spherical braid group in 6 strands by the hyperelliptic involution. The Artin braid group in five strands is precisely the stabilizer in the spherical braid group of one of the marked points, and such a marked point is the image of a Weierstrass point.

The $\mathbb{Z}/4\mathbb{Z}$-spin structure φ preserved by $\Gamma(\mathcal{C})$ descends to a $\mathbb{Z}/2\mathbb{Z}$-spin structure on Σ, defined by a quadratic form ψ on $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$. The above discussion shows that the group $\Gamma(\mathcal{C}')$ is precisely the stabilizer of this spin structure in $\mathrm{Mod}(\Sigma')$. This group acts transitively on the set of curves c with $\varphi(c) = \pm 1$.

Now let $\rho \in \mathrm{Mod}(S)[\varphi]$ be arbitrary. Then $\rho(d)$ is a curve with $\varphi(\rho(d)) = \pm 1$. Connect d to $\rho(d)$ by a path in $\mathcal{CG}_1$. Such a path (d_i) exists by Proposition 11.1. Choose a curve e which is disjoint from $d = d_0, d_1$ and such that $\varphi(e) = 1$. Cut S open along e. We know that the stabilizer of e in $\Gamma(\mathcal{C})$ acts transitively on the curves disjoint from e on which the evaluation of the spin structure is odd. In particular, there is an element $\alpha_0 \in \Gamma(\mathcal{C})$ which moves d_0 to d_1.

Via the action of $\Gamma(\mathcal{C})$ by conjugation, the stabilizer of d_1 in $\Gamma(\mathcal{C})$ is conjugate to $\Gamma(\mathcal{C}')$. In particular, there is an element in this stabilizer which maps d_0 to d_2. Proceeding inductively, we conclude that there is an element $\zeta \in \Gamma(\mathcal{C})$ with $\zeta(d) = \rho(d)$, Then $\zeta^{-1}\rho$ stabiizes both d and the spin structure and hence it coincides with an element of $\Gamma(\mathcal{C}')$. This implies that indeed $\rho \in \Gamma(\mathcal{C})$. □

Corollary 11.4. *Let $\mathcal{Q}$ be the non-hyperelliptic component of the stratum $\mathcal{H}(4)$. Then the components of the preimage of $\mathcal{Q}$ in the Teichmüller space of abelian differentials correspond precisely to the $\mathbb{Z}/4\mathbb{Z}$-spin structures on a surface of genus 3 which project to an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on S. In particular, the stabilizers of two different components are distinct subgroups of* $\mathrm{Mod}(S)$.

Proof. The mapping class group acts on the components of the preimage of $\mathcal{Q}$ by permutation. The number of components of the preimage equals the index of the stabilizer of a component. This stabilizer equals the stabilizer of the $\mathbb{Z}/4\mathbb{Z}$-spin structure defined by a component. Mod 2 reduction maps this spin structure to an odd spin structure on S.

As the mapping class group acts transitively on the different spin structures, the corollary follows. □

12. Completing the proof

From now on we assume that $\mathcal{Q}$ is a non-hyperelliptic component of a stratum of abelian differentials on a surface of genus $g \geq 3$. We view $\mathcal{Q}$ as a complex orbifold. The idea is to proceed by induction on the genus of the surface, using specific degenerations of abelian differentials on S to differentials on surfaces with nodes as described in [EMZ13] and [BCGGM18].

We begin with some general observation on degenerations of abelian differentials to differentials on a surface with nodes.

12.1. **Degeneration to a boundary stratum II.** Consider a non-hyperelliptic component $\mathcal{Q}$ of a stratum of abelian differentials. Let $\mathcal{C}$ be an admissible curve system for $\mathcal{Q}$. Choose a curve $c \in \mathcal{C}$ which corresponds to a leaf of the curve diagram of $\mathcal{C}$ and assume that there exists a curve $d \in \mathcal{C}$ which intersects precisely two other components of $\mathcal{C}$, one of which is c. Write $q = q(\mathcal{C})$.

The differential q contains a cylinder D with core curve d. We may assume that D is a horizontal cylinder. Then D is obtained by glueing two squares R_1, R_2 along their vertical sides to an annulus. These squares correspond to the intersection points of d with c and the second curve $e \in \mathcal{C}$ which intersects d in a single point. The curve c is the core curve of the cylinder which is obtained by identifying the horizontal boundary arcs of the square R_1.

As the vertical sides of the squares R_1, R_2 are identified among each other, we can shrink the vertical sides of R_1, R_2 simultaneously to zero. Then the length of the curve c shrinks to zero as the height of R_1 equals the length of c. The length of the curve e does not shrink to zero since e intersects at least one other of the cylinders which make up $q(\mathcal{C})$.

Now c is an essential non-separating simple closed curve on S. Therefore as the length of c shrinks to zero when the heights of R_1, R_2 shrinks to zero, the differentials obtained by shrinking the height of the cylinder D to zero degenerate to an abelian differential q' on a surface S' of genus $g-1$ with a single non-separating node. The cylinder D degenerates to a saddle connection which joins the node to itself.

Cutting S' open along the node then defines a surface $\hat{S}$ of genus $g-1$ equipped with an abelian differential $\hat{q}$ with two marked points. The core curve d of the cylinder D defines a saddle connection connecting these two marked points. The differential $\hat{q}$ on $\hat{S}$ has one or two zeros, and these singular points are precisely the images of the node.

Let $\mathcal{R}$ be the component of the stratum of abelian differentials on $\hat{S}$ containing $\hat{q}$. It follows as in Section 10.2 that the construction which led to $\hat{q}$ can be reversed. In other words, there exists a neighborhood U of $\hat{q}$ in $\mathcal{Q}$ which is contained in the closure of $\mathcal{Q}$ for the Hausdorff topology. As the construction is compatible with the complex structure, we conclude that $\mathcal{R}$ defines a boundary divisor of $\mathcal{Q}$ in the incidence variety compactifiction of $\mathcal{Q}$ (see [BCGGM18]).

Consider now a component $\tilde{\mathcal{Q}}$ of the preimage of $\mathcal{Q}$ and a lift $\tilde{q}$ of the differential q as above. Denote again by c the marked curve as above. The degeneration described above determines a component $\tilde{\mathcal{R}}$ of the preimage of $\mathcal{R}$ in the Teichmüller space of abelian differentials on $\hat{S}$, where $\hat{S}$ is obtained from S by collapsing the simple closed curve c to a node and opening the node.

The following is well known. For its formulation, note that as $\tilde{\mathcal{R}}$ is a complex manifold in the boundary of $\tilde{\mathcal{Q}}$ of codimension one, a tubular neighborhood N of $\tilde{\mathcal{R}}$ in the closure of $\tilde{\mathcal{Q}}$ is biholomorpic to a disk bundle over $\tilde{\mathcal{R}}$. The fibre of the bundle with the zero point deleted is a punctured disk whose fundamental grouo generates the kernel of the map $\pi_1(N - \mathcal{R}) \to \pi_1(N)$. We have

Lemma 12.1. *The generator of the kernel of the map* $\pi_1(N-\mathcal{R})\to\pi_1(N)$ *is the Dehn twist about* c*.*

The following observation is equally well known; we include it here for easy reference.

Lemma 12.2. *Let* $\mathcal{Q}$ *be a component of a stratum of abelian differentials and let* $\tilde{\mathcal{Q}}_1\neq\tilde{\mathcal{Q}}_2$ *be components of the preimage of* $\mathcal{Q}$ *in the Teichmüller space of abelian differentials; then* $\Pi\mathrm{Stab}(\tilde{\mathcal{Q}}_1)\neq\Pi\mathrm{Stab}(\tilde{\mathcal{Q}}_2)$*.*

Proof. The group $\mathrm{Mod}(S)$ acts by inner automorphisms on itself, and this action induces an action on the left cosets of the subgroup $\Pi\mathrm{Stab}(\tilde{\mathcal{Q}})$. The space $\mathrm{Mod}S/\Pi\mathrm{Stab}(\tilde{\mathcal{Q}})$ of such left cosets can be identified with the set of components of the preimage of $\mathcal{Q}$, and the action of $\mathrm{Mod}(S)$ on the set of such components is naturally identified with the left action of $\mathrm{Mod}(S)$ on $\mathrm{Mod}(S)/\Pi\mathrm{Stab}(\tilde{\mathcal{Q}})$.

If $\mathrm{Stab}(\varphi\tilde{\mathcal{Q}})=\mathrm{Stab}(\tilde{\mathcal{Q}})$ then φ normalizes $\mathrm{Stab}(\tilde{\mathcal{Q}})$. In particular, using again naturality, it maps the generator of the fibre of a deleted tubular neighborhood of $\tilde{\mathcal{Q}}$ to the generator of the fibre of a deleted tubular neighborhood of $\varphi(\tilde{\mathcal{Q}})$. However, this fibre class is just the Dehn twist about a circle enclosing two marked points. But this is impossible.

Alternatively we observe that the following holds true. Let M be a simply connected complex manifold and let $N\subset M$ be a complex submanifold of codimension one. Then we have an exact sequence

$$\dots 0\to\pi_1(M,N)\to\pi_0(N)\to\mathbb{Z}\to 0.$$

Now each fibre of a tubular neighborhood of a generator gives rise an element in the relative homotopy group $\pi_1(M,N)$ and as the map is injective, we conclude that components of N correspond to generators of $\pi_1(M,N)$ by looping around a component different from the one which contains the basepoint.

Now if M is not simply connected then the above is valid for the universal covering $\tilde{M}$ of M. The deck group G acts on $\tilde{M}$. The claim now follows from the homotopy lifting property. ☐

12.2. **Abelian differentials with a single zero.** Consider a non-hyperelliptic component $\mathcal{Q}$ of a stratum of abelian differentials with a single zero on a surface of genus $g\geq 4$. Let $\mathcal{C}$ be an admissible curve system for $\mathcal{Q}$ of type U_{2g} or V_{2g}. Then we can find curves c,d as in Subsection 12.1 with the following property. Let $\mathcal{C}'=\mathcal{C}-\{c,d\}$; then $\mathcal{C}'$ fills a subsurface S' of S with a single boundary component, and it defines an admissible curve system on S' for a component of a stratum with a single zero.

Namely, a tubular neighborhood of $c\cup d$ is a one-holed torus. Shrinking the height of the cylinder D with core curve d to zero yields a surface with two marked points, joined by a saddle connection which is determined by the curve c. Shrinking this saddle connection to a point defines a surface Σ with an abelian differential z with a single zero. This differential is a realization of an admissible curve system which is a projection of $\mathcal{C}'$.

We use this to show

Proposition 12.3. *Let $\mathcal{Q}$ be a non-hyperelliptic component of a stratum with a single zero and let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of abelian differentials.*

1. *For every admissible curve system $\mathcal{C}$ for $\mathcal{Q}$ we have* $\mathrm{Stab}(\tilde{\mathcal{Q}}) = \Gamma(\mathcal{C})$.
2. *Stabilizers in* $\mathrm{Mod}(S)$ *of distinct components of the preimage of $\mathcal{Q}$ are distinct subgroup of* $\mathrm{Mod}(S)$.

Proof. We proceed by induction on the genus g of S. The case $g = 3$ is due to Looijenga and Mondello [LM14].

Now assume that the claim holds true for all non-hyperelliptic components of the minimal stratum for surfaces of genus $g-1 \geq 3$. Let $\mathcal{Q}$ be a such component of a stratum in genus g. Let $\mathcal{C}$ be an admissible curve system for $\mathcal{Q}$ of type U_{2g} or V_{2g}. Let c, d be the two curves from $\mathcal{C}$ with the properties discussed before the statement of the proposition. Let $\tilde{\mathcal{Q}}$ be the component of the preimage of $\mathcal{Q}$ in the Teichmüller space of abelian differentials which is stabilized by $\Gamma(\mathcal{C})$.

Let $\psi \in \mathrm{Stab}(\tilde{\mathcal{Q}})$. By Corollary 9.4, there exists some $\varphi \in \Gamma(\mathcal{C})$ so that $\varphi(c) = \psi(c), \varphi(d) = \psi(d)$. Put $\xi = \varphi^{-1}\psi$; then $\xi(c) = c, \xi(d) = d$.

Let q be a realization of $\mathcal{C}$ and let $z = \xi(q)$. Then ξ preserves the one-holed torus $T \subset S$ which is the tubular neighborhood of $c \cup d$. The above construction shows that shrinking the height of the cylinder D in q with core curve d to zero and collapsing the saddle connection which is the image of the core curve of D yields the realization q' of the projection of $\mathcal{C}' = \mathcal{C} - \{c, d\}$, and the same construction yields the projection of $\xi(\mathcal{C}')$.

The mapping class ξ acts on $\mathrm{Stab}(\tilde{\mathcal{Q}})$ by conjugation, and this action preserves the conjugacy class of the fundamental group of the one-holed torus T and hence it preserves its centralizer, which is the conjugacy class of the fundamental group of $S - T$.

Let q be an abelian differential which realizes the curve system $\mathcal{C}$. Then $\xi(q)$ is a differential which realizes $\xi(\mathcal{C})$. The differentials degenerate to differentials $\hat{q}, \hat{\xi}(q)$ on a surface of genus $g-1$ with a single zero. Furthermore, they are contained in a connected component of the preimage of the same stratum $\mathcal{R}$ of abelian differential on a surface of genus $g-1$. Two applications of Lemma 12.2 show that the differentials $\hat{q}, \hat{\xi}(q)$ are in fact contained in the same component of the preimage of $\mathcal{R}$.

Thus ξ induces an element of $\mathrm{Stab}(\tilde{\mathcal{R}})$. By induction hypothesis, we have $\mathrm{Stab}(\tilde{\mathcal{Q}}) = \Gamma(\mathcal{C}')$. But this shows $\xi \in \Gamma(\mathcal{C})$ and hence the same holds true for ψ. □

12.3. **Opening and closing zeros.** Let us consider an abelian differential q with a finite set Σ of marked points containing all the zeros. Assume that we fix two of these marked points x_1, x_2 which are connected by a saddle connection γ, i.e. by a geodesic segment not passing through another marked point. We always assume that at least one of the marked point is a zero. Furthermore, we assume that γ is the unique shortest saddle connection among all saddle connections in this direction. Let $\delta > 0$ be the length of γ.

There is a transformation of q called the *Schiffer variation* (see [McM13] for a detailed discussion of this classical construction) which contracts this saddle connection to a point without changing the absolute periods of q. This construction is as follows. Let us assume that x_1 is a zero of q and that the saddle connection is horizontal, with its orientation pointing away from the zero. This can be achieved by replacing q by $e^{i\theta}q$ for some $\theta \in [0, 2\pi)$. For a small number $\epsilon < \delta$ cut S open along the initial subsegments of length ϵ of all horizontal saddle connections whose orientation points away from x_1. The result is a surface with polygonal boundary, with sides of length ϵ, which can be reglued in such a way that the endpoints of the slits are all identified and yield a zero of the modified differential. The deformation is local and does not change the differential (and hence the underlying complex structure) outside a neighborhood of the saddle connection. It decreases the length of the horizontal saddle connections whose orientation points away from the zero. It can be continued until the endpoints of γ are identified.

The reverse of this construction consists in opening up a zero. This construction is explained in detail in Section 8 of [EMZ13]. It depends on the choice of a zero x of a differential q of order $k \geq 2$, the choice of a sufficiently small number $\epsilon > 0$ and the choice of a direction at the zero. Choose moreover a decomposition $k = k_1 + k_2$ for some $k_1 \geq k_2 \geq 1$ and a complex parameter $re^{i\theta}$ where $r > 0$ and where $\theta \in [0, 2\pi)$. The construction is as follows.

Choose a horizontal separatrix γ at x whose direction points away from x. Let $\epsilon > 0$ be sufficiently small that the ϵ-neighborhood of x for q consists of $2m+2$ half-disks of radius ϵ glued in cyclic order along segments of length ϵ in their horizontal boundary. For $r < \epsilon$ mark the point on γ of distance r to x and reglue the half-disks in such a way that the resulting differential has two zeros of order k_1, k_2, respectively, connected by a horizontal saddle connection of length r.

The construction depends on the length parameter $r \in (0, \epsilon)$ and the choice of the horizontal separatrix γ. Multiplication of q by $e^{i\theta}$ for some $\theta \in [0, 2\pi)$ can be thought of rotating the separatrix γ counter-clockwise around x. Thus for a fixed choice of a decomposition $k = k_1 + k_2$ for some $k_1 \geq k_2 \geq 1$, the resulting abelian differentials can be parametrized by a complex parameter $re^{i\theta}$ with $0 < r < \epsilon$.

Now let $\mathcal{R}$ be the component of the stratum containing the differential q with a zero of order $k \geq 2$ and let $\mathcal{Q}$ be the component containing the differentials with two zeros of order k_1, k_2 resulting from this construction. There exists an open neighborhood U of q in $\mathcal{R}$, an open set $V \subset \mathcal{Q}$ and a holomorphic surjection $\pi : V \to U$ whose fibre is a punctured disk. The projection π maps a point in V to the differential obtained by contracting the distinguished (shortest) saddle connection connecting the newborn zeros to a point. We refer to Section 8 of [EMZ13] for a detailed discussion of this construction.

Going ones around the fibre of π amounts to the following. Let $\beta \subset S$ be the boundary of the ϵ-disk about q. This circle lifts in an unambiguous way to a circle β' enclosing the two newborn zeros of a differential $z \in \mathcal{Q}$ obtained from q by splitting the zero x into the two zeros x_1, x_2. There exists a distinguished saddle connection for z connecting x_1 to x_2 which is entirely contained in the disk enclosed by β'.

On the other hand, the boundary of a small tubular neighborhood of any embedded arc in $S_{g,m}$ which connects the zeros x_1, x_2 is a circle enclosing the zeros. With this viewpoint, up to homotopy in the punctured surface $S_{g,m}$, the circle β' is determined by the saddle connection connecting x_1 to x_2. Going once around the core curve of the fibre then results in a Dehn twist about the circle β' enclosing the two marked points x_1, x_2.

Thus we can now apply the reasoning in the proof of Subsection 12.2 and complete the proof of Theorem 2.

Remark 12.4. In [LM14] the following is shown. The orbifold fundamental group of the stratum $\mathcal{H}(1,3)$ equals the finite type Artin group of type E_7. Its image under the homomorphism P is the subgroup of $\mathrm{Mod}(S_{3,1})$ which is generated by the Dehn twists about a Humphries system. In particular, $\mathrm{Mod}(S)$ is a quotient of this Artin group E_7. The Wajnryb presentation for the mapping class group shows that $\mathrm{Mod}(S_{g,1})$ is obtained from the Artin group E_7 by adding the so-called *3-chain relations* and the *lantern relations* [FM12]. By [LM14], these relations do not appear in the orbifold fundamental group of the component. In particular, the kernel of the homomorphism $P : \pi_1(\mathcal{H}(1,3)) \to \mathrm{Mod}(S_{3,1})$ is non-trivial.

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