EQUIDISTRIBUTION FOR QUADRATIC DIFFERENTIALS

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1. Introduction

Hermann Weyl was one of the most influential mathematicians of the first half of the twentieth century. He was born in 1885 in Elmshorn. In 1933 he emigrated to the United States, and he died in 1955 in Zürich. The majority of his many fundamental contributions to mathematics belong to the area of analysis in the broadest possible sense. However, one of his earliest and most celebrated results can be viewed as the origin of the study of number theory with tools from dynamical systems.

His theorem, published in 1916 in the "Mathematische Annalen" [W16], is as follows.

Weyl's theorem: *Let $y_0 \in (0,1)$ be irrational. Then the sequence $(u_i)_{i\geq 1}$ defined by $u_i = iy_0$ mod 1 is asymptotically equidistributed: For all $0 < a < b < 1$ we have*

$$\frac{|\{1 \leq i \leq n : a \leq u_i \leq b\}|}{n} \to b - a \quad (n \to \infty).$$

In this note we explain how the idea behind this theorem was used in the last quarter of the twentieth century to gain surprising insights into the interplay between number theory, geometry and dynamical systems.

2. Classical dynamical systems

In this section we discuss how Weyl's theorem can be reformulated in the language of dynamical systems, and we introduce some basic concepts which will be important in the later sections.

Let $S^1 = \{e^{it} \mid t \in [0, 2\pi)\} \subset \mathbb{C}$ be the standard unit circle in the complex plane. Then S^1 is an abelian group with multiplication $e^{it} \times e^{is} = e^{i(t+s)}$. In particular, every angle $\alpha \in (0, 2\pi)$ defines a cyclic group T^n_α of rotations of S^1 $(n \in \mathbb{Z})$ via

$$T^n_\alpha(e^{it}) = e^{it+n\alpha}.$$

Date: July 7, 2007.

Thus the rotation T_α with angle α generates a *dynamical system* with phase space S^1. Global (or asymptotic) properties of a dynamical system can be investigated with the help of *invariant measures.*

Definition 1. A *Radon measure* μ (i.e. a locally finite Borel measure) on a locally compact topological space X is *invariant* under a Borel map T if $\mu(T^{-1}(A)) = \mu(A)$ for every Borel set $A \subset X$.

If X is a compact topological space then the space $\mathcal{P}(X)$ of *Borel probability measures* on X can be equipped with the *weak**-topology. This weak*- topology is the weakest topology such that for every *continuous* function $f : X \to \mathbb{R}$ the function $\mu \in \mathcal{P}(X) \to \int f d\mu$ is continuous. In other words, a sequence $(\mu_i) \subset \mathcal{P}(X)$ converges to $\mu \in \mathcal{P}(X)$ if and only if for every open subset U of X we have $\liminf_{i\to\infty} \mu_i(U) \geq \mu(U)$. The space $\mathcal{P}(X)$ equipped with the weak*-topology is compact.

By compactness, every *continuous* transformation T of X admits an invariant Borel probability measure [Wa82]. Namely, for every point $x \in X$, any weak limit of the sequence of measures

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x}$$

is T-invariant where δ_z is the *Dirac δ-measure* at z, defined by $\delta_z(\{z\}) = 1$ and $\delta_z(X - \{z\}) = 0$.

For our circle rotations, there are now two cases.

Case 1: α is a *rational multiple* of 2π, i.e $\alpha = 2p\pi/q$ for relatively prime $p, q \in \mathbb{N}$.

In this case we have $T_\alpha^q(e^{it}) = e^{it+2p\pi} = e^{it}$ for all t which means the following.

$\boxed{\textit{Every point in } S^1 \textit{ is periodic for } T_\alpha\textit{, with period independent of the point.}}$

In particular, every point $y \in S^1$ is an *atom* of a T_α-invariant probability measure, namely the weighted counting measure on the orbit $\{T_\alpha^i y \mid 0 \leq i \leq q-1\}$ of y. This measure is given by the formula

$$\mu = \frac{1}{q}\sum_{i=0}^{q-1} \delta_{T_\alpha^i y}.$$

Case 2: α is an *irrational multiple* of 2π, i.e. $\alpha = 2\pi\rho$ for an irrational number $\rho \in (0, 1)$.

In this case, T_α does not have periodic points, and Weyl's theorem says precisely the following: For each $y \in S^1$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{T_\alpha^i y} \to \lambda$$

weakly in the space of probability measures on S^1 where λ is the normalized standard Lebesgue measure on S^1 defined by $\lambda\{e^{is} \mid 0 \leq \alpha < s < \beta \leq 2\pi\} = (\beta-\alpha)/2\pi$.

For a continuous map T of a compact space X, the space $\mathcal{P}(X)_T$ of T-invariant Borel probability measures on X is convex: If μ_1, μ_2 are two such measures and if $s \in [0,1]$ then $s\mu_1 + (1-s)\mu_2 \in \mathcal{P}(X)_T$ as well. In other words, $\mathcal{P}(X)_T$ is a compact and convex subset of a topological vector space on which the dual separates points. Hence $\mathcal{P}(X)_T$ is the convex hull of the set of its *extreme* points.

An extreme point $\mu \in \mathcal{P}(X)_T$ is an *ergodic* invariant measure: If $A \subset X$ is a T-invariant Borel set then $\mu(A) = 0$ or $\mu(X - A) = 0$. By the *Birkhoff ergodic theorem* [Wa82], every extreme point $\mu \in \mathcal{P}(X)_T$ is a weak limit of measures of the form $\frac{1}{q}\sum_{i=0}^{q-1} \delta_{T^i y}$ for a suitable choice of $y \in X$. Note that the definition of ergodicity also makes sense for Radon measures on locally compact spaces which are invariant under a continuous transformation.

Definition 2. A continuous transformation T of a compact space X is called *uniquely ergodic* if up to scale, T admits a *unique* invariant Borel probability measure.

An invariant Borel probability measure μ for a uniquely ergodic continuous transformation T on a compact space X is necessarily ergodic. Now Weyl's theorem can be rephrased as follows.

Irrational rotations of the circle are uniquely ergodic.

However, this also means the following.

If α is an irrational multiple of 2π then a measure which is invariant under T_α is invariant under the full *circle group of rotations.*

3. The modular group and hyperbolic geometry

About 1970, the significance of Weyl's theorem became apparent in a somewhat unexpected way and in a different context. This development began with the work of Hillel Furstenberg. Furstenberg was born in 1935 in Berlin and moved shortly later with his family to the United States. He now works at the Hebrew University in Jerusalem (Israel). Furstenberg was interested in lattices in semi-simple Lie groups G of non-compact type and their actions on homogeneous spaces associated to G. A large part of the structure theory for semi-simple Lie groups is due to Hermann Weyl, but it seems that he never attempted to draw a close connection between the structure of Lie groups, their actions on homogeneous spaces and his number theoretic result which we discussed in Section 1.

In this section we explain Furstenberg's work and its generalizations which are entirely in the spirit of Weyl's theorem.

Consider the *modular group*

$$SL(2,\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad - bc = 1\}$$

which acts as a group of linear transformations on $\mathbb{R}^2$ preserving the usual area form. This action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

There is an obvious $SL(2,\mathbb{Z})$-invariant subset of $\mathbb{R}^2$, namely the countable set $\mathbb{R}\mathbb{Q}^2 \subset \mathbb{R}^2$ of points whose coordinates are *dependent over* $\mathbb{Q}$ (which means that their quotient is rational). Since $SL(2,\mathbb{Z})$ preserves the integral lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, each $SL(2,\mathbb{Z})$-orbit of a point whose coordinates are dependent over $\mathbb{Q}$ is a *discrete* subset of $\mathbb{R}^2$. Hence this orbit supports an $SL(2,\mathbb{Z})$-invariant purely atomic ergodic Radon measure. For example, the measure

$$\mu = \sum_{y \in \mathbb{Z}^2} \delta_y$$

is an $SL(2,\mathbb{Z})$-invariant Radon measure. However, it is not ergodic since $\mathbb{Z}^2$ contains countably many orbits for the action of $SL(2,\mathbb{Z})$. Namely, the $SL(2,\mathbb{Z})$-orbit of the point $(1,0) \in \mathbb{R}^2$ consists precisely of all points (p,q) such that $p,q \in \mathbb{Z}$ are relatively prime. As a consequence, there is an uncountable family of $SL(2,\mathbb{Z})$-invariant Radon measures on $\mathbb{R}^2$. Each ergodic measure in this family is a sum of weighted Dirac masses on a single $SL(2,\mathbb{R})$-orbit in $\mathbb{R}\mathbb{Q}^2$.

In contrast, extending earlier work of Furstenberg [F72], Dani [D78] proved in 1978 the following *unique ergodicity result*.

Theorem 1. *(***Unique ergodicity for the standard linear action of $SL(2,\mathbb{Z})$***):* *An $SL(2,\mathbb{Z})$-invariant Radon measure on $\mathbb{R}^2$ which gives full mass to the set of points whose coordinates are independent over $\mathbb{Q}$ coincides with the Lebesgue measure up to scale.*

As a consequence, we have.

A Radon measure on $\mathbb{R}^2$ which is invariant under $SL(2,\mathbb{Z})$ and which gives full measure to points whose coordinates are independent over $\mathbb{Q}$ is invariant under the full *group $SL(2,\mathbb{R})$.*

The proof of this result does not use directly the fact that $SL(2,\mathbb{Z})$ acts on $\mathbb{R}^2$ by linear transformations. Instead, the group $SL(2,\mathbb{Z})$ is viewed as a *lattice* in the simple Lie group $SL(2,\mathbb{R})$. By this we mean that $SL(2,\mathbb{Z})$ is a discrete subgroup of $SL(2,\mathbb{R})$ with the following property. The group $SL(2,\mathbb{R})$ admits a natural Radon measure which is invariant under the action of $SL(2,\mathbb{R})$ on itself by right or left translation. This measure is given by a biinvariant volume form. By biinvariance, this volume form projects to a volume form on the quotient orbifold $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ of finite total volume.

The quotient group $PSL(2,\mathbb{R})$ under the center $\mathbb{Z}/2\mathbb{Z}$ of $SL(2,\mathbb{R})$ admits a natural simply transitive action on the unit tangent bundle $T^1\mathbf{H}^2$ of the *hyperbolic plane* $\mathbf{H}^2$ and hence this unit tangent bundle can be identified with $PSL(2,\mathbb{R})$. Namely, we have $\mathbf{H}^2 = \{z = x+iy \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ with the Riemannian metric

$$Q = \frac{dx^2 + dy^2}{y^2}$$

which is invariant under the action of $SL(2,\mathbb{R})$ by *linear fractional transformations*

$$z \to \frac{az+b}{cz+d} \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}).$$

The subgroup of $SL(2,\mathbb{R})$ acting trivially is just the center of $SL(2,\mathbb{R})$ and hence this action factors to an action of $PSL(2,\mathbb{R})$. The hyperbolic plane $\mathbf{H}^2$ admits a compactification by adding the circle $\partial\mathbf{H}^2 = \mathbb{R} \cup \infty$, and the action of $PSL(2,\mathbb{R})$ on $\mathbf{H}^2$ extends to a transitive action on this circle by homeomorphisms.

There are three characteristic one-parameter subgroups of $SL(2,\mathbb{R})$.

(1) The *diagonal subgroup*
$$A = \{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R}\}$$
(2) The *upper unipotent group*
$$N = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}\}$$
(3) The *lower unipotent group*
$$U = \{\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t \in \mathbb{R}\}$$

These groups project to one-parameter subgroups of $PSL(2,\mathbb{R})$ which we denote by the same symbols.

The right action of the diagonal subgroup A on $PSL(2,\mathbb{R})$ defines the *geodesic flow* on $T^1\mathbb{H}^2$. The right action of the group N of upper triangular matrices of trace two is the *horocycle flow* on $T^1\mathbf{H}^2$. The group $PSL(2,\mathbb{R})$ acts transitively from the left on the *homogeneous space* $PSL(2,\mathbb{R})/N$.

Recall that the linear action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ naturally induces an action of $PSL(2,\mathbb{R})$ on the punctured cone $\mathbb{R}^2 - \{0\}/\pm 1$. We have.

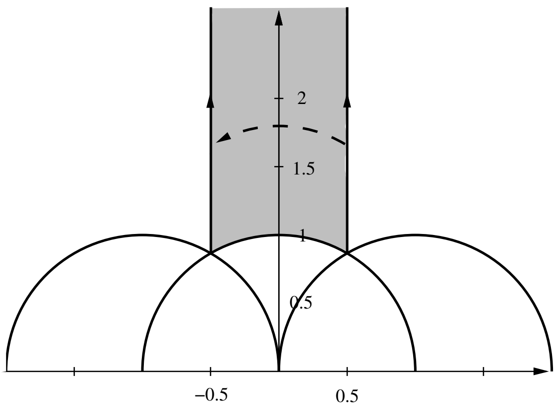
Lemma 3.1. *There is a homeomorphism $F : \mathbb{R}^2 - \{0\}/\pm 1 \to PSL(2,\mathbb{R})/N$ which commutes with the action of $PSL(2,\mathbb{R})$. This means that we have $B(Fz) = F(Bz)$ for all $z \in \mathbb{R}^2 - \{0\}/\pm 1$ and for all $B \in PSL(2,\mathbb{R})$.*

Proof. A homeomorphism as required in the lemma can easily be determined explicitly (see e.g. the paper [LP03]). However, its existence can also be derived as follows. The group $PSL(2,\mathbb{R})$ acts transitively from the left on $\mathbb{R}^2 - \{0\}/\pm 1$ (this is immediate from transitivity of the left linear action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2 - \{0\}$). Moreover, the stabilizer subgroup of the point $(1,0)/\pm 1$ for this action is precisely the group N. $\square$

As a consequence, $PSL(2,\mathbb{Z})$-invariant Radon measures on $\mathbf{R}^2 - \{0\}/\pm 1$ correspond *precisely* to Radon measures on $PSL(2,\mathbb{R})/N$ which are invariant under the left action of $PSL(2,\mathbb{Z})$ or, equivalently, to finite Borel measures on the *unit tangent bundle* $T^1(\mathrm{Mod}) = PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$ of the *modular surface*

Mod $= PSL(2,\mathbb{Z})\backslash\mathbf{H}^2$ which are invariant under the action of the *horocycle flow* h_t defined by the right action of the upper unipotent group N.

There is an obvious family of h_t-invariant Borel probability measures on the homogeneous space $T^1(\mathrm{Mod})$. Namely, a *fundamental domain* for the action of $PSL(2,\mathbb{Z})$ on the hyperbolic plane $\mathbf{H}^2$ by linear fractional transformations is the complement of the euclidean disc of radius one centered at the origin in the strip $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0, \frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}\}$. The stabilizer of ∞ in the group $PSL(2,\mathbb{R})$ is the solvable subgroup G of all upper triangular matrices generated by A and N. This stabilizer is preserved by the action of N by right translation. The orbits of N in S project to the lines Im = const in $\mathbf{H}^2$ and hence they project to *closed* orbits of the horocycle flow on $T^1(\mathrm{Mod})$. In particular, for every such orbit there is a unique h_t-invariant Borel probability measure supported on this orbit. Figure 1 shows a periodic orbit of the horocycle flow about the cusp in the standard fundamental domain of the action of the group $PSL(2,\mathbb{Z})$ on $\mathbf{H}^2$.



To describe the corresponding $PSL(2,\mathbb{Z})$-invariant Radon measure on the cone $\mathbb{R}^2 - \{0\}/\pm 1$, observe that the left action of $PSL(2,\mathbb{R})$ on the cone $\mathbb{R}^2 - \{0\}/\pm 1$ projects to the action of $PSL(2,\mathbb{R})$ on the *real projective line* $\mathbb{R}P^1 \sim S^1$ by projective transformations. This action is transitive, and the stabilizer of the real line $[1,0]$ spanned by the point $(1,0) \in \mathbb{R}^2$ equals the subgroup G of $PSL(2,\mathbb{R})$. Thus the action of $PSL(2,\mathbb{R})$ on $\partial\mathbf{H}^2$ is just the action of $PSL(2,\mathbb{R})$ on $\mathbb{R}P^1$. In particular, the $PSL(2,\mathbb{Z})$-orbit of $[1,0]$ which consists of all points in $\mathbb{R}P^1$ spanned by vectors with integer coordinates (see the above discussion) naturally coincides with the $PSL(2,\mathbb{Z})$-orbit of the point $\infty \in \partial\mathbf{H}^2$. As a consequence, the h_t-invariant Borel probability measures on $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$ supported on the above described closed orbits of the horocycle flow correspond *precisely* to the invariant Radon measures on $\mathbb{R}^2 - \{0\}/\pm 1$ supported on points whose coordinates are dependent over $\mathbb{Q}$. Thus we have.

Borel probability measures supported on closed orbits of h_t on $T^1(\mathrm{Mod})$ correspond to $SL(2,\mathbb{Z})$-invariant Radon measures on $\mathbb{R}^2$ supported on orbits of points whose coordinates are dependent over $\mathbb{Q}$.

On the other hand, the Lebesgue Haar measure is invariant under the action of the whole group $PSL(2,\mathbb{R})$, and it is uniquely determined by this property up to scale. Thus our Theorem 1 is an immediate consequence of the following result of Dani [D78].

Proposition 3.2. *Any h_t-invariant probability measure on $T^1(\mathrm{Mod})$ either is supported on a closed orbit for the horocycle flow or it is invariant under the whole group $PSL(2,\mathbb{R})$.*

In the early nineties, Ratner proved a far-reaching generalization of this result. We refer to the book [BM00] for an introduction to the subject and to [WM05] for a more detailed treatment of Ratner's celebrated work.

4. Uniquely ergodic unipotent flows on some homogeneous spaces of infinite volume

The results explained in Section 3 and their generalizations, in particular the work of Ratner, have many applications. However, they are only applicable in an algebraic setting and to invariant probability measures. The simplest extension of the questions discussed in Section 3 which can not be answered with these methods can be formulated as follows.

Let S be a closed oriented surface of genus $g \geq 2$. Choose a Riemannian metric g on S of constant sectional curvature -1. Then there is a discrete subgroup Γ of $PSL(2,\mathbb{R})$ such that our hyperbolic surface is just $\Gamma\backslash\mathbf{H}^2$, with unit tangent bundle $T^1S = \Gamma\backslash PSL(2,\mathbb{R})$ as before. In particular, the horocycle flow h_t is defined on T^1S. For some $d \leq 2g$ choose a normal subgroup Λ of Γ with factor group Γ/Λ isomorphic to $\mathbb{Z}^d$. An example of such a group is the commutator subgroup of Γ. Consider the regular $\mathbb{Z}^d$-cover $\hat{S}$ of S with fundamental group Λ. Then the horocycle flow h_t on the unit tangent bundle $T^1\hat{S}$ of $\hat{S}$ is defined. By the work of Ratner, h_t-invariant Borel probability measures on $T^1\hat{S}$ can be classified. Namely, in the situation at hand, either they are supported on closed orbits of the horocycle flow or they are invariant under the full group $PSL(2,\mathbb{R})$ on $T^1\hat{S}$. In other words, since the volume of $T^1\hat{S}$ is infinite, such invariant Borel probability measures do not exist. However, the Lebesgue measure (i.e. the measured induced by a Haar measure on $PSL(2,\mathbb{R})$) is a $PSL(2,\mathbb{R})$-invariant *Radon* measure on $T^1\hat{S}$. A natural problem is now to classify all invariant Radon measures for the horocycle flow on $T^1\hat{S}$.

Babillot and Ledrappier [BL98] constructed for every homomorphism $\varphi : \mathbb{Z}^d \to \mathbb{R}$ a Radon measures $\hat{\lambda}^\varphi$ on $T^1\hat{S}$ which is both invariant under the horocycle flow on $T^1\hat{S}$ and under the geodesic flow. The trivial homomorphism corresponds to the Lebesgue measure. Each of these measures is the lift of a Borel probability measure λ^φ on T^1S. If the homomorphism is nontrivial, then the measure λ^φ on T^1S is *not* invariant under the horocycle flow on T^1S. For $\varphi \neq \psi$ the measures $\hat{\lambda}^\varphi, \hat{\lambda}^\psi$ are singular. We call these measures *Babillot-Ledrappier measures.*

The Babillot-Ledrappier measures are all *absolutely continuous* with respect to the *stable foliation*, i.e. the foliation of $T^1\hat{S} = \Lambda\backslash PSL(2,\mathbb{R})$ into the orbits of the right action of the solvable subgroup G generated by the groups A, N. More precisely, the following holds true.

For every point $\xi \in S^1 = \partial \mathbf{H}^2$ there is a *Busemann function* $\theta_\xi : \mathbf{H}^2 \to \mathbb{R}$ at ξ. Such a Busemann function is a one-Lipschitz function for the hyperbolic metric on $\mathbf{H}^2$ which is determined uniquely by ξ up to an additive constant. The function $\theta_\infty(z) = \log \mathrm{Im}(z)$ is a Busemann function at the point ∞. The Busemann functions are invariant under the action of $PSL(2,\mathbb{R})$ on $\partial \mathbf{H}^2 \times \mathbf{H}^2$ and therefore the images under the action of $PSL(2,\mathbb{R})$ of the function θ_∞ determines all Busemann functions on $\mathbf{H}^2$.

For a discrete subgroup Λ of $PSL(2,\mathbb{R})$ and a number $\alpha \geq 0$ define a *conformal density of dimension* $\alpha \geq 0$ for Λ to be an assignment which associates to every $x \in \mathbf{H}^2$ a finite measure μ^x on $\partial \mathbf{H}^2$ with the following properties.

(1) The measures μ^x ($x \in \mathbf{H}^2$) are equivariant under the action of Λ on $\mathbf{H}^2 \times \partial \mathbf{H}^2$.
(2) For $x, y \in \mathbf{H}^2$ the measures μ^x, μ^y are absolutely continuous, with Radon Nikodym derivative $\frac{d\mu^y}{d\mu^x}(\xi) = e^{\alpha(\theta_\xi(y) - \theta_\xi(x))}$ where θ_ξ is a Busemann function at ξ.

The Babillot-Ledrappier measures on $T^1\hat{S}$ are related to conformal densities for the fundamental group Λ of $\hat{S}$ as follows.

Recall that there is a Λ-equivariant canonical projection $PSL(2,\mathbb{R}) \to \partial \mathbf{H}^2 = PSL(2,\mathbb{R})/G$. Let $\tilde{\mu}$ be the Λ-invariant measure on $PSL(2,\mathbb{R})$ which is the lift of a Babillot-Ledrappier measure $\hat{\lambda}^\varphi$ on $T^1\hat{S} = \Lambda \backslash PSL(2,\mathbb{R})$. Then for every relatively compact open subset U of $PSL(2,\mathbb{R})$ the push-forward $\pi_*(\mu|U)$ is contained in the measure class of a conformal density for Λ. This conformal density $\{\mu^x\}$ has the additional property that

$$\mu^x \circ g = e^{\varphi(g)} \circ \mu^x \forall g \in \mathbb{Z}^d.$$

This additional condition determines the conformal density uniquely up to scale. The measure $\hat{\lambda}^\varphi$ in turn is uniquely determined by the conformal density up to scale.

A Radon measure μ on $T^1\hat{S}$ is *quasi-invariant* under the geodesic flow Φ^t on $T^1\hat{S}$ if for every $t \in \mathbb{R}$ the push-forward measure $\Phi^t_*\mu$ is absolutely continuous with respect to μ, i.e. the measures μ and $\Phi^t_*\mu$ have the same set of measure zero. The following result is due to Babillot [B04] and Aaronson, Sarig, Solomyak [ASS02].

Proposition 4.1. *Every h_t-invariant Radon measure on $T^1\hat{S}$ which is quasi-invariant under the geodesic flow is a Babillot-Ledrappier measure.*

Now let λ by any h_t-invariant ergodic Radon measure on $T^1\hat{S}$. Let Φ^t be the geodesic flow on $T^1\hat{S}$ and let $H(\lambda) \subset \mathbb{R}$ be the set of all $t \in \mathbb{R}$ such that the measure $\Phi^t\lambda$ is contained in the measure class of λ. Then $H(\lambda)$ is a *closed* subgroup of $\mathbb{R}$ and hence if $H(\lambda) \neq \mathbb{R}$ then either $H(\lambda)$ is infinite cyclic or trivial. The measure λ is quasi-invariant under the flow Φ^t if and only if we have $H(\lambda) = \mathbb{R}$. Proposition 4.1

then shows that every h_t-invariant Radon measure λ with $H(\lambda) = \mathbb{R}$ is a Babillot-Ledrappier measure.

To classify the h_t-invariant measures with the property that $H(\lambda)$ is either infinite cyclic or trivial, Sarig [S04] proved a general structure theorem for cocycles. He uses this result to show that there is no h_t-invariant Radon measure λ on $T^1\hat{S}$ with $H_\lambda \neq \mathbb{R}$. Thus he obtains.

Theorem 2. *Every h_t-invariant ergodic Radon measure on $T^1\hat{S}$ is a Babillot-Ledrappier measure.*

In fact, the analog of Theorem 2 holds true for the horocycle flow on any $\mathbb{Z}^d$-cover of a closed surface S of higher genus equipped with a Riemannian metric of negative Gauß curvature. In other words, this result does not require any algebraic setting.

5. Moduli space

The first book written by Hermann Weyl is the monograph "Die Idee der Riemannschen Fläche" which appeared in 1913. In this section, a *Riemann surface* will be a closed oriented surface of genus $g \geq 1$ which is equipped with a complex structure. We are going to connect the *moduli space of Riemann surfaces* to the ideas discussed in Section 3 and Section 4.

Define a *marked Riemann surface* to be a Riemann surface M together with a homeomorphism $S \to M$ from a fixed closed oriented surface S of genus $g \geq 1$.

Definition 3. (1) The *Teichmüller space* $\mathcal{T}(S)$ of S is the space of all *marked* Riemann surfaces which are homeomorphic to S up to biholomorphisms isotopic to the identity.

(2) The *mapping class group* $\mathcal{M}(S)$ is the group of all isotopy classes of orientation preserving homeomorphisms of S.

Every element of the mapping class group $\mathcal{M}(S)$ naturally induces a nontrivial automorphism of the fundamental group $\pi_1(S)$ of S. It is easy to see that this automorphism is not *inner*, i.e. it is not induced by a conjugation. Thus there is a natural homomorphism of $\mathcal{M}(S)$ into the group $\mathrm{Out}(\pi_1(S))$ of *outer* automorphisms of $\pi_1(S)$, i.e. the quotient of the group of all automorphisms of $\pi_1(S)$ by the normal subgroup of all inner automorphisms. By an old result of Nielsen, this map is in fact an isomorphism.

The mapping class group naturally acts on Teichmüller space by precomposition, i.e. by changing the markings. It is well known that $\mathcal{T}(S)$ can be equipped with a topology so that with respect to this topology, $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$ and that the mapping class group acts properly discontinuously on $\mathcal{T}(S)$ by homeomorphisms. There is also an $\mathcal{M}(S)$-invariant complex structure on $\mathcal{T}(S)$ which identifies $\mathcal{T}(S)$ with a bounded domain Ω in $\mathbb{C}^{3g-3}$. The mapping class group is then just the group of all biholomorphic automorphisms of Ω.

By *uniformization*, if $g \geq 2$ then the *moduli space* $\text{Mod}(S) = \mathcal{M}(S)\backslash\mathcal{T}(S)$ of S can be identified with the space of all hyperbolic Riemannian metrics on S up to orientation preserving isometries. In the case $g = 1$, it is the space of all euclidean metrics of area one up to orientation preserving isometries.

Example:

In the case $g = 1$ (i.e. in the case of the 2-torus $S = T^2$), the fundamental group $\pi_1(S)$ of S is the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. Then every automorphism of $\mathbb{Z}^2$ is induced by a linear isomorphism of $\mathbb{R}^2$ preserving the lattice $\mathbb{Z}^2$ and hence the mapping class group $\mathcal{M}(S)$ is just the group $SL(2,\mathbb{Z})$. Here the center $\mathbb{Z}/2\mathbb{Z}$ of $SL(2,\mathbb{Z})$ corresponds to the hyperelliptic involution which acts trivially on $\mathcal{T}(T^2)$. More precisely, we have natural identifications as follows.

(1) $\mathcal{T}(T^2) = \mathbf{H}^2 = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$, the hyperbolic plane.
(2) $\mathcal{M}(T^2) = SL(2,\mathbb{Z})$ acting on $\mathbf{H}^2$ by linear fractional transformations.
(3) $\text{Mod}(T^2) = SL(2,\mathbb{Z})\backslash\mathbf{H}^2$, the modular surface.

A Riemann surface S is a one-dimensional complex manifold and hence it admits a natural holomorphic cotangent bundle $T'(S)$ whose fiber at a point x is the one-dimensional $\mathbb{C}$-vector space of all $\mathbb{C}$-linear maps $T_xS \to \mathbb{C}$.

Definition 4. For a Riemann surface S, a *holomorphic quadratic differential* q on S is a holomorphic section of the holomorphic line bundle $T'(S) \otimes T'(S)$.

In a holomorphic coordinate z on S, a holomorphic quadratic differential q can be written in the form $q(z) = f(z)dz^2$ with a local holomorphic function f. The bundle of all holomorphic quadratic differentials over all Riemann surfaces can be viewed as the *cotangent bundle* of Teichmüller space. It is a complex vector bundle of complex dimension $3g-3$. The mapping class group $\mathcal{M}(S)$ acts properly discontinuously on this bundle as a group of bundle automorphisms.

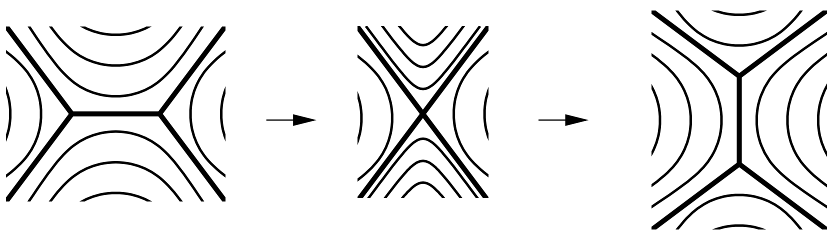
A quadratic differential q defines a *singular euclidean metric* on S as follows. Near a *regular point* z, i.e. away from the zeros of the differential, there is a holomorphic coordinate z on S such that in this coordinate the differential is just dz^2. Such a chart is unique up to translation and multiplication with -1 and hence the euclidean metric defined by this chart is uniquely determined by q. We call such a chart *isometric*. The *area* of a quadratic differential is the area of the singular euclidean metric it defines. The mapping class group preserves the sphere bundle $\tilde{\mathcal{Q}}(S)$ over $\mathcal{T}(S)$ of all holomorphic quadratic differentials of area one and hence this bundle projects to the *moduli space* $\mathcal{Q}(S) = \mathcal{M}(S)\backslash\tilde{\mathcal{Q}}(S)$ of such differentials.

The real line bundles $q > 0, q < 0$ on the complement of the (finitely many) singular points of q define *transverse singular measured foliations* q_h, q_v on S called the *horizontal and vertical measured foliations* of q. By definition, a measured foliation of S is a foliation F with finitely many singularities together with a *transverse measure* which associates to every smooth compact arc which meets the leaves of the foliation F transversely a length which is invariant under a homotopy of the arc moving each endpoint of the arc within a single leaf of the foliation.

There is a natural action of the group $SL(2,\mathbb{R})$ on the space $\tilde{\mathcal{Q}}(S)$ of area one holomorphic quadratic differentials which is given as follows. For each quadratic differential $q \in \tilde{\mathcal{Q}}(S)$ choose a family of isometric charts near the regular points. For $B \in SL(2,\mathbb{R})$ define Bq to be the quadratic differential whose isometric charts are the compositions of the isometric charts for q with B. This collection of charts then defines a new holomorphic quadratic differential on a (different) Riemann surface. The action of $SL(2,\mathbb{R})$ commutes with the action of the mapping class group and hence it descends to an action of $SL(2,\mathbb{R})$ on $\mathcal{Q}(S)$. The diagonal subgroup of $SL(2,\mathbb{R})$ then defines a flow on $\mathcal{Q}(S)$ called the *Teichmüller flow* Φ^t, and the upper unipotent group defines the *horocycle flow* h_t. In the case of the Teichmüller space of surfaces of genus 1, these flows are precisely the geodesic flow and the horocycle flow on the unit tangent bundle of the modular surface. By construction, the horocycle flow preserves the horizontal measured foliation of the quadratic differentials since it preserves the lines $q > 0$ in our charts.

For the moduli space of surfaces of higher genus, the classification of h_t-invariant Borel probability measures is up to date impossible. There are lots of examples of such measures. For example, *Veech surfaces* in moduli space are holomorphically embedded (singular) Riemann surfaces of finite type. They correspond to *closed* $SL(2,\mathbb{R})$-orbits in $\mathcal{Q}(S)$. Any h_t-invariant Borel probability measure on such an orbit then defines a h_t-invariant Borel probability measure on $\mathcal{Q}(S)$.

However, we can ask for the easier question of a classification of $\mathcal{M}(S)$-invariant Radon measures on the space of *equivalence classes* of measured foliations. For this call two measured foliations on S are *equivalent* if they can be transformed into each other by so-called *collapses* of two singular points along a connecting compact singular arc and Whitehead moves. Figure 2 shows a modification of a singular foliation with such a Whitehead move.



The following fundamental fact was discovered by Hubbard and Masur in 1979 [HM79], see also [Hu06].

Theorem 3. *(Hubbard-Masur) Let $\mathcal{H}$ be the natural map which associates to a quadratic differential the equivalence class of its horizontal measured foliation. Then for every Riemann surface M the restriction of $\mathcal{H}$ to the truncated vector space of all nontrivial quadratic differentials on M is a bijection.*

Example: If $S = T^2$ then the space of equivalence classes of measured foliations is $\mathbb{R}^2$.

By the theorem of Hubbard and Masur (in a slightly stronger version than the one we described above), the natural topology on the bundle $\tilde{\mathcal{Q}}(S)$ induces a metrizable topology on the set $\mathcal{MF}$ of equivalence classes of measured foliations on S (which by construction is a purely topologically defined space). The mapping class group $\mathcal{M}(S)$ acts on the space of equivalence classes of measured foliations by homeomorphisms. If $S = T^2$ then this action can naturally be identified with the linear action of $SL(2,\mathbb{R})$ on $\mathbf{R}^2$.

Definition 5. A measured foliation F on S *fills up* S if there is no simple closed curve on S whose F*-length* vanishes.

Here the F-length of a simple closed curve c is the infimum of the transverse lengths of closed curves transverse to the foliation which are freely homotopic to c.

The following classification result which was independently and at the same time shown in [H07a] and in [LM07] extends unique ergodicity for the action of $SL(2,\mathbb{Z})$ on the set of irrational points in $\mathbb{R}^2$ to the framework of Teichmüller theory and moduli spaces. For the formulation of this result, define a *measured multi-cylinder* for a measured foliation of S to be a disjoint union of embedded annuli in S which are foliated by closed leaves of the foliation.

Theorem 4. *Let μ be an $\mathcal{M}(S)$-invariant ergodic Radon measure on $\mathcal{MF}$.*

(1) *If μ gives full mass to measured foliations which fill up S then μ coincides with the Lebesgue measure up to scale.*
(2) *If μ is singular to the Lebesgue measure then there is a measured foliation F containing a nontrivial measured multi-cylinder c such that μ coincides with the translates of a* $\mathrm{Stab}(c)$*-invariant measure on the space of measured foliations on $S - c$.*

Remark: The proof for genus $g \geq 2$ is not valid in the case $g = 1$, i.e. we do not obtain a new proof of the result of Dani. The argument for the first part of the theorem uses the structural result of Sarig discussed in Section 4 in an essential way. The proof of the second part relies on a result of Minsky and Weiss [MW02] which is motivated by an analogous classical result of Dani for the horocycle flow on non-compact hyperbolic surfaces of finite volume.

Finally we discuss some applications. We begin with two classical results of Margulis [M69] and Dani.

Theorem 5. (1) (**Margulis**) *For $\ell > 0$ let $n(\ell)$ be the number of closed geodesics on $M = SL(2,\mathbb{Z})\backslash\mathbf{H}^2$ of length at most ℓ. Then*

$$\lim_{\ell\to\infty} \frac{1}{\ell} \log n(\ell) = 1.$$

(2) (**Dani**) *For a compact subset K of $SL(2,\mathbb{Z})\backslash\mathbf{H}^2$ and for $\ell > 0$ let $n_K(\ell)$ be the number of periodic orbits of length at most ℓ which are entirely contained in K. Then*

$$1 = \sup\{\liminf_{\ell\to\infty} \frac{1}{\ell} \log n_K(\ell) \mid K \subset SL(2,\mathbb{Z})\backslash\mathbf{H}^2 \text{compact }\}.$$

Also, for a hyperbolic surface M a collar lemma holds: There is a compact subset K in M compact such that *every* geodesic in M intersects K.

For closed Teichmüller geodesics, Eskin and Mirzakhani [EM08] obtained recently the analog of Margulis' result.

Theorem 6. *Let $n(\ell)$ be the number of closed Teichmüller geodesics of length at most ℓ. Then*

$$\lim_{\ell\to\infty} \frac{1}{\ell} \log n(\ell) = 6g - 6.$$

We also have [H07b].

Theorem 7. *For a compact subset K of $\mathrm{Mod}(S)$ and for $\ell > 0$ let $n_K(\ell)$ be the number of closed geodesics in $\mathrm{Mod}(S)$ which are entirely contained in K; then*

$$6g - 6 = \sup\{\liminf_{\ell\to\infty} \frac{1}{\ell} \log n_K(\ell) \mid K \subset \mathrm{Mod}(S) \textit{ compact }\}.$$

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