The virtual fibred conjecture V5D4

Lecture 1

Universität Bonn

October 8, 2012
Thurston’s Conjecture 1:
A closed hyperbolic 3-manifold $M$ virtually fibres over the circle.

Means: There is a finite cover $\hat{M} \to M$, a closed oriented surface $S$ of genus $g \geq 2$ and an exact sequence

$$0 \to \pi_1(S) \to \pi_1(\hat{M}) \to \mathbb{Z} \to 0.$$
**Thurston’s Conjecture 2:**
A closed hyperbolic 3-manifold is virtual Haken.

Means: There is a finite cover $\hat{M} \to M$, a closed oriented surface $S$ of genus $g \geq 2$ and an embedding $S \to \hat{M}$ which is injective on $\pi_1$. 
Daniel Wise ~ 2010:
If $M$ is virtual Haken then $M$ virtually fibres over $S^1$.

Ian Agol 2012:
A closed hyperbolic 3-manifold is virtual Haken.
Structure of the proof:

Step 1:

Jeremy Kahn, Vladmir Markovic 2009:
Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold. For every round circle $S^1 \subset S^2 = \partial \mathbb{H}^3$ and every $\epsilon > 0$ there exists a quasi-convex surface subgroup $\Gamma_0 < \Gamma$ whose boundary is contained in the $\epsilon$-neighborhood of $S^1$. 
**Explanation:**

**Fact:** The fundamental group $\Gamma$ of a closed manifold is finitely presented.

**Ex:** $S$ closed oriented surface of genus $g \geq 2$

$$\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g] \rangle$$

A finite symmetric choice of generators defines a word norm $\| \|$ on $\Gamma$

Any two such word norms $\|_1, \|_2$ are equivalent: $\exists \ c > 1:
\|_1/c \leq \|_2 \leq c\|_1.$
The *Cayley graph* of a finite symmetric generating set $\mathcal{G}$ of $\Gamma$ is the graph with
vertices = elements of $\Gamma$
edges: $g \sim h$ iff $h = sg$ for some $s \in \mathcal{G}$.

**Def:** $H < \Gamma$ is quasi-convex if $\exists c \geq 1$: any geodesic in $\Gamma$ between two points of $H$ is contained in the $c$-neighborhood of $H$

Note: In general, this depends on the choice of the generating set.
Problem:
\( H^3 \) is contractible
\[ \iff \text{if } H < \pi_1(M) \text{ is any surface group then there is an immersion } f : S \to M \text{ inducing an isomorphism } \pi_1(S) \to H \]

but: \( f(S) \) may have essential self-intersections which can not be removed.
\[ \iff H \text{ might not be separable in } \pi_1(M). \]

Def: The f.g. subgroup \( H \) is separable in \( \Gamma \) if
\[ H = \cap \{ G \mid H < G, \ G < \Gamma \text{ finite index } \}. \]

Fact: If the surface group \( H < \pi_1(M) \) is separable then \( \exists \) finite cover \( \hat{M} \to M \) containing an embedded surface.
Step 2:

Nicolas Bergeron, Daniel Wise 2010:
The fundamental group $\Gamma$ of $M$ admits a cubulation.

Means: $\Gamma$ acts properly and cocompactly on a $\text{CAT}(0)$-cube complex.
**Def:** Let $I = [-1, 1] \subset \mathbb{R}$. A *cube complex* $X$ is a $CW$-complex s.th. the attaching maps of each $k$-cell is defined on the boundary of $I^k \subset \mathbb{R}^k$.

Restrictions to each $(k - 1)$-face of $\partial I^k$ is an isometry onto $I^{k-1}$ postcomposed with some $(k - 1)$-cell of $X^{k-1}$.

A cube complex is *nonpositively curved* if each link complex is a flag complex.

A $\text{CAT}(0)$-cube complex is a simply connected nonpositively curved cube complex.

**EX:** Cubulations of surface groups
Step 3:

Ian Agol 2012:
Let $\Gamma$ be a word hyperbolic group which acts properly and cocompactly on a CAT(0)-cube complex $X$. Then the action of $\Gamma$ is virtually special.
**Def:** A *midcube* in $I^k$ is the set of points obtained by putting one coordinate 0.

Given a cube complex $Y$, form a new one whose cubes are the midcubes of $X$. Components of $Y$ are *hyperplanes*.

Each edge $a$ of $X$ is *dual* to a unique hyperplane $H(a)$.

Hyperplanes can be
- two-sided
- directly self-osculating
- two hyperplanes can *inter-osculate*
Frederic Haglund, Daniel Wise 2008:

**Def:** A cube complex is *special* if

1. Each hyperplane embeds
2. No hyperplane directly self-osculates
3. No two hyperplanes inter-osculate

The action of a group \( \Gamma \) on a CAT(0)-cube complex \( X \) is *virtually special* if there is a finite index subgroup \( \Gamma' \) s.th. \( X/\Gamma' \) is special.
Def: A f.g. group $\Gamma$ is word hyperbolic if $\exists \delta > 0$: Triangles in $\Gamma$ are $\delta$-thin.

This does not depend on the generating set (Gromov)

Frederic Haglund, Daniel Wise 2008:
Thm: If the word hyperbolic group $\Gamma$ admits a special action on a CAT(0)-cube complex then quasiconvex subgroups are separable.

Moreover, $\Gamma$ embeds into $SL(n, \mathbb{Z})$. 
Content of the course

1. The Rips complex of hyperbolic groups

2. Relatively hyperbolic groups

3. Heights of quasiconvex subgroups

4. Dehn filling- der Satz von Agol, Groves, Manning

5. Special CAT(0)-cube complexes after Haglund and Wise

6. Projection and reassembling- the work of Agol

7. Amalgamation over malnormal quasiconvex subgroups- the work of Haglund and Wise

8. Amalgamation over virtual malnormal subgroups
Homework 1

1. Show that finite groups are hyperbolic.

2. Show that the free group $F_n$ with $n \geq 1$ generators is hyperbolic.

3. Show that surface groups are hyperbolic.

4. Show that free products $A \ast B$ of hyperbolic groups $A, B$ are hyperbolic.

5. Show that $SL(2, \mathbb{Z})$ is hyperbolic.