University Bonn — WS 2017/18 Ursula Hamenstädt

Advanced Geometry I

Exercises 9 5. November 2017

9.1. Let $T = \mathbb{C}/\mathbb{Z}^2$ be a two-torus, equipped with the metric induced by the usual Hermitian scalar product on $\mathbb{C}$.

(a) Show that $H^{0,1}_{\bar\partial}(T) \neq 0$ by constructing an explicit harmonic one-form of type $(0,1)$.

(b) Compute the dimension of the vector space of harmonic one-forms on T of type $(0,1)$ (hint: Use the fact that the complex rank of the bundle $A^{0,1}(T,\mathbb{C})$ is one).

(c) Show that for every harmonic one-form α on T of type $(0,1)$, the complex conjugate $\bar\alpha$ is a harmonic one-form of type $(1,0)$.

9.2. Let P be the regular octagon in $\mathbb{C}$, centered at the origin. Let S be the Riemann surface obtained from P by identifying opposite sides.

(a) Show that a closed form α of type $(0,1)$ can be integrated over a smooth closed oriented curve on S. If the integral is non-trivial, then the Dolbeault cohomology class of α is non-trivial.

(b) Use part (a) to deduce that $H^{0,1}_{\bar\partial}(S) \neq 0$.

9.3. Let M be a compact complex manifold of dimension n with a Hermitian metric h (eg M is Ḱahler).

(a) Show that $\Delta_{\bar\partial} * = * \Delta_{\bar\partial}$.

(b) Deduce that $*$ induces an isomorphism $\mathcal{H}^{p,q}(M) \to \mathcal{H}^{n-p,n-q}(M)$.

9.4. Let M be a compact Ḱahler manifold. Show that the Ḱahler form is harmonic.