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Advanced Geometry I

Exercises 3 | 24. October 2017

3.1. Let E, M be complex manifold and let $\pi : E \to M$ be a continuous map. Show that the following are equivalent.

(a) $\pi : E \to M$ is a holomorphic vector bundle.

(b) The map π is holomorphic, and for each $x \in M$, the fibre $\pi^{-1}(x)$ has the structure of an r-dimensional complex vector space. Furthermore, there exists an open covering $X = \cup U_i$, and biholomorphic maps $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ commuting with the projections to U_i such that the induced map $\pi^{-1}(x) \equiv \mathbb{C}^r$ is $\mathbb{C}$-linear.

3.2. A holomorphic line bundle $L \to M$ is *trivial* if it admits a vector bundle isomorphism onto $M \times \mathbb{C}$.

(a) Let $L \to M$ be a holomorphic line bundle. Show that L^*, the bundle whose fibre at p is the vector space of $\mathbb{C}$-linear functional $L_p \to \mathbb{C}$ is a holomorphic line bundle.

(b) Show that the line bundle L over a compact complex manifold is trivial if and only if L and its dual L^* admit non-trivial holomorphic sections. (Hint: Use the non-trivial sections to construct a non-trivial section of $\mathcal{O} = L \otimes L^*$.)

3.3. Let $f : M \to N$ be a holomorphic map between complex manifolds and let $E \to N$ be a holomorphic vector bundle, with Hermitean metric and Chern connection ∇. Equip the pull-back f^*E with the pull-back Hermitean metric. Show that the curvature Θ_{f^*E}, Θ_E of the Chern connections for f^*E, E satisfy $\Theta_{f^*E} = f^*\Theta_E$.

3.4. (a) Let $E \to M, F \to M$ be holomorphic vector bundles. Show that the Whitney sum $E \oplus F$ and the tensor product $E \otimes F$ are holomorphic vector bundle.

(b) Consider the tautological bundle $\tau_1 \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$. Show that τ_1^* admits a holomorphic section which vanishes precisely on the image of $\mathbb{C}P^{n-1}$, defined by a standard coordinate embedding $\mathbb{C}^n \to \mathbb{C}^{n+1}$.

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