University Bonn — WS 2017/18 Ursula Hamenstädt

Advanced Geometry I

Exercises 12 8. January 2018

12.1. Use the Riemann Roch theorem to show that the degree of the cotangent bundle of a compact Riemann surface of genus g equals $2g-2$.

12.2. Let S be a compact Riemann surface of genus g and let $\sigma : S \to S$ be a biholomorphic *involution* of S, ie ι is biholomorphic, $\iota \neq \mathrm{Id}$ and $\iota^2 = \mathrm{Id}$. Show that ι has at most $2g+2$ fixed points. (Hint: Choose a meromorphic function f with a single pole of order $\leq g+1$ at some $z \in S$ not fixed by σ- show that such a function exists using the Riemann Roch theorem- and study the function $f - f \circ \iota$).

12.3. Let M be a compact Kähler manifold, with Kähler form ω.

(a) Show that $\Delta_d(\omega) = 0$.

(b) Show that a harmonic form on M of type $(p,0)$ is holomorphic.

(c) Let (N_i, ω_i) be compact Kähler manifolds ($i = 1,2$) and let $M = N_1 \times N_2$. Let $\Pi_i : M \to N_i$ be the natural projection. Show that $\Pi_1^*\omega_1 + \Pi_2^*\omega_2$ is a Kähler metric on M so that the following holds true. If α_i are harmonic forms on N_i then $\Pi_i^*\alpha_i$ is harmonic on M, and the same holds true for $\Pi_1^*\alpha_1 \wedge \Pi_2^*\alpha_2$.

12.4. Find an example of a compact Kähler manifold M with the following property. For every Kähler metric g on M, there exists a point $p \in M$ such that for no complex coordinates z_i near p, the matrix $g_{ij} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$ can be represented in the form $I_n + O(\sum_i |z|^3)$ (here I_n is the identity matrix). (Hint: Use problem 12.1.)