University Bonn — WS 2017/18	Ursula Hamenstädt
Advanced Geometry I	
Exercises 11	19. December 2017

11.1. Let $p(z) = \sum a_i z^i$ be a polynomial of degree $d \geq 1$.

(a) Show that p induces a branched self-covering $\mathbb{C}P^1 \to \mathbb{C}P^1$ and compute its degree.

(b) Show that for any compact Riemann surface M and any number $n > 0$, there exists a holomorphic map $M \to \mathbb{C}P^1$ of degree at least n.

11.2. Let M be a compact Riemann surface and let $\mathcal{H}^{1,0}$ be the vector space of holomorphic one-forms on M. Show that the map

$$B : \mathcal{H}^{1,0} \times H^{0,1}_{\bar{\partial}} \to \mathbb{C}$$

defined by $B(\alpha, [\theta]) = \int_M \alpha \wedge \theta$ is well defined and induces an isomorphism $\mathcal{H}^{1,0} \to H^{0,1}_{\bar{\partial}}$.

11.3. Let D be an effective divisor on a compact RIemann surface M and let $L(D)$ be the holomorphic line bundle defined by D. Show that as $k \to \infty$, the dimension of the vector space of holomorphic sections of $L(D)^k$ (k-fold tensor product) tends to infinity.

11.4. An *origami* is defined as follows. Let $k \geq 1$ and consider a collection of k unit squares in $\mathbb{C}$ (side lengths one) so that two sides are parellel to the real axis, two sides are parallel to the imaginary axis and if two squares intersect then they intersect along a common side. Each square has a bottom, top, right and left side. Pair the open right sides with the open left sides and the open top sides with the open bottom sides and glue the sides from a pair by a translation. Assume that the resulting space is connected.

(a) Show that this space M is a Riemann surface, equipped with a natural holomorphic one-form.

(b) Show that M admits a natural holomorphic map onto a two-torus.

(c) Show that for any $g \geq 1$, there exists a Riemann surface of genus g which can be obtained with this construction (computing the Euler characteristic is the easiest way to see this, if you don't know this then draw some pictures!). For a given number $g \geq 2$, can you roughly find out how many squares you need, i.e. what is a rough lower bound on the number?