University Bonn — WS 2017/18	Ursula Hamenstädt
Advanced Geometry I	
Exercises 1	10. October 2017

Part 1: for discussion on Oct. 16

1.1. Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be a continuous function. Show that the following are equivalent.

(a) f is holomorphic.

(b) f is continuously differentiable, and for every $x \in U$, the differential $df : T_x\mathbb{C} = \mathbb{C}^n \to T_{f(x)}\mathbb{C} = \mathbb{C}$ is linear over $\mathbb{C}$.

(c) f is smooth, and for every $x \in U$, the differential $df : T_x\mathbb{C} = \mathbb{C}^n \to T_{f(x)}\mathbb{C} = \mathbb{C}$ is linear over $\mathbb{C}$.

1.2. Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic. Show that for every $x \in U$, either the rank of df at x equals zero or two. Moreover, if f is not constant, then $\{x \in U \mid df(x) = 0\}$ is discrete.

1.3. Let $f : U \subset \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic. Show that if there exists a point $x \in U$ such that the rank of df at x is maximal, then there exists a neighborhood V of $f(x)$ and a holomorphic map $g : V \to U$ such that $f \circ g = \mathrm{Id}|V$. Furthermore, in this case the set $\{y \in U \mid df(y)$ has maximal rank $\}$ is open and dense.

1.4. **Implicit function theorem** Let $U \subset \mathbb{C}^m$ be an open set and let $f : U \to \mathbb{C}^n$ be holomorphic, where $m \geq n$. Suppose that $z_0 \in U$ is a point such that

$$\det(\frac{\partial}{\partial z_j}(z_0))_{1\leq i,j\leq n} \neq 0.$$

Then there exist open subsets $U_1 \subset \mathbb{C}^{m-n}, U_2 \subset \mathbb{C}^n$ and a holomorphic map $g : U_1 \to U_2$ such that $U_1 \times U_2 \subset U$ and $f(z) = f(z_0)$ if and only if $g(z_{n+1}, \ldots, z_m) = (z_1, \ldots, z_n)$.

Part 2: will be collected on Oct. 17

1.1. A map $f : U \to \mathbb{C}$ is *open* if the image of every open set is open.

(a) Show that a non-constant holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$ is open.

(b) Show that for any $n \geq 1$, any non-constant holomorphic function $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is open.

(c) Show that the map $(z, w) \in \mathbb{C}^2 \to (z, zw) \in \mathbb{C}^2$ is holomorphic, but it is not open.

(d) Let $U \subset \mathbb{C}^n$ be a *domain*, ie an open connected set. Let $\mathcal{O}(U)$ be the set of all holomorphic functions on U. Show that $\mathcal{O}(U)$ is a *ring*, ie for all $f, g \in \mathcal{O}(U)$ and all $c, d \in \mathbb{C}$ we have $cf + dg \in \mathcal{O}(U)$ and $fg \in \mathcal{O}(U)$. Determine the neutral element for addition and multiplication.

(e) Show that a function $f \in \mathcal{O}(U)$ is constant if there is a point $a \in U$ such that $f(z) = 0$ for z near a.

1.2. (a) For $a \in \mathbb{C}_* = \mathbb{C} - \{0\}$, show that the map $\phi : z \in \mathbb{C} \to z + a$ is biholomorphic.

(b) Show that the set $\mathrm{Aut}(\mathbb{C})$ of biholomorpic transformations of $\mathbb{C}$ forms a *group* with respect to composition.

(c) Let $\Gamma = \langle \phi \rangle$ be the subgroup of $\mathrm{Aut}(\mathbb{C})$ generated by ϕ. Show that $\mathbb{C}/\langle \phi \rangle$ is a Riemann surface.

(d) Show that all Riemann surfaces as in (c) above are biholomorphic.