University Bonn — SS 2018 Ursula Hamenstädt

Advanced Geometry II

Exercises 2 13. June 2018

2.1. Let S be a Riemann surface of genus $g \geq 1$ and let $0 \neq \omega$ be a holomorphic one-form on S.

(a) Fix a point $x \in S$. Show that for any smooth closed path $\alpha : [0,1] \to S$ with $\alpha(0) = \alpha(1) = x$, the integral $\int_\alpha \omega$ exists and is a number $\omega(\alpha) \in \mathbb{C}$.

(b) $\alpha \to \int_\alpha \omega$ defines a homomorphism $\pi_1(S, \alpha) \to \mathbb{C}$.

2.2. Let S be a Riemann surface of genus $g \geq 1$ and let $0 \neq \omega$ be a holomorphic one-form on S.

(a) Show that for a fixed $x \in S$, $\alpha(\pi_1(S, x)) \subset \mathbb{C}$ is an abelian group isomorphic to $\mathbb{Z}^2$.

(b) The abelian group $\alpha(\pi_1(S, x))$ does not depend on the choice of x.

2.3. Let S be a Riemann surface of genus $g = 1$ and let $0 \neq \omega$ be a holomorphic one-form on S. Show that the group $\alpha(\pi_1(S, x))$ is a discrete subgroup of $\mathbb{C}$.

2.4. Let S be a Riemann surface of genus $g \geq 2$.

(a) Construct an example of a holomorphic one-form ω on S so that $\omega(\pi_1(S, x)) \subset \mathbb{C}$ is not discrete.

(b) **Much harder:** Let now $g = 2$. Let $e_1, e_2 \in \mathbb{C}$ be in the image of $\omega(\pi_1(S, x)) = \Lambda$ and linearly independent over $\mathbb{R}$. Let K be the smallest subfield of $\mathbb{R}$ such that every element of Λ can be written in the form $ae_1 + be_2$ with $a, b \in K$. Can you construct an example so that $K = \mathbb{Z}[\sqrt{5}]$? Hint: A surface of genus 2 can be constructed from an octagon in the plane with parallel opposite sides by glueing opposite sides. There is an obvious holomorphic one-form on this surface, which is the projection of the one-form dz on $\mathbb{C}$. Try to explore that the periods $\omega(\pi_1(S, x))$ of ω vary as the polygon varies in the space of octagons with parallel opposite sides. This is a fun exercise, but I do not expect that you can solve it.