University Bonn — SS 2018	Ursula Hamenstädt
Advanced Geometry II	
Exercises 6	31. May 2018

6.1. Let $E \to M$ be a complex vector bundle over a smooth manifold M and let $\mathfrak{e}^k$ be the k-dimensional trivial bundle. Show that $c(E) = c(E \oplus \mathfrak{e}^k)$ (here and in the sequel, $c(E)$ is the total Chern class of E).

6.2. Let M_1, M_2 be two complex manifolds. Compute the total Chern class of $M_1 \times M_2$ from the total Chern classes $c(M_1), c(M_2)$ of M_1, M_2.

6.3. A Kähler manifold M is called *Kähler Einstein* if there exists a constant $c \in \mathbb{R}$ such that $\omega = c\rho$ (here ω, ρ are the Kähler form and the Ricci form, respectively). Construct an example of a compact complex manifold M of complex dimension 2 which admits two Kähler Einstein metrics with distinct Kähler classes.

6.4. Hopf surfaces. Let $S^3 \subset \mathbb{C}^2$ be the standard unit sphere. Then there exists a projection $\pi : S^3 \to \mathbb{C}P^1$ which is the restriction of the projection $\mathbb{C}^2 - \{0\} \to \mathbb{C}P^1$. Compute the first Chern class of the pullback bundle $\pi^* T\mathbb{C}P^1$. Now let $\alpha \in \mathbb{C}$, $|\alpha| > 1$ and let M be the quotient of $\mathbb{C}^2 - \{0\}$ by the group of biholomorphic automorphisms generated by $z \to \alpha z$ (a Hopf surface). Compute the total Chern class of M. (Hint: Assume first that $\alpha = 2$ and remember some properties of the Hopf surfaces).