University Bonn — SS 2018	Ursula Hamenstädt
Advanced Geometry II	
Exercises 5	9. May 2018

5.1. Let $\mathcal{X}$ be the vector space of smooth vector fields on a smooth manifold M. Let $T : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a bilinear map with the following property. Let $f : M \to \mathbb{R}$ be a smooth function so that $T(fX, Y) = T(X, fY) = fT(X, Y)$ for all $X, Y \in \mathcal{X}$. Show that T is a section of $TM^* \otimes TM^* \otimes TM$, i.e. a tensor field. Now let ∇ be a connection on TM. Show that

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

is a tensor field.

5.2. Construct a compact complex manifold M which admits two distinct K'ahler metrics g_1, g_2 with the following property. The Ricci form for g_1 equals the K'ahler form, but the Ricci form for g_2 is not cohomologous to any multiple of the K'ahler form.

5.3. Let M be a compact complex torus of complex dimension two. Assume that g is a K'ahler metric on M, with K'ahler form ω, so that the Ricci form ρ can be represented by $f\omega$ for a non-negative function f. Show that $f \equiv 0$.

5.4. Let S be a compact Riemann surface and let $D = \sum_{i=1}^{m} k_i x_i$ be a divisor on S (here $b_i \in \mathbb{Z}$ and $x_i \in S$). Let L_D be the corresponding dual line bundle. Now let $s \to x_i(s)$ be a smooth curve through x_i, and for each s consider the divisor $D_s = \sum_i k_i x_i(s)$. Show by an explicit construction that for small enough s, the dual line bundle L_{D_s} is smoothly isomorphic to L. Can you find a deformation (for $m = 1, k_1 = 1$) so that for some arbitrarily small $s > 0$, the line bundle L_{D_s} is not holomorphically equivalent to L_D?

5.5. (Harder) Let M be a compact K'ahler surface (= of complex dimension two) and let $x \in M$. Show that there is a sequence of g_i of K'ahler metrics on the blow-up $\hat{M}$ of M at x with the following properties. Let C be the exceptional divisor (which means that $C = \mathbb{C}P^1$ is the preimage of x under the blowdown map $\hat{M} \to M$). Then for each i, the restriction of the Ricci form ρ_i of g_i is a constant multiple of the restriction of the K'ahler form for a constant $c_i \in \mathbb{R}$ which tends to infinity as $i \to \infty$.