University Bonn — SS 2018 Ursula Hamenstädt

**Advanced Geometry II**

Exercises 4 2. May 2018

**4.1.** Let $M$ be a compact complex manifold and let $\mathcal{F} = \mathcal{O}$ or $\mathcal{F} = \mathcal{O}^*$ be the sheaf of holomorphic functions or non-vanishing holomorphic functions on $M$. Show that any biholomorphic map $F : M \to M$ induces an isomorphism $F^* : H^q(M, \mathcal{F}) \to H^q(M, \mathcal{F})$.

**4.2.** Let again $M$ be a compact complex manifold. Show that $H^1(M, \mathcal{O}^*)$ has the structure of a group, and this group is naturally isomorphic to the group of holomorphic line bundles on $M$, equipped with the tensor product.

**4.3.** With $M$ as in problem (1) and (2), show that the isomorphism $F^* : H^1(M, \mathcal{O}^*) \to H^1(M.\mathcal{O}^*)$ equals pull-back of line bundles.

**4.4.**

(a) Let $M$ be a compact Riemann surface and let $D = \sum_i x_i$ is a divisor on $M$ (with all multiplicities equal to one). Show that there is a holomorphic line bundle $K_M(D)$ on $M$ whose local sections are the holomorphic one-forms on any open set not containing any of the $x_i$, and in a neighborhood of each $x_i$ the meromorphic one-forms which can be written in a local holomorphic coordinate $z_i$ centered at $x_i$ as

$$\phi(z_i)\frac{dz_i}{z_i}$$

with a holomorphic function $\phi_i$.

(b) Let $K_M$ be the canonical bundle of $M$. Show that there is an exact sequence

$$0 \to K_M \to K_M(D) \xrightarrow{\text{Res}} \sum_i \mathbb{C}_{x_i} \to 0$$

where each sheaf $\mathbb{C}_{x_i}$ is a "skyscraper" sheaf supported at $x_i$, whose group of sections over any open set not containing $x_i$ is $\{0\}$, and whose group of sections over any open set containing $x_i$ is $\mathbb{C}$. Here the map $\text{Res}_i : K_M(D) \to \mathbb{C}_{x_i}$ maps a meromorphic form $\omega$ to its residue at $x_i$, defined by

$$\text{Res}_i(\omega) = \int_{\partial D_i} \frac{1}{2i\pi}\omega,$$

where $D_i$ is a disk centered at $x_i$ not containing any of the $x_j$'s, $i \neq j$.