University Bonn — SS 2018	Ursula Hamenstädt
Advanced Geometry II	
Exercises 3	25. April 2018

3.1. Consider the covering $\mathcal{U} = \{U_1, U_2\}$ of $\mathbb{C}$ consisting of the open sets $U_1 = \{z \mid |z| < 1\}$ and $U_2 = \{z \mid |z| > 1/2\}$. Show that $H^1(\mathcal{U}, \mathcal{F}) = 0$ for $\mathcal{F}$ the sheaf of locally constant $\mathbb{C}$-valued functions and the sheaf of smooth functions on $\mathbb{C}$.

3.2. (Harder) Consider the covering $\mathcal{U}$ of $\mathbb{C}$ from the first problem. Show that $H^1(\mathcal{U}, \mathcal{O}) = 0$.
Hint: Use a Laurent series expansion for holomorphic functions in the annulus. You can also try to use the $\overline{\partial}$-Poincare lemma which was discussed in Advanced Geometry I. Check this lemma and try to understand whether or not it can be used here.

3.3. Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}^*$ and let $|\mathcal{O}|$ be the topological realization of the space of all of its stalks, with projection $p : |\mathcal{O}| \to \mathbb{C}^*$. Show that for every $k \geq 1$ there exists a closed subset A of $|\mathcal{O}|$ so that the restriction of p to A is a covering of degree k.

3.4. Let X be a compact Riemann surface of genus $g \geq 2$. Show that for every holomorphic line bundle L over X, the sheaf $\mathcal{F}$ of all holomorphic sections of L has naturally the structure of a sheaf of vector spaces over the locally constant functions $\mathbb{C}$ on X (you may wish to think about what that means). Deduce that $H^0(X, \mathcal{F})$ is a $\mathbb{C}$-vector space. Show that for every $k \geq 0$ there exists a holomorphic vector bundle L so that the dimension of this vector space is at least k (can you achieve equality?)