University Bonn — SS 2018	Ursula Hamenstädt
Advanced Geometry II	
Exercises 10	27. June 2018

10.1. Show that on $\mathbb{C}P^2$ there exists a holomorphic vector bundle of rank two which is not the sum of two holomorphic line bundles.

10.2. Recall that the *Picard group* $\mathrm{Pic}(M)$ of a compact complex manifold M is the group of all holomorphic line bundles on M. Assume that the dimension of M is at least four. Let $Y \subset M$ be a smooth hypersurface with positive dual line bundle. Assume furthermore that $H^2(M,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$ are torsion free. Show that the restriction induces an isomorphism $\mathrm{Pic}(M) \to \mathrm{Pic}(Y)$. (Hint: Use the exponential sequence and the weak Lefschetz theorem).

10.3. A *holomorphic family of elliptic curves* is a complex manifold M and a holomorphic map $\pi : M \to N$ onto a complex manifold N such that for all $a \in N$, the *fibre* $\pi^{-1}(a)$ over a is a Riemann surface of genus one. A biholomorphic transformation of such a family is a biholomorphic map $\phi : M \to M$ which maps fibres to fibres.

(a) Show that smooth cubic curves in $\mathbb{C}P^2$ define a holomorphic family of elliptic curves.

(b) Let $A \in GL(3,\mathbb{C})$. Show that A acts as a biholomorphic transformation on this holomorphic family of elliptic curves.

(c) (Harder) Show that this family M of elliptic curves is not trivial, i.e. there does not exist a fibre preserving biholomorphic map $M \to T^2 \times N'$ where T^2 is a fixed elliptic curve.

10.4. Show that hypersurfaces in $\mathbb{C}P^n$ for $n \geq 3$ do not admit non-trivial holomorphic one-forms. (this is left from the previous sheet).