University Bonn — SS 2018 | Ursula Hamenstädt

Advanced Geometry II

Exercises 1 | 16. April 2018

1.1. Let $T = \mathbb{C}/\mathbb{Z}^2$ be a complex torus. Show that the map $z \to -z$ descends to a holomorphic involution ι of T (i.e. $\iota^2 = \text{Id}$). Is there a ι-invariant holomorphic one-form on T? Justify your answer.

1.2. Let $P \subset \mathbb{C}$ be a polygon consisting of three squares of side length one arranged in the shape of an L. Show that the quotient of P under the equivalence relation which identifies opposite sides by a translation has a natural structure a closed Riemann surface X. Let D be the divisor on X obtained as the image of the vertices of the polygon P and let L be the dual line bundle. Show that L^2 is the canonical bundle of X.

1.3. Let X be a closed Riemann surface of genus 2.

(a) Show that the degree of the canonical bundle of X equals two.

(b) (Much harder- for this you have to remember the Riemann Roch theorem or some other non-trivial result on Riemann surfaces). Show that there is a holomorphic line bundle of degree 2 on X which is not equivalent (as a holomorphic line bundle) to the canonical bundle.

1.4. Let M be any n-dimensional complex manifold.

(a) Show that the canonical bundle K of M is a holomorphic line bundle.

(b) Show that any Hermitian metric on K determines a cohomology class in $H^2_{dR}(M, \mathbb{R})$. Show that this class does not depend on the metric (this is harder).

(c) Construct a compact complex manifold admitting two K'ahler metrics which define cohomology classes in $H^2_{dR}(M, \mathbb{R})$ which span a two-dimensional subspace of $H^2_{dR}(M, \mathbb{R})$.