University Bonn — SS 2017 Ursula Hamenstädt

Advanced Geometry II

Exercises 5 16. May 2017

5.1. Show that the curvature of $\mathbb{H}^3$ equals -1 as follows.

(i) Use the action of the isometry group to verify that it suffices to show the following. Intersect the hyperboloid in $\mathbb{R}^4$ with the subspace $E = \{x_1 = 0\}$ (conventions as in the lecture). Then the length of a circle of radius R about $(0, 0, 0, 1)$ in $E \cap \mathbb{H}^3$ equals $2\pi \sinh(R)$.

(ii) Calculate the length of the circle in (i) using the explicit parametrization of a geodesic in $E \cap \mathbb{H}^3$ through $(0, 0, 0, 1)$.

5.2. Using the explicit form of the action of $PSL(2, \mathbb{C})$ on $S^2 = \mathbb{C} \cup \{\infty\}$, show that the subgroup $PSL(2, \mathbb{R})$ equals the stabilizer of the half-space $\{\Im > 0\}$ in $PSL(2, \mathbb{C})$.

5.3. Let $A \in PSL(2, \mathbb{R})$ acting on $\mathbb{H}^2 = \{\Im > 0\} \subset \mathbb{C} \cup \{\infty\}$. Show that the action of A on $\{\Im < 0\}$ is defined by $z \to \overline{A(\overline{z})}$.

5.4. Consider the light cone $\mathcal{L} \subset \mathbb{R}^4$. Show that there is a quadruple of pairwise distinct lines ℓ_i such that any permutation of these lines is the restriction of an isometry of $\mathbb{H}^3$ (not necessarily orientation preserving). Show that this is not true for every quadruple of pairwise distinct lines in $\mathcal{L}$.