University Bonn — SS 2017 Ursula Hamenstädt

Advanced Geometry II

Exercises 3 2. May 2017

3.1. Show that the area of any convex right angled hexagon in $\mathbb{H}^2$ equals π.

3.2. (i) The curvarture of a smooth curve α in the hyperbolic plane parametrized by arc length (for the hyerpbolic metric) is the norm of the covariant derivative of α'. Show that the (properly parametrized) lines $s \to ir + s$ $(r > 0)$ all have the same constant curvature and caculte this curvature.

(ii) Let γ be a geodesic in $\mathbb{H}^2$ and let $r > 0$. Show that a component of the set $\{z \mid d(z, \gamma) = r\}$ parametrized by arc length is an arc of constant curvature. This curvature does not depend on γ. Show that as $r \to \infty$, the curvature tends to the curvature of the lines in (i).

(iii) (Harder- use (i) and (ii)) Consider the unit half-circle c through i with endpoints on $\mathbb{R} \subset \partial\mathbb{H}^2$. Let $\gamma : [0, \infty) \to \mathbb{H}^2$ be one of the geodesic rays starting at i which are contained in c. Attach a geodesic arc a_t to $\gamma(t)$ which meets γ with angle $\pi/2$ and which is contained in the complement of unit disk. Choose the endpoint of the arc a_t in such a way that its tangent is horizontal (=real) at its endpoint. Let β_t be the vertical line through the endpoint of a_t. Show that the length of the segment $\{z \mid d(a(t), z) = t\}$ which is contained in the vertical strip bounded by β_t and the line $\{\Re = 0\}$ is bounded from above and below by a positive number not depending on t.

3.3. (i) Show that there exists a right angled hyperbolic octagon.

(ii) Show that there exists a regular hyperbolic octagon $\mathcal{O}$ with angles $\pi/4$. Here regular means that there exists a group of isometries of $\mathbb{H}^2$ of order 8 preserving $\mathcal{O}$. Furthermore, such a regular octagon is unique up to isometry.

3.4. Consider the octagon $\mathcal{O}$ from the previous exercise. Identify opposite sides with an isometry which reverses the orientation induced from an orientation of $\mathcal{O}$ (if the octagon were euclidean, such an identification is given by a translation of the euclidean plane). Show that the quotient of this identification admits a natural structure of a smooth closed surface equipped with a hyperbolic metric. Determine the genus of that surface.