University Bonn — SS 2017 | Ursula Hamenstädt

Advanced Geometry II

Exercises 1 | 19.April 2017

1.1. Denote by $d(z,w)$ the hyperbolic distance between $z,w\in\mathbb{H}^2$.

(i) Show
$$\sinh(\frac{1}{2}d(z,w))=\frac{|z-w|}{2(\Im z\Im w)^{1/2}}.$$
Hint: Try first to verify this formula for z,w with $\Re z=\Re w=0$.

(ii) Use (i) to show that
$$d(z,w)=\log\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}.$$

1.2. Show: An euclidean isometry of $\mathbb{R}^2=\mathbb{C}$ which maps the upper half-plane onto itself defines an isometry of $\mathbb{H}^2$. Does every isometry of $\mathbb{H}^2$ arise in this way?

1.3. For $z,w\in\mathbb{H}^2$, $h\in PSL(2,\mathbb{R})$, show that
$$|hz-hw|=|z-w||h'(z)h'(w)|^{1/2}.$$

1.4. (A bit harder) The *inversion* of $\mathbb{C}$ along a circle $\{x\mid |x-x_0|=r\}$ of radius $r>0$ and center x_0 is the map
$$\sigma(z)=x_0+r^2\frac{z-x_0}{|z-x_0|^2}.$$

(i) Show: An inversion maps euclidean circles and straight lines to circles and straight lines.

(ii) Compute the image of the upper half-plane under the inversion along the circle with center $-i$ and radius $r=\sqrt{2}$.

(iii) Deduce that for $z\in\mathbb{H}^2$ and $R>0$, the set $\{w\mid d(z,w)=R\}$ is an euclidean circle.