TYPICAL AND ATYPICAL PROPERTIES OF PERIODIC TEICHMÜLLER GEODESICS

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Abstract. Consider a component $\mathcal{Q}$ of a stratum in the moduli space of area one abelian differentials on a surface of genus g. Call a property $\mathcal{P}$ for periodic orbits of the Teichmüller flow on $\mathcal{Q}$ *typical* if the growth rate of orbits with property $\mathcal{P}$ is maximal. We show that the following properties are typical. The logarithms of the eigenvalues of the symplectic matrix defined by the orbit are arbitrarily close to the Lyapunov exponents of $\mathcal{Q}$, and its trace field is a totally real splitting field of degree g over $\mathbb{Q}$. If $g \geq 3$ then periodic orbits whose $SL(2,\mathbb{R})$-orbit closure equals $\mathcal{Q}$ are typical. We also show that $\mathcal{Q}$ contains only finitely many algebraically primitive Teichmüller curves, and only finitely many affine invariant submanifolds of rank $\ell \geq 2$.

1. Introduction

The *mapping class group* $\mathrm{Mod}(S)$ of a closed surface S of genus $g \geq 2$ acts by precomposition of marking on the *Teichmüller space* $\mathcal{T}(S)$ of marked complex structures on S. The action is properly discontinuous, with quotient the *moduli space* $\mathcal{M}_g$ of complex structures on S.

The goal of this article is to describe properties of this action which are invariant under conjugation and hold true for conjugacy classes of mapping classes which are typical in the following sense.

The *Hodge bundle* $\mathcal{H} \to \mathcal{M}_g$ over moduli space is the bundle whose fibre over a Riemann surface x equals the vector space of holomorphic one-forms on x. This is a holomorphic vector bundle of complex dimension g which decomposes into *strata* of differentials with zeros of given multiplicities. Its sphere subbundle is the moduli space of area one abelian differentials on S. There is a natural $SL(2,\mathbb{R})$-action on this sphere bundle preserving any connected component $\mathcal{Q}$ of a stratum. The action of the diagonal subgroup is called the *Teichmüller flow* Φ^t.

Let Γ be the set of all periodic orbits for Φ^t in $\mathcal{Q}$. The length of a periodic orbit $\gamma \in \Gamma$ is denoted by $\ell(\gamma)$. Let $k \geq 1$ be the number of zeros of the differentials in $\mathcal{Q}$

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and let $h=2g-1+k$. As an application of [EMR12] (see also [EM11]) we showed in [H13] that

$$\sharp\{\gamma\in\Gamma\mid \ell(\gamma)\leq R\}\frac{hR}{e^{hR}}\to 1 \quad (R\to\infty).$$

Call a subset $\mathcal{A}$ of Γ *typical* if

$$\sharp\{\gamma\in\mathcal{A}\mid \ell(\gamma)\leq R\}\frac{hR}{e^{hR}}\to 1 \quad (R\to\infty).$$

Thus a subset of Γ is typical if its growth rate is maximal. The intersection of two typical subsets of Γ is typical.

A periodic orbit $\gamma\in\Gamma$ for Φ^t determines the conjugacy class of a pseudo-Anosov mapping class. The mapping class group acts on the first integral cohomology group $H^1(S,\mathbb{Z})$ of S, and this action preserves the intersection form ι on $H^1(S,\mathbb{Z})$. This defines a natural surjective [FM12] homomorphism

$$\Psi:\mathrm{Mod}(S)\to Sp(2g,\mathbb{Z}).$$

Thus a periodic orbit $\gamma\in\Gamma$ determines the conjugacy class $[A(\gamma)]$ of a matrix $A(\gamma)\in Sp(2g,\mathbb{Z})$.

As a real vector bundle, the Hodge bundle $\mathcal{H}$ equals the flat vector bundle obtained as a quotient of $\mathcal{T}(S)\times H^1(S,\mathbb{R})$ by the standard left diagonal action of the mapping class group $\mathrm{Mod}(S)$, where the action on $H^1(S,\mathbb{R})$ is via the homomorphism Ψ. Thus $\mathcal{H}$, viewed as a real vector bundle, admits a natural symplectic structure as well as a flat connection preserving the symplectic structure which is called the *Gauss Manin connection*. There are other local descriptions of the Gauss Manin connection taking advantage of the integral structure of the first cohomology group of S, but these descriptions will not be important for our discussion.

If we denote by $\Pi:\mathcal{H}\to\mathcal{M}_g$ the canonical projection, then for each component $\mathcal{Q}$ of a stratum, the Gauss Manin connection pulls back to a flat connection on the pullback bundle $\Pi^*\mathcal{H}\to\mathcal{Q}$ which is called again the Gauss Manin connection. Parallel transport for this connection along flow lines of the Teichmüller flow Φ^t on $\mathcal{Q}$ defines a cocycle over Φ^t which is called the *Kontsevich Zorich cocycle* (see Section 4.4 of [Z99] as well as the introduction of [AV07]).

The $SL(2,\mathbb{R})$- action on $\mathcal{Q}$ preserves a Borel probability measure in the Lebesgue measure class, the so-called *Masur-Veech measure* [M82, V86]. The Kontsevich Zorich cocycle is integrable [AV07] for the action of the Teichmüller flow with respect to this measure and hence we can apply the Oseledets multiplicative ergodic theorem [O68] to obtain *Lyapunov exponents* of the cocycle with respect to the Masur-Veech measure on $\mathcal{Q}$. As the cocycle is symplectic, the Lyapunov spectrum is invariant under multiplication with -1. This Lyapunov spectrum is *simple* [AV07], and more precisely, the positive Lyapunov exponents listed in decreasing order form a sequence

$$1=\kappa_1>\kappa_2>\cdots>\kappa_g>0.$$

For $\gamma \in \Gamma$ let $\hat{\alpha}_i(\gamma)$ be the logarithm of the absolute value of the i-th eigenvalue of the matrix $A(\gamma)$, ordered in decreasing order, and write $\alpha_i(\gamma) = \hat{\alpha}_i(\gamma)/\ell(\gamma)$. As $A(\gamma)$ is symplectic, with real leading eigenvalue $e^{\ell(\gamma)}$, we have

$$1 = \alpha_1(\gamma) \geq \cdots \geq \alpha_g(\gamma) \geq 0 \geq -\alpha_g(\gamma) \geq \cdots \geq -\alpha_1(\gamma) = -1.$$

Since eigenvalues of matrices are invariant under conjugation, this does not depend on the choice of a representative in the class $[A(\gamma)]$, and for $-g \leq i \leq g$ we obtain in this way a function $\alpha_i : \Gamma \to [-1, 1]$.

The characteristic polynomial of a symplectic matrix $A \in Sp(2g, \mathbb{Z})$ is a reciprocal polynomial of degree $2g$ with integral coefficients. Its roots define a number field $\mathfrak{k}$ of degree at most $2g$ over $\mathbb{Q}$ which is a quadratic extension of the so-called *trace field* of A. The Galois group of $\mathfrak{k}$ is isomorphic to a subgroup of the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$ where $\mathfrak{S}_g$ is the symmetric group in g variables (see [VV02] for details). The field $\mathfrak{k}$ and the Galois group only depend on the conjugacy class of A.

For $\gamma \in \Gamma$ let $G(\gamma)$ be the Galois group of the number field defined by the conjugacy class $[A(\gamma)]$. We show

Theorem 1. (1) *For $\epsilon > 0$, the set $\{\gamma \in \Gamma \mid |\alpha_i(\gamma) - \kappa_i| < \epsilon\}$ $(1 \leq i \leq g)$ is typical.*
(2) *The set of all $\gamma \in \Gamma$ such that the trace field of $[A(\gamma)]$ is totally real, of degree g over $\mathbb{Q}$, and $G(\gamma) = (\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$ is typical.*

The proof of Theorem 1 uses a result on the Zariski closure of the image under the map Ψ of pseudo-Anosov mapping classes obtained from the first return map of the Teichmüller flow on $\mathcal{Q}$ to a small contractible flow box in $\mathcal{Q}$. We state the result separately as it is of independent interest.

For its precise formulation, recall that the *α-limit set* of an orbit $\{\Phi^t x\}$ of the Teichmüller flow is the set of points y for which there exists a sequence $t_i \to \infty$ so that $\Phi^{t_i} x \to y$. Similarly, the *ω-limit set* is defined to be the set of accumulation points of the backward orbit $t \to \Phi^{-t} x$. A point $x \in \mathcal{Q}$ is *birecurrent* if it is contained in its own α- and ω-limit set.

A component $\mathcal{Q}$ of a stratum is an orbifold, and the manifold points are open, dense and invariant under the action of Φ^t. If $U \subset \mathcal{Q}$ is any open contractible set consisting of manifold points, then the Gauss Manin connection induces a trivialization of the bundle $\Pi^*\mathcal{H} \to \mathcal{Q}$ over U. With respect to such a trivialization, we can investigate the subgroup of $Sp(2g, \mathbb{Z})$ which is generated by the return maps to U of the parallel transport for the Gauss Manin connection over flow lines of the Teichmüller flow. We prove

Theorem 2. *Let $\mathcal{Q}$ be a component of a stratum, let $q \in \mathcal{Q}$ be a birecurrent manifold point and let $U \subset \mathcal{Q}$ be a contractible neighborhood of x consisting of manifold points. Then the sub-semigroup of $Sp(2g, \mathbb{Z})$ which is generated by the return maps to U of the parallel transports along orbits of the Teichmüller flow is Zariski dense in $Sp(2g, \mathbb{R})$.*

There is an easy translation of Theorem 2 into the language of Rauzy induction and the so-called *Rauzy-Veech group* of $\mathcal{Q}$ which yields a solution to a conjecture of Zorich (Conjecture 5 of [Z99]). However, Rauzy induction plays no role in our approach, and we leave this translation to other authors.

For hyperelliptic strata, Avila, Matheus and Yoccoz [AMY16] showed that the Rauzy-Veech group is a subgroup of $Sp(2g,\mathbb{Z})$ of finite index. In [H16] we determine explicitly the Rauzy Veech group for all components of all strata. An independent result along this line is due to Gutierrez-Romo [GR17]. In this article we are interested in *affine invariant manifolds* which are just the orbit closures of the action of $SL(2,\mathbb{R})$ [EMM15], and we prove a version of Theorem 2 which is valid for affine invariant manifolds as well (Theorem 4.6). For components of strata, the result is less precise than the results in [H16].

Theorem 4.6 can be used to analyze *stretch factors* of pseudo-Anosov elements $\varphi \in \mathrm{Mod}(S)$. Here the stretch factor of φ is the unique number $\lambda > 1$ such that there exists a measured foliation ξ on S with $\varphi(\xi) = \lambda\xi$, and it only depends on the conjugacy class of φ. In the case that φ fixes a pair of oriented projective measured foliations, this stretch factor is just the leading eigenvalue for the action of φ on $H^1(S,\mathbb{R})$. The second part of Theorem 1 then states that for a typical pseudo Anosov conjugacy class preserving a pair of oriented projective measured foliations, the stretch factor is an algebraic integer of degree $2g$ over $\mathbb{Q}$.

The maximal degree over $\mathbb{Q}$ of the stretch factor for arbitrary pseudo-Anosov elements is known to be $6g-6$. This was claimed by Thurston in [Th88] and was verified in [St15]. The article [St15] shows more precisely that a number d is the algebraic degree of the stretch factor of a pseudo-Anosov mapping class if and only if either d is at most $3g-3$, or d is even and at most $6g-6$.

Counting conjugacy classes of all pseudo-Anosov elements amounts to counting periodic orbits for the Teichmüller flow acting on a component $\mathcal{D}$ of a stratum in the moduli space of area one quadratic differentials on S. If this component consists of quadratic differentials which are not squares of holomorphic one-forms and with $k \geq 1$ zeros, then the number of conjugacy classes of translation length (=logarithmic stretch factor) at most R is asymptotic to $e^{(2g-2+k)R}/(2g-2+k)R$ [H13]. Note that if $\mathcal{D}$ consists of quadratic differentials whose zeros are all of odd order then k is even. As before, we can speak of typical properties for such conjugacy classes. As an application of Theorem 4.6, we obtain

Corollary. *Let $\mathcal{D}$ a component of a stratum of quadratic differentials consisting of differentials with $k \geq 2$ zeros of odd order. Then the algebraic degree of the stretch factor of a pseudo-Anosov conjugacy class defined by a typical periodic orbit in $\mathcal{D}$ equals $2g-2+k$.*

Thus together with the second part of Theorem 1, we obtain for every even number $2g \leq d \leq 6g-6$ countably many conjugacy classes of pseudo-Anosov elements with stretch factor of algebraic degree d.

By the groundbreaking work of Eskin, Mirzakhani and Mohammadi [EMM15], affine invariant manifolds in a component $\mathcal{Q}$ of a stratum are precisely the closures of orbits for the $SL(2,\mathbb{R})$-action. Examples of non-trivial orbit closures are arithmetic *Teichmüller curves.* They arise from branched covers of the torus, and they are dense in any component of a stratum of abelian differentials. Other examples of orbit closures different from entire components of strata can be constructed using more general branched coverings. A more exotic orbit closure was recently discovered by McMullen, Mukamel and Wright [MMW16].

Period coordinates for a component $\mathcal{Q}$ of a stratum of abelian differentials, with set $\Sigma \subset S$ of zeros, are obtained by integration of a holomorphic one-form $q \in \mathcal{Q}$ over a basis of the relative homology group $H_1(S,\Sigma;\mathbb{Z})$. Thus a tangent vector of $\mathcal{Q}$ defines a point in $H_1(S,\Sigma;\mathbb{Z})^*$. The *rank* of an affine invariant manifold $\mathcal{C}$ is defined by

$$\mathrm{rk}(\mathcal{C}) = \frac{1}{2}\dim_{\mathbb{C}}(pT\mathcal{C})$$

where p is the projection of $H_1(S,\Sigma;\mathbb{R})^*$ into $H_1(S,\mathbb{R})^* = H^1(S,\mathbb{R})$ [W15]. The rank of a component of a stratum equals g, and *Teichmüller curves*, ie closed orbits of the $SL(2,\mathbb{R})$-action, are affine invariant manifolds of rank one and real dimension three.

We establish a finiteness result for affine invariant submanifolds of rank at least two which is independently due to Eskin, Filip and Wright [EFW17].

Theorem 3. *Let $g \geq 2$ and let $\mathcal{Q}$ be a component of a stratum in the moduli space of abelian differentials. For every $2 \leq \ell \leq g$, there are only finitely many proper affine invariant submanifolds in $\mathcal{Q}$ of rank ℓ.*

As an application, we obtain

Theorem 4. *Let $\mathcal{Q}$ be any component of a stratum in genus $g \geq 3$. Then the set of all $\gamma \in \Gamma$ whose $SL(2,\mathbb{R})$-orbit closure equals $\mathcal{Q}$ is typical.*

For $g = 2$, Theorem 4 is false in a very strong sense. Namely, McMullen [McM03a] showed that in this case, the orbit closure of any periodic orbit is an affine invariant manifold of rank one. If the trace field $\mathfrak{k}$ of the periodic orbit is quadratic, then $\mathfrak{k}$ defines a Hilbert modular surface in the moduli space of principally polarized abelian varieties which contains the image of the orbit closure under the *Torelli map.* Such a Hilbert modular surface is a quotient of $\mathbf{H}^2 \times \mathbf{H}^2$ by the lattice $SL(2,\mathfrak{o}_{\mathfrak{k}})$ where $\mathfrak{o}_{\mathfrak{k}}$ is an order in $\mathfrak{k}$. This insight is the starting point of a complete classification of orbit closures in genus 2 [Ca04, McM03b].

In higher genus, Apisa [Ap15] classified all orbit closures in hyperelliptic components of strata. For other components of strata, a classification of orbit closures is not available. However, there is substantial recent progress towards a geometric understanding of orbit closures. In particular, Mirzakhani and Wright [MW16] showed that all affine invariant manifolds of maximal rank either are components of strata or are contained in the hyperelliptic locus. We refer to the work [LNW15] of Lanneau, Nguyen and Wright for an excellent recent overview of what is known and for a structural result for rank one affine invariant manifolds.

To each Teichmüller curve is associated a *trace field* which is an algebraic number field of degree at most g over $\mathbb{Q}$. This trace field coincides with the trace field of every periodic orbit contained in the curve [KS00]. The Teichmüller curve is called *algebraically primitive* if the algebraic degree of its trace field equals g.

The stratum $\mathcal{H}(2)$ of abelian differentials with a single zero on a surface of genus 2 contains infinitely many algebraically primitive Teichmüller curves [Ca04, McM03b]. Recently, Bainbridge, Habegger and Möller [BHM14] showed finiteness of algebraically primitive Teichmüller curves in any stratum in genus 3. Finiteness of algebraically primitive Teichmüller curves in strata of differentials with a single zero for surfaces of prime genus $g \geq 3$ was established in [MW15]. Our final result generalizes this to every stratum in every genus $g \geq 3$, with a different proof. A stronger finiteness result covering Teichmüller curves whose field of definition is of degree at least three over $\mathbb{Q}$ is contained in [EFW17].

Theorem 5. *Any component $\mathcal{Q}$ of a stratum in genus $g \geq 3$ contains only finitely many algebraically primitive Teichmüller curves.*

Plan of the paper and strategy of the proofs: The proofs of the above results use tools from hyperbolic and non-uniform hyperbolic dynamics, differential geometry and algebraic groups. We embark from the foundational results of Eskin, Mirzakhani and Mohammadi [EMM15] and Filip [F16], but we do not use any methods developed in these works. We also do not use methods from the theory of flat surfaces, nor from algebraic geometry, although we apply several recent results from these areas, notably a foundational insight of Wright [W15] and a structural result of Möller [Mo06]. Instead we initiate a study of differential geometric properties of the moduli space of abelian differentials using the geometry of the moduli space of principally polarized abelian differentials and the Torelli map. We hope that such ideas together with the use of algebraic geometry will lead to a better understanding of the Schottky locus in the future.

In Section 2 we introduce the dynamical setup which is used throughout the article. We summarize the relevant properties of the Hodge bundle as well as some results from [H13], and we establish the first part of Theorem 1 as a fairly easy consequence.

In Section 3 we begin the investigation of affine invariant submanifolds. Its first part is devoted to a study of the so-called *absolute period foliation*. This foliation has extensively been studied for components of strata. We will only need some fairly elementary properties discovered in [McM13] (see also [H15]), and we generalize these properties to affine invariant manifolds. We also extend some dynamical results for the Teichmüller flow on strata from [H13] to general affine invariant manifolds.

Section 4 contains the main algebraic results of this article. We look at the local monodromoy group of an affine invariant manifold and show that it is Zariski dense in the symplectic group of rank corresponding to the rank of the manifold. The proof uses a result of Wright [W15] on flat structures on surfaces defined by abelian differentials which are contained in an affine invariant manifold (this is the only part of this work which refers to flat structures defined by abelian differentials), non-uniform hyperbolicity of the Teichmüller flow as explained in Section 2 and mod p-reduction for the integral symplectic group $Sp(2g,\mathbb{Z})$. Theorem 2 is obtained as a corollary as well as the existence of periodic orbits whose monodromy maps are semisimple and such that no product of two eigenvalues is an eigenvalue.

Theorem 2 and some aspect of its proof is used in Section 5 together with group sieving and tools from hyperbolic dynamics to deduce the second part of Theorem 1 and Corollary 1.

In Section 6 we begin with the differential geometric analysis of the moduli space of abelian differentials. We compare the Chern connection on the Hodge bundle to the Gauss Manin connection and establish a rigidity result using the results from Section 4. This then leads to the proof of the first part of Theorem 3 in Section 7. As a byproduct, we obtain that the Oseledets splitting of the Hodge bundle over a component of a stratum is not of class C^1, however our methods are insufficient to deduce that this splitting is not continuous.

Section 8 is based on the ideas developed in Section 6, but relies on precise information on the absolute period foliation. It contains the completion of the proof of Theorem 3. The proofs of Theorem 4 and Theorem 5 are contained in Section 9.

The article concludes with an appendix which collects some differential geometric properties of the moduli space of principally polarized abelian differentials which are used in Section 6 and Section 8.

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2. Lyapunov exponents

The goal of this section is to establish some properties of the Kontsevich Zorich cocycle and use this together with a refined understanding of the dynamics of the Teichmüller flow established in [H13] to show the first part of Theorem 1.

2.1. **The Hodge bundle.**

In this subsection we introduce the geometric setup which will be used throughout the remainder of this article.

A point in *Siegel upper half-space* $\mathfrak{D}_g = \mathrm{Sp}(2g,\mathbb{R})/U(g)$ is a principally polarized abelian variety of complex dimension g. Here as usual, $U(g)$ denotes the unitary group of rank g. Let $\omega = \sum_i dx_i \wedge dy_i$ be the standard symplectic form on the real vector space $\mathbb{R}^{2g}$. A point in $\mathfrak{D}_g$ can be viewed as a complex structure J on $(\mathbb{R}^{2g},\omega)$ which is *compatible* with the symplectic structure, ie such that $\omega(\cdot, J\cdot)$ is an inner product on $\mathbb{R}^{2g}$. The Siegel upper half-space is a Hermitean symmetric space, in particular it is a complex manifold, and the symmetric metric is Kähler.

There is a natural rank g complex vector bundle $\tilde{\mathcal{V}} \to \mathfrak{D}_g$ whose fibre over y is just the complex vector space defining y. This bundle is holomorphic. The polarization (ie the symplectic structure) and the complex structure define a Hermitean metric h on $\tilde{\mathcal{V}}$. The group $Sp(2g,\mathbb{R})$ acts from the left on the bundle $\tilde{\mathcal{V}}$ as a group of bundle automorphisms preserving the polarization and the complex structure, and hence this action preserves the Hermitean metric. Thus the bundle $\tilde{\mathcal{V}}$ projects to a holomorphic Hermitean (orbifold) vector bundle

$$\mathcal{V} \to Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/U(g) = \mathcal{A}_g.$$

We refer to the appendix for a more detailed information on this bundle.

Let $\mathcal{M}_g$ be the moduli space of closed Riemann surfaces of genus g. The *Torelli map*

$$\mathcal{I}_g : \mathcal{M}_g \to \mathcal{A}_g = Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/U(g)$$

which associates to a Riemann surface its Jacobian is holomorphic. The *Hodge bundle*

$$\Pi : \mathcal{H} \to \mathcal{M}_g$$

is the pullback of the holomorphic vector bundle $\mathcal{V} \to \mathcal{A}_g$ by the Torelli map. As the Torelli map is holomorphic, $\mathcal{H}$ is a g-dimensional holomorphic Hermitean vector bundle on $\mathcal{M}_g$ (in the orbifold sense). Its fibre over $x \in \mathcal{M}_g$ can be identified with the vector space of holomorphic one-forms on x. The Hermitean inner product on $\mathcal{H}$ is given by

$$(\omega,\zeta) = \frac{i}{2}\int \omega \wedge \overline{\zeta}.$$

Here the integration is over the basepoint, which is a Riemann surface. With this interpretation, the sphere bundle in $\mathcal{H}$ for the inner product $(,)$ is just the moduli space of area one abelian differentials.

As a real vector bundle, the Hodge bundle $\mathcal{H}$ has the following additional description. The action of the mapping class group $\mathrm{Mod}(S)$ on the first real cohomology group $H^1(S,\mathbb{R})$ defines the homomorphism $\Psi : \mathrm{Mod}(S) \to Sp(2g,\mathbb{Z})$. As a real vector bundle, the Hodge bundle is then the flat orbifold vector bundle

$$\mathcal{N} = \mathrm{Mod}(S)\backslash\mathcal{T}(S) \times H^1(S,\mathbb{R}) \to \mathcal{M}_g \tag{1}$$

for the standard action of $\mathrm{Mod}(S)$ on *Teichmüller space* $\mathcal{T}(S)$ and the action on $H^1(S,\mathbb{R})$ via Ψ. This description determines a flat connection on $\mathcal{N}$ which is called the *Gauss Manin* connection. We use the notation $\mathcal{N}$ to emphasize that we consider a flat real vector bundle. The bundle $\mathcal{N}$ has a natural real analytic structure induced by the complex structure on $\mathcal{M}_g$ so that the Gauss Manin connection is real analytic.

The Hodge bundle $\mathcal{H}$ is the real vector bundle $\mathcal{N}$ equipped with the following complex structure. Each point $x \in \mathcal{M}_g$ determines a complex structure J_x on $H^1(S,\mathbb{R})$. Namely, every cohomology class $\alpha \in H^1(S,\mathbb{R})$ can be represented by a unique harmonic one-form for the complex structure x, and this one-form is the real part of a unique holomorphic one-form ζ on x. The imaginary part of ζ is a harmonic one-form which represents the cohomology class $J_x\alpha$. The complex structure J_x is compatible with the symplectic structure defined by the intersection form ι on $H^1(S,\mathbb{R})$.

The assignment $x \to J_x$ defines a real analytic section J of the endomorphism bundle $\mathcal{N}^* \otimes \mathcal{N} \to \mathcal{M}_g$ of $\mathcal{N}$ which satisfies $J^2 = -\mathrm{Id}$. Thus the flat vector bundle $\mathcal{N} \otimes \mathbb{C} \to \mathcal{M}_g$ can be decomposed as

$$\mathcal{N} \otimes \mathbb{C} = \mathcal{H} \oplus \overline{\mathcal{H}}$$

where the *holomorphic* bundle $\mathcal{H} = \{\alpha + iJ\alpha \mid \alpha \in \mathcal{N}\}$ admits a natural identification with the bundle of holomorphic one-forms on $\mathcal{M}_g$, ie $\mathcal{H}$ is just the Hodge bundle over $\mathcal{M}_g$. The antiholomorphic bundle $\overline{\mathcal{H}}$ is defined by $\overline{\mathcal{H}} = \{\alpha - iJ\alpha \mid \alpha \in \mathcal{N}\}$.

Denote by $\mathcal{H}_+ \subset \mathcal{H}$ the complement of the zero section in the Hodge bundle $\mathcal{H}$. This is a complex orbifold. The pull-back

$$\Pi^*\mathcal{H} \to \mathcal{H}_+$$

to $\mathcal{H}_+$ of the Hodge bundle on $\mathcal{M}_g$ is a holomorphic vector bundle on $\mathcal{H}_+$. As a real vector bundle, it coincides with the pull-back $\Pi^*\mathcal{N}$ of the flat bundle $\mathcal{N}$. The pull-back of the Gauss-Manin connection on $\mathcal{N}$ is a flat connection on $\Pi^*\mathcal{N}$ which we call again the Gauss Manin connection. In the sequel we identify the real vector bundles $\Pi^*\mathcal{N}$ and $\Pi^*\mathcal{H}$ at leisure, using mainly the notation $\Pi^*\mathcal{H}$. However, sometimes we are only interested in the flat structure of $\Pi^*\mathcal{N}$ and then we write $\Pi^*\mathcal{N}$ to avoid confusion.

Let $\mathcal{Q}_+ \subset \mathcal{H}_+$ be a component of a stratum of abelian differentials. We use the notation $\mathcal{Q}_+$ to indicate that we do not normalize the area of an abelian differential. Then $\mathcal{Q}_+$ is a complex suborbifold of $\mathcal{H}_+$. *Period coordinates* for $\mathcal{Q}_+$ define the complex structure. Such coordinates are obtained by integration of a closed complex valued one-form $\alpha \in \mathcal{Q}_+$ over a basis of the relative homology group of (S,Σ) where Σ is the set of zeros of a differential in $\mathcal{Q}_+$.

The component $GL^+(2,\mathbb{R})$ of the identity of the full linear group $GL(2,\mathbb{R})$ acts on $\mathcal{H}_+$ as a group of real analytic transformations, and this action preserves $\mathcal{Q}_+$.

2.2. **Non-uniform hyperbolic dynamics.** Let $\mathcal{Q}$ be a component of a stratum of area one abelian differentials on S. Recall from the introduction that the Teichmüller flow Φ^t acts on $\mathcal{Q}$ preserving a Borel probability measure λ in the Lebesgue measure class, the so-called Masur-Veech measure. Let $k \geq 1$ be the number of zeros of a differential in $\mathcal{Q}$ and let $h = 2g - 2 + k$ be the entropy of Φ^t with respect to the measure λ.

The Gauss Manin connection on $\Pi^*\mathcal{N} \to \mathcal{Q}$ is symplectic, but it is not trivial. More precisely, its monodromy group is a nontrival subgroup of the group $Sp(2g,\mathbb{Z}) < Sp(2g,\mathbb{R})$. Parallel transport for the Gauss Manin connection defines a lift of the Teichmüller flow Φ^t to a flow Θ^t on $\Pi^*\mathcal{N} \to \mathcal{Q}$, and the corresponding cocycle over the Teichmüller flow on $\mathcal{Q}$ with values in the flat bundle $\Pi^*\mathcal{N}$ is called the *Kontsevich Zorich cocycle*.

The Kontsevich Zorich cocycle is integrable with respect to the Masur Veech measure λ on $\mathcal{Q}$ (see [AV07] for more and for references), and therefore its *Lyapunov* *exponents* are defined. These exponents measure the asymptotic growth rate of vectors along orbits of Φ^t which are generic for λ. Since the Gauss Manin connection is symplectic, the exponents are invariant under multiplication with -1. Let

$$1 = \kappa_1 > \cdots > \kappa_g > 0$$

be the largest g Lyapunov exponents of the Teichmüller flow on $\mathcal{Q}$. That these exponents are all positive and pairwise distinct is the main result of [AV07].

Remark 2.1. For more general *affine invariant manifolds*, the Lyapunov spectrum of the Kontsevich Zorich cocycle need not be simple. We refer to [Au15] for a discussion and examples.

Let

$$\Gamma \subset \mathcal{Q}$$

be the countable collection of all periodic orbits for Φ^t contained in $\mathcal{Q}$. Denote by $\ell(\gamma)$ the period of $\gamma \in \Gamma$. The orbit $\gamma \in \Gamma$ determines a conjugacy class in $\mathrm{Mod}(S)$ of pseudo-Anosov elements. Let $\varphi \in \mathrm{Mod}(S)$ be an element in this conjugacy class; then

$$A(\gamma) = \Psi(\varphi) \in \mathrm{Sp}(2g, \mathbb{Z})$$

is determined by γ up to conjugation. The matrix $A(\gamma)$ is Perron Frobenius, with leading eigenvalue $e^{\ell(\gamma)}$, and the eigenspace for the eigenvalue $e^{\ell(\gamma)}$ is spanned by the real cohomology class which is defined by intersection with the *vertical measured foliation* of any point on the cotangent line of the unique φ-invariant Teichmüller geodesic. Namely, this measured foliation μ is orientable by assumption, and hence it defines a first cohomology class on S by associating to a smooth closed curve α on $S - \Sigma$ its vertical length, obtained by integrating the transverse invariant measure of μ over α. Here as before, Σ denotes the set of zeros of a differential on this cotangent line.

If we define $1 = \alpha_1(\gamma) > \cdots \geq \alpha_g(\gamma) \geq 0$ to be the quotients by $\ell(\gamma)$ of the logarithms of the g largest absolute values of the eigenvalues of the matrix $A(\gamma)$, ordered in decreasing order and counted with multiplicities, then the numbers $\alpha_i(\gamma)$ only depend on γ but not on any choices made.

Let $\epsilon > 0$. For $\gamma \in \Gamma$ define $\chi_\epsilon(\gamma) = 1$ if $|\alpha_i(\gamma) - \kappa_i| < \epsilon$ for every $i \in \{1, \dots, g\}$, and define $\chi_\epsilon(\gamma) = 0$ otherwise.

For $R_1 < R_2$ let $\Gamma(R_1, R_2) \subset \Gamma$ be the set of all periodic orbits for Φ^t of prime period contained in the interval (R_1, R_2) (asking for prime period means that we do not consider multiply covered orbits). For an open or closed subset V of $\mathcal{Q}$ denote by $\chi(V)$ the characteristic function of V and define

$$H(V, R_1, R_2) = \sum_{\gamma \in \Gamma(R_1, R_2)} \int_\gamma \chi(V) \quad \text{and}$$
$$H_\epsilon(V, R_1, R_2) = \sum_{\gamma \in \Gamma(R_1, R_2)} \int_\gamma \chi(V)\chi_\epsilon(\gamma).$$

Clearly we have

$$H_\delta(V, R_1, R_2) \leq H_\epsilon(V, R_1, R_2) \leq H(V, R_1, R_2)$$

for all $\epsilon > \delta > 0$. Note that $H(V, R_1, R_2)$ is the measure of V for the Φ^t-invariant measure on $\mathcal{Q}$ which is the sum of the standard length measures over all periodic orbits of prime period contained in (R_1, R_2).

There are some technical difficulties due to nontrivial point stabilizers for the action of the mapping class group on the Teichmüller space of marked abelian differentials. To avoid dealing with this issue (although this can be done with some amount of care) we define the *good subset* $\mathcal{Q}_{\text{good}}$ of $\mathcal{Q}$ to be the set of all points $q \in \mathcal{Q}$ with the following property. Let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of marked abelian differentials and let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of q; then an element of $\text{Mod}(S)$ which fixes $\tilde{q}$ acts as the identity on $\tilde{\mathcal{Q}}$ (compare [H13] for more information on this technical condition). Thus $\mathcal{Q}_{\text{good}}$ is precisely the subset of $\mathcal{Q}$ of manifold points. Lemma 4.5 of [H13] shows that the good subset $\mathcal{Q}_{\text{good}}$ of $\mathcal{Q}$ is open, dense and Φ^t-invariant.

Call a point $q \in \mathcal{Q}$ *birecurrent* if it is contained in its own α- and ω limit set. By the Poincaré recurrence theorem, the set of birecurrent points in $\mathcal{Q}$ has full Masur-Veech measure. In Corollary 4.8 of [H13] we showed

Proposition 2.2. *For every good birecurrent point $q \in \mathcal{Q}_{\text{good}}$, for every neighborhood U of q in $\mathcal{Q}$ and for every $\delta > 0$ there is an open neighborhood $V \subset U$ of q in $\mathcal{Q}$ and a number $t_0 > 0$ such that*

$$H(V, R - t_0, R + t_0)e^{-hR} \leq 2t_0\lambda(V)(1 + \delta)$$

for all sufficiently large $R > 0$.

The proof of Proposition 2.2 is based on a more technical result which will be used several times in the sequel. For its formulation, we say that a closed curve η in $\mathcal{Q}_{\text{good}}$ *defines the conjugacy class of a pseudo-Anosov mapping class* $\varphi \in \text{Mod}(S)$ if the following holds true. Let $\tilde{\eta}$ be a lift of η to an arc in the Teichmüller space of abelian differentials, parametrized one some interval $[0, a] \subset \mathbb{R}$; note that such a lift exists and is unique up to translation by an element of $\text{Mod}(S)$ since we require that $\eta \subset \mathcal{Q}_{\text{good}}$. Then $\tilde{\eta}(a) = \psi(\tilde{\eta}(0))$ for a mapping class ψ, and we require that ψ is conjugate to φ. As two different lifts of η determine conjugate elements in $\text{Mod}(S)$, this definition does not depend on any choices made.

For an Anosov flow Φ^t on a closed manifold M, Margulis [Ma04] calculated the asymptotic growth rate of the number of periodic orbits sorted by their length using that the flow is mixing with respect to the unique invariant measure μ of maximal entropy and that moreover each point in M admits a particularly nice basis of neighborhoods. These neighborhoods are flow boxes with a local product structure. Using uniform contraction and expansion, respectively, of strong stable and strong unstable manifolds under the action of Φ^t, for large $R > 0$ the measure of a connected component of the intersection of such a flow box B with the image of a slightly smaller flow box under the time-R-map Φ^R can effectively be estimated. These estimates are combined with an Anosov closing lemma resulting from expansiveness of the flow to relate the asymptotic growth rate of periodic orbits to the topological entropy of the flow.

The Teichmüller flow is not Anosov, but its enjoys sufficient non-uniform hyperbolicity that a similar statement can be established [H13]. Such a statement is a bit more complicated than the corresponding statement for Anosov flows, with all estimates local near birecurrent points, ie the constants depend on the specific birecurrent point we are interested in. Lemma 2.3 below combines Lemma 4.7 and Proposition 5.4 of [H13]. In its statement, λ is as before the Masur-Veech measure on $\mathcal{Q}$.

Lemma 2.3. *Let $q \in \mathcal{Q}_{\rm good}$ be a good birecurrent point, let $\delta > 0$ and let U be any neighborhood of q. Then U contains a nested sequence of neighborhoods $Z_0 \subset Z_1 \subset Z_2 \subset V$ of q, and there are numbers $R_0 > 0, t_0 > 0$ with the following properties.*

- (a) *V is open and contractible, and Z_i is closed, with dense interior, and contained in the interior of Z_{i+1}.*
- (b) *$\lambda(Z_0) > (1-\delta)\lambda(V)$.*
- (c) *The length of a connected subsegment of the intersection with Z_1 or Z_2 of an orbit of Φ^t equals $2t_0$.*
- (d) *If $R > R_0$ and $z \in Z_1$ are such that $\Phi^R z = z$ and if $\hat{E}$ denotes the connected component containing z of the intersection $\Phi^R V \cap V$, then the Masur Veech measure of the intersection $\Phi^R Z_1 \cap Z_2 \cap \hat{E}$ is contained in the interval*
$$[e^{-hR}\lambda(V)(1-\delta), e^{-hR}\lambda(V)(1+\delta)].$$
- (e) *Let $z \in Z_0$ and $R > R_0$ be such that $\Phi^R z \in Z_0$. Connect $\Phi^R z$ to z by an arc in V and let η be the concatenation of the orbit segment $\cup_{0 \le t \le R}\Phi^t z$ with this arc. We call η a* characteristic curve *of the orbit segment $\cup_{t\in[0,R]}\Phi^t z$. There is a unique periodic orbit γ for Φ^t of length contained in the interval $[R - t_0 - \delta, R + t_0 + \delta]$ which intersects $\Phi^R Z_1 \cap Z_1 \cap \hat{E}$ where $\hat{E}$ equals the connected component containing z of the intersection $\Phi^R V \cap V$. The closed curve η and the periodic orbit γ define the same conjugacy class in* $\mathrm{Mod}(S)$.

Note that in the above statement, we slightly adjusted the choice of the sets Z_i compared to the terminology in [H13] for clarity of exposition. The more refined construction of a nested chain of sets in [H13] was necessary for a more precise volume control but is unimportant for the purpose of this article.

We use Lemma 2.3 to show an improved version of Proposition 5.4 of [H13].

Proposition 2.4. *For every birecurrent point $q \in \mathcal{Q}_{\rm good}$, for every neighborhood U of q in $\mathcal{Q}$ and for every $\delta > 0$ there is an open neighborhood $V \subset U$ of q in $\mathcal{Q}$ and a number $t_0 > 0$ with the properties stated in Lemma 2.3 such that for every $\epsilon > 0$ we have*
$$\liminf_{R\to\infty} H_\epsilon(V, R - t_0 - \delta, R + t_0 + \delta)e^{-hR} \ge 2t_0\lambda(V)(1-\delta).$$

Proof. Let $\| \, \|$ be any smooth Riemannian norm on the flat vector bundle $\mathcal{N} \to \mathcal{Q}$. Let Θ^t be the lift of the Teichmüller flow to a flow on $\mathcal{N}$ defined by parallel transport for the Gauss Manin connection. Recall that Θ^t preserves the symplectic structure on $\mathcal{N}$, but it may not preserve the norm $\| \, \|$.

For $z \in \mathcal{Q}$ let $\mathcal{N}_z$ be the fibre of $\mathcal{N}$ at z. For $1 \leq i \leq g$ and for $t > 0$ let

$$\zeta_i(t, z)$$

be the infimum of the operator norms of the restriction of $\Theta^t(z)$ to a symplectic subspace of $\mathcal{N}_z$ of real dimension $2(g - i + 1)$. Define

$$\kappa_i(t, z) = \frac{1}{t} \log \zeta_i(t, z).$$

Let $\epsilon > 0, \delta > 0$ and let U be a neighborhood of a birecurrent point $q \in \mathcal{Q}_{\rm good}$. Since the Kontsevich Zorich cocycle is locally constant (or, equivalently, the Gauss Manin connection is flat), we can find a collection of nested neighborhoods $Z_0 \subset Z_1 \subset Z_2 \subset V \subset U$ with the properties in Lemma 2.3 and such that furthermore, with the notations from part (e) of the lemma, if $z \in Z_0, R > R_0$ and if $\Phi^R z \in Z_0$ then the periodic orbit γ for Φ^t determined by a characteristic curve η of the orbit segment $\cup_{t\in[0,R]} \Phi^t z$ satisfies

$$|\kappa_i(R, z) - \alpha_i(\gamma)| \leq \epsilon/2. \tag{2}$$

Namely, for a sufficiently small contractible neighborhood V of q in $\mathcal{Q}_{\rm good}$, the trivialization of $\mathcal{N}|V$ defined by the Gauss Manin connection almost preserves the norm $\| \, \|$. Then the estimate (2) holds true if we replace $\alpha_i(\gamma)$ be the i-th absolute value in decreasing order of an eigenvalue of the symplectic transformation A_η of $\mathcal{N}_z$ which is defined by parallel transport for the Gauss Manin connection along a characteristic curve η for the orbit segment $\cup_{t\in[0,R]} \Phi^t z$. But by property (e) in Lemma 2.3, the characteristic curve η defines the same conjugacy class of a pseudo-Anosov mapping class as γ. This means that the numbers $\alpha_i(\gamma)$ are precisely the absolute values of the eigenvalues of the transformation A_η. Thus the estimate for the transformation A_η implies the estimate (2).

As the Teichmüller flow is ergodic for the Masur-Veech measure λ [M82, V86] and as the Gauss Manin connection preserves the symplectic structure, the Oseledets multiplicative ergodic theorem [O68] states that for λ-almost every point $z \in \mathcal{Q}$, the numbers $\kappa_i(R, z)$ converge as $R \to \infty$ to the i-th positive Lyapunov exponent κ_i of the Kontsevich Zorich cocycle. Then there is a number $R(\epsilon) > R_0$ and a Borel subset B of Z_0 of measure $\lambda(B) > \lambda(Z_0)(1 - \delta)$ with the following property. Let $u \in B$ and let $R > R(\epsilon)$; then $|\kappa_i(R, u) - \kappa_i| \leq \epsilon/2$.

Since the Masur-Veech measure is mixing for the Teichmüller flow [M82, V86], there is a number $R_1 > R(\epsilon)$ such that

$$\lambda(\Phi^R B \cap B) \geq \lambda(B)^2(1 - \delta) \geq \lambda(Z_0)^2(1 - \delta)^3 \geq \lambda(V)^2(1 - \delta)^5$$

for all $R \geq R_1$. Property (e) in Lemma 2.3 implies that for each $R \geq R_1$ and each connected component $\hat{E}$ of $V \cap \Phi^R V$ which contains a point in $B \cap \Phi^R B$, there is a periodic orbit of Φ^t passing through $Z_1 \subset V$. Furthermore, the length ℓ of this periodic orbit is contained in the interval $[R - t_0 - \delta, R + t_0 + \delta]$, and if $z \in Z_1$ is a point on this orbit then by the estimate (d) in Lemma 2.3, the Masur Veech measure of the intersection $\Phi^\ell Z_1 \cap Z_2 \cap \hat{E}$ is contained in the interval $[e^{-h\ell}\lambda(V)(1 - \delta), e^{-h\ell}\lambda(V)(1 + \delta)]$.

This implies that the number of components of the intersection $\Phi^R V \cap V$ containing points in $\Phi^R B \cap B$ is at least $e^{h(R-t_0-\delta)}\lambda(V)(1-\delta)^5(1+\delta)^{-1}$. Each such component determines a periodic orbit γ of Φ^t of length at most $R+t_0+\delta$, and it also determines a component of the intersection of γ with the set Z_1. The estimate (2) together with the choice of the set B yields that each such periodic orbit γ satisfies $\chi_\epsilon(\gamma) = 1$. Thus by (c) of Lemma 2.3, any component of $\Phi^R V \cap V$ which contains a point in $\Phi^R B \cap B$ contributes at least $2t_0$ to the value $H_\epsilon(V, R-t_0-\delta, R+t_0+\delta)$. Together this shows that

$$H_\epsilon(V, R-t_0-\delta, R+t_0+\delta) \geq 2t_0 e^{h(R-t_0-\delta)}\lambda(V)(1-\delta)^5(1+\delta)^{-1}.$$

Up to adjusting the constant δ (which was arbitrarily prescribed at the beginning of this proof), this implies the proposition. We refer to the proof of Proposition 5.4 of [H13] and [Ma04] for more details on this construction. □

As a corollary, we obtain the first part of Theorem 1. As before, κ_i denotes the i-th Lyapunov exponent of the Kontsevich Zorich cocycle with respect to the Masur-Veech measure.

Corollary 2.5. *For $\epsilon > 0$, the set $\{\gamma \in \Gamma \mid |\alpha_i(\gamma) - \kappa_i| < \epsilon\}$ $(1 \leq i \leq g)$ is typical.*

Proof. The main result of [H13] states the following. As $R \to \infty$, the measures

$$\mu_R = e^{-hR} \sum_{\gamma \in \Gamma, \ell(\gamma) \leq R} \delta(\gamma)$$

converge weakly to the Masur-Veech measure on $\mathcal{Q}$. Here $\delta(\gamma)$ denotes the Φ^t-invariant length measure on the periodic orbit γ. Furthermore, by the main result of [EM11, EMR12], there is a compact subset K of $\mathcal{Q}$ such that the growth rate of all periodic orbits which do not intersect K is strictly smaller than h.

The Masur-Veech measure of $\mathcal{Q}-\mathcal{Q}_{\rm good}$ vanishes. As orbits which do not intersect the compact set K do not contribute towards the asymptotic counting of all periodic orbits, it follows that periodic orbits γ with $\chi_\epsilon(\gamma) > 0$ are typical if we can show that for any $\epsilon > 0$, the measures

$$\mu_R = e^{-hR} \sum_{\gamma \in \Gamma, \ell(\gamma) \leq R} \chi_\epsilon(\gamma)\delta(\gamma)$$

converge as $R \to \infty$ weakly to the Lebesgue measure on $\mathcal{Q}_{\rm good}$ (we refer to [H13] for a comprehensive discussion). However, this is a consequence of Proposition 2.2 and Proposition 2.4. We refer again to [Ma04, H13] for more details on this construction. □

Remark 2.6. As the results of [H13] equally hold for components of strata of quadratic differentials, Corollary 2.5 is valid without modification in this case as well. Even more generally, Corollary 2.5 is valid for the Lyapunov spectrum of an integrable cocycle for the Teichmüller flow on a component $\mathcal{Q}$ of abelian or quadratic differentials which is defined by parallel transport with respect to a flat connection on an arbitrary flat bundle over $\mathcal{Q}$.

3. The local structure of affine invariant manifolds

In this section we begin the investigation of affine invariant manifolds. Our first goal is to gain some understanding of their local structure. Most of the results in this section are known to the experts. As we did not find precise references in the literature, we include a detailed discussion.

An *affine invariant manifold* $\mathcal{C}_+$ in a component $\mathcal{Q}_+$ of a stratum of abelian differentials with fixed number and multiplicities of zeros is the closure in $\mathcal{Q}_+$ of an orbit of the $GL^+(2,\mathbb{R})$-action. Such an affine invariant manifold is complex affine in period coordinates [EMM15]. In particular, $\mathcal{C}_+ \subset \mathcal{Q}_+$ is a complex suborbifold. Period coordinates determine a projection

$$p : T\mathcal{C}_+ \to \Pi^*(\mathcal{H} \oplus \overline{\mathcal{H}})|\mathcal{C}_+ = \Pi^*\mathcal{N} \otimes \mathbb{C}|\mathcal{C}_+$$

to absolute periods (see [W14] for a clear exposition). The image $p(T\mathcal{C}_+)$ is flat, ie it is invariant under the restriction of the Gauss Manin connection to a connection on $\Pi^*(\mathcal{H} \oplus \overline{\mathcal{H}})|_{\mathcal{C}_+} = \Pi^*\mathcal{N} \otimes \mathbb{C}|_{\mathcal{C}_+}$.

By the main result of [F16], there is a *holomorphic* subbundle $\mathcal{Z}$ of $\Pi^*\mathcal{H}|_{\mathcal{C}_+}$ such that

$$p(T\mathcal{C}_+) = \mathcal{Z} \oplus \overline{\mathcal{Z}}.$$

We call $\mathcal{Z}$ the *absolute holomorphic tangent bundle* of $\mathcal{C}_+$. As a consequence, the bundle $p(T\mathcal{C}_+)$ is invariant under the complex structure on $\Pi^*\mathcal{N} \otimes \mathbb{C}$.

As a real vector bundle, $\mathcal{Z}$ is isomorphic to $p(T\mathcal{C}_+) \cap \Pi^*\mathcal{N}|\mathcal{C}_+$. Since $\mathcal{Z}$ is invariant under the compatible complex structure J, $p(T\mathcal{C}_+) \cap \Pi^*\mathcal{N}$ is symplectic [AEM12]. Moreover, $\mathcal{Z} \oplus \overline{\mathcal{Z}} \subset \Pi^*\mathcal{N} \otimes \mathbb{C}|\mathcal{C}_+$ viewed as the real part of $p(T\mathcal{C}_+)$ is flat [EMM15, F16].

Define the *rank* of the affine invariant manifold $\mathcal{C}_+$ as

$$\mathrm{rk}(\mathcal{C}_+) = \frac{1}{2}\dim_{\mathbb{C}} p(T\mathcal{C}_+) = \dim_{\mathbb{C}} \mathcal{Z}.$$

With this definition, components of strata are affine invariant manifolds of rank g.

When we investigate dynamical properties it is as before more convenient to consider the intersection $\mathcal{C}$ of an affine invariant manifold $\mathcal{C}_+ \subset \mathcal{H}_+$ with the moduli space of area one abelian differentials. This intersection $\mathcal{C}$ is invariant under the action of the group $SL(2,\mathbb{R}) < GL^+(2,\mathbb{R})$. Throughout we always denote such an affine invariant manifold by $\mathcal{C}$, and we let $\mathcal{C}_+$ be its natural extension to $\mathcal{H}_+$.

3.1. **The absolute period foliation.** Every component $\mathcal{Q}$ of a stratum in the bundle of area one abelian differentials which consists of differentials with at least two zeros admits a foliation $\mathcal{AP}(\mathcal{Q})$ whose leaves locally consist of differentials with the same absolute periods. This foliation is called the *absolute period foliation* (we adopt this terminology from [McM13], other authors call it the relative period foliation). The leaves of this foliation admit a complex affine structure (see e.g. [McM13]).

If $\mathcal{C} \subset \mathcal{Q}$ is an affine invariant manifold whose complex dimension is strictly bigger than twice its rank then $\mathcal{C}$ intersects the leaves of the absolute period foliation of $\mathcal{Q}$ nontrivially. This fact alone does not imply that $\mathcal{C} \cap \mathcal{AP}(\mathcal{Q})$ is a foliation of $\mathcal{C}$. The main goal of this subsection is to verify that indeed, this is always the case.

To this end we need some more detailed information on the absolute period foliation of the component of a stratum $\mathcal{Q}$. Its tangent bundle $T\mathcal{AP}(\mathcal{Q})$ has an explicit description via so-called *Schiffer variations* [McM13] which we explain now.

Let first ω be an abelian differential with a simple zero p. Then ω defines a singular euclidean metric on S which has a cone point of cone angle 4π at p. There are four horizontal separatrices at p for this metric. In a complex coordinate z near p so that $\omega = (z/2)dz$, the horizontal separatrices are the four rays contained in the real or the imaginary axis. The restriction of ω to these rays defines an orientation on the rays. With respect to this orientation, the two rays contained in the real axis are outgoing from p, while the rays contained in the imaginary axis are incoming. The *Schiffer variation* of ω with weight one at p is the tangent at ω of the following arc of deformations of ω. For small $u > 0$ cut the surface S open along the initial subsegment of length $2u$ of the two separatrices whose orientations point towards p and refold the resulting four-gon so that the singular point p slides backwards along the incoming rays in the imaginary axis. We refer to [McM13] and [H15] for a more detailed description.

If ω has a zero of order $n \geq 2$ at p then the Schiffer variation of ω with weight one at p is defined as follows (see p.1235 of [McM13]). Choose a coordinate z near p so that $\omega = (z^n/2)dz$ in this coordinate. This choice of coordinate is unique up to multiplication with $e^{\ell 2\pi i/(n+1)}$ for some $\ell \leq n$. There are $n+1$ horizontal separatrices at p for the flat metric defined by ω whose orientations point towards p. For small $u > 0$ cut the surface S open along the initial subsegments of length $2u$ of these $n+1$ segments. The result is a $2n+2$-gon which we refold as in the case of a simple zero. The tangent at ω of this arc of deformations of ω is called the Schiffer variation of ω with weight one at p.

Now let $\mathcal{Q}$ be an arbitrary component of a stratum of abelian differentials consisting of differentials with $k \geq 2$ zeros. By passing to a finite cover $\hat{\mathcal{Q}}$ of $\mathcal{Q}$ we may assume that the zeros are *numbered.* For $\omega \in \hat{\mathcal{Q}}$ let $Z(\omega)$ be the set of numbered zeros of ω. Let moreover $V(\omega) \sim \mathbb{C}^k$ be the complex vector space freely generated by the set $Z(\omega)$. Then the tangent space $T\mathcal{AP}(\hat{\mathcal{Q}})$ of the absolute period foliation of $\hat{\mathcal{Q}}$ at ω is naturally isomorphic to the hyperplane in $V(\omega)$ of all points whose coordinates sum up to zero [McM13, H15], ie of points with zero mean.

More explicitly, let $\mathfrak{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ be any k-tuple of *real* numbers with $\sum_i a_i = 0$. Then $\mathfrak{a}$ defines a smooth vector field $X_{\mathfrak{a}}$ on $\hat{\mathcal{Q}}$ as follows. For each $\omega \in \hat{\mathcal{Q}}$, the value of $X_{\mathfrak{a}}$ at ω is the Schiffer variation for the tuple $(a_1, \dots, a_k)$ of weight parameters at the numbered zeros of ω. Thus $X_{\mathfrak{a}}$ is tangent to the absolute period foliation. The $k-1$-dimensional real subbundle of the tangent bundle of $\hat{\mathcal{Q}}$ spanned by these vector fields is the tangent bundle of the *real rel foliation* $\mathcal{R}$ of $\hat{\mathcal{Q}}$ which is the intersection of the absolute period foliation with the *strong unstable foliation* W^{su} of $\hat{\mathcal{Q}}$. Recall that the leaf of the strong unstable foliation through $q \in \hat{\mathcal{Q}}$ locally consists of all differentials with the same horizontal *measured foliation* as q. In period coordinates, the local leaf of W^{su} through q is just the set of all differentials q' whose imaginary part defines the same relative cohomology class as the imaginary part of q, taken relative to the zeros of q (or q').

Similarly, we define the *imaginary rel foliation* of $\hat{\mathcal{Q}}$ to be the intersection of the absolute period foliation with the *strong stable foliation* W^{ss} of $\hat{\mathcal{Q}}$. The leaf of the foliation W^{ss} through q locally consists of all differentials with the same vertical measured foliations as q. Exchanging the roles of the horizontal and the vertical foliation in the definition of the Schiffer variations identifies the tangent bundle of the imaginary rel foliation of $\hat{\mathcal{Q}}$ with the purely imaginary weight vectors of zero mean on the numbered zeros of the differentials in $\hat{\mathcal{Q}}$. As the tangent bundle of the absolute period foliation is spanned by its intersection with the tangent bundle of the strong stable and the strong unstable foliation, mapping a real weight vector to its multiple with $i = \sqrt{-1}$ defines a natural almost complex structure on $T\mathcal{AP}(\hat{\mathcal{Q}})$. This almost complex structure is in fact integrable [McM13] and equals the complex structure defined by period coordinates.

The Teichmüller flow Φ^t preserves the absolute period foliation. The following is Lemma 2.2 of [H15].

Lemma 3.1. *$d\Phi^t X_{\mathfrak{a}} = e^t X_{\mathfrak{a}}$ and $d\Phi^t X_{i\mathfrak{a}} = e^{-t} X_{i\mathfrak{a}}$ for every $\mathfrak{a} \in \mathbb{R}^k$ with zero mean.*

We observe next that an affine invariant submanifold $\mathcal{C}$ of $\mathcal{Q}$ intersects the absolute period foliation of $\mathcal{Q}$ in a real analytic foliation $\mathcal{AP}(\mathcal{C})$ with complex affine leaves.

For the formulation, denote again by $\hat{\mathcal{Q}}$ a finite cover of $\mathcal{Q}$ on which the zeros of the differentials are numbered. Then any choice of numbering of these zeros determines an identification of the complex vector space freely generated by the zeros of one (and hence any) differential in $\hat{\mathcal{Q}}$ with $\mathbb{C}^k$. This yields an identification of the tangent bundle $T\mathcal{AP}(\hat{\mathcal{Q}})$ of the absolute period foliation of $\hat{\mathcal{Q}}$ with the trivial bundle over $\hat{\mathcal{Q}}$ whose fibre is the hyperplane of $\mathbb{C}^k$ of vectors of zero mean.

Let now $\mathcal{C} \subset \mathcal{Q}$ be an affine invariant manifold. As before, $\mathcal{C}_+$ denotes the extension of $\mathcal{C}$ to a $GL^+(2,\mathbb{R})$-invariant subspace of $\mathcal{H}_+$. We define the *deficiency* $\mathrm{def}(\mathcal{C})$ as

$$\mathrm{def}(\mathcal{C}) = \dim_{\mathbb{C}}(\mathcal{C}_+) - 2\mathrm{rk}(\mathcal{C}_+).$$

Let moreover $\hat{\mathcal{C}}$ be a lift of $\mathcal{C}$ to an affine invariant manifold in $\hat{\mathcal{Q}}$.

The following lemma is a concrete and global version of Remark 1.4.(ii) of [F16]. As before, $k \geq 1$ denotes the number of zeros of a differential in $\mathcal{Q}$.

Lemma 3.2. *Let $\mathcal{C}$ be an affine invariant submanifold of $\mathcal{Q}$ of deficiency $r = \mathrm{def}(\mathcal{C}) > 0$ and let $\hat{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in $\hat{\mathcal{Q}}$. Then $\hat{\mathcal{C}}$ intersects the real rel foliation (or the imaginary rel foliation) of $\hat{\mathcal{Q}}$ in a real analytic foliation of real dimension r. Furthermore, if $q \in \hat{\mathcal{C}}$ and if $\mathfrak{a} \in \mathbb{R}^k$ is a vector of zero mean such that $X_{\mathfrak{a}}(q) \in T\mathcal{AP}(\hat{\mathcal{C}})$, then $X_{\mathfrak{a}}(z) \in T\mathcal{AP}(\hat{\mathcal{C}})$, $X_{i\mathfrak{a}}(z) \in T\mathcal{AP}(\hat{\mathcal{C}})$ for every $z \in \hat{\mathcal{C}}$.*

Proof. Let $\hat{\mathcal{C}} \subset \hat{\mathcal{Q}}$ be an affine invariant manifold of deficiency $r = \mathrm{def}(\mathcal{C}) > 0$. Then for each $q \in \hat{\mathcal{C}}$ there is a vector $0 \neq X \in T_q\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to $\hat{\mathcal{C}}$. By invariance of $\hat{\mathcal{C}}$ and of the absolute period foliation of $\hat{\mathcal{Q}}$ under the Teichmüller flow, we have $d\Phi^t(X) \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ for all t.

A vector $X \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ decomposes as $X = X^u + X^s$ where $X^u \in T\mathcal{AP}(\hat{\mathcal{Q}})$ is real (and hence tangent to the strong unstable foliation) and X^s is imaginary (and hence tangent to the strong stable foliation). We claim that we can find a vector $Y \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which either is tangent to the strong unstable or to the strong stable foliation. To this end we may assume that $X^u \neq 0$. Since this is an open condition and since the Teichmüller flow on $\hat{\mathcal{C}}$ is topologically transitive, we may furthermore assume that the Φ^t-orbit of the footpoint q of X is dense in $\hat{\mathcal{C}}$. Then there is a sequence $t_i \to \infty$ such that $\Phi^{t_i}(q) \to q$.

Choose any smooth norm $\| \|$ on $T\hat{\mathcal{Q}}$. As $X^u \neq 0$, Lemma 3.1 and its analog for imaginary vectors and the inverse $t \to \Phi^{-t}$ of the Teichmüller flow shows that up to passing to a subsequence,

$$d\Phi^{t_i}(X)/\|d\Phi^{t_i}(X)\|$$

converges to a vector $Y \in T_q\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to the strong unstable foliation. Now $T\hat{\mathcal{C}}$ is a smooth $d\Phi^t$-invariant subbundle of the restriction of the tangent bundle of $\hat{\mathcal{Q}}$ to $\hat{\mathcal{C}}$ (this is meant in the orbifold sense) and hence we have $Y \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which is what we wanted to show.

Using Lemma 3.1 and density of the Φ^t-orbit of q, if $0 \neq \mathfrak{a} \in \mathbb{R}^k$ is a vector of zero mean such that $Y = X_{\mathfrak{a}}(q)$, then $X_{\mathfrak{a}}(u) \in T\hat{\mathcal{C}}$ for all $u \in \hat{\mathcal{C}}$.

By invariance of $T\mathcal{C}_+$ under the complex structure defined by period coordinates, if $r = 1$ then

$$T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}}) = \mathbb{R}X_{\mathfrak{a}} \oplus \mathbb{R}X_{i\mathfrak{a}}$$

and we are done. Otherwise there is a tangent vector $Y \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}}) - \mathbb{C}X_{\mathfrak{a}}$. Apply the above argument to Y, perhaps via replacing the Teichmüller flow by its inverse. In finitely many such steps we conclude that there is a smooth subbundle $\mathcal{B}$ of $T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to the strong unstable foliation (ie real for the real structure), of real rank r, and such that $T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}}) = \mathbb{C}\mathcal{B}$. Moreover, if $z \in \hat{\mathcal{C}}$ and if $\mathfrak{a} \in \mathbb{R}^k$ is such that $X_{\mathfrak{a}}(z) \in \mathcal{B}$ then $X_{\mathfrak{a}}(q) \in \mathcal{B}$ for every $q \in \hat{\mathcal{C}}$.

To summarize, there exists an r-dimensional real linear subspace V of the hyperplane of $\mathbb{R}^k$ of vectors of zero mean, and for each $\mathfrak{a} \in V$ and every $q \in \hat{\mathcal{C}}$, the vectors $X_{\mathfrak{a}}(q), X_{i\mathfrak{a}}(q)$ are both tangent to $\hat{\mathcal{C}}$ at q. Then $\hat{\mathcal{C}}$ is invariant under the flows $\Lambda^t_{\mathfrak{a}}$ generated by the vector fields $X_{\mathfrak{a}}$ for $\mathfrak{a} \in V$. However the flow lines of these flows define an affine structure on the leaves of the absolute period foliation. As a consequence, the intersection of $\hat{\mathcal{C}}$ with a leaf of the absolute period foliation is locally an affine submanifold of the corresponding leaf of $\mathcal{AP}(\hat{\mathcal{Q}})$. This completes the proof of the lemma. $\square$

3.2. **Dynamical properties of the Teichmüller flow.** The goal of this subsection is to generalize some dynamical properties of the Teichmüller flow on components of strata as recorded in Section 2 to affine invariant manifolds.

As before, we will concentrate on the manifold points in an affine invariant manifold $\mathcal{C}_+$. Namely, there is an obvious notion of a good point in $\mathcal{C}_+$ extending the notion of a good point in a component of a stratum. Denote by $\mathcal{C}_{+,\mathrm{good}}$ the good subset of $\mathcal{C}_+$ and by $\mathcal{C}_{\mathrm{good}}$ the intersection of the good subset with the hypersurface of area one differentials. The good set is precisely the set of manifold points. The proof of Lemma 4.5 of [H13] is equally valid for affine invariant manifolds and shows that the set $\mathcal{C}_{\mathrm{good}} \subset \mathcal{C}$ of good points is open, dense and invariant under the Teichmüller flow.

Lemma 3.2 can be viewed as a global version of a local structural result for affine invariant manifolds which is a consequence of the fact that such an affine invariant manifold $\mathcal{C}_+ \subset \mathcal{H}_+$ is described in period coordinates as the set of solutions of a system of linear equations [EMM15]. In particular, each manifold point of $\mathcal{C}_+$ has a neighborhood U which is mapped by period coordinates homeomorphically onto an open subset V of an affine subspace of $H_1(S,\Sigma;\mathbb{R})^* \otimes \mathbb{C}$. Here as before, Σ is the set of zeros of a differential in the stratum containing $\mathcal{C}_+$. The period coordinates are obtained by integration of a holomorphic one-form over a basis of the relative homology group $H_1(S,\Sigma;\mathbb{Z})$. The tangent bundle of $\mathcal{C}_+$ is invariant under the complex structure induced from the complex structure on $H_1(S,\Sigma;\mathbb{R})^* \otimes \mathbb{C}$.

Recall the definition of the foliation W^{ss}, W^{su} of a component $\mathcal{Q}_+$ of a stratum of abelian differentials. In period coordinates, a local leaf of the strong unstable foliation W^{su} through a point $w \in H_1(S,\Sigma;\mathbb{R})^* \otimes \mathbb{C}$ consists of all differentials whose imaginary parts coincide with the imaginary part of w, and the local leaf of the strong stable foliation consists of all differentials whose real parts coincide with the real part of w. This discussion immediately implies

Corollary 3.3. *Let $\mathcal{C}_+$ be an affine invariant manifold. Then $\mathcal{C}_+ \cap W^i$ is a smooth foliation of $\mathcal{C}_+$ (in the orbifold sense) into leaves of real dimension* $\dim_{\mathbb{C}}(\mathcal{C}_+)$ *($i = ss, su$).*

Remark 3.4. The dimension formula in Corollary 3.3 stems from the fact that we look at differentials whose area is not normalized. If we denote as before by $\mathcal{C} \subset \mathcal{C}_+$ the hyperplane of differentials of area one, then the intersection of $\mathcal{C}$ with a local leaf of the strong unstable or strong stable foliation, respectively, is a submanifold of $\mathcal{C}$ of real dimension $\dim_{\mathbb{C}}(\mathcal{C}_+) - 1$.

Corollary 3.3 implies that for every affine invariant manifold $\mathcal{C}$, every point $q \in \mathcal{C}_{\mathrm{good}}$ has a neighborhood *with a product structure*. We next define a set with a product structure formally. To this end let again $\Sigma \subset S$ be the set of zeros of a differential in the component $\mathcal{Q}$ of a stratum containing $\mathcal{C}$. An abelian differential $\omega \in \mathcal{Q}$ then determines an euclidean metric on $S - \Sigma$, given by a system of complex local coordinates z on $S - \Sigma$ for which ω assumes the form $\omega = dz$. Chart transitions are translations. The foliations of S into horizontal and vertical line segments, respectively, are equipped with a transverse invariant measure by integration of the imaginary or real part, respectively, of ω. These measured foliations are oriented, and they are called the *horizontal and vertical measured foliation* of ω. Via integration with respect to the transverse measure and respecting orientation, these foliations define points in $H_1(S,\Sigma;\mathbb{R})^*$.

Definition 3.5. A closed contractible set $V \subset \mathcal{C}_{\rm good}$ with dense interior *admits a product structure* if for some component $\tilde V$ of V of the preimage of V in the Teichmüller space of abelian differentials, there are two disjoint compact subsets D, K of the set of (marked) projective measured foliations on S, viewed as projective classes of points in $H_1(S,\Sigma;\mathbb{R})^*$, with the following properties.

(1) The sets D, K are homeomorphic to closed balls of dimension
$$m = \dim_{\mathbb{C}}(\mathcal{C}_+) - 1.$$
(2) There is a continuous map
$$\Lambda : D \times K \to \tilde V$$
such that for any pair $(\xi,\nu) \in D \times K$, the horizontal projective measured foliation of $\Lambda(\xi,\nu)$ equals ξ, and its vertical projective measured foliation equals ν.
(3) There is some $\epsilon > 0$ such that
$$\tilde V = \cup_{-\epsilon \le t \le \epsilon} \cup_{(\xi,\nu)\in D\times K} \Phi^t \Lambda(\xi,\nu).$$

We say that an open subset U of $\mathcal{C}_{\rm good}$ *has a product structure* if its closure has a product structure in the sense of Definition 3.5. We refer to Section 3.1 of [H13] for a detailed description of this construction for strata which carries over word by word to affine invariant manifolds. The requirement (1) in Definition 3.5 is made for convenience of exposition; we will occasionally talk about a set with a product structure which only has properties (2) and (3) above.

We use sets with a product structure in $\mathcal{C}_{\rm good}$ to establish a local version of the so-called *shadowing property for pseudo-orbits* for the Teichmüller flow on $\mathcal{C}$. A global shadowing property is a distinguished feature of hyperbolic flows as discovered by Bowen [Bw73], and it holds true for the restriction of the Teichmüller flow to compact invariant sets [H10].

Definition 3.6. Let $\mathcal{V} = \{V_i \mid i \in \mathcal{I}\}$ be a non-empty finite or infinite collection of open contractible subsets of $\mathcal{C}$. For some $n > 0$, an $(n,\mathcal{V})$*-pseudo-orbit* for the Teichmüller flow Φ^t on $\mathcal{C}$ consists of a sequence of points $q_0, q_1, \dots, q_m \in \mathcal{C}$ and a sequence of numbers $t_0, \dots, t_{m-1} \in [n,\infty)$ with the following property. For every $j \le m$, there exists some $\kappa(j) \in \mathcal{I}$ such that $\Phi^{t_{j-1}} q_{j-1}, q_j \in V_{\kappa(j)}$. The pseudo-orbit is called *periodic* if $q_m = q_0$.

The shadowing property [Bw73] for hyperbolic flows on a compact Riemannian manifold states that for sufficiently large n and sufficiently small ϵ, if $\mathcal{V}_\epsilon$ is the collection of all open balls of radius ϵ then an $(n,\mathcal{V}_\epsilon)$-pseudo-orbit is fellow-traveled by an orbit: For an arbitrarily prescribed number $\sigma > 0$, there are $n > 0, \epsilon > 0$ such that for any $(n,\mathcal{V}_\epsilon)$-pseudo-orbit η, there exists an orbit segment whose Hausdorff distance to η is less than σ. This orbit segment can be chosen to be a periodic orbit in the case that the pseudo-orbit is periodic.

For the Teichmüller flow on components of strata or, more generally, on affine invariant manifolds, we can not expect that the shadowing property for all small balls holds true. However, Lemma 2.3 indicates that there should be a local, non-uniform version of shadowing. Proposition 3.7 below establishes such a local version for periodic pseudo-orbits on affine invariant manifolds.

For its formulation, assume that the collection $\mathcal{V}$ of open contractible subsets V_i ($i \in \mathcal{I}$) of $\mathcal{C}$ is finite and that we are given a periodic $(n, \mathcal{V})$-pseudo-orbit, specified by points $q_0, q_1, \dots, q_m = q_0 \in \mathcal{C}$, numbers $t_0, \dots, t_{m-1} \in [n, \infty)$ and indices $\kappa(j) \in \mathcal{I}$. Connect $\Phi^{t_i} q_i$ to q_{i+1} by an arc α_{i+1} in $V_{\kappa(i+1)}$. The concatenation of the orbit segments connecting q_i to $\Phi^{t_i} q_i$ with the arcs α_{i+1} defines a closed curve η in $\mathcal{C}$ which we call a *characteristic curve* of the pseudo-orbit. Note that such a characteristic curve is by no means unique, but as the sets V_i have been selected a priori, any other such curve can be obtained by a collection of small deformations in a fixed contractible set.

Recall from Section 2 that a closed curve in $\mathcal{C}_{\rm good}$ defines the conjugacy class of some mapping class.

Proposition 3.7. *Let $\mathcal{C}$ be an affine invariant manifold, let $q_1, \dots, q_m \in \mathcal{C}_{\rm good}$ be good birecurrent points, and for each i let U_i be a neighborhood of q_i. Then there are neighborhoods $Y_i \subset V_i \subset U_i$ of q_i, and there is a number $R_0 > 0$ with the following property. Suppose that $y_0, \dots, y_{s-1}$ is a sequence of points with $y_i \in Y_{\kappa(i)}$ for all i and some $\kappa(i) \in \{1, \dots, m\}$, and that there are numbers $t_i > R_0$ such that $\Phi^{t_i} y_i \in Y_{\kappa(i+1)}$ (indices are taken modulo s). Let η be a characteristic curve of the corresponding periodic pseudo-orbit of Φ^t. Then there is a periodic orbit $\gamma \subset \mathcal{C}$ for Φ^t which passes through each of the sets $V_{\kappa(i)}$ at times close to $\sum_{j \le i-1} t_j$ and which defines the same conjugacy class in* $\mathrm{Mod}(S)$ *as η.*

Proof. Using the notation from the proposition, for each i choose a closed contractible neighborhood $V_i \subset U_i$ of q_i with a product structure. Recall that such a product structure is determined by a choice $\tilde{V}_i$ of a component of the preimage of V_i in the Teichmüller space of marked abelian differentials, of two closed disjoint subsets D_i, K_i of the space of projective measured foliations which are homeomorphic to closed balls of dimension $d = \dim_{\mathbb{C}}(\mathcal{C}_+) - 1$, an embedding $\Lambda_i : D_i \times K_i \to \tilde{V}_i$ and a number $\epsilon > 0$ with the properties stated in Definition 3.5.

For each j and each $\tilde{z} \in \tilde{V}_j$, the product structure determines a closed local strong unstable manifold $W^{su}_{\rm loc}(\tilde{z})$ containing $\tilde{z}$ which is homeomorphic to a closed ball of dimension d. This set consists of all points whose marked horizontal measured foliation coincides with the marked horizontal measured foliation of $\tilde{z}$, and whose vertical projective measured foliation is contained in K_j. Similarly we obtain a local strong stable manifold $W^{ss}_{\rm loc}(z)$. The sets $W^i_{\rm loc}(\tilde{z})$ $(i = ss, su)$ need not be contained in $\tilde{V}_j$, but every $\tilde{y} \in W^i_{\rm loc}(\tilde{z})$ can be moved into V_j with a small translate along the flow line of Φ^t through $\tilde{y}$. We require that the projection into $\mathcal{C}$ of the union of all these local manifolds are contained in a fixed contractible subset of U_j. For $z \in V_j$ we denote by $W^i_{\rm loc}(z)$ the projection to $\mathcal{C}$ of the set $W^i_{\rm loc}(\tilde{z})$ for the preimage $\tilde{z}$ of z in $\tilde{V}_j$; this does not depend on the choice of the component $\tilde{V}_j$.

The tangent bundle of the strong stable or strong unstable foliation of the component $\mathcal{Q}$ can be equipped with the so-called *modified Hodge norm* which induces a *Hodge distance* d_H on these leaves. By Theorem 8.12 of [ABEM12], there exists a number $c_H > 0$ not depending on choices such that for any $q \in V_j$, any $q' \in W^{ss}_{\rm loc}(q)$ and all $t > 0$ we have

$$d_H(\Phi^t q, \Phi^t q') \leq c_H d_H(q, q'). \tag{3}$$

By the definition of a product structure, for any two points $\tilde{u}, \tilde{z} \in \tilde{V}_j$ there exists a *holonomy homemorphism* $W^{su}_{\rm loc}(\tilde{u}) \to W^{su}_{\rm loc}(\tilde{z})$ which maps a point $u' \in W^{su}_{\rm loc}(\tilde{u})$ to the intersection of $W^{su}_{\rm loc}(\tilde{z})$ with the local *stable manifold* through u'; this local stable manifold consists of translates under Φ^t of $W^{ss}_{\rm loc}(u')$. These holonomy homeomorphisms are smooth and depend smoothly on $\tilde{u}, \tilde{z}$. In particular, they are bilipschitz for the Hodge distance. By decreasing the size of the sets V_j we may assume that the bilipschitz constants for these holonomy maps is at most 2.

Choose a neighborhood $Y_j \subset V_j$ of q_j with a product structure so that for $z \in Y_j$, the local strong stable and strong unstable manifolds $W^i_{{\rm loc},Y_j}(z)$ $(i = su, ss)$ constructed as above for Y_j have the following additional property. There exists a number $r > 0$ such that for any $z \in Y_j$, the d_H-distance between the set $W^i_{{\rm loc},Y_j}(z)$ and the boundary of $W^i_{\rm loc}(z)$ is at least r. We assume furthermore that for each $y \in Y_j$ the d_H-diameter of the local strong stable or strong unstable manifold $W^i_{{\rm loc},Y_j}(y)$ $(i = ss, su)$ is at most r. Let $\chi > 2r$ be an upper bound for the d_H-diameter of the sets $W^i_{\rm loc}(V_j)$.

The growth estimate (3) for the Hodge distance and birecurrence of the points q_i is used in the proof of Proposition 5.4 of [H13] to show the following. Up to decreasing the size of the sets Y_j, for each j, there exists a number $R_j > 0$ with the following property. If $z \in Y_j$ and if $T > R_j$ then

$$\begin{aligned} d_H(\Phi^T z, \Phi^T z') &\leq \frac{r}{4\chi} d_H(z, z') \text{ for all } z' \in W^{ss}_{\rm loc}(z) \text{ and} \\ d_H(\Phi^{-T} z, \Phi^{-T} z') &\leq \frac{r}{4\chi} d_H(z, z') \text{ for all } z' \in W^{su}_{\rm loc}(z). \end{aligned} \tag{4}$$

Let $R = \max_j R_j$, let $\mathcal{Y} = \{Y_j\}$ and let η be the characteristic curve of a periodic $(R, \mathcal{Y})$-pseudo-orbit. By definition, η is determined by points $y_j \in Y_{\kappa(j)}$ and numbers $t_j > 0$ $(j \geq 0)$. Parametrize η in such a way that $\eta(\sum_{j<s} t_j + s) = y_s$. For simplicity of notation, assume that $\eta(0) \in Y_1$. Let $T > 0$ be such that $\eta(T) = \eta(0)$.

Let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of area one abelian differentials, contained in the component $\tilde{\mathcal{Q}}$ of the preimage of $\mathcal{Q}$. Let $\tilde{\eta}$ be a lift of η to $\tilde{\mathcal{C}}$ which begins at $\tilde{\eta}(0) = \tilde{y}_0$. Using the notation from the beginning of this proof, we may assume that $\tilde{y}_0 \in \tilde{V}_1$. Then there is an element $\varphi \in {\rm Mod}(S)$ which maps the endpoint $\tilde{\eta}(T)$ of $\tilde{\eta}$ back to $\tilde{y}_0$. As any element of ${\rm Mod}(S)$ either stabilizes $\tilde{\mathcal{C}}$ or maps $\tilde{\mathcal{C}}$ to a disjoint component of the preimage of $\mathcal{C}$, we know that $\varphi \in {\rm Stab}(\tilde{\mathcal{C}})$.

The proof of Proposition 5.4 of [H13] shows that the mapping class φ is pseudo-Anosov. Our goal is to show that it defines a periodic orbit γ in $\mathcal{C}$ with the properties stated in the proposition. To this end we use the argument in the proof of Proposition 5.4 of [H13].

Let $\tilde{\gamma} \subset \tilde{\mathcal{Q}}$ be the cotangent line of the axis in Teichmüller space of the pseudo-Anosov element φ. The curve $\tilde{\gamma}$ is a φ-invariant orbit of the Teichmüller flow in $\tilde{\mathcal{Q}}$ which projects to the periodic orbit γ. The (biinfinite) lift $\tilde{\eta}$ of the characteristic curve η is contained in a uniformly bounded neighborhood of $\tilde{\gamma}$.

The pseudo-Anosov element φ acts with north-south dynamics on the Thurston sphere $\mathcal{PML}$ of projective measured foliations of the surface S. This means that φ has precisely two fixed points in $\mathcal{PML}$, one is attracting, the other repelling. Furthermore, if $\tilde{u} \in \tilde{\gamma}$ is arbitrary, then the vertical projective measured foliation ξ of $\tilde{u}$ equals the attracting fixed point of φ, and the horizontal projective measured foliation ν of $\tilde{u}$ equals the repelling fixed point of φ.

It now suffices to verify that with the above notation, we have $\xi \in D_1, \nu \in K_1$. Namely, every flow line of the Teichmüller flow in the Teichmüller space of abelian differentials which is defined by a differential with horizonal measured foliation in D_1 and vertical measured foliation in K_1 passes through the set $\tilde{V}_1$ and hence it is entirely contained in $\tilde{\mathcal{C}}$. As the choice of initial point of the periodic pseudo-orbit is arbitrary among the starting points of the orbit segments which determine the pseudo-orbit, the periodic orbit then has the properties stated in the proposition.

Using the reasoning on p.524 of [H13], we show that indeed $\xi \in D_1$. Consider the point $\tilde{\eta}(t_0+1)$. It is contained in the same component $\tilde{Y}_{\kappa(1)}$ of the preimage of $Y_{\kappa(1)}$ as $\tilde{\eta}(t_0)$. Moreover, we have $\tilde{\eta}(t_0) = \Phi^{t_0}\tilde{z}_0$.

By the choice of the sets Y_j, if we denote by $\tilde{v}_1$ the point on $W^{su}_{\rm loc}(\tilde{\eta}(t_0))$ which is the image of $\tilde{\eta}(t_0+1)$ under the holonomy map, then $\tilde{v}_1$ is contained in the local strong unstable manifold $W^{su}_{{\rm loc},\tilde{Y}_{\kappa(1)}}(\tilde{\eta}(t_0)) \subset W^{su}_{\rm loc}(\tilde{\eta}(t_1))$ by the very definition of these sets and the properties of the pseudo-orbit. In particular, the Hodge distance between $\tilde{\eta}(t_0)$ and $\tilde{v}_1$ is at most r.

As $t_0 > R_0$, the image of the set $W^{su}_{\rm loc}(\tilde{q}_1)$ of diameter at most χ under the map Φ^{-t_0} is of diameter at most $r/4$. In particular, the Hodge distance between $\Phi^{-t_0}\tilde{v}_1$ and $\tilde{y}_0 = \Phi^{-t_0}\tilde{\eta}(t_0)$ is at most $r/4$. But $\tilde{y}_0 \in \tilde{Y}_1 \subset \tilde{V}_1$ where $\tilde{Y}_i$ is the component of the preimage of Y_i contained in $\tilde{V}_i$ and hence the Hodge distance of $\Phi^{-t_0}\tilde{v}_1$ to any boundary point of $W^{su}_{\rm loc}(\tilde{y}_0)$ is at least $3r/4$. This implies that $\Phi^{-t_0}W^{su}_{\rm loc}(\tilde{\eta}(t_0)) \subset W^{su}_{\rm loc}(\tilde{z}_0)$. In particular, if we denote by $K_{\kappa(1)}$ the closed set of all vertical projective measured foliations for points in the component of the preimage of $V_{\kappa(1)}$ containing $\tilde{v}_1$ then we have $K_{\kappa(1)} \subset K_1$.

For $s \geq 1$ let now $\hat{K}_{\kappa(s)}$ be the set of all projective measured foliations of all marked abelian differentials which are contained in the component $\tilde{V}_{\kappa(s)}$ of the preimage of $V_{\kappa(s)}$ containing $\tilde{\eta}(\sum_{j<s} t_j + s)$. By induction on s, we now show that for any $s \geq 1$, the set $\hat{K}_{\kappa(s)}$ is entirely contained in K_1. So let us assume that

this holds true for all $s < s_0$. Then $\hat{K}_{\kappa(s_0)} \subset \hat{K}_{\kappa(1)}$ by the induction hypothesis, however we showed above that $\hat{K}_{\kappa(1)} \subset K_1$. This yields the induction step.

To summarize, for each $t > 0$ the vertical projective measured foliation of $\tilde{\eta}(t)$ is contained in the compact set K_1. As the attracting fixed point of φ is the limit as $t \to \infty$ of the vertical projective measured foliation of $\tilde{\eta}(t)$ (see the proof of Proposition 5.4 of [H13] for details), this attracting fixed point of φ is indeed contained in K_1.

The same argument applies to the repelling fixed point of φ and shows that this repelling fixed point is contained in D_1. In particular, the periodic orbit of Φ^t defined by φ is contained in $\mathcal{C}$, and it passes through V_1. As remarked earlier on, this suffices for the proof of the proposition. □

Remark 3.8. The proof of Proposition 3.7 follows the strategy from the proof of Proposition 5.4 of [H13]. However, the argument in [H13] is more involved as it contains the proof of the most important property used in the above argument: The non-uniform contraction of the Hodge distance on the strong stable manifold of a birecurrent point.

Remark 3.9. Let $\mathcal{C}$ be an affine invariant manifold, contained in a component $\mathcal{Q}$ of a stratum, and let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials. If $\varphi \in \mathrm{Mod}(S)$ defines a periodic orbit of the Teichmüller flow on $\mathcal{C}$, then φ is a pseudo-Anosov mapping class which is conjugate to an element of $\mathrm{Stab}(\tilde{\mathcal{C}})$. However, it is not true that any pseudo-Anosov mapping class in $\mathrm{Stab}(\tilde{\mathcal{C}})$ determines a periodic orbit for Φ^t contained in the closure of $\mathcal{C}$. An example of this situation is the case that $\mathcal{C}$ equals a non-principal stratum of abelian differentials with at least one simple zero. In this case the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials is connected [H16] and hence the stabilizer of this preimage equais the entire mapping class group. However, the set of periodic orbits for the Teichmüller flow contained in the closure of $\mathcal{C}$ is a proper subset of the set of all periodic orbits.

The last goal of this subsection is to establish a parametrized version of Proposition 3.7. This is needed to associate to a periodic orbit of Φ^t on $\mathcal{C}$ an element of $\mathrm{Mod}(S)$ rather than a conjugacy class in such a way that adjunction of orbit segments in a pseudo-orbit corresponds to multiplication of group elements.

To this end let again $q \in \mathcal{C}_{\mathrm{good}}$ be a good birecurrent point. Let $U \subset \mathcal{C}_{\mathrm{good}}$ be a neighborhood of q and let $Z \subset V \subset U$ be a nested family of neighborhoods of q in $\mathcal{C}_{\mathrm{good}}$ as in Proposition 3.7. We may assume that V is contractible, that Z has a product structure and that any connected component of the intersection with Z of an orbit segment of the Teichmüller flow is an arc of fixed length $2t_0$.

For sufficiently large $R_0 > 0$ let $z \in Z$ and let $T > R_0$ be such that $\Phi^T z \in Z$. A characteristic curve of this orbit segment determines uniquely a periodic orbit γ of Φ^t which intersects V in an arc of length $2t_0$. There may be more than one such intersection arc, but there is a unique arc determined by the component of the intersection $V \cap \Phi^T V$ containing the point z similar to the statement of Lemma

2.3. Choose the midpoint of this intersection arc as a basepoint for γ and as an initial point for a parametrization of γ.

Let Γ_0 be the set of all parametrized periodic orbits of this form for points $z \in Z$ with $\Phi^T z \in Z$. The map which associates to a component of $\Phi^T V \cap V$ containing points in $\Phi^T Z \cap Z$ the corresponding parametrized periodic orbit in Γ_0 is a bijection.

Fix once and for all a lift $\tilde{V}$ of the contractible set V to a component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials. A periodic orbit γ which intersects V in an arc of length $2t_0$ lifts to a subarc of a flow line of the Teichmüller flow on $\tilde{\mathcal{Q}}$ with starting point in $\tilde{V}$. The endpoints of this arc are identified by a pseudo-Anosov element $\Omega(\gamma) \in \mathrm{Mod}(S)$. The conjugacy class of $\Omega(\gamma)$ is uniquely determined by γ, and the element $\Omega(\gamma)$ only depends on the choice of $\tilde{V}$ (and the component of $\gamma \cap V$ as explained above). In particular, following Proposition 3.7, a characteristic curve of a sufficiently long orbit segment beginning and ending in Z determines an element in $\mathrm{Mod}(S)$.

The following proposition is a parametrized version of shadowing as established in Proposition 3.7.

Proposition 3.10. *For $\gamma_1, \dots, \gamma_m \in \Gamma_0$, there is a point $z \in V$, and there are numbers $0 < t_1 < \cdots < t_m$ with the following properties.*

(1) $\Phi^{t_i} z \in V$.
(2) *For each $i \leq m$, a characteristic curve of the orbit segment $\{\Phi^t z \mid t_{i-1} \leq t \leq t_i\}$ defines the element $\Omega(\gamma_i)$ in* $\mathrm{Mod}(S)$.
(3) *A characteristic curve of the orbit segment $\{\Phi^t z \mid 0 \leq t \leq t_m\}$ determines a parametrized periodic orbit γ for Φ^t with initial point in V, and $\Omega(\gamma) = \Omega(\gamma_k) \circ \cdots \circ \Omega(\gamma_1)$.*

Proof. In the case that the arcs γ_i are contained in a fixed compact invariant subset K for Φ^t and that the set V is chosen small in dependence of K, the lemma is identical with the slight weakening of Theorem 4.3 of [H10]. That the statement holds true in the form presented here is immediate from Proposition 3.7. $\square$

As a consequence, the subsemigroup $\langle \Omega(\Gamma_0) \rangle$ of $\mathrm{Mod}(S)$ generated by $\{\Omega(\gamma) \mid \gamma \in \Gamma_0\}$ consists of pseudo-Anosov elements whose corresponding periodic orbits are contained in the affine invariant manifold $\mathcal{C}$ and pass through the set V. This can be viewed as a version of Rauzy-Veech induction as used in [AV07, AMY16] which is valid for all affine invariant manifolds, in particular for strata of quadratic differentials, or as a version of symbolic dynamics for the Teichmüller flow on strata as in [H16].

4. Local Zariski density for affine invariant manifolds

The goal of this section is to prove a version of Theorem 2 for arbitrary affine invariant manifolds in the moduli space of abelian differentials of a surface of genus $g \geq 3$. Throughout this section we assume that $g \geq 3$, and we use the assumptions and notations from Section 3.

Let $\mathcal{Q}_+ \subset \mathcal{H}_+$ be a component of a stratum and let $\mathcal{C}_+ \subset \mathcal{H}_+$ be an affine invariant manifold. Recall from Section 3 that the image of the projection $p : T\mathcal{C}_+ \to \Pi^*\mathcal{N} \otimes \mathbb{C}|\mathcal{C}_+ = \Pi^*(\mathcal{H} \oplus \overline{\mathcal{H}})$ to absolute periods is a flat subbundle $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ of $\Pi^*\mathcal{N} \otimes \mathcal{C}|\mathcal{C}_+$ which is invariant under both the complex structure defined by multiplication with i as well as the complex structure of the Hodge bundle. We denote by $2\ell \geq 1$ its complex dimension. Then $p(T\mathcal{C}_+) \cap \Pi^*\mathcal{N}|\mathcal{C}_+$ is a flat bundle whose fibre is a symplectic subspace of $H^1(S, \mathbb{R})$ of real dimension 2ℓ. As before, by a flat subbundle of the bundle $\Pi^*\mathcal{N}|\mathcal{C}_+$ we mean a bundle which is invariant under the restriction of the Gauss Manin connection.

As a real vector bundle, $p(T\mathcal{C})_+ \cap \Pi^*\mathcal{N}|\mathcal{C}_+$ can be identified with the holomorphic bundle $\mathcal{Z}$ and we will use this identification throughout this section for convenience of notation. In particular, the Gauss Manin connection induces a flat connection on $\mathcal{Z}$. The monodromy of the restriction of the Gauss Manin connection to $\mathcal{Z}$ is defined as the subgroup of $GL(2\ell, \mathbb{R})$ which is generated by parallel transport along loops based at some fixed point p. As the Gauss Manin connection is symplectic, this monodromy group is a subgroup of $Sp(2\ell, \mathbb{R})$. Its conjugacy class does not depend on any choices made.

Definition 4.1. The *monodromy group* of the affine invariant manifold $\mathcal{C}_+$ of rank ℓ is the subgroup of $Sp(2\ell, \mathbb{R})$ which is the monodromy of the absolute holomorphic tangent bundle $\mathcal{Z}$ of $\mathcal{C}_+$ for the restriction of the Gauss Manin connection.

A geometric description of the monodromy group of $\mathcal{C}_+$ is as follows. Observe first that the monodromy coincides with the monodromy of the restriction of the bundle $\mathcal{Z}$ to the intersection $\mathcal{C}$ of $\mathcal{C}_+$ with the moduli space of area one abelian differentials. Let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials. The stabilizer $\mathrm{Stab}(\tilde{\mathcal{C}})$ of $\tilde{\mathcal{C}}$ in the mapping class group maps via the natural surjective projection $\Psi : \mathrm{Mod}(S) \to Sp(2g, \mathbb{Z})$ to a subgroup of $Sp(2g, \mathbb{Z})$. There is a linear symplectic subspace $H \subset \mathbb{R}^{2g}$ of dimension 2ℓ which is preserved by $\Psi(\mathrm{Stab}(\tilde{\mathcal{C}}))$. The monodromy group of $\mathcal{C}$ then is the projection of $\Psi(\mathrm{Stab}(\tilde{\mathcal{C}}))$ to the group of symplectic automorphisms of H. This description is immediate from the description of the Gauss Manin connection in Section 2.

Example 4.2. If $\mathcal{C}_+$ is a Teichmüller curve then the monodromy group of $\mathcal{C}_+$ is just the Veech group of $\mathcal{C}_+$. Thus this monodromy group is a lattice in $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$, in particular it is Zariski dense in $SL(2, \mathbb{R})$.

The monodromy group of a component of a stratum is a subgroup of $Sp(2g, \mathbb{Z})$.

To investigate the monodromy group of arbitrary affine invariant manifolds we will make use of the fact that an abelian differential on S defines a singular euclidean metric on S with cone points of cone angle a multiple of 2π at the zeros of the differential. This singular euclidean metric is given by a family of charts, defined on the complement of the zeros of the differential, with chart transitions being translations. As it is customary in the literature, if we view an abelian differential on S as a singular euclidean metric, we refer to these data as a *translation surface*. We denote such a translation surface by X or by a pair (X,ω) if we like to specify the abelian differential ω which defines the translation structure. Note that ω can be read off from the horizontal and vertical measured foliations of the translation surface.

We begin with evoking a result of Wright [W15]. He introduced the following two deformations of a translation surface (X,ω).

The *horocycle flow* is defined as part of the $SL(2,\mathbb{R})$-action,

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \subset SL(2,\mathbb{R}),$$

and the *vertical stretch* is defined by

$$a_t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \subset GL^+(2,\mathbb{R}).$$

For a collection $\mathcal{Y}$ of horizontal cylinders on a translation surface X, define the *cylinder shear* $u_t^{\mathcal{Y}}(X)$ to be the translation surface obtained by applying the horocycle flow to the cylinders in $\mathcal{Y}$ but not to the rest of X. Similarly, the *cylinder stretch* $a_t^{\mathcal{Y}}(X)$ is obtained by applying the vertical stretch only to the cylinders in $\mathcal{Y}$.

The following lemma is a fairly easy consequence of the work of Wright [W15]. For its formulation, a translation surface (X,ω) is called *horizontally periodic* if it is a union of horizontal cylinders.

Lemma 4.3. *Let $\mathcal{C}_+$ be an affine invariant manifold of rank ℓ. Suppose that $(X,\omega)\in\mathcal{C}_+$ is horizontally periodic, and that there is a decomposition of X into ℓ collections $\mathcal{Y}_1,\dots,\mathcal{Y}_\ell$ of horizontal cylinders so that for each i, the cylinder shear $u_t^{\mathcal{Y}_i}(X)$ remains in $\mathcal{C}_+$. Then for each i, the moduli of the cylinders in the collection $\mathcal{Y}_i$ are rationally dependent.*

Proof. For each i, the collection $\mathcal{Y}_i$ consists of $r_i\geq 1$ cylinders. By assumption, the cylinder shear $u_t^{\mathcal{Y}_i}(X)$ remains entirely in $\mathcal{C}_+$. A local version of Lemma 3.1 of [W15], applied to this cylinder shear rather than to the full horocycle flow (note that the proof of this local version is identical to the proof given in Section 3 of [W15]) shows the following. If the moduli $m_1^i,\dots,m_{r_i}^i$ of the cylinders in $\mathcal{Y}_i$ are not rationally dependent, then there is a proper subcollection $\mathcal{V}$ of $\mathcal{Y}_i$ consisting of $1\leq s<r_i$ cylinders so that the cylinder shear $u_t^{\mathcal{V}}$ for this subcollection is contained in $\mathcal{C}_+$.

But then there are at least $\ell+1$ pairwise distinct collections of horizontal cylinders in (X,ω) with the property that the cylinder shear of X for each of these collections is contained in $\mathcal{C}_+$ (see Section 3 of [W15] for details). This violates Theorem 1.10 of [W15] and yields that indeed, for fixed i the moduli m_j^i $(1\leq j\leq r_i)$ of the cylinders in $\mathcal{Y}_i$ are rationally dependent. ☐

Define a *piecewise affine transformation* of a translation surface (X,ω) to be a continuous self-map $F : X \to X$ with the following property. There exists a decomposition $X = \cup_i X_i$ into finitely many components with geodesic boundary for the singular euclidean metric which is preserved by F, and the restriction of F to each of these components is affine. In contrast to an affine automorphism of (X,ω), we allow that the restriction of F to some of the components X_i equals the identity. A cylinder shear of a collection $\mathcal{Y}$ of horizontal cylinders with non-empty complement is such a piecewise affine transformation. If the result of such a transformation is isometric to (X,ω) then we call the piecewise affine transformation a *piecewise affine automorphism* of (X,ω).

A *transvection* in a rank 2ℓ symplectic vector space (H,ι) is a symplectic automorphism of H which fixes a subspace of H of (real) codimension one pointwise and has determinant one. Any map of the form $x \to x + \iota(x,z)z$ for some $0 \neq z \in H$ is a transvection. We call this map a *transvection by z*. The main consequence of Lemma 4.3 we are going to use is the following

Corollary 4.4. *Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 2$. Then there is a horizontally periodic surface $(X,\omega) \in \mathcal{C}_+$, and there is a free abelian group of rank ℓ of piecewise affine transformations of (X,ω) which preserves $\mathcal{C}_+$. This group of affine transformations contains a lattice H, ie a subgroup isomorphic to $\mathbb{Z}^\ell$, which acts on (X,ω) as a group of piecewise affine automorphisms consisting of Dehn-multitwists. The group H acts on $H_1(S,\mathbb{R})$ as a group of transvections of rank ℓ.*

Proof. By Theorem 1.10 of [W15] and its proof (more precisely, the results in Section 8 of [W15]), the affine invariant manifold $\mathcal{C}_+$ contains a horizontally periodic surface (X,ω) which admits a decomposition into ℓ cylinder families $\mathcal{Y}_1,\dots,\mathcal{Y}_\ell$ with the properties stated in Lemma 4.3. Moreover, for each i and each t the image of X under the vertical stretch $a_t^{\mathcal{Y}_i}(X)$ is contained in $\mathcal{C}_+$. These vertical stretches commute.

The vertical stretch $a_t^{\mathcal{Y}_i}$ changes the heights of the horizontal cylinders in the family $\mathcal{Y}_i$ while keeping their circumferences fixed. The image translation surfaces are horizontally periodic. Using Lemma 4.3, this implies that we can find $t_1,\dots,t_\ell \in \mathbb{R}$ so that the modulus of *every* horizontal cylinder in

$$Z = a_{t_1}^{\mathcal{Y}_1} \cdots a_{t_\ell}^{\mathcal{Y}_\ell}(X)$$

is rational.

Using again the results in Section 8 of [W15], the affine invariant manifold $\mathcal{C}_+$ contains the images of the translation surface Z under the cylinder shears $u_t^{\mathcal{Y}_i}(Z)$ where by abuse of notation, we denote again by $\mathcal{Y}_i$ the cylinder family on Z which is the image of the horizontal cylinder family $\mathcal{Y}_i$ on (X,ω). As the moduli of all cylinders in the family $\mathcal{Y}_i$ are rational, these cylinder shears are eventually periodic. This means that for each i there exists some number $r_i > 0$ such that for some fixed marking of the surface Z, the surface $u_{r_i}^{\mathcal{Y}_i}(Z)$ is the image of Z by a Dehn multitwist T_i about the core curves of the cylinders in $\mathcal{Y}_i$.

Since the core curves of the horizontal cylinders in Z are pairwise disjoint, the Dehn multitwists T_i commute. Therefore these multitwists generate a free abelian group of rank ℓ of piecewise affine automorphisms of Z. The multitwist T_i acts as a transvection on $H_1(S,\mathbb{R})$ by a homology class of the form $\sum_s b_i^s \zeta_i^s$ where $b_i^s \in \mathbb{Z}$ and where ζ_i^s runs through the homology classes of the waist curves of the oriented cylinders in the family $\mathcal{Y}_i$.

Using the natural pairing between homology and cohomology and pull-back under the projection $p : T\mathcal{C}_+ \to \Pi^*\mathcal{N} \otimes \mathbb{C}$, the homology class $a_i = \sum_s b_i^s \zeta_i^s$ induces a linear functional on the fibre of $T\mathcal{C}_+$ at Z. The proposition now follows from another application of Theorem 1.10 of [W15]: The rank of the subspace of $T\mathcal{C}_+^*$ spanned by these homology classes equals ℓ. But this just means that the subgroup of $\mathrm{Mod}(S)$ generated by the Dehn multitwists T_i $(i = 1, \dots, \ell)$ acts on $H_1(S,\mathbb{R})$ as a group of transvections of rank ℓ. $\square$

Definition 4.5. An affine invariant manifold $\mathcal{C}_+$ of rank ℓ is called *locally Zariski dense* if for every open contractible subset U of $\mathcal{C}_+$ the subsemigroup of $Sp(2\ell,\mathbb{R})$ generated by the monodromy of those periodic orbits for Φ^t in $\mathcal{C}_+$ which pass through U is Zariski dense in $Sp(2\ell,\mathbb{R})$.

Here as before, monodromy means monodromy of the restriction of the Gauss Manin connection to the bundle $\mathcal{Z} \to \mathcal{C}_+$, and this is computed with respect to a fixed trivialization of $\mathcal{Z}$ over U which is parallel for the Gauss Manin connection. Replacing such a trivialization by another one changes the local monodromy group by a conjugation.

Our goal is to show

Theorem 4.6. *An affine invariant manifold is locally Zariski dense.*

We begin with reducing the statement of the theorem to a statement on local Zariski density near a single point. To this end we consider again the hypersurface $\mathcal{C}$ of $\mathcal{C}_+$ of area one abelian differentials. Local Zariski density is defined in the same way, and it is equivalent to local Zariski density of $\mathcal{C}_+$.

Lemma 4.7. *An affine invariant manifold $\mathcal{C}$ of rank ℓ is locally Zariski dense if and only if there exists a birecurrent point $q \in \mathcal{C}$ with the following property. For every open neighborhood U of q, the subgroup of $Sp(2\ell,\mathbb{R})$ generated by the monodromy of those periodic orbits for Φ^t in $\mathcal{C}$ which pass through U is Zariski dense in $Sp(2\ell,\mathbb{R})$.*

Proof. That the condition stated in the lemma is necessary is obvious from the definition of local Zariski density. We have to show that it is also sufficient.

To this end let $q \in \mathcal{C}$ be a birecurrent point as in the statement of the lemma. Let U be any open subset of $\mathcal{C}$. Let $z \in U$ be an arbitrary birecurrent point; such a point exists since the set of birecurrent points is dense. Write $U = U_z$. By Proposition 3.7, for every neighborhood U_q of q we can find neighborhoods $Y_z \subset V_z \subset U_z$ of z, $Y_q \subset V_q \subset U_q$ of q and a number $n > 0$ with the following property.

The sets V_z, V_q are contractible. Write $\mathcal{Y} = \{Y_q, Y_z\}$ and let u_0, u_1, u_2, u_3 be a periodic $(n,\mathcal{Y})$-pseudo-orbit for Φ^t, with $u_0 = u_3 \in Y_z$ and $u_1, u_2 \in Y_q$. There are numbers $t_i > n$ such that $\Phi^{t_i} u_i \in Y_{\kappa(i+1)}$ where $\kappa(i+1) = q$ for $i = 0, 1$ and $\kappa(i+1) = z$ otherwise. Such a pseudo-orbit exists since the Teichmüller flow on $\mathcal{C}$ is topologically transitive.

Let η be a characteristic curve for the pseudo-orbit. Then η determines a parametrized periodic orbit ν for Φ^t beginning in V_z, and this orbit passes through V_q.

Choose a component $\tilde{V}_z$ of the preimage of V_z in the Teichmüller space of abelian differentials. This choice determines a pseudo-Anosov element $\Omega(\nu) \in \mathrm{Mod}(S)$. Let $\tilde{u}_0$ be the preimage of u_0 in $\tilde{v}_z$ and let $\tilde{V}_q$ be the component of the preimage of V_z which contains $\Phi^{t_0}\tilde{u}_0$.

Our goal is to show that the subsemigroup of $Sp(2\ell, \mathbb{R})$ generated by the elements $\Psi(\Omega(\nu))$ for periodic orbits ν of the above form is Zariski dense in $Sp(2\ell, \mathbb{R})$. To this end note that if η' is a characteristic curve of a pseudo-orbit defined by points $u_0, u_1', u_2, u_3 = u_0$, with $u_1' \in Z_q$, and times t_0, t_1', t_2, and if ν' is the corresponding periodic orbit, then the element $\Omega(\nu')^{-1} \circ \Omega(\nu)$ of $\mathrm{Mod}(S)$ is defined by the lift beginning in $\tilde{V}_z$ of the concatentation $(\eta')^{-1} \circ \eta$ (recall that this makes sense since η, η' begin at the same point $u_0 \in Z_z$). Thus $\Omega(\nu')^{-1} \circ \Omega(\nu)$ is conjugate to $(g\Omega(\xi'))^{-1} \circ g\Omega(\xi)$ where $\Omega(\xi), \Omega(\xi')$ are the elements of $\mathrm{Mod}(S)$ constructed in the same way from $\tilde{V}_q$ and from parametrized periodic orbits of Φ^t through V_q determined by the one-segment periodic pseudo-orbits $u_1 = u_2$ and $u_1' = u_2'$ and return times t_1, t_1' and where $g \in \mathrm{Mod}(S)$ is defined by the periodic pseudo-orbit given by the points $v_0 = u_2, v_1 = u_0, v_2 = v_0$ and times $s_0 = t_2, s_1 = t_0$.

To complete the proof just note that a subsemigroup G of $Sp(2\ell, \mathbb{R})$ is Zariski dense if and only if for any $h \in Sp(2\ell, \mathbb{R})$ the conjugate hGh^{-1} is Zariski dense if and only if there exists an element $g \in G$ such that $g^{-1}G \subset Sp(2\ell, \mathbb{R})$ is not contained in any proper algebraic subvariety of $Sp(2\ell, \mathbb{R})$. Thus Zariski density of the semigroup constructed from V_q implies Zariski density of the semigroup constructed from V_z. $\square$

Our criterion for Zariski density relies on a result of Hall [Hl08]. For its formulation, for a prime $p \geq 2$ let F_p be the field with p elements. Then $Sp(2g, F_p)$ is a finite group. Therefore for every $A \in Sp(2g, F_p)$ there is some $\ell \geq 1$ such that $A^\ell = A^{-1}$. As a consequence, if $G < Sp(2g, F_p)$ is any sub-semigroup then for all $x, y \in G$ we have $xy^{-1} \in G$ as well and hence $G < Sp(2g, F_p)$ is a group.

Define a *transvection* in $Sp(2g, F_p)$ to be a map $A \in Sp(2g, F_p)$ which fixes a subspace of F_p^{2g} of codimension one and has determinant one (see [Hl08]). Any map of the form

$$\alpha \to \alpha + \iota(\alpha, \beta)\beta$$

for some $0 \neq \beta \in F_p^{2g}$ (here as before, ι is the symplectic form) is a transvection. We call this map a *transvection by* β.

Lemma 4.8. *Let $p \geq 3$ be an odd prime and let $G < Sp(2g, F_p)$ be a subgroup generated by $2g$ transvections by the elements of a set $\mathcal{E} = \{e_1, \ldots, e_{2g}\} \subset F_p^{2g}$ which spans F_p^{2g}. Assume that there is no nontrivial partition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ so that $\iota(e_{i_1}, e_{i_2}) = 0$ for all $e_{i_j} \in \mathcal{E}_j$. Then $G = Sp(2g, F_p)$.*

Proof. For each i write $A_i(x) = x + \iota(x, e_i)e_i$. Let $G < Sp(2g, F_p)$ be the subgroup generated by the transvections $A_1, \dots, A_{2g}$. Since the vectors $e_1, \dots, e_{2g}$ span F_p^{2g}, the intersection of the invariant subspaces of the transvections A_i $(i \leq 2g)$ is trivial.

We claim that the standard representation of G on F_p^{2g} is irreducible. Namely, assume to the contrary that there is an invariant proper linear subspace $W \subset F_p^{2g}$. Let $0 \neq w \in W$; then there is at least one i so that $\iota(w, e_i) \neq 0$. By invariance, we have $w + \iota(w, e_i)e_i \in W$ and hence $e_i \in W$ since F_p is a field.

As a consequence, W is spanned by some of the e_i, say by $e_{i_1}, \dots, e_{i_k}$, and if j is such that $\iota(e_{i_s}, e_j) \neq 0$ for some $s \leq k$ then $e_j \in W$. However, this implies that $W = F_p^{2g}$ by the assumption on the set $\mathcal{E} = \{e_i\}$.

To summarize, G is an irreducible subgroup of $Sp(2g, F_p)$ generated by transvections (where irreducible means that the standard representation of G on F_p^{2g} is irreducible). Furthermore, as p is an odd prime by assumption, the order of each of these transvections is not divisible by 2. Theorem 3.1 of [Hl08] now yields that $G = Sp(2g, F_p)$ which is what we wanted to show. $\square$

Remark 4.9. By Proposition 6.5 of [FM12], Lemma 4.8 is not true for $p = 2$.

We use Lemma 4.8 to establish a criterion for Zariski density of a subgroup of $Sp(2\ell, \mathbb{R})$ acting on a 2ℓ-dimensional symplectic subspace of $H^1(S, \mathbb{R})$. In its formulation, we use the standard pairing

$$\langle , \rangle : H^1(S, \mathbb{R}) \times H_1(S, \mathbb{R}) \to \mathbb{R}$$

between homology and cohomology to view a class in $H_1(S, \mathbb{R})$ as an element of $H^1(S, \mathbb{R})^*$. Then a symplectic automorphism of $H_1(S, \mathbb{R})$ can be viewed as a symplectic automorphism of $H^1(S, \mathbb{R})^*$. Recall also that the real part $\mathrm{Re}(\tilde{q})$ and the imaginary part $\mathrm{Im}(\tilde{q})$ of a marked abelian differential $\tilde{q}$ define a cohomology class $[\mathrm{Re}(\tilde{q})], [\mathrm{Im}(\tilde{q})] \in H^1(S, \mathbb{R})$.

For a symplectic subspace V of $H^1(S, \mathbb{R})$ denote by $Sp(V^*)$ the group of symplectic automorphisms of its dual V^*. Recall that the image of $\mathrm{Mod}(S)$ under the homomorphism Ψ is the integral symplectic group $Sp(2g, \mathbb{Z})$ and hence reduction of coefficients modulo a prime p makes sense. By a weighed oriented simple multicurve c on S we mean a simple oriented multicurve with integral weights. Such a weighted oriented multicurve then defines a homology class $[c] \in H_1(S, \mathbb{Z})$.

Proposition 4.10. *Let $\mathcal{C}$ be an affine invariant manifold of rank ℓ, let $\tilde{\mathcal{C}}_+$ be a component of the preimage of $\mathcal{C}_+$ in the Teichmüller space of abelian differentials and let $V = p(T\tilde{\mathcal{C}}_+) \cap H^1(S, \mathbb{R})$. Let $c_1, \dots, c_\ell$ be pairwise disjoint simple oriented weighted multicurves whose homology classes $[c_i]$ generate a subspace of V^* of rank ℓ. Let $U \subset \mathcal{C}$ be an open contractible set and assume that there is component $\tilde{U}$ of the preimage of U in $\tilde{\mathcal{C}}_+$ such that $\langle [\mathrm{Re}(\tilde{z})], [c_i] \rangle > 0$ for all $\tilde{z} \in \tilde{U}$, all $1 \leq i \leq \ell$. Let $\Omega(\Gamma_0) \subset \mathrm{Mod}(S)$ be the subsemigroup determined by U and the choice of its preimage $\tilde{U}$. Then the subsemigroup of $Sp(V^*)$ generated by $\Psi(\Omega(\Gamma_0))$ and the Dehn multitwists T_{c_i} about the multicurves c_i is Zariski dense in $Sp(V^*)$. If $\ell = g$ then for all but finitely many primes $p \geq 3$, this semigroup surjects onto $Sp(2g, F_p)$.*

Proof. Let $\mathcal{C}$ be an affine invariant manifold of rank ℓ. Let $U \subset \mathcal{C}$ be an open contractible set and let $\tilde{U}$ be a component of the preimage of U in the Teichmüller space of abelian differentials. Via perhaps decreasing the size of U we may assume that $\tilde{U}$ has a product structure, defined by disjoint compact balls of dimension $\dim_{\mathbb{C}}(\mathcal{C}_+) - 1$ in the sphere of projective measured foliations on S. The real parts $\mathrm{Re}(\tilde{z})$ of the differentials $\tilde{z} \in \tilde{U}$ project to an open subset of the 2ℓ-dimensional subspace V of $H^1(S,\mathbb{R})$ as defined in the proposition.

Let $c_1,\dots,c_\ell$ be pairwise disjoint simple oriented weighted multicurves and denote by $[c_i] \in H_1(S,\mathbb{Z})$ the homology class of c_i. With respect to some fixed marking of S, used for the choice of the lift $\tilde{U}$, assume that the cohomology classes $[c_i]$ define a subspace of V^* of dimension ℓ. Then the projection which associates to a marked abelian differential $\tilde{z} \in \tilde{U}$ the cohomology class $[\mathrm{Re}(\tilde{z})] \in H^1(S,\mathbb{R})$ of its real part $\mathrm{Re}(\tilde{z})$ maps the open subset $\tilde{U}$ of $\tilde{\mathcal{C}}$ to an open subset of the dual L^* of the ℓ-dimensional linear subspace L of $H_1(S,\mathbb{R})$ spanned by the classes $[c_i]$.

Let $\tilde{z} \in \tilde{U}$ be the lift of a periodic point $z \in U$ for Φ^t; such a point exists by Proposition 3.7. Let $\varphi \in \Gamma_0 < \mathrm{Mod}(S)$ be the pseudo-Anosov element which preserves the Φ^t-orbit of $\tilde{z}$. Recall the assumption $\langle[\mathrm{Re}(\tilde{z})],[c_i]\rangle > 0$ for all i.

There is a number $\kappa > 1$ such that $\varphi^*\mathrm{Re}(\tilde{z}) = \kappa\mathrm{Re}(\tilde{z})$, moreover κ is the Perron Frobenius eigenvalue for the action of φ on $H^1(S,\mathbb{R})$. By invariance of the natural pairing $\langle,\rangle$ under φ, as $k \to \infty$ the homology classes $\varphi^k([c_i])$ converge up to rescaling to a class $u \in H_1(S,\mathbb{R})$ whose contraction with the intersection form ι defines $\pm[\mathrm{Re}(\tilde{z})]$, viewed as a linear functional on $H_1(S,\mathbb{R})$. By this we mean that $\iota(u,a) = \langle \pm[\mathrm{Re}(\tilde{z})], a\rangle$ for all $a \in H_1(S,\mathbb{R})$. As a consequence, for sufficiently large j and all i,ℓ we have $\iota([\varphi^j c_i],[c_\ell]) \neq 0$.

Let $G_1 < \mathrm{Mod}(S)$ be the group generated by the pseudo-Anosov mapping class φ as well as the Dehn multitwists $T_i = T_{c_i}$ $(i \leq \ell)$. Then G_1 also contains the multitwists $\varphi^j T_i \varphi^{-j} = T_{\varphi^j c_i}$. Moreover, $\iota([\varphi^j c_i],[c_u]) \neq 0$ for all i,u adn sufficiently large j.

Let $A_1 < V^*$ be the linear subspace of rank ℓ which is the common fixed set in V^* for the transvections $\Psi(T_{c_i})$ of V^*. Then A_1 is a Lagrangian subspace of V^*. Let $A_2 \subset A_1$ be the common fixed set in V^* of the transvections which are the images under the map Ψ of all multitwists $T_i, \varphi^j T_u \varphi^{-j}$. Then A_2 is a linear subspace of A_1, and for large enough j its codimension in A_1 is $s \geq 1$. Let $i_1,\dots,i_s \subset \{1,\dots,\ell\}$ be such that the homology classes $[c_i],[\varphi^j c_{i_u}] \in H_1(S,\mathbb{Z})$ $(i \leq \ell, u \leq s)$ are independent over $\mathbb{R}$ and that the common fixed set in V^* of the transvections defined by the corresponding Dehn multitwists is A_2.

Using again the fact that the set of real parts of differentials in $\tilde{U}$ define an open subset of the symplectic vector space V, we can find some $\tilde{u} \in \tilde{U}$ and some $a \in A_2$ so that $\langle[\mathrm{Re}(\tilde{u})],a\rangle > 0$. As before, we may assume that $\tilde{u}$ is the preimage of a periodic point of U. Argue now as in the previous paragraph and find a multitwist β in the subgroup G of $\mathrm{Mod}(S)$ generated by $\Omega(\Gamma_0)$ so that the common fixed set of the subgroup generated by $\Psi(\beta)$ and A_2 has codimension at least one in A_2.

Repeat this construction. In at most ℓ steps we find integral homology classes $a_1,\dots,a_\ell,a_{\ell+1},\dots,a_{2\ell}\in H_1(S,\mathbb{Z})$ (where for $i\leq \ell$ the class a_i is the class $[c_i]$ of the weighted multicurve c_i) with the following properties.

(1) Let $E\subset H_1(S,\mathbb{R})$ be the real vector space spanned by the classes a_i. Then the dimension of E equals 2ℓ. Each element $a\in E$ defines a linear functional on $H^1(S,\mathbb{R})$, and the restriction to V of this linear subspace of $H^1(S,\mathbb{R})^*$ is non-degenerate. In particular, E is a symplectic subspace of $H_1(S,\mathbb{R})$.
(2) $\iota(a_j,a_i)\neq 0$ for all $i\leq \ell$, $j\geq \ell+1$.
(3) For each i the transvection $b\to b+\iota(b,a_i)a_i$ is contained in the group generated by $\Psi(\Omega(\Gamma_0))$ and the Dehn multitwists $\Psi(T_{c_i})$.

By the choice of the homology classes a_i, the $(2\ell,2\ell)$-matrix $(\iota(a_i,a_j))$ whose (i,j)-entry is the intersection $\iota(a_i,a_j)$ is integral and of maximal rank. Choose a prime $p\geq 5$ so that each of the entries of $(\iota(a_i,a_j))$ is prime to p. All but finitely many primes will do. Then the reduction mod p of the matrix $(\iota(a_i,a_j))$ is of maximal rank as well. In particular, if F_p denotes the field with p elements then the reductions mod p of the homology classes a_i span a 2ℓ-dimensional symplectic subspace E_p of $H_1(S,F_p)$.

Let $L<\mathrm{Sp}(E)$ be the subgroup of the symplectic group of E which is generated by the transvections with the elements a_i. Its reduction L_p mod p acts on E_p as a group of symplectic transformations. Lemma 4.8 shows that $L_p=Sp(2\ell,F_p)$. Note that property (2) above guarantees that all conditions in Lemma 4.8 are fulfilled. Then L is a Zariski dense subgroup of the group of symplectic automorphisms of E [Lu99]. By duality, this just implies that the subgroup G of $Sp(V^*)$ generated by $\Psi(T_{c_i})$ and $\Psi(\Omega(\Gamma_0))$ is Zariski dense in $Sp(V^*)$.

Now assume that $\ell=g$. The Dehn twists T_{c_i} define elements of $Sp(2g,\mathbb{Z})$. All elements of $Sp(2g,\mathbb{R})$ constructed in the above way are integral, and the above proof shows that the subgroup of $Sp(2g,\mathbb{Z})$ constructed in the above way surjects onto $Sp(2g,F_p)$ for all but finitely many p. ☐

Now we are ready to show Theorem 4.6.

Proof. Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell\geq 1$, and let $\mathcal{C}\subset\mathcal{C}_+$ be its subset of differentials of area one. By Lemma 4.7, it suffices to show the existence of a single birecurrent point $q\in\mathcal{C}$ with the following property. For every open neighborhood U of q, the subgroup of $Sp(2\ell,\mathbb{R})$ generated by the monodromoy of those periodic orbits for Φ^t in $\mathcal{C}$ which pass through U is Zariski dense in $Sp(2\ell,\mathbb{R})$.

Choose a translation surface $(X,\omega)\in\mathcal{C}$ with the properties stated in Corollary 4.4. Denote by H the free abelian group of rank ℓ of Dehn multitwists which is contained in the group of piecewise affine automorphisms of X whose existence was shown in Corollary 4.4.

Let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials and let $\tilde\omega$ be a preimage of ω in $\tilde{\mathcal{C}}$. The cylinder shears of the translation surface (X,ω) which are used to construct the Dehn multitwists T_i generating the group H preserve the horizontal projective measured foliation of ω, but they deform the vertical projective measured foliation. These cylinder shears define ℓ smooth paths c_i $(i = 1, \dots, \ell)$ in $\mathcal{C}$ which lift to smooth paths $\tilde{c}_i$ in $\tilde{\mathcal{C}}$ beginning at the preimage $\tilde{\omega}$ of ω in $\tilde{U}$ and connecting $\tilde{\omega}$ to $T_i\tilde{\omega}$.

Fix $j \leq \ell$ and write $T = T_j$ and $\tilde{c} = \tilde{c}_j$ for simplicity. Our first goal is to show that there is a neighborhood of the entire path $\tilde{c} \subset \tilde{\mathcal{C}}$ with a product structure. Recall that such a neighborhood $\tilde{A}$ is determined by closed disjoint subsets D, K of the Thurston sphere $\mathcal{PML}$ of projective measured foliations, a number $\epsilon > 0$ and a map $\Lambda : D \times K \to \tilde{\mathcal{C}}$ with the properties stated in Definition 3.5 so that

$$\tilde{A} = \cup_{-\epsilon \leq t \leq \epsilon} \cup_{(\mu,\nu)\in D\times K} \Phi^t\Lambda(\mu, \nu).$$

Although this is an easy consequence of the fact that in period coordinates, an affine invariant manifold is a solution of a linear system of equations, we give a detailed proof. Cover the compact path $\tilde{c}$ by finitely many open subsets W_j $(i = 0, \dots, k)$ of $\tilde{\mathcal{C}}$ whose closures $\overline{W_i}$ have a product structure as described above. These product structures are defined by compacts balls D_i, K_i of dimension $m = \dim_{\mathbb{C}}(\mathcal{C}_+) - 1$ in the space of projective measured foliations, a map $\Lambda_i : D_i \times K_i \to \overline{W_i}$ and a number $\epsilon_i > 0$. For each i, the set D_i is homeomorphic to an m-dimensional compact ball and coincides with the set of all horizontal projective measured foliations of all points in $\overline{W_i}$. Let $\mathrm{int}(D_i)$ be the interior of D_i. As the horizontal projective measured foliation of any point on $\tilde{c}$ coincides with the horizontal projective measured foliation μ of $\tilde{\omega}$, for each i the set $\mathrm{int}(D_i)$ contains μ.

Up to renumbering, we may assume that $W_i \cap W_{i+1} \cap \tilde{c} \neq \emptyset$ for all i. We may also assume that $W_k = T(W_i)$. We now show by induction on $j \leq k$ that the set $\cap_{i\leq j}\mathrm{int}(D_i)$ is an open neighborhood of μ in each of the sets D_i $(i \leq j)$.

The case $j = 0$ is obvious, so assume that the claim is known for some $0 \leq j < k$. This means that $E_j = \cap_{i\leq j}\mathrm{int}(D_i)$ is an open neighborhood of μ in each of the sets $\mathrm{int}(D_i)$ for $i \leq j$. Note however that E_j is not an open subset of $\mathcal{PML}$.

Let $\overline{E}_j$ be the closure of E_j in D_j. As E_j is an open neighborhood of μ in $\mathrm{int}(D_j)$, the subset Z_j of W_j with a product structure which is defined by $\overline{E}_j, K_j$ contains an open neighborhood of $\tilde{c} \cap W_j$ (compare the remark after Definition 3.5). Thus by the assumption on the sets W_u, the intersection $Z_j \cap W_{j+1}$ contains an open neighborhood in $\tilde{\mathcal{C}}$ of $W_j \cap W_{j+1} \cap \tilde{c}$. But this is only possible if $E_j \cap \mathrm{int}(D_{j+1})$ is an open neighborhood of μ in both E_j, D_{j+1}. The induction step follows.

Let $E \subset E_k$ be a compact neighborhood of μ in E_k which is homeomorphic to a closed ball of dimension m. Then E is a compact neighborhood of μ in each of the sets $\mathrm{int}(D_i)$, and by construction,

$$\tilde{c} \subset \cup_i \cup_{-\epsilon_i \leq t \leq \epsilon_i} \Phi^t\Lambda_i(E \times K_i).$$

It now follows from the definition of a subset of $\tilde{\mathcal{C}}$ with a product structure that there is a neighborhood of $\tilde{c}$ in $\tilde{\mathcal{C}}$ with a product structure which is of the form $\cup_{-\delta \leq t \leq \delta}\Phi^t\Lambda(E \times \cup_i K_i)$. Here for a point $(\xi, \eta) \in E \times K_j$, the point $\Lambda(\xi, \eta)$ is obtained from $\Lambda_i(\xi,\eta)$ by postcomposition with $\Phi^{\sigma_i(\xi,\eta)}$ where $\sigma_i : E \times K_i \to \mathbb{R}$ is a continuous function.

Do this construction for all $j \leq \ell$ and as well for the maps T_j^{-1}. This results in a neighborhood W of $\tilde{\omega}$ in $\tilde{\mathcal{C}}$ with a product structure with the following properties.

(1) There are compact disjoint balls D, K in the sphere $\mathcal{PML}$ of projective measured foliations, there is a number $\epsilon > 0$ and there is a map $\Lambda : D \times K \to W$ with the properties stated in Definition 3.5 such that $W = \cup_{-\epsilon \leq t \leq \epsilon} \Phi^t \Lambda(D \times K)$.
(2) There exists a compact neighborhood R of μ in D homeomorphic to a closed ball so that $T_j R \subset D$ for all $j \leq \ell$.
(3) There is a compact neighborhood $B \subset K$ of the vertical projective measured foliation of $\tilde{\omega}$ such that $T_j^{-1}(B) \subset K$ for all j.

Let A be the projection to $\mathcal{C}$ of the set

$$\tilde{A} = \cup_{-\delta \leq t \leq \delta} \Phi^t \Lambda(R \times B).$$

Then U is a closed neighborhood of ω. We may adjust U in such a way that U is contractible; this is always possible in spite of the fact that ω may not be contained in $\mathcal{C}_{\rm good}$. Up to passing to a finite branched cover of $\mathcal{C}$, we then may assume that the holomorphic tangent bundle $\mathcal{Z}$ of $\mathcal{C}$ admits a trivialization over U which is parallel for the Gauss Manin connection. To this end recall that there is a finite branched cover of $\mathcal{M}_g$ which is the quotient of Teichmüller space by a torsion free subgroup of ${\rm Mod}(S)$ of finite index.

Since the Teichmüller flow on $\mathcal{C}$ is topologically transitive, Proposition 3.7 shows that there exists a periodic orbit γ for Φ^t which passes through the interior of A. A point $q \in A \cap \gamma$ is birecurrent. We will show that for every neighborhood U of q, the subgroup of $Sp(2\ell, \mathbb{R})$ generated by the monodromy of those periodic orbits which pass through U is Zariski dense in $Sp(2\ell, \mathbb{R})$.

Note first that it suffices to consider neighborhoods $V \subset A$ of q. Thus let $V \subset A$ be such a neighborhood. We may assume that the component $\tilde{V} \subset \tilde{A}$ of the preimage of V in $\tilde{\mathcal{C}}$ which is contained in $\tilde{A}$ equals the interior of the set

$$\overline{V} = \cup_{-\delta' \leq t \leq \delta'} \Phi^t \Lambda(R' \times B')$$

for some closed balls $R' \subset R$, $B' \subset B$ and for a number $\delta' < \delta$. Construct a set Γ_0 of periodic orbits passing through V as in Proposition 3.10. Denote by $\langle \Omega(\Gamma_0) \rangle$ the corresponding subsemigroup of ${\rm Mod}(S)$ constructed with the above component $\tilde{V}$ of the preimage of V and let $G < {\rm Mod}(S)$ be the subgroup generated by $\langle \Omega(\Gamma_0) \rangle$.

Let $\tilde{q}$ be the lift of q to $\tilde{A}$. The preimage of γ passing through $\tilde{q}$ is the cotangent line of a pseudo-Anosov mapping class φ. We claim that for each j there is a number $k > 0$ such that for the Dehn multitwist $T = T_j$, we have $\varphi^k \circ T \circ \varphi^k \in \Omega(\Gamma_0)$. Since G is a group, this implies that $T_j \in G$ for all j.

We establish the existence of a number $k > 0$ with the above property using a fixed point argument for the action of ${\rm Mod}(S)$ on the sphere of projective measured foliations which is motivated by the argument in the proof of Proposition 5.4 of [H13] (compare the proof of Proposition 3.7).

Let $\tau > 0$ be the period of γ; then $\varphi(\tilde{q}) = \Phi^\tau(\tilde{q})$. Write again $T = T_j$. The horizontal projective measured foliation $\zeta \in R'$ of $\tilde{q}$ is the attracting fixed point for the action of the map φ on the sphere $\mathcal{PML}$ of projective measured foliations of S. As φ preserves the component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ containing $\tilde{q}$, by possibly replacing φ by a large power we may assume that $\varphi(D)$ is contained in the interior of R', and $T\varphi(D)$ is contained in the interior of D. Recall to this end that $T\zeta$ is contained in the interior of D by assumption. Then $\varphi \circ T \circ \varphi(D)$ is contained in the interior of D.

The proof of Proposition 5.4 of [H13] shows that replacing φ by another power will guarantee that $\varphi \circ T \circ \varphi$ is pseudo-Anosov. As $\varphi \circ T \circ \varphi(D)$ is contained in the interior of D, the attracting fixed point of $\varphi \circ T \circ \varphi$ is contained in D.

We show next that the repelling fixed point of $\varphi \circ T \circ \varphi$ (which is the attracting fixed point of $\varphi^{-1} \circ T^{-1} \circ \varphi^{-1}$) is contained in the interior of B', possibly after replacing φ by another power. Namely, the attracting fixed point for φ^{-1} is contained in interior of B'. Moreover, $T^{-1}(B') \subset K$ by construction. But a large enough iterate of φ^{-1} maps K into the interior of B' and hence as before, we conclude that the repelling fixed point of $\varphi \circ T \circ \varphi$ is contained in the interior of B'. In particular, via replacing φ by a sufficiently large power, we may assume that the periodic orbit of Φ^t defined by $\varphi \circ T \circ \varphi$ passes through V.

As a consequence, the pseudo-Anosov elements φ and $\varphi \circ T \circ \varphi$ are contained in the group G and hence G contains the multiwist $T = T_j$. As this argument is valid for each $i \leq \ell$, we deduce that the group G contains each of the multi-twists T_i.

Theorem 4.6 now follows from Proposition 4.10 if we can make sure that for each $\tilde{z} \in \tilde{V}$ and each i we have $\langle [\mathrm{Re}(\tilde{z}), [c_i] \rangle > 0$. But by construction, we have $\langle [\mathrm{Re}(\tilde{\omega}), [c_i] \rangle > 0$ for all i, and the set $D \subset \mathcal{PML}$ in the definition of the neighborhood W of $\tilde{\omega}$ as constructed above can be chosen to project to an arbitrarily small neighborhood of the projective class of $[Re(\tilde{\omega})]$. Thus by continuity, we may choose the set D in such a way that indeed, $\langle [\mathrm{Re}(\tilde{u}), [c_i] \rangle > 0$ for all j and all $\tilde{u} \in W$. Theorem 4.6 now follows from Proposition 4.10. $\square$

For a prime p let $\Lambda_p : Sp(2g, \mathbb{Z}) \to Sp(2g, F_p)$ be reduction mod p. Recall from the remark before Lemma 4.8 that a semi-subgroup of a finite group G is a subgroup of G. The proof of Theorem 4.6 together with Proposition 4.10 shows more precisely the following strenthening of Theorem 2 from the introduction.

Corollary 4.11. *Let $\mathcal{C}$ be an affine invariant manifold of rank g. Then for all but finitely many primes $p \geq 3$, we have $\{\Lambda_p \Psi(\Omega(\gamma)) \mid \gamma \in \Gamma_0\} = Sp(2g, F_p)$.*

The following corollary is the main consequence of Theorem 2 used in the proof of Theorem 3.

Corollary 4.12. *Let $\mathcal{C}$ be the hyperplane of area one abelian differentials in an affine invariant manifold $\mathcal{C}_+$ of rank $\ell \geq 1$, with absolute holomorphic tangent bundle $\mathcal{Z}$. Then for every open subset U of $\mathcal{C}$ there exists a periodic orbit γ for Φ^t through U with the following properties.*

1. *The eigenvalues of the matrix $A = \Psi(\Omega(\gamma))|\mathcal{Z}$ are real and pairwise distinct.*
2. *No product of two eigenvalues of A is an eigenvalue.*

Proof. By Proposition 4.6, for every small open contractible subset U of $\mathcal{C}$, the image under Ψ of the subsemigroup $\Psi(\Omega(\Gamma_0))$ defined as in Proposition 3.10 by the monodromy along periodic orbits through U is Zariski dense in $Sp(2\ell, \mathbb{R})$. The statement of the corollary is now an immediate consequence of the main result of [Be97]. □

Let again $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \leq g$ and let $\tilde{\mathcal{C}}_+$ be a component of the preimage of $\mathcal{C}_+$ in the Teichmüller space of marked abelian differentials. Then the projected tangent space $p(T\mathcal{C}_+)$ can be identified with the complexification of a 2ℓ-dimensional symplectic subspace V of $\mathbb{R}^{2g} = H^1(S, \mathbb{R})$. The stabilizer in $Sp(2g, \mathbb{R})$ of this subspace is the subgroup $G = Sp(V) \times Sp(V^\perp)$ of $Sp(2g, \mathbb{R})$ where $V^\perp$ is the orthogonal complement of V with respect to the symplectic form. Thus the group G is isomorphic to $Sp(2\ell, \mathbb{R}) \times Sp(2(g - \ell), \mathbb{R})$.

Let $P : G \to Sp(V) = Sp(2\ell, \mathbb{R})$ be the natural projection. Proposition 4.5 shows that $P(G \cap Sp(2g, \mathbb{Z}))$ is a Zariski dense subgroup of $Sp(2\ell, \mathbb{R})$. The following consequence of this fact was communicated to me by Yves Benoist. Although it is not used in the sequel, we include it here since it relates affine invariant manifolds to proper subvarieties of $\mathcal{A}_g$.

Proposition 4.13. *If $P(G \cap Sp(2g, \mathbb{Z}))$ is Zariski dense in $Sp(2\ell, \mathbb{R})$ then either $P(G \cap Sp(2g, \mathbb{Z}))$ is a lattice in $Sp(2\ell, \mathbb{R})$ or dense.*

Proof. Using the above notations, write $G_\mathbb{Z} = Sp(2g, \mathbb{Z}) \cap G$ and let $F < Sp(2\ell, \mathbb{R})$ be the Zariski closure of $P(G_\mathbb{Z})$.

The group F is defined over $\mathbb{Q}$. Namely, the set of polynomials P which vanish on $G_\mathbb{Z}$ is invariant under the Galois action. As a consequence, either $F_\mathbb{Z} = G_\mathbb{Z}$ is a lattice in F, or there is a nontrivial character on F defined over $\mathbb{Q}$.

Assume for contradiction that there exists a nontrivial character on F defined over $\mathbb{Q}$. Define

$$F^0 = \cap\{\ker(\chi) \mid \chi \text{ is a character on } F \text{ defined over } \mathbb{Q}\}.$$

Then $F^0 = F$ since up to multiplication with an integer, the evaluation on $G_\mathbb{Z}$ of a nontrivial character χ defined over $\mathbb{Q}$ has to be integral in $\mathbb{C}^*$ which is impossible. This contradiction yields that $F_\mathbb{Z}$ is a lattice in F.

The group $G_1 = Sp(2\ell, \mathbb{R})$ is simple, and $\Delta = P(G_\mathbb{Z}) < G_1$ is Zariski dense. Then $\Delta < G_1$ either is discrete or dense. We have to show that if Δ is discrete then Δ is a lattice.

Thus assume that Δ is discrete. Consider the surjective homomorphism $\varphi : F \to G_1$. Its kernel K is a locally compact group which intersects the lattice $F_{\mathbb{Z}}$ in a discrete subgroup. The exact sequence

$$1 \to K \to F \to G_1 \to 1$$

induces a sequence

$$K/K \cap F_{\mathbb{Z}} \to F/F_{\mathbb{Z}} \to G_1/\varphi(F_{\mathbb{Z}}).$$

Now the Haar measure on F can locally be represented as a product of the Haar measure on the orbits of K and the quotient Haar measure. If the volume of $G_1/\varphi(F_{\mathbb{Z}})$ is infinite then this shows that the volume of $F/F_{\mathbb{Z}}$ has to be infinite. But $F_{\mathbb{Z}}$ is a lattice in F which is a contradiction. $\square$

Remark 4.14. If $\mathcal{C}$ is a Teichmüller curve, then the group $P(G \cap Sp(2g,\mathbb{Z}))$ is just the Veech group of $\mathcal{C}$, which is a lattice in $SL(2,\mathbb{R})$. The image under the Torelli map of the projection to moduli space of such a Teichmüller curve is a *Kobayashi geodesic* in Siegel upper half-space.

If $\mathcal{C}$ is *algebraically primitive*, then this Kobayashi geodesic is contained in a Hilbert modular variety defined by an order $\mathfrak{o}$ in the trace field of $\mathcal{C}$. This Hilbert modular variety is the quotient of an embedded copy of $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$ in Siegel upper half-space $\mathfrak{D}_g$ by its stabilizer $SL(2,\mathfrak{o})$ in $Sp(2g,\mathbb{Z})$. Moreover, the finite area Riemann surface Σ obtained by projecting $\mathcal{C}$ to the moduli space of curves admits a *modular embedding* into $SL(2,\mathfrak{o})\backslash\mathbf{H}^2 \times \cdots \times \mathbf{H}^2$ whose composition with the first factor projection $\mathbf{H}^2 \times \cdots \times \mathbf{H}^2 \to \mathbf{H}^2$ is a finite area Riemann surface.

Proposition 4.13 states that we may expect a similar picture for affine invariant manifolds of higher rank. However, the example in [MMW16] is the only example of an higher dimensional affine invariant manifold not arising from a construction related to a branched covering.

5. Galois groups

In this section we consider an arbitrary component $\mathcal{Q}$ of a stratum of abelian differentials. We continue to use the assumptions and notations from Section 2-4. In view of the results in Section 3, we expect that appropriate versions of the results in this section should also hold true for arbitrary affine invariant manifolds. However, the discussion in this section builds on the main result of [H13] which at the moment has not been completely verified for arbitrary affine invariant manifolds, and we leave the verification in this case to other authors.

Recall from Proposition 3.10 the construction of the set Γ_0 of parametrized periodic orbits in $\mathcal{Q}_{\rm good}$ defined by suitable small neighborhoods $Z_0 \subset Z_1$ of a point $q \in \mathcal{Q}_{\rm good}$ which is birecurrent under the Teichmüller flow. By Proposition 3.10, this set determines a subsemigroup $\Omega(\Gamma_0)$ of $\mathrm{Mod}(S)$ consisting of pseudo-Anosov elements. By Theorem 4.6, the image of $\Omega(\Gamma_0)$ under the homomorphism $\Psi : \mathrm{Mod}(S) \to Sp(2g,\mathbb{Z})$ is Zariski dense in $Sp(2g,\mathbb{R})$.

Let $p \geq 3$ be an odd prime and let as before F_p be the field with p elements. Denote by $N(p)$ the cardinality of $Sp(2g,F_p)$. Let $\Lambda_p : Sp(2g,\mathbb{Z}) \to Sp(2g,F_p)$ be the reduction modulo p. By Corollary 4.11, for all but finitely many primes p we have $\Lambda_p(\Psi\Omega(\Gamma_0)) = Sp(2g, F_p)$.

Our next goal is to make this statement quantitative. To this end denote for a periodic orbit γ for Φ^t by δ_γ the Φ^t-invariant measure supported on γ whose total mass equals the period $\ell(\gamma)$ of γ.

Let $k > 0$ be the number of zeros of a differential in $\mathcal{Q}$ and let $h = 2g - 1 + k$. In the sequel, for functions $f, g : [0, \infty) \to \mathbb{R}$ we write $f \sim g$ if $f(t)/g(t) \to 1$ as $t \to \infty$. Recall that periodic orbits in the set Γ_0 are parametrized, so a single unparametrized periodic orbit may give rise to many different elements of Γ_0.

The following proposition holds true for any finite group G of order N with the property that there is a homomorphism $\rho : \mathrm{Mod}(S) \to G$ whose restriction to the semigroup $\Omega(\Gamma_0)$ is surjective. It can be viewed as an equidistribution result for elements in $Sp(2g, F_p)$ defined by periodic orbits of the Teichmüller flow which parallels the familiar equidistribution of random walks on finite connected graphs. To this end recall that $2t_0$ is the length of a connected component of the intersection with Z_1 of an orbit segment of Φ^t.

Proposition 5.1. *Let $p \geq 3$ be an odd prime such that $\Lambda_p(\Psi \circ \Omega(\Gamma_0)) = Sp(2g, F_p)$. Let $B \in Sp(2g, F_p)$ be arbitrary and for $R > 0$ define*

$$\mathcal{B}(R, B) = \{\gamma \in \Gamma_0 \mid \ell(\gamma) \leq R, \Lambda_p(\Psi\Omega(\gamma)) = B\}.$$

Then as $R \to \infty$,

$$\sharp\mathcal{B}(R, B) \sim c\frac{e^{hR}\lambda(Z_0)}{2ht_0 N(p)}$$

independent of B where $c > 0$ depends on Z_0 and can be arranged to be arbitrarily close to one.

Proof. We show first that there is a number $a > 0$ such that

$$\sharp\mathcal{B}(R, B) \geq ae^{hR}$$

for all $B \in Sp(2g, F_p)$ and for all sufficiently large R.

To this end recall from Lemma 2.3 the choice of the nested sets $Z_0 \subset Z_1 \subset Z_2 \subset V \subset U$ which are neighborhoods of a birecurrent point $q \in \mathcal{Q}_{\text{good}}$ as well as the choice of the numbers $R_0 > 0, \delta > 0$. The sets are used to construct the sub-semigroup $\Omega(\Gamma_0)$ of $\mathrm{Mod}(S)$. By Proposition 3.10, this semigroup consists of pseudo-Anosov elements. Furthermore, each $\rho \in \Omega(\Gamma_0)$ is represented by a parametrized periodic orbit ζ for Φ^t which intersects the set Z_1 in a segment of length $2t_0$ containing $\zeta(0)$ as its midpoint. Vice versa, every periodic orbit which passes through Z_0 admits a parametrization so that the corresponding element of $\mathrm{Mod}(S)$ is contained in $\Omega(\Gamma_0)$. The subsemigroup $\Psi(\Omega(\Gamma_0))$ of $Sp(2g, \mathbb{Z})$ is mapped by Λ_p onto the finite group $Sp(2g, F_p)$.

Since $Sp(2g, F_p)$ is a *finite* group, there is a number $\hat{R} > R_0$ with the following property. Let $A \in Sp(2g, F_p)$ be arbitrary. Then there is a parametrized periodic orbit $\gamma(A) \in \Gamma_0$ of length $\ell(\gamma) \leq \hat{R} - t_0 - \delta$ with $\Lambda_p\Psi(\Omega(\gamma(A))) = A$. Here $\delta > 0$ coincides with the constant which entered in the construction of the set Z_0.

Let $v,w\in Z_0$ be such that $\Phi^Tv\in Z_0,\Phi^Uw\in Z_0$ for some $T,U>R_0$. By Lemma 2.3 and Proposition 3.10, the orbit segments $\{\Phi^tv\mid 0\leq t\leq T\}$ and $\{\Phi^tw\mid 0\leq t\leq U\}$ determine two parametrized periodic orbits γ_1,γ_2 for Φ^t which define elements $\Lambda_p\Psi(\Omega(\gamma_1)),\Lambda_p\Psi(\Omega(\gamma_2))\in Sp(2g,F_p)$. Let $\gamma=\gamma_2\hat{\circ}\gamma_1$ be the periodic orbit corresponding to the pseudo-orbit consisting of the two orbit segments $\{\Phi^tv\mid 0\leq t\leq T\}$ and $\{\Phi^tw\mid 0\leq t\leq U\}$ in this order. The notation $\gamma=\gamma_2\hat{\circ}\gamma_1$ indicates that the element $\Omega(\gamma)$ of $\mathrm{Mod}(S)$ defined by γ is the product of the elements of $\mathrm{Mod}(S)$ defined by γ_1 and γ_2. We have

$$\Lambda_p\Psi(\Omega(\gamma_2\hat{\circ}\gamma_1))=\Lambda_p\Psi(\Omega(\gamma_2))\circ\Lambda_p\Psi(\Omega(\gamma_1)).$$

Fix an element $B\in Sp(2g,F_p)$. If $A\in Sp(2g,F_p)$ is arbitrary and if $\zeta\in\Gamma_0$ is such that $\Lambda_p\Psi(\Omega(\zeta))=A$ then $\Lambda_p\Psi(\Omega(\gamma(BA^{-1})\hat{\circ}\zeta))=B$. In particular, by the choice of $\hat{R}$ and by Proposition 3.10, for sufficiently large $R>R_0$ the number of parametrized periodic orbits $\gamma\in\Gamma_0$ with $\ell(\gamma)\leq R+\hat{R}$ and $\Lambda_p\Psi(\Omega(\gamma))=B$ is not smaller than the number of orbits in Γ_0 of length at most R.

For large enough R, the number of orbits in Γ_0 of length at most R can be estimated as in [H13]. Namely, by Proposition 2.2 and the choice of the set Z_0, for the fixed number $\delta>0$ used in the construction of Z_0 and for large R, the volume of the set Z_1 with respect to the sum of the measures supported on periodic orbits of Φ^t of length in the interval $[R-t_0-\delta,R+t_0+\delta]$ is contained in the interval

$$[2t_0e^{hR}\lambda(Z_0)(1-\delta),2t_0e^{hR}\lambda(Z_0)(1+\delta)].$$

Each component of intersection (in the sense explained in Lemma 2.3) of $\Phi^RZ_1\cap Z_2$ containing points in $\Phi^RZ_0\cap Z_0$ defines an element of Γ_0 (recall that Γ_0 consists of parametrized orbits). Up to adjusting the a priori chosen constant δ, the number of such components is contained in the interval $[e^{hR}\lambda(Z_0)(1-\delta),e^{hR}\lambda(Z_0)(1+\delta)]$ (compare [H13] for details). Thus for large enough R, the number of elements of Γ_0 of length contained in $[R-t_0-\delta,R+t_0+\delta]$ equals $e^{hR}\lambda(Z_0)$ up to a factor contained in the interval $[1-\delta,1+\delta]$.

Summation yields that for sufficiently large k, up to a factor contained in the interval $[1-2\delta,1+2\delta]$, the number of elements of Γ_0 of length at most $2k(t_0+\delta)$ equals

$$\sum_{i=1}^k e^{h(2i-1)(t_0+\delta)}\lambda(Z_0)\sim\frac{e^{(2k+1)h(t_0+\delta)}}{e^{2h(t_0+\delta)}-1}\lambda(Z_0).$$

As a consequence, there is a number $a>0$ not depending on B such that up to passing to a subsequence, the measures

$$he^{-hR}\sum_{\gamma\in\mathcal{B}(R,B)}\delta_{[\gamma(-t_0),\gamma(t_0)]}$$

converge to a measure $\hat{\lambda}_B$ on Z_1 of total mass contained in $[a\lambda(Z_1),\lambda(Z_1)(1+\sigma)]$. Here $\delta_{[\gamma(-t_0),\gamma(t_0)]}$ is the restriction to $\gamma[-t_0,t_0]$ of the Φ^t-invariant measure δ_γ supported on γ and σ depends on δ and tends to zero as $\delta\to 0$ and $t_0\to 0$.

It is immediate from the construction that $\hat{\lambda}_B$ is the restriction to Z_1 of a Φ^t-invariant Borel measure λ_B on $\mathcal{Q}$ which is a weak limit as $R \to \infty$ of the measures

$$\nu(B,R) = he^{-hR} \sum_{\gamma \in \mathcal{B}(R,B)} \delta_\gamma.$$

To guarantee the existence of such a limit we may have to pass to a subsequence, and we will continue passing to suitable subsequences in the course of this proof. Note again that a single unparametrized periodic orbit γ may appear many times in this sum. The interpretation is as follows. Let γ be a periodic orbit for Φ^t of length at most R. For each connected component I of the intersection $\gamma \cap Z_1$, check whether there is a point $z \in Z_0$ with $\Phi^s z \in Z_0$ for some $s \in [R - t_0 - \delta, R + t_0 + \delta]$ whose characteristic curve determines the parametrization $\hat{\gamma}$ of γ whose starting point is the midpoint of I. If such a point $z \in Z_0$ exists and if moreover $\Lambda_p(\Omega(\hat{\gamma})) = B$ where $\Omega(\hat{\gamma})$ is obtained from this specific parametrization of γ, then we add a copy of δ_γ to the measure $\nu(B,R)$. Other components of $\gamma \cap Z_1$ do not contribute towards $\nu(B,R)$.

By the main result of [H13] (see also Proposition 2.2), the measure λ_B is contained in the measure class of the Masur Veech measure λ. Furthermore, for large R the total mass of the measure

$$he^{-hR} \sum_{\gamma \in \Gamma_0, \ell(\gamma) \le R} \delta_\gamma = \sum_B \nu(B,R)$$

is bounded from above and below by a positive constant not depending on R.

Choose a sequence $R_i \to \infty$ such that each of the finitely many measures $\nu(B,R_i)$ ($B \in Sp(2g,F_p)$) converges as $i \to \infty$ to measure λ_B. Let $b > a$ be the total mass of $\sum_B \lambda_B$. By ergodicity of λ under the Teichmüller flow, we have $\lambda_B = c(B)\lambda$ for a number $c(B) \in [a,b]$. Via rescaling all measures with b^{-1} we may assume that in fact $b = 1$.

Our goal is to show that $c(B)$ is independent of B. To this end recall that the Masur-Veech measure λ is mixing of all orders [M82]. In particular, for numbers $R, S > 0$ we have

$$\lambda(Z_0 \cap \Phi^R Z_0 \cap \Phi^{R+S}(Z_0)) \to \lambda(Z_0)^3 \quad (R, S \to \infty)$$

and therefore $\lambda_B(Z_0 \cap \Phi^R Z_0 \cap \Phi^{R+S} Z_0) \to c(B)\lambda(Z_0)^3$.

For $R > 0, T > 0$ and $B \in Sp(2g,F_p)$ let $\Gamma(R,T,B,Z_0)$ be the set of all parametrized periodic orbits $\gamma \in \Gamma_0$ for Φ^t with the following properties.

(1) There exists some $v \in Z_0$ with $\Phi^{R+T} v \in Z_0$ such that γ is determined by a characteristic curve of the periodic pseudo-orbit $\cup_{t \in [0,R+T]} \Phi^t v$. In particular, the length of γ is contained in the interval $[R + T - t_0 - \delta, R + T + t_0 + \delta]$.
(2) $\Phi^R v \in Z_0$.
(3) $\Lambda_p \Psi(\Omega(\gamma)) = B$.

Define similarly a set $\Gamma(R+T,B,Z_0)$ containing all orbits with properties (1), (2) and (4) above. It follows from the above discussion (compare [H13]) that

$$\sharp\Gamma(R+T,B,Z_0)\sim c(B)\lambda(Z_1)e^{h(R+T)}/2ht_0$$

for large enough R and similarly

$$\sharp\Gamma(R,T,B,Z_0)\sim c(B)\lambda(Z_1)^2e^{h(R+T)}/(2ht_0)^2.$$

Note that in this asymptotic formula, the term $\lambda(Z_1)$ rather than $\lambda(Z_0)$ occurs due to our renormalization of the limiting measure $\sum_B\lambda_B$.

By definition, each orbit $\gamma\in\Gamma(R,T,B,Z_0)$ can be represented in the form $\gamma=\gamma_2\hat{\circ}\gamma_1$ for some $\gamma_1\in\Gamma(R,A,Z_0)$ and some $\gamma_2\in\Gamma(T,BA^{-1},Z_0)$.

As a consequence, for sufficiently large $R>0$, as $T\to\infty$ we observe that

$$\begin{aligned}\sharp\Gamma(R,T,B,Z_0)&\leq\sum_{A\in Sp(2g,F_p)}\sharp\Gamma(T,BA^{-1},Z_0)\sharp\Gamma(R,A,Z_0)\\&\sim\frac{1}{(2ht_0)^2}\lambda(Z_1)^2e^{h(R+T)}\sum_{A\in Sp(2g,F_p)}c(BA^{-1})c(A).\end{aligned}$$

Now let $B\in Sp(2g,F_p)$ be such that $c(B)=\max\{c(A)\mid A\}$. Such an element exists since $Sp(2g,F_p)$ is finite. Since $\sum_A c(BA^{-1})=\sum_A c(A)=1$ for all B and since each of the terms $c(A)$ is positiv, we have

$$\sum_A c(BA^{-1})c(A)\leq\sum_A c(BA^{-1})c(B)=c(B)$$

with equality only if $c(A)=c(B)$ for all A. This implies that $\sharp\Gamma(R,T,B,Z_0)\sim c(B)\lambda(Z_1)^2e^{h(R+T)}/(2ht_0)^2$ only if $c(A)=c(B)=\frac{1}{N(p)}$ for all A. The proposition follows. □

Now we are ready to complete the proof of the second part of Theorem 1. To this end recall that the characteristic polynomial of a symplectic matrix $A\in Sp(2g,\mathbb{Z})$ is reciprocal of degree $2g$. The roots of such a polynomial come in pairs. The Galois group of the number field defined by the polynomial is a subgroup of the semidirect product

$$(\mathbb{Z}/2\mathbb{Z})^g\rtimes\mathfrak{S}_g$$

where $\mathfrak{S}_g$ is the symmetric group in g elements [VV02], and $\mathfrak{S}_g$ acts on $(\mathbb{Z}/2\mathbb{Z})^g$ by permutation of the factors.

In the sequel we call the Galois group of the field defined by the characteristic polynomial of a matrix $A\in Sp(2g,\mathbb{Z})$ simply the Galois group of A. It only depends on the conjugacy class of A. We say that the Galois group of A is *full* if it coincides with $(\mathbb{Z}/2\mathbb{Z})^g\rtimes\mathfrak{S}_g$.

Having a full Galois group makes also sense for an element in $Sp(2g,F_p)$. We use this as in [R08] as follows.

Let $p_0\geq 5$ be large enough so that $\Psi(\Omega(\Gamma_0))$ surjects onto $Sp(2g,F_p)$ for all $p\geq p_0$. Let $p\geq p_0$ and let $N(p)$ be the number of elements of $Sp(2g,F_p)$. By Proposition 5.1, for large enough R the proportion of the elements $\gamma\in\Gamma_0$ of length at most R which satisfy $\Lambda_p \circ \Psi \circ \Omega(\gamma) = B$ roughly equals $\frac{1}{N(p)}$. On the other hand, if we denote by $R_p(2g)$ the subset of $Sp(2g, F_p)$ of elements with reducible characteristic polynomial then

$$\frac{|R_p(2g)|}{N(p)} < 1 - \frac{1}{3g}$$

(see Theorem 6.2 of [R08] for a reference to this classical result of Borel).

We follow the proof of Theorem 6.2 of [R08]. Let $p_1, \dots, p_k$ be k distinct primes bigger than p_0, and let $K = p_1 \cdots p_k$. Then the reduction $\Lambda_K(A)$ modulo K of any element $A \in Sp(2g, \mathbb{Z})$ is defined, and we have

$$\Lambda_K(A) = \Lambda_{p_1}(A) \times \cdots \times \Lambda_{p_k}(A).$$

Namely, for distinct primes $p \neq q \geq 5$, the groups $Sp(2g, F_p)$ and $Sp(2g, F_q)$ are non-isomorphic simple groups. This implies that if Γ is any group and if $\rho_p : \Gamma \to Sp(2g, F_p)$ and $\rho_q : \Gamma \to Sp(2g, F_q)$ are surjective homomorphisms, then the homomorphism $\rho_p \times \rho_q : \Gamma \to Sp(2g, F_p) \times Sp(2g, F_q)$ is surjective. Then by the discussion preceding Proposition 5.1, reduction mod K defines a surjection of the semigroup $\Psi\Omega(\Gamma_0)$ onto the finite group $\Lambda_{p_1}(\Psi\Omega(\Gamma_0)) \times \cdots \times \Lambda_{p_k}(\Psi\Omega(\Gamma_0)) = \Lambda_K(\Psi\Omega(\Gamma_0))$.

Now if $A \in Sp(2g, \mathbb{Z})$ has a reducible characteristic polynomial, then the same holds true for $\Lambda_{p_i}(A)$ for all i. The discussion in the previous paragraph yields that the proportion of the number of elements in $Sp(2g, K)$ with this property is at most $(1 - \frac{1}{3g})^k$.

By Proposition 5.1 (taking into account the comment preceding the proposition), this implies that for large enough R, the proportion of all orbits γ of length at most R with the property that the Galois group $G(\gamma)$ of $A(\gamma)$ is *not* full is at most of the order of $(1 - \frac{1}{3g})^k$. As $k \to \infty$, we conclude that the Galois group of a typical periodic orbit for Φ^t is full. Thus we have shown

Corollary 5.2. *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials. The set of all $\gamma \in \Gamma$ such that the trace field of $[A(\gamma)]$ is of degree g over $\mathbb{Q}$, and $G(\gamma) = (\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$ is typical.*

For a periodic orbit $\gamma \subset \mathcal{Q}$, the trace field of $A(\gamma)$ can also be read off directly from a point ω on γ. Namely, let $\tilde{\omega}$ be a lift of ω to a marked abelian differential. The periods of $\tilde{\omega}$ define an abelian subgroup $\Lambda = \tilde{\omega}(H_1(S, \mathbb{Z}))$ of $\mathbb{C}$ of rank two. Let $e_1, e_2 \in \Lambda$ be two points which are linearly independent over $\mathbb{R}$. Let K be the smallest subfield of $\mathbb{R}$ such that every element of Λ can be written as $ae_1 + be_2$, with $a, b \in K$; then $\Lambda \otimes_K K = K^2$. If we write $T = A(\gamma) + A(\gamma)^{-1}$, then the field K also is the field of the characteristic polynomial of T. We call K the *trace field* of γ (see the appendix of [KS00] for more details).

Definition 5.3. The periodic orbit γ is called *algebraically primitive* if the trace field K of γ is a totally real number field of degree g over $\mathbb{Q}$, with maximal Galois group.

The following corollary completes the proof of the second part of Theorem 1.

Corollary 5.4. *Algebraically primitive periodic orbits for Φ^t are typical.*

Proof. Corollary 5.2 shows that for a typical periodic orbit γ, the trace field of $[A(\gamma)]$ is of degree g over $\mathbb{Q}$. We have to show that it is also totally real.

As the Lyapunov spectrum of $\mathcal{Q}$ is simple [AV07], the first part of Theorem 1 implies that for a typical periodic orbit γ, the absolute values of the eigenvalues of $[A(\gamma)]$ are pairwise distinct and hence all eigenvalues are real. Thus by the discussion following Proposition 5.1, we only have to show that for a symplectic matrix $A \in Sp(2g, \mathbb{R})$ with $2g$ distinct real eigenvalues r_i, r_i^{-1} $(i \leq g, r_i > 1)$ the field defined by $A + A^{-1}$ is totally real. However, this is immediate from the fact that the roots of the polynomial defining the trace field of A are of the form $r_i + r_i^{-1}$ where r_i are the roots of the characteristic polynomial of A. □

We are left with the proof of Corollary 1 from the introduction. We repeat its formulation. We require that strata of quadratic differentials are not strata of squares of holomorphic one-forms. As the counting results for periodic orbits hold in this setting as well [H13], a typical property is defined for periodic orbits on such strata as well.

Corollary 5.5. *Let $\mathcal{D}$ be a component of a stratum of quadratic differentials with k zeros of odd order. Then for a typical periodic orbit $\gamma \subset \mathcal{D}$, the algebraic degree of the stretch factor of $\Omega(\gamma) \in \mathrm{Mod}(S)$ equals $2g - 2 + k$.*

Proof. Let $\mathcal{D}$ be a component of a stratum in the moduli space of area one quadratic differentials for the surface S with k zeros of odd order. Then $\mathcal{D}$ is a complex orbifold of dimension $2g - 2 + k$.

For each quadratic differential q which is not the square of a holomorphic one-form, there is a two-sheeted cover S' of S, ramified at some of the zeros of q, such that q lifts to an abelian differential on S'. Each singular point of q whose cone angle is an odd multiple of π, ie which is a zero of odd order, is a ramification point. Thus if $q \in \mathcal{D}$ then the covering is ramified at each singular point. By the Riemann Hurwitz formula, the genus g' of S' equals $2g - 1 + \frac{k}{2}$, and S is obtained from S' by taking the quotient by an involution ι which exchanges the two sheets in the cover.

The component $\mathcal{D}$ lifts to an affine invariant manifold $\mathcal{C}$ in a component $\mathcal{Q}$ of a stratum in the moduli space of abelian differentials on S', consisting of abelian differentials with k zeros.

The involution ι acts on the real cohomology $H^1(S', \mathbb{R})$ of S'. This cohomology decomposes over $\mathbb{R}$ as

$$H^1(S', \mathbb{R}) = \mathcal{E}_1 \oplus \mathcal{E}_2$$

where $\mathcal{E}_1$ is the eigenspace for ι with respect to the eigenvalue 1, and $\mathcal{E}_2$ is the eigenspace for ι with respect to the eigenvalue -1. As the action of ι on the first cohomology is a symplectic transformation, this decomposition is orthogonal for the symplectic form on $H^1(S', \mathbb{R})$. The vector space $\mathcal{E}_1$ is precisely the pull-back of $H^1(S, \mathbb{R})$ under the branched covering map and hence its dimension equals $2g$. Thus $\dim(\mathcal{E}_2) = 2g - 2 + k$ which coincides with the complex dimension of the extension $\mathcal{D}_+$ of $\mathcal{D}$ to a stratum of quadratic differentials of arbitrary area. In other words, we have $\dim(\mathcal{E}_2) = \dim_{\mathbb{C}}(\mathcal{C}_+)$.

Since ι is an involution, the decomposition $H^1(S', \mathbb{R}) = \mathcal{E}_1 \oplus \mathcal{E}_2$ is defined over $\mathbb{Z}[\frac{1}{2}]$. As a consequence, the stabilizer of this decomposition in the group $Sp(2g', \mathbb{Z})$ projects to a lattice in the group of symplectic automorphisms of $\mathcal{E}_2$.

In [Th88], Thurston observed that in the case of the principal stratum $\mathcal{P}_+$ of quadratic differentials (ie in the case $k = 4g-4$), the intersection form on $\mathcal{E}_2$ induces a symplectic structure on the space of measured foliations on S. This implies that the rank of the affine invariant manifold $\mathcal{C}_+$ which is the pull-back of $\mathcal{P}_+$ is equal to half its dimension, ie that the deficiency of $\mathcal{C}_+$ vanishes. Then the tangent bundle of $\mathcal{C}_+$ can be identified with $\mathcal{E}_2 \otimes \mathbb{C}$. The same also holds true for an arbitrary component $\mathcal{D}$ of a stratum consisting of differentials with only singularities of odd order. Namely, let Σ be the set of zeros of a differential in $\mathcal{D}$. As the covering $S' \to S$ is ramified at every point of Σ, the preimage in S' of any arc in S with endpoints at two distinct points of Σ is a closed curve which defines an absolute homology class in S'. Thus period coordinates for $\mathcal{D}_+$ lift to coordinates for $\mathcal{C}_+$ obtained by integration of a one-form over absolute homology classes.

Via lifting a mapping class of S to the branched cover S', for each periodic orbit γ for the Teichmüller flow in $\mathcal{D}$, the monodromy of its lift to $\mathcal{C}$ is a Perron Frobenius automorphism of $\mathcal{E}_2$ whose Perron Frobenius eigenvalue is just the stretch factor of the pseudo-Anosov element of $\mathrm{Mod}(S)$ defining γ.

By Theorem 4.6, the affine invariant manifold $\mathcal{C}$ is locally Zariski dense. Using dynamical properties of the Teichmüller flow on $\mathcal{D}$ as used before for strata of abelian differentials and as described in [H13], the proofs of Proposition 5.1 and Corollary 5.2 apply verbatim and show that the algebraic degree of the stretch factor of a pseudo-Anosov mapping class defined by a typical periodic orbit for Φ^t on $\mathcal{D}$ equals $2g - 2 + k$. This is what we wanted to show. $\square$

Remark 5.6. The proof of the above result also shows the following. Let $\mathcal{D}$ be any component of a stratum of quadratic differentials consisting of differentials with k zeros. Then the algebraic degree of the stretch factor of any pseudo-Anosov mapping class defining a periodic orbit in $\mathcal{D}$ is at most $2g - 2 + k$.

Furthermore, the degree of the stretch factor for a typical such element can be explicitly computed: It equals twice the rank of the affine invariant manifold obtained from the double orientation cover of a differential in $\mathcal{D}$.

These result do not answer however any of the more specific questions on stretch factors one might ask, and in contrast to the second part of Theorem 1, they do not imply that the extension of $\mathbb{Q}$ by a typical stretch factor is a totally real number field.

6. Connections on the Hodge bundle

In this section we begin the investigation of differential geometric properties of an affine invariant manifold $\mathcal{C}_+$, with tangent bundle $T\mathcal{C}_+$. We study the Gauss Manin connection on the projection $p(T\mathcal{C}_+)$ of $T\mathcal{C}_+$ to the flat bundle $\Pi^*\mathcal{N}\otimes\mathbb{C}|\mathcal{C}_+$. Recall that $p(T\mathcal{C}_+)$ is a subbundle of $\Pi^*\mathcal{N}\otimes\mathbb{C}|\mathcal{C}_+$ which is invariant under the Gauss Manin connection [EMM15] and invariant under the complex structure i [F16]. We establish a first rigidity result geared towards Theorem 3 from the introduction. We always assume that $g\geq 3$.

Recall from Section 4 that the Hodge bundle $\mathcal{H}$ on the moduli space $\mathcal{M}_g$ of curves of genus g is the pull-back under the Torelli map of the Hermitean holomorphic (orbifold) vector bundle $\mathcal{V}\to\mathcal{A}_g$ (see also the appendix).

The complement $\mathcal{H}_+$ of the zero section in $\mathcal{H}$ is a complex orbifold. Let as before $\Pi:\mathcal{H}\to\mathcal{M}_g$ be the canonical projection. The pull-back $\Pi^*\mathcal{H}\to\mathcal{H}_+$ to $\mathcal{H}_+$ of the Hodge bundle on $\mathcal{M}_g$ is a holomorphic vector bundle on $\mathcal{H}_+$. The Hermitean metric on $\mathcal{H}$ which is determined by the complex structure J on $\mathcal{H}$ and a natural symplectic structure (see the appendix for more details) pulls back to a Hermitean structure on $\Pi^*\mathcal{H}$. The bundle $\Pi^*\mathcal{H}$ splits as a direct sum

$$\Pi^*\mathcal{H}=\mathcal{T}\oplus\mathcal{L}$$

of complex vector bundles. Here the fibre of $\mathcal{T}$ over a point $q\in\mathcal{H}_+$ is just the $\mathbb{C}$-span of q, and the fibre of $\mathcal{L}$ is the orthogonal complement of $\mathcal{T}$ for the natural Hermitean metric, or, equivalently, the orthogonal complement of $\mathcal{T}$ for the symplectic form. The complex line bundle $\mathcal{T}$ is holomorphic. Via identification of $\mathcal{L}$ with the quotient bundle $\Pi^*\mathcal{H}/\mathcal{T}$, we may assume that $\mathcal{L}$ is holomorphic. Its complex dimension equals $g-1\geq 2$.

The group $GL^+(2,\mathbb{R})$ acts on $\mathcal{H}_+$ as a group of real analytic transformations, and this action pulls back to an action on $\Pi^*\mathcal{H}\to\mathcal{H}_+$ as a group of real analytic bundle automorphisms.

Recall that the bundle $\Pi^*\mathcal{H}$ can be equipped with the flat Gauss Manin connection. We say that a splitting of $\Pi^*\mathcal{H}$ over a subset V of $\mathcal{H}_+$ is *flat* if it is invariant under parallel transport for the Gauss Manin along paths in V.

Lemma 6.1. *The restriction of the bundle $\mathcal{L}$ to the orbits of the $GL^+(2,\mathbb{R})$-action is flat.*

Proof. Let $q\in\mathcal{H}_+$ and let $A\subset H^1(S,\mathbb{R})$ be the $\mathbb{R}$-linear span of the real and the imaginary part of q. Then A is locally constant along the orbit $GL^+(2,\mathbb{R})q$ and hence it defines a subbundle of $\Pi^*\mathcal{H}\to GL^+(2,\mathbb{R})q$ which is locally invariant under parallel transport for the Gauss-Manin connection.

Now in a neighborhood of q in $GL^+(2,\mathbb{R})q$, the subspace A coincides with the fibre of the bundle $\mathcal{T}\to\mathcal{H}_+$, viewed as a subbundle of the bundle $\Pi^*\mathcal{N}$ which is invariant under the complex structure J on $\mathcal{N}$. Therefore the restriction of the bundle $\mathcal{T}$ to any orbit of $GL^+(2,\mathbb{R})$ is flat. As the Gauss Manin connection preserves the symplectic structure on $\Pi^*\mathcal{H}$, the restriction of its symplectic complement $\mathcal{L}$ to an orbit of the action of the group $GL^+(2,\mathbb{R})$ is flat as well. □

The foliation $\mathcal{F}$ of $\mathcal{H}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action is real analytic in period coordinates since the action of $GL^+(2,\mathbb{R})$ is affine in period coordinates. Furthermore, its leaves are complex suborbifolds of the complex orbifold $\mathcal{H}_+$. In particular, the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$ is a real analytic subbundle of the tangent bundle of the complex orbifold $\mathcal{H}_+$.

By Lemma 6.1, the Gauss-Manin connection on the flat bundle $\Pi^*\mathcal{H}\to\mathcal{H}_+$ restricts to a real analytic flat leafwise connection ∇^{GM} on the bundle $\mathcal{L}\to\mathcal{H}_+$. Here a leafwise connection is a connection whose covariant derivative is only defined for vectors tangent to the foliation $\mathcal{F}$. In other words, a leafwise connection associates to a smoth section of $\mathcal{L}$ and a tangent vector $X\in T\mathcal{F}$ a point in $\mathcal{L}$.

The leafwise connection ∇^{GM} is real analytic, which means that it is defined by a connection matrix which is real analytic in period coordinates. A connection matrix is defined as follows (see [GH78] for the use of a connection matrix in algebraic geometry). Choose a real analytic local basis of the bundle $\mathcal{L}$. The connection matrix for this local basis is the matrix whose entries are one-forms on $\mathcal{F}$, ie sections of the bundle $T\mathcal{F}^*$. The evaluation of these one-forms on a tangent vector $Y\in T\mathcal{F}$ expresses the covariant derivatives of the local basis of $\mathcal{L}$ in direction of Y as a linear combination of the basis elements. The leafwise connection ∇^{GM} preserves the symplectic structure of $\mathcal{L}$ as this is true for the Gauss Manin connection, but there is no information on the complex structure.

For each $k\leq g-2$, the leafwise connection ∇^{GM} extends to a flat leafwise connection on the bundle $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$ whose fibre at q is the $2k$-th exterior power of the fibre of $\mathcal{L}$ at q, viewed as a real vector space.

The Hermitean holomorphic vector bundle $\Pi^*\mathcal{H}\to\mathcal{H}_+$ admits a unique *Chern connection* ∇ (see e.g. [GH78]). The Chern connection defines parallel transport of the fibres of $\Pi^*\mathcal{H}$ along smooth curves in $\mathcal{H}_+$. This parallel transport preserves the Hermitean metric. The complex structure J on $\Pi^*\mathcal{H}$ is parallel for ∇. Since the $GL^+(2,\mathbb{R})$-orbits on $\mathcal{H}_+$ are complex suborbifolds of $\mathcal{H}_+$ and the restriction of the bundle $\mathcal{T}$ to each leaf of the foliation $\mathcal{F}$ can locally be identified with the pull-back to $GL^+(2,\mathbb{R})$ of the tangent bundle of the complex homogeneous space $GL^+(2,\mathbb{R})/(\mathbb{R}^+\times S^1)=\mathbf{H}^2$, by naturality the restriction of the Chern connection to the leaves of the foliation $\mathcal{F}$ of $\mathcal{H}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action preserves the decomposition $\Pi^*\mathcal{H}=\mathcal{T}\oplus\mathcal{L}$.

For every $k\leq g-2$, the complex structure J on $\mathcal{L}$ can be viewed as a real vector bundle automorphism of $\mathcal{L}$, and such a bundle automorphism extends to an automorphism of the real tensor bundle $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$. The restriction of the connection ∇ to the orbits of the $GL^+(2,\mathbb{R})$-action extends to a leafwise connection on $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$ which commutes with this automorphism.

The Hermitean metric which determines the Chern connection is defined by the polarization and the complex structure. These data are real analytic in period coordinates (recall that the Torelli map is holomorphic) and consequently the connection matrix for the Chern connection is real analytic in period coordinates (see [GH78]).

To summarize, for every $k \leq g-2$, both the Chern connection and the Gauss Manin connection restrict to leafwise connections of the restriction on the bundle $\wedge_{\mathbb{R}}^{2k}\mathcal{L} \to \mathcal{H}_+$ to the orbits of the $GL^+(2,\mathbb{R})$-action. Furthermore, $\nabla - \nabla^{GM}$ defines a real analytic tensor field

$$\Xi \in \Omega((T\mathcal{F})^* \otimes \mathcal{L}^* \otimes \mathcal{L})$$

where we denote by $\Omega((T\mathcal{F})^* \otimes \mathcal{L}^* \otimes \mathcal{L})$ the vector space of real analytic sections of the real analytic vector bundle $(T\mathcal{F})^* \otimes \mathcal{L}^* \otimes \mathcal{L}$. For every $k \leq g-2$ we obtain in the same way a real analytic tensor field

$$\Xi^k \in \Omega((T\mathcal{F})^* \otimes (\wedge_{\mathbb{R}}^{2k}\mathcal{L})^* \otimes \wedge_{\mathbb{R}}^{2k}\mathcal{L}). \tag{5}$$

If $\mathcal{C}_+ \subset \mathcal{H}_+$ is an affine invariant manifold of rank $2 \leq \ell \leq g-1$, with absolute holomorphic tangent bundle $\mathcal{Z}$, then the restriction of the tensor field $\Xi^{\ell-1}$ to $\mathcal{C}_+$ preserves the J-invariant section of $\wedge_{\mathbb{R}}^{2\ell-2}\mathcal{L}|\mathcal{C}_+$ which is defined by $p(T\mathcal{C}_+) \cap \Pi^*\mathcal{N}$. This section associates to a point $q \in \mathcal{C}_+$ the exterior product of a normalized oriented basis of the (real) $2\ell-2$-dimensional vector space $p(T_q\mathcal{C}_+) \cap \mathcal{L}$. Note that as $p(T\mathcal{C}_+) \cap \mathcal{L}$ is equipped with a complex structure, it also is equipped with an orientation.

The next proposition is a key step towards Theorem 3. For its formulation, denote again by $\mathcal{Q}_+ \subset \mathcal{H}_+$ a component of a stratum. Recall that the group $GL^+(2,\mathbb{R})$ acts on the bundle $\mathcal{L}$ by parallel transport for the Gauss Manin connection.

Proposition 6.2. *Let $\mathcal{C}_+ \subset \mathcal{Q}_+$ be an affine invariant manifold of rank $\ell \geq 3$, with absolute holomorphic tangent bundle $\mathcal{Z}$. Then one of the following two possibilities holds true.*

1. *There are finitely many proper affine invariant submanifolds of $\mathcal{C}_+$ which contain every affine invariant submanifold of $\mathcal{C}_+$ of rank $2 \leq k \leq \ell-1$.*
2. *Up to passing to a finite cover of $\mathcal{C}_+$, the restriction of the bundle $\mathcal{L} \cap \mathcal{Z}$ to an open dense $GL^+(2,\mathbb{R})$-invariant subset of $\mathcal{C}_+$ admits a non-trivial $GL^+(2,\mathbb{R})$-invariant real analytic splitting $\mathcal{L}\cap\mathcal{Z} = \mathcal{E}_1 \oplus \mathcal{E}_2$ into two complex subbundles.*

Proof. Let $\mathcal{Q}_+ \subset \mathcal{H}_+$ be a component of a stratum and let $\mathcal{C}_+ \subset \mathcal{Q}_+$ be an affine invariant manifold of rank $\ell \geq 3$, with absolute holomorphic tangent bundle $\mathcal{Z} \to \mathcal{C}_+$. An affine invariant manifold is affine in period coordinates and hence it inherits from $\mathcal{Q}_+$ a real analytic structure. As before, there is a splitting

$$\mathcal{Z} = \mathcal{T} \oplus (\mathcal{L} \cap \mathcal{Z}).$$

The bundle

$$\mathcal{W} = \mathcal{L} \cap \mathcal{Z} \to \mathcal{C}_+$$

is holomorphic. It also can be viewed as a real analytic real vector bundle with a real analytic complex structure J (which is just a real analytic section of the real analytic endomorphism bundle of $\mathcal{W}$ with $J^2 = -\mathrm{Id}$).

For $1 \leq k \leq \ell - 2$ denote by $\mathrm{Gr}(2k) \to \mathcal{C}_+$ the fibre bundle whose fibre over q is the Grassmannian of oriented $2k$-dimensional real subspaces of $\mathcal{W}_q$. This is a real analytic fibre bundle with compact fibre. It contains a real analytic subbundle $\mathcal{P}(k) \to \mathcal{C}_+$ whose fibre over q is the Grassmannian of *complex* k-dimensional subspaces of $\mathcal{W}_q$ (for the complex structure J).

The real part of the Hermitean metric on $\mathcal{W}$ naturally extends to a real analytic Riemannian metric on $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$. The bundle $\mathrm{Gr}(2k)$ can be identified with the set of pure vectors in the sphere subbundle of $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$ for this metric. Namely, an oriented $2k$-dimensional real linear subspace E of $\mathcal{W}_q$ defines uniquely a pure vector in $\wedge_{\mathbb{R}}^{2k}\mathcal{W}_q$ of norm one which is just the exterior product of an orthonormal basis of E with respect to the inner product on $\mathcal{W}_q$. The points in $\mathcal{P}(k)$ correspond precisely to those pure vectors which are invariant under the extension of J to an automorphism of $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$.

From now on, we work on the real analytic hyperplane $\mathcal{C} \subset \mathcal{C}_+$ of abelian differentials in $\mathcal{C}_+$ of area one, and we replace the action of $GL^+(2,\mathbb{R})$ by the action of $SL(2,\mathbb{R})$. The tangent bundle of the foliation of $\mathcal{C}$ into the orbits of the $SL(2,\mathbb{R})$-action is naturally trivialized by the following elements of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6}$$

defining the generator X of the Teichmüller flow, the generator Y of the horocycle flow, and the generator Z of the circle group of rotations.

Let B^k (or C^k, D^k) be the contraction of the tensor field Ξ^k defined in equation (5) with the vector field X (or Y, Z). Since these vector fields are real analytic and since the bundle $\mathcal{W} \to \mathcal{C}$ is invariant under both the Gauss Manin connection and the Chern connection, B^k (or C^k, D^k) can be viewed as a real analytic section of the endomorphism bundle $(\wedge_{\mathbb{R}}^{2k}(\mathcal{W}))^* \otimes \wedge_{\mathbb{R}}^{2k}(\mathcal{W})$ of $\wedge_{\mathbb{R}}^{2k}(\mathcal{W})$.

Define a *real analytic subset* of $\mathcal{P}(k)$ to be the intersection of the zero sets of a finite or countable number of real analytic functions on $\mathcal{P}(k)$. Recall that this is well defined since $\mathcal{P}(k)$ has a natural real analytic structure. We allow such functions to be constant zero, ie we do not exclude that such a set coincides with $\mathcal{P}(k)$. The real analytic set is *proper* if it does not coincide with $\mathcal{P}(k)$. Then there is at least one defining function which is not identically zero, and the set is closed and nowhere dense in $\mathcal{P}(k)$. We do not exclude the possibility that the set is empty.

For $1 \leq k \leq \ell - 2$ and $q \in \mathcal{C}$ let

$$\mathcal{R}_0^k(q,0) \subset \mathcal{P}(k)_q$$

be the set of all k-dimensional complex linear subspaces L of $\mathcal{W}_q$ with $B^k L = 0 = C^k L = D^k L$ (here we view as before a k-dimensional complex subspace of $\mathcal{W}_q$ as a pure J-invariant vector in $\wedge_{\mathbb{R}}^{2k}\mathcal{W}_q$). By linearity of the contractions B^k, C^k, D^k of the tensor field Ξ^k, the set $\mathcal{R}_0^k(q,0)$ can be identified with the set of all J-invariant pure vectors in $\wedge_{\mathbb{R}}^{k}(\mathcal{W}_q)$ which are contained in some (perhaps trivial) linear subspace of $\wedge_{\mathbb{R}}^{k}(\mathcal{W}_q)$. This subspace is the intersection of the kernels of the endomorphisms B^k, C^k, D^k.

Since the tensor field Ξ^k and the vector fields X, Y, Z are real analytic, $\cup_q \mathcal{R}_0^k(q, 0)$ is a real analytic subset of $\mathcal{P}(k)$, defined as the common zero set of three real analytic functions (the function which associates to a pure vector of square norm one the square norm of its image under the bundle map B^k, C^k, D^k).

Let ρ_t be the flow on $\mathcal{C}$ induced by the action of the circle group of rotations in $SL(2,\mathbb{R})$, obtained by a standard parametrization as a one-parameter subgroup of $SL(2,\mathbb{R})$. For $t \in \mathbb{R}$ define

$$\mathcal{R}_0^k(q, t) \subset \mathrm{Gr}(2k)_q$$

to be the preimage of $\mathcal{R}_0^k(\rho_t q, 0)$ under parallel transport for the Gauss Manin connection along the flow line $s \to \rho_s q$ $(s \in [0, t])$. By the previous paragraph and the fact that parallel transport is real analytic, $\cup_q \mathcal{R}_0^k(q, t)$ is a real analytic subset of $\mathrm{Gr}(2k)_q$ and hence the same holds true for

$$\mathcal{A}_0^k = \cap_{t \in \mathbb{R}} (\cup_q \mathcal{R}_0^k(q, t)) \subset \mathcal{P}(k)$$

(take the intersections for all $t \in \mathbb{Q}$).

By construction, the set $\mathcal{A}_0^k$ is invariant under the extension of the circle group of rotations by parallel transport with respect to the Gauss Manin connection to the fibres of the bundle $\mathrm{Gr}(2k) \to \mathcal{C}$. Here as before, we view $\mathrm{Gr}(2k)$ as a subset of the bundle $\wedge_{\mathbb{R}}^{2k}(\mathcal{W})$. It also is invariant under parallel transport with respect to the Chern connection: Namely, by definition, if $Z \in \mathcal{A}_0^k(q)$ and if $Z(t)$ is the parallel transport of $Z = Z(0)$ for the Gauss Manin connection along the orbit $t \to \rho_t(q)$ through q, then the covariant derivative of the section $t \to Z(t)$ for the Chern connection vanishes since for each t the vector $Z(t)$ is contained in the kernel of the contraction of the tensor field Ξ^k with the generator of the flow.

Similarly, for $t \in \mathbb{R}$ define

$$\mathcal{R}_1^k(q, t) \subset \mathrm{Gr}(2k)_q$$

to be the preimage of $\mathcal{A}_0^k(\Phi^t q)$ under parallel transport for the Gauss Manin connection along the flow line $s \to \Phi^s q$ $(s \in [0, t])$ of the Teichmüller flow and let

$$\mathcal{A}_1^k = \cap_{t \in \mathbb{R}} \big(\cup_q \mathcal{R}_1^k(q, t)\big) \subset \mathcal{P}(k).$$

Then $\mathcal{A}_1^k$ is invariant under the extension of the Teichmüller flow by parallel transport both for the Gauss Manin connection and the Chern connection. Furthermore, if $\alpha : [0, 1] \to SL(2,\mathbb{R})$ is any path which is a concatenation of an orbit segment of the Teichmüller flow, ie an orbit segment of the action of the diagonal group, with an orbit segment of the circle group of rotations, then for every $L \in \mathcal{A}_1^k$, the parallel transport of L along α for the Gauss Manin connection coincides with the parallel transport for the Chern connection, and it consists of points in the kernels of the tensor fields B^k, C^k, D^k.

Repeat this construction once more with the circle group of rotations to find a real analytic set

$$\mathcal{A}^k \subset \mathcal{P}(k).$$

This set is invariant under the action of $SL(2,\mathbb{R})$ defined by parallel transport for the Gauss Manin connection. Namely, let $\alpha \subset SL(2,\mathbb{R})$ be a path which is a concatentation of three segments $\alpha_1 \circ \alpha_2 \circ \alpha_3$, where α_1, α_3 are orbit segments of the circle group of rotations and α_2 is an orbit segment of the diagonal group. Then for each $z \in \mathcal{C}$ and each point $L \in \mathcal{A}^k(z)$, the parallel transport of L along the path $t \to z\alpha(t)$ for the Gauss-Manin connection coincides with the parallel transport of L for the Chern connection, and it consists of points contained in $\mathcal{A}^k$. In particular, this parallel transport consists of points whose contractions with B^k, C^k, D^k vanish.

Now using the Cartan decomposition of $SL(2,\mathbb{R})$, each point on $zSL(2,\mathbb{R})$ is the endpoint of a path of the above form beginning at z. As the restriction of ∇^{GM} to the orbits of the action of $SL(2,\mathbb{R})$ is flat, this implies the following. If $z \in \mathcal{C}$ and if $L \in \mathcal{A}^k(z)$, then the image of L under parallel transport for the Gauss Manin connection along the $SL(2,\mathbb{R})$-orbit $zSL(2,\mathbb{R})$ defines a section of the bundle $\wedge^k_{\mathbb{R}}(\mathcal{W})|zSL(2,\mathbb{R})$ which is parallel for the Chern connection and contained in $\mathcal{A}^k$.

If $\mathcal{D} \subset \mathcal{C}$ is a proper affine invariant manifold of rank $2 \leq k+1 < \ell$, then it follows from the discussion preceding this proof (see [F16]) that for every $q \in \mathcal{D}$ the projected tangent space $p(T_q\mathcal{D})$ defines a point in $\mathcal{A}^k(q) \subset \mathcal{P}(k)_q$. In particular, we have $\mathcal{A}^k \neq \emptyset$.

Let $\pi : \mathcal{P}(k) \to \mathcal{C}$ be the natural projection and let

$$M(k) = \pi(\mathcal{A}^k).$$

The fibres of π are compact and hence π is closed. Therefore $M(k)$ is a closed $SL(2,\mathbb{R})$-invariant subset of $\mathcal{C}$ which contains all affine invariant submanifolds of $\mathcal{C}$ of rank $k+1$.

There are now two possibilities. In the first case, the set $M(k)$ is nowhere dense in $\mathcal{C}$. Theorem 2.2 of [EMM15] then shows that $M(k)$ is a finite union of proper affine invariant submanifolds of $\mathcal{C}$. By construction, the union of these affine invariant submanifolds contains each affine invariant submanifold of $\mathcal{C}$ of rank $k+1$. Thus the first possibility in the proposition is fulfilled for affine invariant manifolds of rank $k+1$.

It now suffices to show the following. If there is some $k \leq \ell - 2$ such that the set $M(k)$ contains an open subset of $\mathcal{C}$, then there is a splitting of the bundle $\mathcal{W}$ over an open dense invariant subset of a finite cover of $\mathcal{C}$ as predicted in case (2) of the proposition.

Thus assume that the set $M = M(k) = \pi(\mathcal{A}^k)$ contains an open subset of $\mathcal{C}$. By invariance and topological transitivity of the action of $SL(2,\mathbb{R})$ on $\mathcal{C}$ [EMM15], M contains an open dense invariant set. On the other hand, M is closed and hence we have $M = \mathcal{C}$. This is equivalent to stating that for every $q \in \mathcal{C}$ the set $\mathcal{A}^k(q) \subset \mathcal{P}(k)_q$ is non-empty. In particular, for every $q \in \mathcal{C}$ there is a line in $\wedge^{2k}_{\mathbb{R}}(\mathcal{W}_q)$ spanned by a pure vector L which is an eigenvector for the extension of the complex structure J and which is contained in the kernel of B^k, C^k, D^k. Moreover, the same holds true for the parallel transport of L with respect to the Gauss Manin connection along the orbits of the $SL(2,\mathbb{R})$-action (compare the above discussion).

With respect to a real analytic local trivialization of the bundle $\mathcal{P}(k)$ over an open set $V \subset \mathcal{C}$, the set $\mathcal{A}^k$ is of the form $(q, \mathcal{A}^k(q))$ where $\mathcal{A}^k(q)$ is a real analytic subset of the compact projective variety $\mathcal{P}(k)_q$ of k-dimensional complex linear subspaces of $\mathcal{W}_q$ depending in a real analytic fashion on q. Even more is true: $\mathcal{A}^k(q)$ can be identified with the space of all J-invariant pure vectors which are contained in some J-invariant linear subspace R_q of $\wedge_{\mathbb{R}}^{2k}(\mathcal{W}_q)$ of positive dimension which depends in a real analytic fashion on q. The subspaces R_q are equivariant with respect to the action of $SL(2,\mathbb{R})$ by parallel transport for the Gauss Manin connection. Thus the set of all $q \in \mathcal{C}$ so that the dimension of R_q is minimal is an open $SL(2,\mathbb{R})$-invariant subset V of $\mathcal{C}$.

We claim that for $q \in V$, the set $\mathcal{A}^k(q)$ consists of only finitely many points. To this end choose a periodic orbit $\gamma \subset V$ for the Teichmüller flow so that for $q \in \gamma$, the restriction B to $\mathbb{R}^{2\ell} = \mathcal{Z}_q$ of the transformation $\Psi(\Omega(\gamma)) \in Sp(2\ell,\mathbb{R})$ has 2ℓ distinct real eigenvalues (the notations are as in Section 3). Such an orbit exists by Corollary 4.12. The linear map B is the return map for parallel transport of $\mathcal{Z}$ along γ with respect to the Gauss Manin connection, and it preserves the decomposition $\mathcal{Z}_q = \mathcal{T}_q \oplus \mathcal{W}_q$.

We are looking for k-dimensional complex linear subspaces L of $\mathcal{W}_q$ with the property that B^jL is complex for all $j \in \mathbb{Z}$. Now the linear map B preserves the symplectic form on $\mathcal{W}_q$, and the complex structure and the symplectic structure define an inner product $\langle,\rangle$ on $\mathcal{W}_q$ and hence on $\wedge_{\mathbb{R}}^{2k}(\mathcal{W}_q)$. Thus the set $\mathcal{A}^k(q)$ consists of pure vectors $Y \in \wedge_{\mathbb{R}}^{2k}(\mathcal{W}_q)$ whose norms are preserved by the symplectic linear map B.

As the eigenvalues of B are all real, nonzero and of multiplicity one, the eigenvalues for the action of B on $\wedge_{\mathbb{R}}^{2k}(\mathcal{W}_q)$ are all real as well. Each such eigenvalue τ is the product of $2k$ eigenvalues of the linear map B. The corresponding eigenspace is spanned by all pure vectors which are the exterior product of a basis of a $2k$-tuple of eigenspaces for B such that the product of the corresponding eigenvalues equals τ. In particular, the action of B on $\wedge_{\mathbb{R}}^{2k}(\mathcal{W}_q)$ is diagonalizable over $\mathbb{R}$. A vector $Y \in \wedge_{\mathbb{R}}^{2k}(\mathcal{W}_q)$ whose norm is preserved by the action of B is an eigenvector for the eigenvalue ± 1.

Since the complex structure defines an orientation, a point in $\mathcal{A}^k(q)$ is a $2k$-dimensional linear subspace of $\mathcal{W}_q$ which corresponds to a J-invariant pure eigenvector of B (acting on $\wedge_{\mathbb{R}}^{2k}(\mathcal{W}_q)$) for the eigenvalue one. In particular, such a linear subspace is invariant under the map B acting on $\mathcal{W}_q$.

A B-invariant linear subspace of $\mathcal{W}_q$ is a direct sum of subspaces of eigenspaces. As all eigenvalues are simple, it is in fact a direct sum of eigenspaces, and there are only finitely many such subspaces of $\mathcal{W}_q$. In other words, the number of points in $\mathcal{A}^k(q)$ is finite.

By Corollary 4.12 and the above discussion, the set of all points $q \in V$ such that $\mathcal{A}^k(q) \subset \mathcal{P}(k)_q$ is a finite set is dense in V. But $\mathcal{A}^k$ is a real analytic subset of $\mathcal{P}(k)$ and therefore by perhaps decreasing the size of V we may assume that $\mathcal{A}^k(q)$ is finite for all $q \in V$. Furthermore, the cardinality of $\mathcal{A}^k(q)$ is locally constant and hence constant on V since we may assume that V is connected.

As the dependence of $\mathcal{A}^k(q)$ on $q \in V$ is real analytic, any choice of a point in $\mathcal{A}^k(q)$ defines locally near q an analytic section of $\mathcal{P}(k)|V$ and hence a real analytic J-invariant subbundle of $\mathcal{W}|V$. This subbundle is invariant under parallel transport for the Gauss Manin connection along the orbits of the $SL(2,\mathbb{R})$-action. In the case that this local section is globally invariant under parallel transport for the Gauss Manin connection along the orbits of the $SL(2,\mathbb{R})$-action, it defines a real analytic J-invariant $SL(2,\mathbb{R})$-invariant subbundle of $\mathcal{W}|V$.

Otherwise parallel transport along the orbits of the $SL(2,\mathbb{R})$-action acts as a finite group of permutations on the finite set $\mathcal{A}^k(q)$. Thus we can pass to a finite cover of $\mathcal{C}$ so that on the covering space, using the same notation, the induced local subbundles of $\mathcal{W}$ are globally defined.

In other words, up to passing to a finite cover of $\mathcal{C}$, $\mathcal{A}^k$ defines a real analytic $SL(2,\mathbb{R})$-invariant complex k-dimensional vector bundle over the open dense invariant subset V of $\mathcal{C}$. By invariance of the symplectic structure under parallel transport along the orbits of the $SL(2,\mathbb{R})$-action, $\mathcal{A}^k$ then defines a splitting of the bundle $\mathcal{W}|V$ as predicted in the second part of the proposition. $\square$

Remark 6.3. By Proposition 4.5 (see also [W14]), a real analytic splitting of the bundle $\mathcal{L}$ as stated in the second part of Proposition 6.2 can not be flat, i.e. invariant under the Gauss Manin connection. However, the second part of Proposition 6.2 does not claim the existence of a flat subbundle of the projected tangent bundle of $\mathcal{C}$. Namely, the splitting is only required to be invariant under parallel transport along the orbits of the $SL(2,\mathbb{R})$-action.

7. Invariant splittings of the lifted Hodge bundle

In this section we use information on the moduli space of principally polarized abelian varieties to rule out the second case in Proposition 6.2. We continue to use all assumptions and notations from Section 6.

Recall the splitting $\Pi^*\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$ of the lifted Hodge bundle $\Pi^*\mathcal{H} \to \mathcal{H}_+$. Let $\mathcal{C}_+$ be an affine invariant manifold with absolute holomorphic tangent bundle $\mathcal{Z}$.

Consider again the intersection $\mathcal{C}$ of $\mathcal{C}_+$ with the moduli space of abelian differentials of area one. We shall argue by contradiction. As our discussion does not change by replacing $\mathcal{C}$ by a finite cover, we assume that there is an open dense $SL(2,\mathbb{R})$-invariant subset V of $\mathcal{C}$, and there is an $SL(2,\mathbb{R})$-invariant real analytic splitting $\mathcal{L} \cap \mathcal{Z}|V = \mathcal{E}_1 \oplus \mathcal{E}_2$ into complex orthogonal subbundles as in the second part of Proposition 6.2. The restriction of the splitting to an orbit of the $SL(2,\mathbb{R})$-action is invariant under the Gauss Manin connection.

By Lemma 3.2, if $r = \dim_{\mathbb{C}}(\mathcal{C}_+) - 2\mathrm{rk}(\mathcal{C}_+) > 0$ then the absolute period foliation $\mathcal{AP}(\mathcal{C})$ of $\mathcal{C}$ is defined, and it is a real analytic foliation with complex leaves of dimension r. Furthermore, as differentials contained in a leaf of this foliation locally have the same absolute periods, they define locally the same complex one-dimensional linear subspace of $\mathcal{H}$. This means that the splitting $\mathcal{Z} = \mathcal{T} \oplus \mathcal{W}$ where $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$ is invariant under the restriction of the Gauss Manin connection to the leaves of the absolute period foliation of $\mathcal{C}$ in the sense described in Section 6.

Our first goal is to show that the real analytic splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is invariant under the restriction of the Gauss Manin connection to the leaves of the absolute period foliation as well.

Lemma 7.1. *The restriction of the bundle $\mathcal{E}_i \to V$ to a leaf of $\mathcal{AP}(\mathcal{C})$ is invariant under the Gauss Manin connection.*

Proof. We may assume that the dimension r of $\mathcal{AP}(\mathcal{C})$ is positive. Furthermore, by passing to a finite cover of $\mathcal{C}$, we may assume that the zeros of the differentials in $\mathcal{C}$ are numbered. By abuse of notation, we will ignore these modifications in our notations as they do not alter the argument.

Write again $\mathcal{W}|V = \mathcal{Z} \cap \mathcal{L} = \mathcal{E}_1 \oplus \mathcal{E}_2$. By assumption, the bundles $\mathcal{E}_i \to V$ are real analytic and invariant under parallel transport for the Gauss Manin connection along the leaves of the foliation of V into the orbits of the action of $SL(2, \mathbb{R})$.

The splitting $\mathcal{Z}|V = \mathcal{T} \oplus \mathcal{E}_1 \oplus \mathcal{E}_2$ can be used to project the Gauss Manin connection ∇^{GM} on $p(T\mathcal{C}_+) = \mathcal{Z}$ to a real analytic connection $\hat{\nabla}$ on $\mathcal{Z}$ along the leaves of the absolute period foliation which preserves this decomposition. Namely, given a tangent vector $Z \in T\mathcal{AP}(\mathcal{C})$ and a local smooth section Y of $\mathcal{E}_i$, define

$$\hat{\nabla}_Z Y = P_i(\nabla^{GM}_Z Y)$$

where

$$P_i : \mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{E}_i$$

is the natural projection. Recall that this makes sense since the Gauss Manin connection restricted to a leaf of $\mathcal{AP}(\mathcal{C})$ preserves the bundle $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$ and hence $\nabla^{GM}_Z Y \in \mathcal{W}$.

We now use the assumptions and notations from Section 3. Let k be the number of zeros of a differential in $\mathcal{C}$. Choose once and for all a numbering of the zeros of a differential in $\mathcal{C}$. With respect to such a numbering, every vector $\mathfrak{a} \in \mathbb{C}^k$ of zero mean defines a vector field $X_{\mathfrak{a}}$ which is tangent to the absolute period foliation of the component $\mathcal{Q}$ of the stratum containing $\mathcal{C}$.

By Lemma 3.2, there exists a complex linear subspace $\mathcal{O}$ of $\mathbb{C}^k$ of complex dimension r contained in the complex hyperplane of vectors with zero mean so that for every $\mathfrak{a} \in \mathcal{O}$, the vector field $X_{\mathfrak{a}}$ is tangent to $\mathcal{C}$ at every point of $\mathcal{C}$. Furthermore, for every $\mathfrak{a} \in \mathcal{O}$, the affine invariant manifold $\mathcal{C}$ is invariant under the flow $\Lambda^t_{\mathfrak{a}}$ generated by $X_{\mathfrak{a}}$. For every $q \in V \subset \mathcal{C}$, every $\mathfrak{a} \in \mathcal{O}$ and every $Y \in \mathcal{E}_1(q)$ we can extend Y by parallel transport for $\hat{\nabla}$ along the flow line of the flow $\Lambda^t_{\mathfrak{a}}$.

Let us denote this extension by $\hat{Y}$; then

$$\beta(X_{\mathfrak{a}}, Y) = \frac{\nabla^{GM}}{dt} \hat{Y}(\Lambda^t_{\mathfrak{a}}(q))|_{t=0} \in \mathcal{E}_2(q)$$

only depends on $X_{\mathfrak{a}}$ and Y, moreover this dependence is linear in both variables. In this way we obtain a real analytic tensor field

$$\beta \in \Omega(T\mathcal{AP}(\mathcal{C})^* \otimes \mathcal{E}_1^* \otimes \mathcal{E}_2).$$

Here as before, $\Omega(T\mathcal{AP}(\mathcal{C})^* \otimes \mathcal{E}_1^* \otimes \mathcal{E}_2)$ is the vector space of real analytic sections of the bundle $T\mathcal{AP}(\mathcal{C})^* \otimes \mathcal{E}_1^* \otimes \mathcal{E}_2$. The splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is invariant under the restriction of the Gauss Manin connection to the leaves of the absolute period foliation if and only if β vanishes identically.

The Teichmüller flow Φ^t acts on the bundle $\mathcal{W}$ by parallel transport with respect to the Gauss Manin connection, and by assumption, this action preserves the bundles $\mathcal{E}_i$ $(i = 1, 2)$. The Teichmüller flow also preserves the absolute period foliation of $\mathcal{C}$. Thus the tensor field β is equivariant under the action of Φ^t.

Assume to the contrary that β does not vanish identically. As β is real analytic and bilinear and the vector space $\mathcal{O}$ is invariant under the complex structure, there is then an open subset U of $V \subset \mathcal{C}$ and either a real or a purely imaginary vector $\mathfrak{a} \in \mathcal{O}$ such that the contraction of β with $X_{\mathfrak{a}}$ does not vanish on U.

Assume that $\mathfrak{a} \in \mathbb{R}^k \cap \mathcal{O}$ is real, the case of a purely imaginary vector is treated in the same way; then $d\Phi^t X_{\mathfrak{a}} = e^t X_{\mathfrak{a}}$ by Lemma 3.1. Let now $\gamma \subset \mathcal{C}$ be a periodic orbit with the properties stated in Corollary 4.12 which passes through U. Let $q \in \gamma \cap U$. The eigenvalues of the matrix $A = \Psi(\Omega(\gamma))|\mathcal{Z}_q$ (where we identify $\mathcal{Z}_q$ with the symplectic subspace of $\Pi^*\mathcal{H}_q$ it defines) are real and of multiplicity one. The largest eigenvalue of A equals $e^{\ell(\gamma)}$ where $\ell(\gamma)$ is the length of the orbit γ, and the eigenspace for this eigenvalue is contained in the fibre $\mathcal{T}_q$ of the bundle $\mathcal{T}$.

The subspace $\mathcal{W}_q$ of $\mathcal{Z}_q$ is invariant under A and hence $\mathcal{W}_q$ is a sum of eigenspaces for A (viewed as a transformation of $\mathcal{Z}_q$) for eigenvalues whose absolute values are strictly smaller than $e^{\ell(\gamma)}$. Furthermore, by invariance of the splitting of $\mathcal{W}$ under parallel transport for the Gauss Manin connection along flow lines of the Teichmüller flow, the decomposition $\mathcal{W}_q = \mathcal{E}_1(q) \oplus \mathcal{E}_2(q)$ $(i = 1, 2)$ is invariant under the map A. Then $\mathcal{E}_i(q)$ is a direct sum of eigenspaces for A.

For clarity of exposition, write for the moment $||_\gamma^{GM}$ for parallel transport along γ with respect to the Gauss Manin connection. By equivariance of the tensor field β under the action of Φ^t, for $Y \in \mathcal{E}_1(q)$ we have

$$\beta(d\Phi^{\ell(\gamma)} X_{\mathfrak{a}}, ||_\gamma^{GM} Y) = ||_\gamma^{GM} \beta(X_{\mathfrak{a}}, Y) \in \mathcal{E}_2(q).$$

Now if $Y \in \mathcal{E}_1(p)$ is an eigenvector of A for the eigenvalue $b \neq 0$, then from $d\Phi^{\ell(\gamma)} X_{\mathfrak{a}} = e^{\ell(\gamma)} X_{\mathfrak{a}}$ we obtain

$$\beta(d\Phi^{\ell(\gamma)} X_{\mathfrak{a}}, AY) = e^{\ell(\gamma)} b \beta(X_{\mathfrak{a}}, Y) = A\beta(X_{\mathfrak{a}}, Y) \in \mathcal{E}_2(p).$$

In other words, the contraction $Y \in \mathcal{E}_1(p) \to \beta(X_{\mathfrak{a}}, Y) \in \mathcal{E}_2(p)$ of β with $X_{\mathfrak{a}}$ maps an eigenspace of A contained of A contained in $\mathcal{E}_2(p)$ for the eigenvalue $e^{\ell(\gamma)} b$.

But $e^{\ell(\gamma)}$ is an eigenvalue of the matrix A (for an eigenvector contained in the bundle $\mathcal{T}$) and by the choice of γ, no product of two eigenvalues of A is an eigenvalue. By the discussion in the previous paragraph, this implies that the contraction of β with $X_{\mathfrak{a}}$ vanishes at q, contradicting the assumption that this contraction does not vanish at q.

As a consequence, the tensor field β vanishes identically, and parallel transport for $\hat{\nabla}$ of a vector $Y \in \mathcal{E}_1$ along a path which is entirely contained in a leaf of the absolute period foliation of $V \subset \mathcal{C}$ coincides with parallel transport with respect to the Gauss Manin connection. Equivalently, the restriction of the Gauss Manin connection to a leaf of the absolute period foliation preserves the splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$. This is what we wanted to show. $\square$

Remark 7.2. Lemma 7.1 is valid for Φ^t-invariant splittings of the bundle $\mathcal{W}$ of class C^1, but the case of a continuous splitting can not be deduced in the same way. We expect nevertheless that the lemma holds true for continuous splittings as well. A possible strategy towards this end is to use methods from hyperbolic dynamics to show that a continuous Φ^t-invariant splitting has to be of class C^1 along the leaves of the real real foliation and the leaves of the imaginary rel foliation and then use Lemma 7.1 and its proof to deduce that it is parallel along these leaves.

It is an interesting question whether it is possible to deduce from Lemma 7.1, Proposition 4.13 and Moore's theorem, applied to the right action of $SL(2,\mathbb{R})$ on $Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})$, that a splitting as in the second part of Proposition 6.2 does not exist. The main difficulty is that the global structure of the absolute period foliation of an affine invariant manifold is poorly understood. Moreover, we do not know whether there is a leaf of the foliation of the bundle $\mathcal{S} \to \mathfrak{D}_g$ as described in the appendix which intersects the image of the period map in more than one component.

We saw so far that a splitting of the subbundle $\mathcal{W}$ of the bundle $\Pi^*\mathcal{H}$ over an affine invariant manifold $\mathcal{C}$ as predicted by the second part of Proposition 6.2 has to be parallel for the Gauss Manin connection along the leaves of the absolute period foliation. Our final goal is to use the curvature of the projection of the Gauss Manin connection to the bundle $\mathcal{L}$ to derive a contradiction. Note that this projected connection is not flat (see below). To compute its curvature we take advantage of the geometry of the tautological vector bundle $\mathcal{V} \to \mathcal{A}_g$. We will use some differential geometric properties of this bundle described in the appendix.

Proposition 7.3. *Let $\mathcal{Z}$ be the absolute holomorphic tangent bundle of an affine invariant manifold $\mathcal{C}_+$ of rank at least three. There is no open dense $GL^+(2,\mathbb{R})$-invariant subset V of $\mathcal{C}_+$ such that the bundle $\mathcal{Z}\cap\mathcal{L}|V$ admits a $GL^+(2,\mathbb{R})$-invariant real analytic splitting $\mathcal{Z}\cap\mathcal{L} = \mathcal{E}_1 \oplus \mathcal{E}_2$ into two complex subbundles.*

Proof. As before, we write $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$. Furthermore, we restrict our attention to the intersection $\mathcal{C}$ of $\mathcal{C}_+$ with the moduli space of area one abelian differentials.

We argue by contradiction, and we assume that an open dense invariant set V and a splitting $\mathcal{W}|V = \mathcal{E}_1 \oplus \mathcal{E}_2$ with the properties stated in the proposition exists. Lemma 7.1 shows that this splitting is invariant under the restriction of the Gauss Manin connection to the leaves of the absolute period foliation of $\mathcal{C}$. Furthermore, it naturally induces an invariant splitting of the bundle $\mathcal{W} \oplus \overline{\mathcal{W}} \subset p(T\mathcal{C}_+)$ into two subbundles which are complex for the flat complex structure on $H^1(S,\mathbb{C}) = H^1(S,\mathbb{R})\otimes\mathbb{C}$. Namely, recall that via the identifications used earlier, the bundle $\mathcal{W}$ can be represented as $\mathcal{W} = \{X + iJX \mid X \in \mathcal{W}_\mathbb{R}\}$ where $\mathcal{W}_\mathbb{R}$ is a (real) subbundle of the flat vector bundle $\Pi^*\mathcal{N} \to \mathcal{C}_+$ which is globally invariant under the Gauss Manin connection.

Let $\mathcal{T}(S)$ be the Teichmüller space of the surface S and let $\mathcal{I}_g < \mathrm{Mod}(S)$ be the Torelli group. The group $\mathcal{I}_g$ acts properly and freely from the left on $\mathcal{T}(S)$, with quotient the *Torelli space* $\mathcal{I}_g\backslash\mathcal{T}(S)$. Let $\mathcal{D} \to \mathcal{I}_g\backslash\mathcal{T}(S)$ be the bundle of area one homology-marked abelian differentials. The *period map* F maps the bundle $\mathcal{D}$ into the sphere subbundle $\mathcal{S}$ of the tautological vector bundle $\mathcal{V}$ over the Siegel upper half-space $\mathfrak{D}_g$ (see the beginning of Section 4 and the appendix for the notations).

By the discussion in the appendix, the composition of the map F with the projection

$$\Pi : \mathcal{S} = Sp(2g, \mathbb{R}) \times_{U(g)} S^{2g-1} \to \Omega = Sp(2g, \mathbb{R})/Sp(2g-2, \mathbb{R})$$

is equivariant for the standard right $SL(2, \mathbb{R})$-actions on $\mathcal{D}$ and on Ω. Furthermore, we have

$$\Omega = \{x + iy \mid x, y \in \mathbb{R}^{2g}, \omega(x, y) = 1\}$$

where $\omega = \sum_i dx_i \wedge dy_i$ is the standard symplectic form on $\mathbb{R}^{2g}$.

Let $\hat{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the bundle $\mathcal{D} \to \mathcal{I}_g \backslash \mathcal{T}(S)$. The projection $p(T\mathcal{C}_+)$ determines a subbundle of the trivial bundle $\hat{\mathcal{C}} \times H^1(S, \mathbb{C}) \to \hat{\mathcal{C}}$ which is locally constant and invariant under the complex structure i induced from the representation $H^1(S, \mathbb{C}) = H^1(S, \mathbb{R}) \otimes \mathbb{C}$. Hence $\hat{\mathcal{C}}$ determines a complex subspace $\mathbb{C}^{2\ell}$ of $H^1(S, \mathbb{C})$ whose real part is symplectic. The composition of the map $\Pi \circ F$ with symplectic orthogonal projection then defines a map

$$\Upsilon : \hat{\mathcal{C}} \to \Omega_\ell = \{x + iy \mid x, y \in \mathbb{R}^{2\ell}, \omega(x, y) = 1\}.$$

We refer to the appendix for more details of this construction.

The manifold Ω_ℓ is a hyperplane in the open subset

$$\mathcal{O}_\ell = \{x + iy \mid x, y \in \mathbb{R}^{2\ell}, \omega(x, y) > 0\}$$

of $\mathbb{C}^{2\ell}$. By naturality (see the appendix for details), the Gauss Manin connection on the bundle $p(T\hat{\mathcal{C}}_+)$ with fibre $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ is just the pull-back via Υ of the natural flat connection $\nabla^{\mathcal{O}}$ on $T\mathcal{O}_\ell$.

As a consequence, using the notations from the appendix, we obtain the following. The restriction of the tangent bundle $T\mathcal{O}_\ell$ of $\mathcal{O}_\ell$ to Ω_ℓ decomposes as

$$T\mathcal{O}_\ell|\Omega_\ell = \mathcal{T}_{SL} \oplus \mathcal{R} \oplus \mathbb{R}$$

where $\mathcal{T}_{SL}$ is the tangent bundle of the foliation of Ω_ℓ into the orbits of the right action of the group $SL(2, \mathbb{R})$, $\mathcal{T}_{SL} \oplus \mathcal{R}$ is the tangent bundle of Ω_ℓ and $\mathbb{R}$ is the normal bundle of Ω_ℓ in $\mathcal{O}_\ell$. The standard flat connection $\nabla^{\mathcal{O}_\ell}$ on $T\mathcal{O}_\ell$ projects to a connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$. The restriction of $\nabla^{\mathcal{O}_\ell}$ to the foliation of Ω_ℓ into the orbits of the $SL(2, \mathbb{R})$-action preserves the bundle $\mathcal{R}$ and hence the restriction of $\nabla^{\mathcal{R}}$ to this foliation coincides with the restriction of $\nabla^{\mathcal{O}_\ell}$. The leafwise connection ∇^{GM} on the bundle $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$ is the pull-back of the restriction of the connection $\nabla^{\mathcal{O}_\ell}$ (see the appendix for more details).

By Lemma 7.1, the splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is real analytic, invariant under the action of $SL(2, \mathbb{R})$ and parallel with respect to the restriction of the Gauss Manin connection to the leaves of the absolute period foliation. By Lemma A.7 in the appendix, this implies that the splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is the pull-back by Υ of a real analytic local splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ of the bundle $\mathcal{R}$ into a sum of two complex vector bundles, defined on the image of the map Υ. That this image is open follows from the description of affine invariant manifolds via period coordinates. Furthermore, this splitting is invariant under the right action of the group $SL(2,\mathbb{R})$.

The curvature form Θ for the connection $\nabla^{\mathcal{R}}$ is a two-form on Ω_ℓ with values in the bundle $\mathcal{R}^*\otimes\mathcal{R}$. We claim that Θ preserves the decomposition $\mathcal{R}=\mathcal{R}_1\oplus\mathcal{R}_2$ on the image of the map Υ. This means that for any point x in the image of Υ, any two tangent vectors $u,v\in T_x\Omega_\ell$ and any $Y\in\mathcal{R}_i$, we have $\Theta(u,v)(Y)\in\mathcal{R}_i$.

To this end let $\gamma\subset\mathcal{C}$ be a periodic orbit for Φ^t with the properties as in Corollary 4.12 and let $\hat\gamma$ be a lift of γ to $\hat{\mathcal{C}}$. By Lemma A.7, $\Upsilon(\hat\gamma)$ is an orbit in Ω_ℓ for the action of the diagonal subgroup of $SL(2,\mathbb{R})$. This orbit is periodic under the action of an element $A\in Sp(2\ell,\mathbb{R})$ (which is the restriction of an element of $Sp(2g,\mathbb{Z})$ stabilizing the subspace $\mathbb{R}^{2\ell}$) whose eigenvalues are all real, of multiplicity one, and such that no product of two eigenvalues is an eigenvalue.

Since the local splitting $\mathcal{R}=\mathcal{R}_1\oplus\mathcal{R}_2$ is invariant under the action of $SL(2,\mathbb{R})$ and is complex and hence symplectic, for any choice of a point $z\in\Upsilon(\hat\gamma)$, the subspaces $(\mathcal{R}_i)_z$ are direct sums of eigenspaces for A, containing with an eigenspace for the eigenvalue a the eigenspace for a^{-1}.

We now follow the proof of Lemma 7.1. Let $\nabla^{\mathcal{R}_1}$ be the projection of the connection $\nabla^{\mathcal{R}}$ to a connection on $\mathcal{R}_1$. Then $\nabla^{\mathcal{R}}-\nabla^{\mathcal{R}_1}$ is a real analytic (locally defined) tensor field $\beta\in\Omega(T^*\Omega_\ell\otimes\mathcal{R}_1^*\otimes\mathcal{R}_2)$. Since $\mathcal{R}_1$ and $\nabla^{\mathcal{R}}$ are invariant under the action of the diagonal flow $\Psi^t\subset SL(2,\mathbb{R})$ (we use the notation Ψ^t here to indicate that we are looking at a flow on the space Ω_ℓ), this tensor field is equivariant under the action of Ψ^t. Now no product of two eigenvalues of the matrix A is an eigenvalue and hence this implies that the restriction of β to $\Upsilon(\hat\gamma)$ vanishes (compare the proof of Lemma 7.1).

By Corollary 4.12, the set of points $q\in V\subset\mathcal{C}$ which are contained in a periodic orbit with the above properties is dense in V. Hence the image of this set under the restriction of the map Υ to a small contractible open subset of V is a dense subset of a nonempty open subset E of Ω_ℓ where the splitting $\mathcal{R}=\mathcal{R}_1\oplus\mathcal{R}_2$ is defined. As the real analytic tensor field β vanishes on this dense subset of E, it vanishes identically on E. Hence the splitting $\mathcal{R}=\mathcal{R}_1\oplus\mathcal{R}_2$ of $\mathcal{R}$ on E is invariant under the connection $\nabla^{\mathcal{R}}$.

As a consequence, the curvature form Θ of $\nabla^{\mathcal{R}}$ preserves the decomposition $\mathcal{R}=\mathcal{R}_1\oplus\mathcal{R}_2$ on E. Using the terminology from the appendix, this means that Θ splits $\mathcal{R}$ as a complex vector bundle. This contradicts Lemma A.4 and shows the proposition. ∎

Remark 7.4. The reasoning in the proof of Lemma 7.1 and Proposition 7.3 also implies that the Lyapunov filtration for the action of the Teichmüller flow on a stratum of abelian differentials is not smooth (or, less restrictive, is not of the class C^1). As we use covariant differentiation in our argument, mere continuity of the filtration can not be ruled out in this way.

Corollary 7.5. (1) *Let $\mathcal{Q}$ be a component of a stratum; then for every $2\leq\ell\leq g-1$ there are finitely many affine invariant submanifolds of $\mathcal{Q}$ of rank ℓ which contain every affine invariant submanifold of rank ℓ.*

(2) *The smallest stratum of differentials with a single zero contains only finitely many affine invariant submanifolds of rank at least two.*

Proof. Let $\mathcal{C}$ be an affine invariant manifold of rank $\ell \geq 3$. By Proposition 6.2 and Proposition 7.3, there are finitely many proper affine invariant submanifolds of $\mathcal{C}$ which contain every affine invariant submanifold of $\mathcal{C}$ of rank $2 \leq k \leq \ell - 1$.

An application of this fact to a component $\mathcal{Q}$ of a stratum shows that for $2 \leq \ell \leq g-1$, there are finitely many proper affine invariant submanifolds $\mathcal{C}_1, \dots, \mathcal{C}_m$ of $\mathcal{Q}$ which contain every affine invariant submanifold of $\mathcal{Q}$ of rank ℓ. The dimension of $\mathcal{C}_i$ is strictly smaller than the dimension of $\mathcal{Q}$.

By reordering we may assume that there is some $u \leq m$ such that for all $i \leq u$ the rank $\mathrm{rk}(\mathcal{C}_i)$ of $\mathcal{C}_i$ is at most ℓ, and that for $i > u$ the rank $\mathrm{rk}(\mathcal{C}_i)$ of $\mathcal{C}_i$ is bigger than ℓ. Apply the first paragraph of this proof to each of the affine invariant manifolds $\mathcal{C}_i$ $(i > u)$. We conclude that for each i there are finitely many proper affine invariant submanifolds of $\mathcal{C}_i$ of rank $r \in [\ell, \mathrm{rk}(\mathcal{C}_i))$ which contain every affine invariant submanifold of $\mathcal{C}_i$ of rank ℓ. The dimension of each of these submanifolds is strictly smaller than the dimension of $\mathcal{C}_i$. In finitely many such steps, each applied to all affine invariant submanifolds of rank strictly bigger than ℓ found in the previous step, we deduce the statement of the first part of the corollary.

Now let $\mathcal{H}(2g-2)$ be a stratum of differentials with a single zero. Period coordinates for $\mathcal{H}(2g-2)$ are given by absolute periods, and the dimension of an affine invariant manifold $\mathcal{C} \subset \mathcal{H}(2g-2)$ of rank ℓ equals 2ℓ. Thus $\mathcal{C}$ does not contain any proper affine invariant submanifold of rank ℓ.

By Proposition 6.2 and the first part of this proof, there are finitely many proper affine invariant submanifolds $\mathcal{C}_1, \dots, \mathcal{C}_s$ of $\mathcal{H}(2g-2)$ which contain every affine invariant submanifold of $\mathcal{H}(2g-2)$ of rank at most $g-1$. In particular, there are only finitely many such manifolds of rank $g-1$.

To show finiteness of affine invariant manifolds of any rank $2 \leq \ell \leq g-1$, apply Proposition 6.2 and the first part of this proof to each of the finitely many affine invariant manifolds constructed in some previous step and proceed by inverse induction on the rank. □

Remark 7.6. The proof of the second part of Corollary 7.5 immediately extends to the following statement. An affine invariant manifold $\mathcal{C}$ with trivial absolute period foliation contains only finitely many affine invariant manifolds of rank at least two.

8. Nested affine invariant submanifolds of the same rank

The goal of this section is to analyze affine invariant submanifolds of affine invariant manifolds $\mathcal{C}_+$ of the same rank $\ell \geq 2$ and to complete the proof of Theorem 3. Our strategy is a variation of the strategy used in Section 7. Namely, given an affine invariant manifold $\mathcal{C}_+$ with nontrivial absolute period foliation, we observe first that either $\mathcal{C}_+$ contains only finitely many affine invariant manifolds of the same rank, or there is a $GL^+(2,\mathbb{R})$-invariant real analytic splitting of the tangent bundle $T\mathcal{C}_+$ of $\mathcal{C}_+$ over a $GL^+(2,\mathbb{R})$-invariant open dense subset V of $\mathcal{C}_+$ into two subbundles, where one of these subbundles is contained in the tangent bundle of the absolute period foliation. In a second step, we show that these subbundles can be chosen to be integrable, with integral manifolds which are flat. We then use the assumption that the rank of $\mathcal{C}_+$ is at least two to deduce that these manifolds determine a flat $GL^+(2,\mathbb{R})$-invariant splitting of $T\mathcal{C}_+$ which is impossible [W14].

Denote as before by $\mathcal{AP}(\mathcal{C}_+)$ the absolute period foliation of an affine invariant manifold $\mathcal{C}_+$. By perhaps passing to a finite cover we may assume that the zeros of a differential $q \in \mathcal{C}_+$ are numbered.

The following proposition is analogous to Proposition 6.2 and carries out the first and second step of the above outline. Recall that the Teichmüller flow Φ^t acts on $T\mathcal{C}_+$ as a group of bundle automorphisms.

Proposition 8.1. *Let $\mathcal{C}_+ \subset \mathcal{Q}_+$ be an affine invariant manifold of rank $\ell \geq 1$. Then one of the following two possibilities holds true.*

(1) *There are at most finitely many proper affine invariant submanifolds of $\mathcal{C}_+$ of rank ℓ.*
(2) *Up to passing to a finite cover, the tangent bundle $T\mathcal{C}_+$ of $\mathcal{C}_+$ admits a non-trivial Φ^t-invariant real analytic splitting $T\mathcal{C}_+ = \mathcal{A} \oplus \mathcal{E}$ where $\mathcal{A}$ is a flat complex subbundle of $T\mathcal{AP}(\mathcal{C}_+)$ and where $\mathcal{E}$ contains the tangent bundle of the orbits of the $GL^+(2,\mathbb{R})$-action. Furthermore, the bundle $\mathcal{E}$ is integrable, and it defines a foliation of $\mathcal{C}_+$ with locally flat leaves.*

Proof. By Theorem 2.2 of [EMM15], it suffices to show the following. Let $m = \dim_{\mathbb{C}}(\mathcal{C}_+)$ and write $\ell = \mathrm{rk}(\mathcal{C}_+)$. Assume that there is a number $k \in [1, m-2\ell]$, and there is an open subset V of $\mathcal{C}_+$ such that the set of all affine invariant submanifolds of $\mathcal{C}_+$ of complex codimension k whose rank coincide with the rank of $\mathcal{C}_+$ is dense in V; then the second property in the proposition holds true.

Assume from now on that a nonempty open subset V of $\mathcal{C}_+$ with the properties stated in the previous paragraph exists. Note that we may assume that V is dense by $GL^+(2,\mathbb{R})$-invariance and topological transitivity of the action of $GL^+(2,\mathbb{R})$.

The leaves of the foliation $\mathcal{F}$ of $\mathcal{C}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action are complex suborbifolds of $\mathcal{C}_+$, ie the tangent bundle $T\mathcal{F}$ of this foliation is invariant under the complex structure i obtained from period coordinates. Let $\mathcal{Y}$ be an i-invariant $GL^+(2,\mathbb{R})$-invariant real analytic subbundle of the tangent bundle $T\mathcal{C}_+$ of $\mathcal{C}_+$ which is complementary to the bundle $T\mathcal{F}$. Using the notations from Section 6, such a bundle can be constructed as follows.

Let $\mathcal{Z}$ be the absolute holomorphic tangent bundle of $\mathcal{C}_+$ and write $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$. Let moreover $T\mathcal{C}$ be the tangent bundle of the foliation of $\mathcal{C}_+$ into the hypersurfaces of differentials with fixed area and let i be the standard complex structure in period coordinates; then we can take $\mathcal{Y} = p^{-1}(\mathcal{W} \oplus \overline{\mathcal{W}}) \cap T\mathcal{C} \cap iT\mathcal{C}$.

For the number $k \in \{1, \dots, m-2\ell\}$ as specified above let $\mathcal{P} \to \mathcal{C}_+$ be the real analytic fibre bundle whose compact fibre at a point $q \in \mathcal{C}_+$ equals the Grassmannian of all *complex* subspaces of $\mathcal{Y}_q$ of complex codimension k. This is a real analytic subbundle of the fibre bundle whose fibre at q equals the Grassmannian of all oriented real linear subspaces of codimension $2k$ in $\mathcal{Y}_q$. The bundle $\mathcal{P}$ admits a natural decomposition $\mathcal{P} = \cup_{i=0}^{k}\mathcal{P}_i$ where $\mathcal{P}_i$ consists of all subspaces which intersect $T\mathcal{AP}(\mathcal{C}_+)$ in a subspace of complex codimension $k - i$. Thus $\mathcal{P}_0$ is the bundle of complex subspaces of complex codimension k which intersect $T\mathcal{AP}(\mathcal{C}_+)$ in a subspace of smallest possible dimension. In particular, $\mathcal{P}_0 \subset \mathcal{P}$ is open and $GL^+(2,\mathbb{R})$-invariant, and $\cup_{i\geq 1}\mathcal{P}_i$ is a closed nowhere dense subvariety of $\mathcal{P}$.

Our strategy is similar to the strategy used before. We begin with investigating the action of the Teichmüller flow Φ^t on the bundle $\mathcal{P}$, where for convenience of exposition, we restrict this flow to the real hypersurface $\mathcal{C}$ of differentials in $\mathcal{C}_+$ of area one, but we let its derivative act on the tangent bundle $T\mathcal{C}_+$ of $\mathcal{C}_+$.

Recall that the action of the flow Φ^t on $T\mathcal{C}_+$ preserves the bundle $\mathcal{Y}$. For $q \in \mathcal{C}$ and $t \in \mathbb{R}$ let $\rho(q,t)$ be the image of $\mathcal{P}(\Phi^t q)$ under the map $d\Phi^{-t}$. Then

$$\mathcal{R}_\infty = \cap_t \cup_q \rho(q,t)$$

is a (possibly empty) real analytic subset of $\mathcal{P}$. By construction, this subset is invariant under the action of Φ^t.

The tangent bundle of an affine invariant submanifold $\mathcal{D}_+$ of $\mathcal{C}_+$ intersects the complex vector bundle $\mathcal{Y}$ in a complex subbundle $\mathcal{Y} \cap T\mathcal{D}_+|\mathcal{D}_+$. This subbundle is invariant under the action of the flow Φ^t. Hence if $q \in \mathcal{C}$ is contained in an affine invariant submanifold $\mathcal{D}$ of $\mathcal{C}$ of the same rank as $\mathcal{C}$ and of complex codimension k, then $\mathcal{R}_\infty \cap \mathcal{P}_0(q) \neq \emptyset$.

Thus under the assumption on the existence of a nonempty open Φ^t-invariant subset V of $\mathcal{C}$ containing a dense set of points which lie on an affine invariant submanifold of $\mathcal{C}$ of rank ℓ and complex codimension k, the real analytic subset $\mathcal{R}_\infty$ of $\mathcal{P}$ is not empty, and its image under the canonical projection $\pi : \mathcal{P} \to V$ is dense in the open set V. Since $\mathcal{R}_\infty \subset \mathcal{P}$ is closed and the canonical projection π is closed as well, this implies that the restriction of π to $\mathcal{R}_\infty$ maps $\mathcal{R}_\infty$ onto V. We refer to the proof of Proposition 6.2 for details on this construction.

Now $\mathcal{P}_0 \subset \mathcal{P}$ is an open subset of $\mathcal{P}$, and $\mathcal{R}_\infty \cap \mathcal{P}_0(q) \neq \emptyset$ for a dense set of points $q \in V$. As $\mathcal{R}_\infty$ is a real analytic set, this implies that up to perhaps decreasing the set V, we may assume that $\mathcal{R}_\infty \cap \mathcal{P}_0(q)$ is not empty for every $q \in V$. As the tangent bundle of the absolute period foliation is invariant under the action of Φ^t, the set $\mathcal{R}_\infty \cap \mathcal{P}_0$ is Φ^t-invariant as well. Thus

$$\mathcal{K} = \mathcal{R}_\infty \cap \mathcal{P}_0$$

is a real analytic subset of the (open) suborbifold $\mathcal{P}_0$ of $\mathcal{P}$ which is invariant under the natural action of the Teichmüller flow Φ^t and which projects onto an open dense Φ^t-invariant subset of $\mathcal{C}$ which we denote again by V.

For each $q \in V$, each point $z \in \mathcal{K}(q)$ is a complex linear subspace of $\mathcal{Y}_q$ of complex codimension k which intersects $T\mathcal{AP}(\mathcal{C})$ in a subspace of complex codimension k. Define

$$E(q) = \cap_{z\in\mathcal{K}(q)} z \subset \mathcal{Y}_q \subset T_q\mathcal{C}_+.$$

Then $E(q)$ is a (possibly trivial) complex linear subspace of $\mathcal{Y}_q$. As $\mathcal{K} \subset \mathcal{P}_0$ is a real analytic subset of $\mathcal{P}_0$ which projects to V and which is invariant under the action of the Teichmüller flow Φ^t, by possibly replacing the set V by a proper open

Φ^t-invariant subset we may assume that the dimension of $E(q)$ (which may be zero) does not depend on $q \in V$. If this dimension is positive, then $\cup_{q\in V} E(q)$ is a real analytic complex subbundle of $\mathcal{Y}|V$. Furthermore, if $\mathcal{D} \subset \mathcal{C}$ is an affine invariant submanifold of rank ℓ and complex codimension k which intersects the set V, then for every point $q \in V \cap \mathcal{D}$, the tangent space $T_q\mathcal{D}_+$ of $\mathcal{D}_+$ at q contains $E(q)$.

Our next goal is to show that for $q \in V$, the complex dimension of $E(q)$ is at least $\ell - 1$. To this end let $\gamma \subset \mathcal{C}$ be a periodic orbit with the properties stated in Corollary 4.12 which intersects V in a point q. Let $\ell(\gamma)$ be the length of γ. The return map $d\Phi^{\ell(\gamma)}(q)$ acts on $\mathcal{Y}_q$.

Let again p be the projection of $T\mathcal{C}_+$ into absolute periods, and let $\mathcal{Z}$ be the absolute holomorphic tangent bundle of $\mathcal{C}_+$. The map $d\Phi^{\ell(\gamma)}$ commutes with p and hence it descends to a linear map A on the vector space $(\mathcal{Z} \oplus \overline{\mathcal{Z}})_q$, ie we have

$$p \circ d\Phi^{\ell(\gamma)} = A \circ p.$$

The map A is just the monodromy map obtained from parallel transport for the Gauss Manin connection on the flat bundle $\Pi^*\mathcal{N} \otimes \mathbb{C} \to \mathcal{Q}_+$.

By the choice of γ, the map A is semi-simple, with real eigenvalues, and the eigenspaces are complex lines (recall that we look here at the action of the pseudo-Anosov map $\Omega(\gamma)$ on $H^1(S,\mathbb{C}) = H^1(S,\mathbb{R}) \otimes \mathbb{C}$). Let $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$ be as before. Then the complex subspace $\mathcal{W} \otimes \overline{\mathcal{W}}$ is a direct sum of eigenspaces for eigenvalues whose absolute values are contained in the open interval $(e^{-\ell(\gamma)}, e^{\ell(\gamma)})$.

Together with Lemma 3.1, we conclude that the restriction F of the map $d\Phi^{\ell(\gamma)}$ to $\mathcal{Y}_p$ is semi-simple. The eigenspaces for eigenvalues of absolute value contained in $(e^{-\ell(\gamma)}, e^{\ell(\gamma)})$ are complex lines. The remaining eigenvalues are $e^{-\ell(\gamma)}, e^{\ell(\gamma)}$. The eigenspace for the eigenvalue $e^{\ell(\gamma)}$ is the intersection of $\mathcal{Y}_q$ with the tangent space of the real rel foliation, and the eigenspace for the eigenvalue $e^{-\ell(\gamma)}$ is the intersection of $\mathcal{Y}_q$ with the tangent space of the imaginary rel foliation. Furthermore, the image under the complex structure i induced by period coordinates of an eigenvector for the eigenvalue $e^{\ell(\gamma)}$ is an eigenvector for the eigenvalue $e^{-\ell(\gamma)}$.

By definition, a point $z \in \mathcal{K}(q)$ is a complex subspace of $\mathcal{Y}_q \subset T_q\mathcal{C}_+$ of complex codimension k which is complementary to some k-dimensional complex subspace of $T_q\mathcal{AP}(\mathcal{C})$, and the image of z under the map F is complex as well. We claim that such a subspace has to contain the sum of the eigenspaces for A with respect to the eigenvalues of absolute value different from $e^{\ell(\gamma)}, e^{-\ell(\gamma)}$.

To this end recall that the fibre $\mathcal{P}(q)$ of the bundle $\mathcal{P}$ at q is a closed subset of the Grassmann manifold of all oriented linear subspaces of $\mathcal{Y}_q$ of real codimension $2k$. Furthermore, $\mathcal{R}_\infty \cap \mathcal{P}(q)$ is a non-empty closed F-invariant subset containing the non-empty set $\mathcal{K}(q)$. If $z \in \mathcal{K}(q)$, then any limit of a subsequence of the sequence $F^i z$ $(i \to \pm\infty)$ is complex. Such a limit y is a fixed point for the action of F on $\mathcal{P}(q)$ and hence it is a direct sum of subspaces of eigenspaces of F (compare the proof of Proposition 6.2 for details on this fact).

Now for $z \in \mathcal{K}(q)$, the complex dimension of the intersection of z with $T_q\mathcal{AP}(\mathcal{C}_+)$ equals $\dim_{\mathbb{C}} T_q(\mathcal{AP}(\mathcal{C}_+)) - k$. As the image of z under arbitrary iterates by the map F remains complex and the complex structure pairs an eigenvector for the eigenvalue $e^{\ell(\gamma)}$ with an eigenvector for the eigenvalue $e^{-\ell(\gamma)}$, we conclude that z contains the sum of the eigenspaces for F with respect to the eigenvalues different from $e^{\ell(\gamma)}, e^{-\ell(\gamma)}$. As this discussion is valid for each $z \in \mathcal{K}(q)$, the sum of the eigenspaces for F with respect to eigenvalues of absolute value different from $e^{\pm\ell(\gamma)}$ is contained in $\cap_{z\in\mathcal{K}(q)} z = E(q)$. In particular, we have $\dim_{\mathbb{C}} E(q) \geq \ell - 1$ and hence

$$\dim_{\mathbb{C}} E(q) \in [\ell - 1, \dim_{\mathbb{C}}(\mathcal{Y}_q) - k],$$

moreover $\mathcal{Y}_q = T\mathcal{AP}(\mathcal{C})_q + E(q)$ (this sum may not be direct). Now $E(q)$ depends in a real analytic fashion on $q \in V$ and hence the assignment $q \to E(q)$ is a real analytic Φ^t-invariant subbundle of $\mathcal{Y}|V$.

Let as before $\mathcal{F} \subset \mathcal{C}_+$ be the foliation into the orbits of the $GL^+(2,\mathbb{R})$-action and let $\hat{\mathcal{E}} \to V$ be the real analytic vector bundle whose fibre at $q \in V$ equals $T\mathcal{F} \oplus E(q)$. Clearly $\hat{\mathcal{E}}$ is invariant under the Teichmüller flow Φ^t. Moreover, if $\mathcal{D} \subset \mathcal{C}$ is an affine invariant manifold of rank ℓ and complex codimension k which intersects V, then for every $q \in \mathcal{D}$, the fibre $\hat{\mathcal{E}}(q)$ of $\hat{\mathcal{E}}$ at q is contained in the tangent space $T_q\mathcal{D}$ of $\mathcal{D}$ at q.

We use the bundle $\hat{\mathcal{E}}$ to construct a bundle $\mathcal{E}$ with the properties stated in (2) of the proposition. To this end note that as $\hat{\mathcal{E}}$ is a real analytic subbundle of the tangent bundle of $\mathcal{C}_+$, for $q \in V$ we can consider the linear subspace $\mathcal{B}(q) \supset \hat{\mathcal{E}}(q)$ of $T_q\mathcal{C}_+$ spanned by $\hat{\mathcal{E}}(q)$ and the values of all Lie brackets of sections of $\hat{\mathcal{E}}$. Let $q \in V$ be a point such that the dimension of $\mathcal{B}(q)$ is maximal, say that this dimension equals n. Then in a small neighborhood U of q, this dimension is constant and hence the assignment $u \to \mathcal{B}(u) \subset T_u\mathcal{C}_+$ defines an integrable real analytic subbundle of $T\mathcal{C}_+$ of real dimension n which contains the bundle $\hat{\mathcal{E}}$. In particular, we have $\mathcal{B} + T\mathcal{AP}(\mathcal{C}_+) = T\mathcal{C}_+$. On the other hand, for any point $q \in U$ with the property that q is contained in an affine invariant submanifold $\mathcal{D}$ of $\mathcal{C}_+$ of rank ℓ and complex codimension k, we have $T_q\mathcal{D}_+ \supset \mathcal{B}_q$. As the set of these points is dense in V by assumption, this shows that the (real) codimension of $\mathcal{B}$ is at least $2k \geq 2$.

As Φ^t acts on $\mathcal{C}$ as a group of diffeomorphisms, the set U constructed above is invariant under Φ^t and hence it is open and dense in $\mathcal{C}$ by topological transitivity of the action of Φ^t. To facilitate the notation we assume that in fact $U = V$.

Let $\hat{\mathcal{B}} = \mathcal{B} + i\mathcal{B}$; then for each $q \in V$, $\hat{\mathcal{B}}(q)$ is a complex subspace of $T_p\mathcal{C}_+$, and the above reasoning shows that its complex codimension is at least k. There exists an open Φ^t-invariant subset U of V such that the complex dimension of $\hat{\mathcal{B}}(q)$ is maximal for every $q \in U$. Then the restriction of $\hat{\mathcal{B}}$ to U is a real analytic complex subbundle of $T\mathcal{C}_+$ containing $\mathcal{B}$. Using again the fact that $\mathcal{B}$ is tangent to each affine invariant submanifold $\mathcal{D}_+$ of $\mathcal{C}_+$ of rank ℓ and complex codimension k, the complex codimension of $\hat{\mathcal{B}}$ is at least k.

Now if $\hat{\mathcal{B}} = \hat{\mathcal{E}}$ in U then $\hat{\mathcal{E}}$ is integrable and we put $\mathcal{E} = \hat{\mathcal{E}}$. Otherwise the complex dimension of $\hat{\mathcal{B}}$ is strictly larger than the complex dimension of $\hat{\mathcal{E}}$. Repeat the above construction with the bundle $\hat{\mathcal{B}}$ instead of $\hat{\mathcal{E}}$. In finitely many such steps, the complex dimension of the bundles constructed in this way has to stabilize. As a consequence, in finitely many such steps we find an integrable subbundle $\mathcal{E} \subset T\mathcal{C}_+$, defined on an open Φ^t-invariant subset V of $\mathcal{C}$, of complex codimension at least k, and such that for each affine invariant manifold $\mathcal{D} \subset \mathcal{C}$ of rank ℓ and complex codimension k which intersects V and each point $q \in \mathcal{D} \cap V$, a local integral manifold of $\mathcal{E}$ through q is contained in $\mathcal{D}$.

Recall that the bundle $\mathcal{E}$ contains the tangent bundle $T\mathcal{F}$ of the foliation of $\mathcal{C}$ into the orbits of the action of $GL^+(2,\mathbb{R})$. This means that its integral manifolds are locally saturated for the foliation of $\mathcal{C}$ into the orbits of the action of the group $GL^+(2,\mathbb{R})$. Then $\mathcal{E}$ is invariant under the action of $GL^+(2,\mathbb{R})$.

We show next that we can choose the bundle $\mathcal{E}$ in such a way that its the integral manifolds are locally affine. Note that as $\mathcal{E}$ is tangent to each of the affine invariant manifolds of rank ℓ and complex codimension k which intersects V and such affine invariant manifolds are affine in period coordinates, the integral manifolds of $\mathcal{E}$ are locally affine if the complex codimension of $\mathcal{E}$ in $T\mathcal{C}_+|V$ equals k.

Otherwise let $q \in V$ be a point which is contained in an affine invariant manifold $\mathcal{D}$ of rank ℓ and complex codimension k. Define $\mathcal{G}(q) \subset T\mathcal{C}_+$ to be the intersection of $T\mathcal{D}$ with all limits $T_{q_i}\mathcal{D}_i$ as $i \to \infty$ where q_i is a point on an affine invariant manifold $\mathcal{D}_i$ of rank ℓ and complex codimension k and $q_i \to q$. Since $\mathcal{E}$ is a real analytic vector bundle and since $\mathcal{E}(q_i) \subset T_{q_i}\mathcal{D}_i$ for all i, we have $\mathcal{G}(q) \supset \mathcal{E}(q)$. Furthermore, in the case that $\dim_{\mathbb{C}}(\mathcal{G}(q) \cap T_q\mathcal{D}) = \dim_{\mathbb{C}}\mathcal{E}(q)$ then in period coordinates, the local leaf M through q of the local foliation of $\mathcal{C}$ into integral manifolds of the bundle $\mathcal{E}$ equals the intersection of $\mathcal{D}$ with a collection of local limits of affine invariant manifolds $\mathcal{D}_i$ and hence this local leaf is affine.

It now suffices to observe that via perhaps decreasing the set V, we may assume that there exists a real analytic complex vector bundle $\mathcal{G} \supset \mathcal{E}$ whose fibre at a dense set of points $q \in V$ lying on an affine invariant manifold $\mathcal{D}$ as above coincides with the complex vector space constructed in the previous paragraph. To this end choose q so that the dimension of the complex vector space $\mathcal{G}(q) \supset \mathcal{E}(q)$ is minimal. As before, locally near q there exists a vector bundle $\mathcal{G} \supset \mathcal{E}$ with fibre $\mathcal{G}_q$ at q such that for a dense set of points z in a neighborhood of q, $\mathcal{G}_z$ is tangent to an affine submanifold of $\mathcal{C}$. Thus via perhaps replacing the bundle $\mathcal{E}$ by the bundle $\mathcal{G}$, we may assume that the local integral manifolds of $\mathcal{E}$ are affine.

We are left with showing that there is a flat subbundle of $T\mathcal{AP}(\mathcal{C})$ which is complementary to $\mathcal{E}$. Namely, let m be the number of zeros of a differential in $\mathcal{C}$. Let $q \in V$ and let $\mathfrak{a}_1, \dots, \mathfrak{a}_{m-1} \in \mathbb{R}^m$ be linearly independent with zero mean such that for some $u \leq m-1$, the vector fields $X_{\mathfrak{a}_1}, \dots, X_{\mathfrak{a}_u}$ are tangent to $\mathcal{C}_+$ and such that moreover their complex span is a linear subspace of $T\mathcal{AP}(\mathcal{C}_+)$ complementary to $\mathcal{E}(q)$. By invariance and Lemma 3.2, the complex span of these vector fields defines a flat invariant complex subbundle of $T\mathcal{AP}(\mathcal{C})|V$ which is complementary to the bundle $\mathcal{E}$. This is what we wanted to show. □

Remark 8.2. Proposition 8.1 is valid for affine invariant manifolds $\mathcal{C}_+$ of rank one, but in this case, property (2) just states that $\mathcal{C}_+$ is foliated into the orbits of the action of $GL^+(2,\mathbb{R})$, and these leaves are flat. Thus for rank one affine invariant manifolds, property (2) above always holds true for straightforward reason.

Remark 8.3. Lemma A.6 discusses real analytic $SL(2,\mathbb{R})$-invariant splittings of the tangent bundle of the sphere subbundle of the tautological vector bundle $\mathcal{V}$ over the moduli space $\mathcal{A}_g$ of principally polarized abelian differentials. In contrast to the statement of Proposition 8.4, such splittings can explicitly be constructed. This witnesses the fact that orbits of the action of $SL(2,\mathbb{R})$ on the Teichmüller space of abelian differentials project to Kobayashi geodesics which in general do not map to totally geodesic complex curves in the Siegel upper half-space, equipped with the symmetric metric. In other words, in spite of Lemma A.7, the actions of $SL(2,\mathbb{R})$ on the moduli space of abelian differentials and on the sphere subbundle of $\mathcal{V}$ are not compatible in any obvious geometric way.

Our final goal is to show that for $\ell \geq 2$, an affine invariant manifold $\mathcal{C}_+$ of rank ℓ does not admit a nontrivial $GL^+(2,\mathbb{R})$-invariant foliations into locally affine leaves which is transverse to the absolute period foliation. We refer to Theorem 5.1 of [W14] for a related result.

Proposition 8.4. *Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 2$; then there is no nontrivial Φ^t-invariant real analytic splitting $T\mathcal{C}_+ = \mathcal{A} \oplus \mathcal{E}$ over an open dense Φ^t-invariant subset of $\mathcal{C}$ with property (2) of Proposition 8.1.*

Proof. We proceed as in the proof of Lemma 7.1 and Proposition 7.3. Let $\mathcal{C} \subset \mathcal{C}_+$ be the hyperplane of area one differentials. Assume to the contrary that there is an open dense Φ^t-invariant set $V \subset \mathcal{C}$, and there is a Φ^t-invariant real analytic splitting $T\mathcal{C}_+|V = \mathcal{A} \oplus \mathcal{E}$ as in the statement of the proposition. As before, we pass to a finite cover $\hat{\mathcal{C}}$ of $\mathcal{C}$ such that the zeros of a differential in this cover are numbered. Our goal is to show that the bundle $\mathcal{E}$ is flat; this then contradicts Theorem 5.1 of [W14].

An affine invariant manifold is locally defined by real linear equations in period coordinates (see [W14]). The affine structure of $\mathcal{C}_+$ defines a flat connection $\nabla^{\mathcal{C}}$ on $T\mathcal{C}_+$ which is invariant under affine transformations. In particular, this connection is invariant under the $GL^+(2,\mathbb{R})$-action. The bundle $\mathcal{A} \subset T\mathcal{AP}(\mathcal{C}_+)$ is flat, ie invariant under parallel transport for $\nabla^{\mathcal{C}}$. Namely, it is trivialized by globally defined vector fields $X_{\mathfrak{a}_i}$ where $\mathfrak{a}_i \in \mathbb{C}^m$ (compare the proof of Proposition 8.1), and these vector fields are parallel for $\nabla^{\mathcal{C}}$ (compare [W14]).

Recall from the proof of Proposition 8.1 that there is a real analytic complex subbundle $\mathcal{Y} \subset T\mathcal{C}_+$ which is invariant under the $GL^+(2,\mathbb{R})$-action and transverse to the tangent bundle $T\mathcal{F}$ of the foliation $\mathcal{F}$ of $\mathcal{C}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action. Let $\mathcal{K} = \mathcal{E} \cap \mathcal{Y}$. Since the rank ℓ of $\mathcal{C}_+$ is at least two, $\mathcal{K}$ is a complex subbundle of $\mathcal{Y}$ of positive dimension.

By passing to a finite cover, assume that the zeros of the differentials in $\mathcal{C}$ are numbered. Let $k \geq 2$ be the number of these zeros. Using the notation from the proof of Lemma 7.1, let $\mathcal{O} \subset \mathbb{C}^k$ be the complex vector space of vectors $\mathfrak{a}$ with zero mean which are tangent to $\mathcal{C}_+$. For $\mathfrak{a} \in \mathcal{O}$ let $X_{\mathfrak{a}} \subset T\mathcal{AP}(\mathcal{C})$ be the vector field defined by the Schiffer variation with weight $\mathfrak{a}$. Then for each $\mathfrak{a} \in \mathcal{O}$, the affine invariant manifold $\mathcal{C}$ is invariant under the flow $\Lambda^t_{\mathfrak{a}}$ generated by $X_{\mathfrak{a}}$ (Lemma 3.2).

Furthermore, the bundle $\mathcal{A}$ is defined by a linear subspace of $\mathcal{O}$ which is invariant under the complex structure.

We claim that the bundle $\mathcal{K}$ is invariant under the flows generated by the vector fields $X_{\mathfrak{a}}$ for $\mathfrak{a} \in \mathcal{A}$. This is equivalent to stating that for all $\mathfrak{a} \in \mathcal{A}$ and every $q \in \mathcal{C}$, the Lie derivative $L_{X_{\mathfrak{a}}}Y(q)$ of every local section Y of $\mathcal{K}$ near q in direction of $X_{\mathfrak{a}}$ is contained in $\mathcal{K}$ at the point q.

We proceed as in the proof of Lemma 7.1. Use the $SL(2,\mathbb{R})$-invariant decomposition $T\mathcal{C}_+ = T\mathcal{F} \oplus \mathcal{Y}$ to project the flat connection $\nabla^{\mathcal{C}}$ on $T\mathcal{C}_+$ to a connection $\nabla^{\mathcal{Y}}$ on $\mathcal{Y}$. Let $q \in V$, let $Y \in \mathcal{K}$ and let $\hat{Y}$ be the vector field along the flow line of the flow $\Lambda^t_{\mathfrak{a}}$ obtained by parallel transport of Y for the connection $\nabla^{\mathcal{Y}}$. Then the Lie derivative $L_{X_{\mathfrak{a}}}(\hat{Y})$ is defined at q, and we have to show that $L_{X_{\mathfrak{a}}}(\hat{Y}) \in \mathcal{Y}$.

To this end define $\beta(X_{\mathfrak{a}}, Y) \in T\mathcal{F} \oplus \mathcal{A}$ to be the component of $L_{X_{\mathfrak{a}}}\hat{Y}$ in $T\mathcal{F} \oplus \mathcal{A}$ with respect to the decomposition $T\mathcal{C}_+ = T\mathcal{F} \oplus \mathcal{A} \oplus \mathcal{K}$. Then β is a real analytic section of $\mathcal{A}^* \otimes \mathcal{Y}^* \otimes (T\mathcal{F} \oplus \mathcal{A})$. By invariance of the decomposition of $T\mathcal{C}_+$ under the Teichmüller flow and equivariance of the flat connection $\nabla^{\mathcal{C}}$, the tensor field β is equivariant under the action of the Teichmüller flow.

As in the proof of Lemma 7.1, it now suffices to show that β vanishes at any point $q \in \mathcal{C}$ contained in a periodic orbit γ for Φ^t with the properties stated in Corollary 4.12. Let F be the differential of the map $d\Phi^{\ell(\gamma)}$; then the fibre $\mathcal{K}_q$ can be represented in the form

$$\mathcal{K}_q = L_q \oplus (\mathcal{K}_q \cap T\mathcal{AP}(\mathcal{C}_+)$$

where L_q is a direct sum of eigenspaces of the map F for eigenvalues which are different from $e^{\ell(\gamma)}, e^{-\ell(\gamma)}, \pm 1$.

Now if $Z \in T\mathcal{AP}(\mathcal{C}_+) \cap \mathcal{K}_q$ then $\beta(\cdot, Z) = 0$ as a leaf of the absolute period foliation is flat. On the other hand, $(T\mathcal{F} \oplus \mathcal{A})_q$ is a direct sum of eigenspaces of F for the eigenvalues $e^{\ell(\gamma)}, e^{-\ell(\gamma)}, \pm 1$ and hence vanishing of $\beta(\cdot, Z)$ for $Z \in L_q$ follows as in the proof of Lemma 7.1.

We showed so far that the bundle $\mathcal{K}$ is invariant under each of the flows $\Lambda^t_{\mathfrak{a}}$ generated by a vector field $X_{\mathfrak{a}} \subset \mathcal{A}$. Then the (locally defined) bundle $\hat{\mathcal{K}}$ generated by $\mathcal{K}$ and all Lie brackets of sections of $\mathcal{K}$ is invariant under such a flow as well (compare the proof of Proposition 8.1 for details of this construction). But $\mathcal{K}$ projects to a complex subbundle of rank at least one in the bundle $\mathcal{W}$. Hence by Lemma A.2 in the appendix, the bundle $\hat{\mathcal{K}}$ contains the generator Y of the Teichmüller flow. However, this contradicts the fact that for each $\mathfrak{a} \in \mathcal{O} \cap \mathbb{R}^k$ we have

$$L_{X_{\mathfrak{a}}}(Y) = [X_{\mathfrak{a}}, Y] = -L_Y(X_{\mathfrak{a}}) = -X_{\mathfrak{a}}$$

by Lemma 3.1. This is a contradiction which completes the proof of the proposition. $\square$

As an immediate consequence of Proposition 6.2 and Lemma 8.4, we obtain

Corollary 8.5. *An affine invariant manifold $\mathcal{C}$ of rank at least two contains only finitely many affine invariant submanifolds of the same rank.*

Theorem 3 from the introduction is now an immediate consequence of Proposition 7.5 and Corollary 8.5.

9. Algebraically primitive Teichmüller curves

A point in the moduli space of area one abelian differentials on a closed surface S of genus $g \geq 2$ defines an euclidean metric on S whose singularities are cone points of cone angle a multiple of 2π at the zeros of the differential. Such a metric is called a *translation structure* on S. An *affine automorphism* of such a translation structure (X, ω) is a homeomorphism $f : S \to S$ which takes singularities of (X, ω) to singularities and and is locally affine in the nonsingular part of S. Let Γ be the group of affine automorphisms of (X, ω). The function which takes an affine automorphism f to its derivative Df gives a homomorphism from Γ into $GL(2, \mathbb{R})$. The image $D(\Gamma)$ is called the *Veech group* of the translation surface. It is contained in the subgroup $SL^{\pm}(2, \mathbb{R})$ of all elements with determinant ± 1.

If the affine automorphism group of the translation surface (X, ω) contains a pseudo-Anosov element φ then the trace field of φ is defined. Recall that φ acts on $H^1(S, \mathbb{R})$ as a Perron Frobenius automorphism, and if μ is the leading eigenvalue for this action, then the trace field of φ equals $\mathbb{Q}[\mu + \mu^{-1}]$.

By Theorem 28 in the appendix of [KS00], the trace field of φ coincides with the so-called *holonomy field* of (X, ω). The holonomy field is defined for any translation surface, however we will not make use of this fact in the sequel. Instead we refer to the appendix of [KS00] for more information. By Lemma 2.10 of [LNW15], if $\mathcal{C}$ is a rank one affine invariant manifold then for all $(X, \omega) \in \mathcal{C}$, the holonomy field of (X, ω) equals the *field of definition* of $\mathcal{C}$ [W14]. In particular, the trace field of a pseudo-Anosov element whose conjugacy class corresponds to a periodic orbit in a rank one affine invariant manifold $\mathcal{C}$ only depends on $\mathcal{C}$ but not on the periodic orbit. As we will not use any other information on the field of definition, we will not define it here.

For the proof of Theorem 4 we have a closer look at rank one affine invariant manifolds $\mathcal{C}$ whose field of definition $\mathfrak{k}$ is of degree g over $\mathbb{Q}$. Then $\mathfrak{k}$ is a totally real [F16] number field of degree g, with ring of integers $\mathcal{O}_{\mathfrak{k}}$. Via the g field embeddings $\mathfrak{k} \to \mathbb{R}$, the group $SL(2, \mathcal{O}_{\mathfrak{k}})$ embeds into $G = SL(2, \mathbb{R}) \times \cdots \times SL(2, \mathbb{R}) < Sp(2g, \mathbb{R})$ and in fact, $SL(2, \mathcal{O}_{\mathfrak{k}})$ is a lattice in G. The trace field of every periodic orbit γ in $\mathcal{C}$ equals $\mathfrak{k}$ and hence the image of a corresponding pseudo-Anosov element $\Omega(\gamma)$ under the homomorphism $\Psi : \mathrm{Mod}(S) \to Sp(2g, \mathbb{R})$ is contained in a conjugate of $SL(2, \mathcal{O}_{\mathfrak{k}})$.

The following observation is immediate from Theorem 4.6 and [G12]. For its formulation, define the *extended local monodromy group* of an open contractible subset U of $\mathcal{C}$ to be the subgroup of $SL(2, \mathcal{O}_{\mathfrak{k}})$ which is generated by the monodromy of those (parametrized) periodic orbits for Φ^t in $\mathcal{C}$ which pass through U. Compare with Theorem 4.6.

Lemma 9.1. *For a rank one affine invariant manifold $\mathcal{C}$ whose field of definition is of degree g over $\mathbb{Q}$, the extended local monodromoy group of any open set is Zariski dense in $SL(2, \mathbb{R}) \times \cdots \times SL(2, \mathbb{R})$.*

Proof. By Theorem 4.6, the projection of the extended local monodromy group of an open set $U \subset \mathcal{C}$ to the first factor $SL(2,\mathbb{R})$ of $G = SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$ is Zariski dense in $SL(2,\mathbb{R})$ and hence it is non-elementary. Moreover, by definition and [KS00, LNW15], the invariant trace field of the extended local monodromy group equals $\mathfrak{k}$ (compare [G12] for the definition of the invariant trace field). Thus by Corollary 2.2 of [G12], the extended local monodromy group of U is Zariski dense in G. □

In the statement of the next corollary, the affine invariant manifold $\mathcal{B}_+$ may be a component of a stratum. As before, we put a lower index $+$ whenever we do not normalize the area of a holomorphic differential.

Corollary 9.2. *Let $\mathcal{C}_+$ be a rank one affine invariant manifold whose field of definition is of degree g over $\mathbb{Q}$. Assume that $\mathcal{C}_+$ is properly contained in an affine invariant manifold $\mathcal{B}_+$ of rank at least three. Let $\mathcal{Z} \to \mathcal{B}_+$ be the absolute holomorphic tangent bundle of $\mathcal{B}_+$; then $\mathcal{Z}|\mathcal{C}_+$ splits as a sum of holomorphic line bundles which are invariant under both the Chern connection and the Gauss Manin connection.*

Proof. In the case that the rank of $\mathcal{B}_+$ equals g (and hence $\mathcal{Z} = \Pi^*\mathcal{H}|\mathcal{B}_+$), the statement is immediate from Theorem 1.5 of [W14]. Thus assume that the rank of $\mathcal{B}_+$ is at most $g-1$.

Since $\mathcal{C}_+ \subset \mathcal{B}_+$, the restriction of $\Pi^*\mathcal{H}$ to $\mathcal{C}_+$ has two splittings which are invariant under the extended local monodromy of $\mathcal{C}_+$. The first splitting is the splitting into g line bundles obtained from the different field embeddings of the field of definition of $\mathcal{C}_+$ into $\mathbb{R}$ (see Theorem 1.5 of [W14]). The second splitting is the splitting into the absolute holomorphic tangent bundle $\mathcal{Z}$ of $\mathcal{B}_+$ (which is a holomorphic subbundle of $\Pi^*\mathcal{H}|_{\mathcal{B}_+}$ whose complex rank equals the rank of $\mathcal{B}_+$) and its symplectic complement. Since by Lemma 9.1 the extended local monodromy group of $\mathcal{C}_+$ is Zariski dense in $SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$, the bundle $\mathcal{Z}|\mathcal{C}_+$ is a sum of invariant line bundles. □

Corollary 9.3. *For a component $\mathcal{Q}_+$ of a stratum in genus $g \geq 3$, all affine invariant submanifolds of rank one whose fields of definition are of degree g over $\mathbb{Q}$ are contained in a finite collection of affine invariant submanifolds of rank at most two.*

Proof. Let $\mathfrak{C}$ be the collection of all rank one affine invariant submanifolds of $\mathcal{Q}$ whose field of definition is a number field of degree g over $\mathbb{Q}$. Recall the invariant decomposition $\Pi^*\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$. For each $\mathcal{C}_+ \in \mathfrak{C}$, the restriction of the bundle $\mathcal{L}$ to $\mathcal{C}_+$ splits as a sum of holomorphic line bundles which are invariant under the Gauss-Manin connection in the sense discussed in Section 6. Thus by Proposition 6.2 and its proof, there exists a finite collection of affine invariant submanifolds of $\mathcal{Q}$ of rank at most $g-1$ which contain each element of $\mathfrak{C}$.

Now if $\mathcal{B}_+$ is an affine invariant manifold of rank contained in $[3, g-1]$ which contains some $\mathcal{C}_+ \in \mathfrak{C}$ then by Corollary 9.2, the above reasoning can be applied to $\mathcal{B}_+$. In finitely many steps we find finitely many proper affine invariant manifolds

$\mathcal{C}_1, \ldots, \mathcal{C}_k \subset \mathcal{Q}_+$ of rank at most two which contain every $\mathcal{C}_+ \in \mathfrak{C}$. This is the statement of the corollary. □

Now we are ready to complete the proof of Theorem 4.

Corollary 9.4. *For $g \geq 3$ the $SL(2,\mathbb{R})$-orbit closure of a typical periodic orbit in any component of a stratum is the entire stratum.*

Proof. Let $\mathcal{Q}$ be a component of a stratum and let $U \subset \mathcal{Q}$ be a non-empty open set. Then a typical periodic orbit for Φ^t passes through U [H13]. Thus by Theorem 3 (see also [MW15, MW16]), the $SL(2,\mathbb{R})$-orbit closure of a typical periodic orbit either equals the entire stratum, or it is an affine invariant manifold of rank one.

By the second part of Theorem 1, the trace field of a typical perodic orbit γ is totally real and of degree g over $\mathbb{Q}$. If the rank of the $SL(2,\mathbb{R})$-orbit closure $\mathcal{C}$ of γ equals one then this trace field is the field of definition of $\mathcal{C}$ [LNW15]. Thus the corollary follows from Corollary 9.3. □

We complete the main body of this article with the proof of Theorem 5. We begin with

Proposition 9.5. *Let $g \geq 3$ and let $\mathcal{B}_+ \subset \mathcal{Q}_+$ be a rank two affine invariant manifold. Then the union of all algebraically primitive Teichmüller curves which are contained in $\mathcal{B}_+$ is nowhere dense in $\mathcal{B}_+$.*

Proof. Let $\mathcal{B}_+ \subset \mathcal{Q}_+$ be a rank two affine invariant manifold. We argue by contradiction, and we assume that the closure of the union of all algebraically primitive Teichmüller curves $\mathcal{C}_+ \subset \mathcal{B}_+$ contains some open subset V of $\mathcal{B}_+$.

Let $\mathcal{Z} \to \mathcal{B}_+$ be the absolute holomorphic tangent bundle of $\mathcal{B}_+$. Let U be a small contractible subset of V so that there is a trivialization of the Hodge bundle over U defined by the Gauss Manin connection. The extended local monodromy group of U preserves $\mathcal{Z}$. Let $\mathcal{C}_i \subset \mathcal{B}_+$ be a sequence of algebraically primitive Teichmüller curves which pass through U and whose closures contain a compact subset of U with non-empty interior W.

Let $\Pi : \mathcal{Q}_+ \to \mathcal{M}_g$ be the canonical projection and let $\mathcal{I}_g : \mathcal{M}_g \to \mathcal{A}_g$ be the Torelli map. The image under Π of the curve $\mathcal{C}_i$ is an algebraic curve (see [F16]) which admits a *modular embedding*. Namely, by the main result of [Mo06], there is a totally real number field K_i of degree g over $\mathbb{Q}$, there is an order $\mathfrak{o}_{K_i}$ in K_i, and there is an embedding

$$SL(2,\mathfrak{o}_{K_i}) \to SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R}) \to Sp(2g,\mathbb{R})$$

which maps $SL(2,\mathfrak{o}_{K_i})$ into $Sp(2g,\mathbb{Z})$ and such that the image of $\mathcal{C}_i$ under the Torelli map is contained in the Hilbert modular variety $H(\mathfrak{o}_{K_i})$. This Hilbert modular variety is the quotient of $\mathbf{H}^2 \times \cdots \times \mathbf{H}^2$ under the lattice $SL(2,\mathfrak{o}_{K_i})$ in a Lie subgroup G_i of $Sp(2g,\mathbb{R})$ which is isomorphic to $SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$.

We claim that $G_i = G_j = G$ for all i. Namely, assume otherwise. Then there are algebraically primitive Teichmüller curves $\mathcal{C}_i, \mathcal{C}_j$ which intersect U and for which the groups G_i, G_j are distinct. By Lemma 9.1, the extended local monodromy groups of $\mathcal{C}_i \cap U$ and $\mathcal{C}_j \cap U$ are Zariski dense in G_i, G_j. Therefore the Zariski closure in $Sp(2g, \mathbb{R})$ of the extended local monodromy group of $U \subset \mathcal{B}_+$ contains $G_i \cup G_j$. But as $G_i \neq G_j$, a subgroup of $Sp(2g, \mathbb{R})$ which contains $G_i \cup G_j$ can not preserve the subspace $\mathcal{Z}$. This is a contractiction and implies that indeed, $G_i = G_j = G$ for all i.

Write $SL(2, \mathfrak{o}) = SL(2, \mathfrak{o}_{K_i})$. The Hilbert modular variety $H(\mathfrak{o}) = H(\mathfrak{o}_{K_i}) \subset \mathcal{A}_g$ consists of abelian varieties with real multiplication with the field $K = K_i$. The image of $\mathcal{C}_i$ under the map $\mathcal{I}_g \circ \Pi$ is contained in $H(\mathfrak{o})$. As a consequence, the set of points in $\mathcal{B}_+$ which are mapped by the composition of the foot-point projection $\Pi : \mathcal{B}_+ \to \mathcal{M}_g$ with the Torelli map $\mathcal{I}_g$ into $H(\mathfrak{o})$ contains a dense subset of the open set W. But $H(\mathfrak{o})$ is a complex submanifold of $\mathcal{A}_g$ and this composition map is holomorphic and therefore the image of $\mathcal{B}_+$ is contained in $H(\mathfrak{o})$.

We showed so far that each point in $\mathcal{B}_+$ is an abelian differential whose Jacobian has real multiplication with K. Now a point on an algebraically primitive Teichmüller curve is mapped to an eigenform for real multiplication [Mo06] and hence the closure of the set of differentials in $\mathcal{B}_+$ which are mapped to eigenforms for real multiplication with K contains an open set. This implies as before that each point in $\mathcal{B}_+$ corresponds to such an eigenform and hence $\mathcal{B}_+$ is a rank one affine invariant manifold, contrary to our assumption. The proposition follows $\square$

Proof of Theorem 5:

Let $\mathcal{Q}$ be a component of a stratum in genus $g \geq 3$. By Corollary 9.3, there are finitely many affine invariant submanifolds $\mathcal{B}_1, \dots, \mathcal{B}_k$ of rank two which contain all but finitely many algebraically primitive Teichmüller curves.

Let $\mathcal{B}_i$ be such an affine invariant manifold of rank two. Assume that its dimension equals r for some $r \geq 4$. By Proposition 9.5, the closure of the union of all algebraically primitive Teichmüller curves which are contained in $\mathcal{B}_i$ is nowhere dense in $\mathcal{B}_i$. As this closure is invariant under the action of $GL(2, \mathbb{R})$, it consists of a finite union of affine invariant manifolds. The dimension of each of these invariant submanifolds is at most $r - 1$.

If there are submanifolds of rank two in this collection then we can repeat this argument with each of these finitely many submanifolds. By inverse induction on the dimension, this yields that all but finitely many algebraically primitive Teichmüller curves are contained in one of finitely many affine invariant manifolds of rank one. The field of definition of such a manifold coincides with the field of definition of the Teichmüller curve, in particular it is of degree g [LNW15].

By the main result of [LNW15], a rank one affine invariant manifold with field of definition of degree g over $\mathbb{Q}$ only contains finitely many Teichmüller curves. Thus the number of algebraically primitive Teichmüller curves in $\mathcal{Q}$ is finite as promised.

Appendix A. Structure of the homogeneous space $Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})$

In this appendix we collect some geometric properties of the Siegel upper half-space $\mathfrak{D}_g = Sp(2g,\mathbb{R})/U(g)$ and its quotient $\mathcal{A}_g = Sp(2g,\mathbb{Z})\backslash\mathfrak{D}_g$ which are either directly or indirectly used in the proofs of our main results.

The tautological vector bundle

$$\mathcal{V} \to \mathfrak{D}_g$$

over the Hermitean symmetric space $\mathfrak{D}_g = Sp(2g,\mathbb{R})/U(g)$ is obtained as follows.

Via the right action of the unitary group $U(g)$, the symplectic group $Sp(2g,\mathbb{R})$ is an $U(g)$-principal bundle over $\mathfrak{D}_g$. The bundle $\mathcal{V}$ is the associated vector bundle

$$\mathcal{V} = Sp(2g,\mathbb{R}) \times_{U(g)} \mathbb{C}^g$$

where $U(g)$ acts from the right by $(x,y,\alpha) \to (x\alpha,\alpha^{-1}y)$. The bundle $\mathcal{V}$ is holomorphic, and it is equipped with a hermitean metric obtained from an $U(g)$-invariant hermitean inner product on $\mathbb{C}^g$. As $U(g)$ acts transitively on the unit sphere in $\mathbb{C}^g$, with isotropy group $U(g-1)$, the associated sphere bundle

$$\mathcal{S} = Sp(2g,\mathbb{R}) \times_{U(g)} S^{2g-1}$$

in $\mathcal{V} \to \mathfrak{D}_g$ can naturally be identified with the homogeneous space

$$\mathcal{S} = Sp(2g,\mathbb{R})/U(g-1)$$

(Proposition I.5.5 of [KN63]).

The group $Sp(2g-2,\mathbb{R})$ is the isometry group of Siegel upper half-space

$$\mathfrak{D}_{g-1} = Sp(2g-2,\mathbb{R})/U(g-1).$$

Since the action of $Sp(2g-2,\mathbb{R})$ on $\mathfrak{D}_{g-1}$ is transitive, with isotropy group $U(g-1)$, the bundle $\mathcal{S} = Sp(2g,\mathbb{R})/U(g-1) \to \mathfrak{D}_g$ can also be identified with the associated bundle

$$\mathcal{S} = Sp(2g,\mathbb{R}) \times_{Sp(2g-2,\mathbb{R})} \mathfrak{D}_{g-1}$$

where $Sp(2g-2,\mathbb{R})$ acts via

$$(g,x)h = (gh, h^{-1}(x)).$$

The first factor projection then defines a projection

$$\Pi : \mathcal{S} \to Sp(2g,\mathbb{R})/Sp(2g-2,\mathbb{R}).$$

Let $\omega = \sum_i dx_i \wedge dy_i$ be the standard symplectic form on $\mathbb{R}^{2g}$. The standard representation of $Sp(2g,\mathbb{R})$ on $(\mathbb{R}^{2g},\omega)$ naturally extends to an action of $Sp(2g,\mathbb{R})$ on $\mathbb{R}^{2g}\otimes\mathbb{C} = \mathbb{C}^{2g}$. The open subset

$$\mathcal{O} = \{x+iy \mid x,y \in \mathbb{R}^{2g}, \omega(x,y) > 0\} \subset \mathbb{C}^{2g}$$

is $Sp(2g,\mathbb{R})$-invariant. It contains the invariant hypersurface

$$\Omega = \{x+iy \in \mathbb{C}^{2g} \mid \omega(x,y) = 1\}.$$

Lemma A.1. *Ω can naturally and $Sp(2g,\mathbb{R})$-equivariantly be identified with the homogeneous space*

$$Sp(2g,\mathbb{R})/Sp(2g-2,\mathbb{R}).$$

Proof. Observe that the diagonal action of the group $Sp(2g,\mathbb{R})$ on Ω is transitive. The stabilizer in $Sp(2g,\mathbb{R})$ of a point $x+iy\in\Omega$ is isomorphic to a standard embedded

$$\mathrm{Id}\times Sp(2g-2,\mathbb{R})<Sp(2,\mathbb{R})\times Sp(2g-2,\mathbb{R})<Sp(2g,\mathbb{R}).$$

□

For a more explicit description of Ω we use the standard basis $(x_1,y_1,\cdots,x_g,y_g)$ of the symplectic vector space $\mathbb{R}^{2g}$. With respect to this basis, the symplectic form ω is given by the matrix

$$\mathfrak{J}=\begin{pmatrix} & 1 & & & \\ -1 & & & & \\ & & \ddots & & \\ & & & & 1 \\ & & & -1 & \end{pmatrix}$$

The Lie algebra $\mathfrak{sp}(2g,\mathbb{R})$ of $Sp(2g,\mathbb{R})$ is then the algebra of $(2g,2g)$-matrices A with $A\mathfrak{J}+\mathfrak{J}A=0$. The Lie algebra $\mathfrak{h}$ of the subgroup $Sp(2,\mathbb{R})\times Sp(2g-2,\mathbb{R})$ consists of matrices in block form

$$\begin{pmatrix} A & \\ & B \end{pmatrix}$$

where $A\in\mathfrak{sl}(2,\mathbb{R})$ and $B\in\mathfrak{sp}(2g-2,\mathbb{R})$.

Let $\mathfrak{p}$ be the linear subspace of $\mathfrak{sp}(2g,\mathbb{R})$ of matrices whose only non-trivial entries are entries a_{ij} with $i=1,2$ and $3\leq j\leq 2g$ or $j=1,2$ and $3\leq i\leq 2g$. This subspace can explicitly be computed as follows. Let ι be the complex structure on $\mathbb{R}^{2g}$ defined informally by $\iota x_i=y_i,\iota y_i=-x_i$; then a matrix in $\mathfrak{p}$ is of the form

$$\begin{pmatrix} & & x & \\ & & -Jx & \\ & & & \\ y^t & -Jy^t & & \\ & & & \end{pmatrix}$$

where $x,y\in\mathbb{R}^{2g-2}\subset\mathbb{R}^{2g}$ are vectors with vanishing first and second coordinate. Thus $\mathfrak{p}=\mathfrak{a}\oplus\mathfrak{b}$ where $\mathfrak{a},\mathfrak{b}$ are abelian subalgebras of dimension $2g-2$. Here $\mathfrak{a}$ is the intersection with $\mathfrak{p}$ of the vector space of matrices whose only non-zero entries are contained in the first and second line, and $\mathfrak{b}$ is the intersection with $\mathfrak{p}$ of the vector space of matrices whose only non-zero entries are contained in the first and second row. Note that the transpose of a matrix in the subspace $\mathfrak{a}$ is contained in $\mathfrak{b}$.

The group $SL(2,\mathbb{R})=Sp(2,\mathbb{R})$ acts from the right on Ω. Namely, the real and imaginary part of a point $x+iy\in\Omega$ define the basis of a two-dimensional symplectic subspace V of $\mathbb{R}^{2g}$. The group $SL(2,\mathbb{R})$ acts by basis transformation on this subspace, preserving the symplectic form. Furthermore, the action of $SL(2,\mathbb{R})$ fixes preserves pointwise the symplectic complement of the subspace V of $\mathbb{R}^{2g}$ spanned by x and y.

If we identify the Lie algebra $\mathfrak{sp}(2g,\mathbb{R})$ with the vector space of right invariant vector fields on $Sp(2g,\mathbb{R})$ then as $\Omega = Sp(2g,\mathbb{R})/Sp(2g-2,\mathbb{R})$, the tangent bundle $T\Omega$ of Ω is trivialized by the linear subspace $\mathfrak{u} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{p}$ of $\mathfrak{sp}(2g,\mathbb{R})$. The decomposition of $T\Omega$ resulting from this decomposition of $\mathfrak{u}$ is just the splitting

$$T\Omega = \mathcal{T} \oplus \mathcal{R}$$

where for a point $x+iy \in \Omega$ we have

$$\mathcal{R}_{x+iy} = \{u+iv \mid \omega(u,x) = \omega(u,y) = \omega(v,x) = \omega(v,y) = 0\}$$

and where $\mathcal{T}$ is tangent to the orbits of the right action of $SL(2,\mathbb{R})$.

Let J be the complex structure on $\mathfrak{p}$ defined by $JA = A^t$ for $A \in \mathfrak{a}$ and $JB = -B^t$ for $B \in \mathfrak{b}$. In the above identification of $T\Omega$ with $\mathfrak{u}$, the complex structure i on $\mathbb{C}^{2g} \supset \Omega$ restricts to the complex structure J on $\mathfrak{p}$, viewed as the subbundle $\mathcal{R}$ of $T\Omega$. Furthermore, this complex structure pairs the matrix $A \in \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sp}(2g,\mathbb{R})$ with entries $a_{12} = 1$ and $a_{ij} = 0$ otherwise (ie the generator of the horocycle flow) with its transpose A^t.

Let $X \in \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sp}(2g,\mathbb{R})$ be the matrix given by $x_{11} = 1, x_{22} = -1$ and $x_{ij} = 0$ otherwise. Then for any $0 \neq A \in \mathfrak{p}$ we have $[A, JA] = aX$ for some $a \neq 0$. The above Lie algebra computation now shows

Lemma A.2. *$\Omega \subset \mathbb{C}^{2g}$ is a CR-hypersurface: If θ is the one-form on Ω with $\theta(X) = 1$ and $\theta(T\Omega \cap iT\Omega) = 0$ then $\theta \wedge d\theta^{2g-1}$ is a volume form on Ω.*

Proof. All we need to show is that $d\theta(Y, iY) > 0$ for $Y \in T\Omega \cap iT\Omega$. Thus let $\mathfrak{y}$ be the local section of $T\Omega \cap JT\Omega$ obtained by the right action of the one-parameter subgroup of $Sp(2g,\mathbb{R})$ generated by the element in $\mathfrak{p} \oplus \mathfrak{sl}(2,\mathbb{R})$ which projects to Y. As $[\mathfrak{y}, J\mathfrak{y}] = aX$ for some $a > 0$ we have $d\theta(Y, iY) = -\theta([\mathfrak{y}, J\mathfrak{y})]) < 0$. ☐

The left action of $Sp(2g,\mathbb{R})$ on $\mathcal{O}$ is the restriction of a linear action on $\mathbb{C}^{2g}$. Therefore the tangent bundle of $\mathcal{O}$ admits an $Sp(2g,\mathbb{R})$-invariant flat connection. Namely, we can write $T\mathcal{O} = \mathcal{O} \times \mathbb{C}^{2g}$, and the standard vector fields on $\mathcal{O}$ which are the tangent lines of one-parameter groups of translations are invariant under the linear action of $Sp(2g,\mathbb{R})$. These vector fields define a parallel trivialization of $T\mathcal{O}$. This flat connection restricts to a left $Sp(2g,\mathbb{R})$-invariant flat connection ∇^{GM} on $T\mathcal{O}|\Omega$.

The bundle $T\mathcal{O}|\Omega$ splits as a sum

$$T\mathcal{O}|\Omega = T\Omega \oplus \mathbb{R}$$

where the trivial line bundle $\mathbb{R}$ is the tangent bundle of the orbits of the one-parameter group of deformations $((x+iy), t) \to e^t x + ie^t y$ transverse to Ω. Lemma A.2 shows that this splitting is not flat, i.e. it is not invariant under the connection ∇^{GM}.

The subbundles $\mathcal{R}$ and $\mathcal{T}$ of $T\Omega$ are invariant under both the left action of $Sp(2g,\mathbb{R})$ and the right action of $SL(2,\mathbb{R})$. Thus the flat left $Sp(2g,\mathbb{R})$-invariant connection ∇^{GM} on $T\mathcal{O}|\Omega$ projects to a left $Sp(2g,\mathbb{R})$-invariant right $SL(2,\mathbb{R})$-invariant connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ defined as follows. Let

$$P : T\mathcal{O}|\Omega = \mathcal{R} \oplus \mathcal{T} \oplus \mathbb{R} \to \mathcal{R}$$

be the canonical projection, and for $X \in T\Omega$ and a local section Y of $\mathcal{R}$ define $\nabla^{\mathcal{R}}_X Y = P\nabla^{GM}_X(Y)$.

Lemma A.3. *The flat left $Sp(2g,\mathbb{R})$-invariant connection on $T\mathcal{O}$ projects to a connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ which is invariant under both the left $Sp(2g,\mathbb{R})$ action and the right $SL(2,\mathbb{R})$ action.*

The curvature of the connection $\nabla^{\mathcal{R}}$ is a two-form on Ω with values in the Lie algebra $\mathfrak{sp}(2g-2,\mathbb{R})$ of $Sp(2g-2,\mathbb{R})$, acting as an algebra of transformation on $\mathcal{R}$. The restriction of this two-form to the tangent bundle of the orbits of the $SL(2,\mathbb{R})$-action vanishes. Moreover, the two-form is equivariant with respect to the left action of $Sp(2g,\mathbb{R})$ and the right action of $SL(2,\mathbb{R})$.

We say that the curvature form Θ for a connection ∇ on a complex vector bundle $E \to M$ *splits E as a complex vector bundle* if there is a nontrivial Θ-invariant decomposition $E = E_1 \oplus E_2$ as a Whitney sum of two complex vector bundles. This means that for any $x \in M$ and any two vectors $Y, Z \in T_xM$ the map $\Theta(Y,Z)$ preserves the decomposition $E = E_1 \oplus E_2$.

Since Ω is not locally affine, the curvature form of the connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ does not vanish identically. Furthermore, the stabilizer in $Sp(2g,\mathbb{R})$ of a point $z \in \Omega$ can be identified with the subgroup $Sp(2g-2,\mathbb{R})$, which act on the fibre of $\mathcal{R}$ at z via the standard representation of $Sp(2g-2,\mathbb{R})$ on $\mathbb{C}^{2g-2}$, viewed as the complexification of the standard representation on $\mathbb{R}^{2g-2}$. Since the standard representation of $Sp(2g-2,\mathbb{R})$ on the complex vector space $\mathbb{C}^{2g-2}$ is irreducible and since the curvature form of $\nabla^{\mathcal{R}}$ is equivariant with respect to the left action of $Sp(2g,\mathbb{R})$, by equivariance we have the following analog of Lemma A.2.

Lemma A.4. *The curvature form of $\nabla^{\mathcal{R}}$ does not split $\mathcal{R}$ as a complex vector bundle.*

The complement of the zero section $\mathcal{V}_+ \subset \mathcal{V}$ of the bundle $\mathcal{V} \to \mathcal{D}_g$ is a complex manifold. The fibration $\mathcal{S} \to \Omega$ extends to a holomorphic fibration $\mathcal{V}_+ \to \mathcal{O}$ of complex manifolds. The fibres of the fibration define a foliation $\mathcal{U}$ of $\mathcal{V}_+$.

The following is immediate from the definition of the complex structure on $\mathcal{V}_+$ and on $\mathcal{O} \subset \mathbb{C}^{2g}$.

Lemma A.5. *The foliation $\mathcal{U}$ is holomorphic. A leaf is biholomorphic to $\mathfrak{D}_{g-1}$.*

The foliation $\mathcal{U}$ on $\mathcal{V}_+$ can be viewed as the analog of the absolute period foliation on the bundle $\mathcal{H}_+ \to \mathcal{M}_g$ which is the pull-back of $\mathcal{V}_+$ by the Torelli map. Recall that the restriction of the absolute period foliation to any component of a stratum has a complex affine and hence a complex structure. However, this affine structure is singular at the boundary points of the strata (which are contained in lower dimensional strata).

Let for the moment G be an arbitrary Lie group. A *G-connection* for a G-principal bundle $X \to Y$ is given by an $\mathrm{Ad}(G)$-invariant subbundle of the tangent bundle of X which is transverse to the tangent bundle of the fibres. Such a bundle is called *horizontal*.

The following observation contrasts the case of the absolute period foliation on $\mathcal{H}_+$ and reflects the fact that the right $SL(2,\mathbb{R})$-action on the bundle $\mathcal{S}$ does not pull back to the $SL(2,\mathbb{R})$-action on $\mathcal{H}_+$. Namely, orbits of the $SL(2,\mathbb{R})$-action on $\mathcal{H}_+$ define *orientable Teichmüller curves* which are mapped by the Torelli map to geodesics in $\mathfrak{D}_g$ for the Kobayashi metric. However, these Kobayashi geodesics are in general not totally geodesic for the symmetric metric. We refer to [BM14] for more and for references.

The group $Sp(2g,\mathbb{R})$ is an $Sp(2g-2,\mathbb{R})$-principal bundle over Ω. In the statement of the following Lemma, the type $(2g, 2g-1)$ stems from the fact that Ω is a hypersurface in the manifold $\mathcal{O}$ with invariant indefinite metric of type $(2g, 2g)$.

Lemma A.6. *The $Sp(2g-2,\mathbb{R})$-principal bundle $Sp(2g,\mathbb{R}) \to \Omega$ admits a natural real analytic $Sp(2g-2,\mathbb{R})$-connection which is invariant under the left action of $Sp(2g,\mathbb{R})$ and the right action of $SL(2,\mathbb{R})$. The horizontal bundle $\mathcal{Z}_0$ contains the tangent bundle $\mathcal{T}$ of the orbits of the $SL(2,\mathbb{R})$-action, and it admits an $SL(2,\mathbb{R})$-invariant $Sp(2g,\mathbb{R})$-invariant pseudo-Riemannian metric h of type $(2g, 2g-1)$. The h-orthogonal complement $\mathcal{Y}_0$ of $\mathcal{T}$ in $\mathcal{Z}_0$ is a real analytic $SL(2,\mathbb{R})$-invariant $Sp(2g,\mathbb{R})$invariant bundle.*

Proof. The fibre containing the identity induces an embedding of Lie algebras
$$\mathfrak{sp}(2g-2,\mathbb{R}) \to \mathfrak{sp}(2g,\mathbb{R}).$$
The restriction of the Killing form B of $\mathfrak{sp}(2g,\mathbb{R})$ to the Lie algebra $\mathfrak{sp}(2g-2,\mathbb{R})$ is non-degenerate. Thus the B-orthogonal complement $\mathfrak{z}$ of $\mathfrak{sp}(2g-2,\mathbb{R})$ is a linear subspace of $\mathfrak{sp}(2g,\mathbb{R})$ which is complementary to $\mathfrak{sp}(2g-2,\mathbb{R})$ and invariant under the restriction of the adjoint representation Ad of $Sp(2g,\mathbb{R})$ to $Sp(2g-2,\mathbb{R})$. The restriction to $\mathfrak{z}$ of the Killing form is a non-degenerate bilinear form of type $(2g, 2g-1)$.

The group $Sp(2g,\mathbb{R})$ acts by left translation on itself, and this action commutes with the right action of $Sp(2g-2,\mathbb{R})$. Hence $Sp(2g,\mathbb{R})$ acts as a group of automorphisms on the principal bundle $Sp(2g,\mathbb{R}) \to \Omega$.

Define a $\mathfrak{sp}(2g-2,\mathbb{R})$-valued one-form θ on $Sp(2g,\mathbb{R})$ by requiring that $\theta(e)$ equals the canonical projection
$$T_e Sp(2g,\mathbb{R}) = \mathfrak{z} \oplus \mathfrak{sp}(2g-2,\mathbb{R}) \to \mathfrak{sp}(2g-2,\mathbb{R})$$
and
$$\theta(g) = \theta \circ dg^{-1}.$$
Then for every $h \in Sp(2g-2,\mathbb{R})$ we have
$$\theta(gh) = \mathrm{Ad}(h^{-1}) \circ \theta(g)$$
and hence this defines an $Sp(2g,\mathbb{R})$-invariant connection on the $Sp(2g-2,\mathbb{R})$-principal bundle $Sp(2g,\mathbb{R}) \to \Omega$. Denote by $\mathcal{Z}_0$ the horizontal bundle. It is invariant

under the left action of $Sp(2g,\mathbb{R})$ and the right action of $Sp(2g-2,\mathbb{R})$, and it is equipped with an invariant pseudo-Riemannian metric of type $(2g, 2g-1)$.

Now $\mathfrak{sp}(2,\mathbb{R}) \subset \mathfrak{z}$, and hence the tangent bundle for the right action of $Sp(2,\mathbb{R})$ is contained in the horizontal bundle $\mathcal{Z}_0$. Thus the subbundle $\mathcal{Y}_0$ of $\mathcal{Z}_0$ defined by the B-orthogonal complement $\mathfrak{y}$ in $\mathfrak{z}$ of the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$ is invariant as well. The lemma follows. ☐

Since $\mathcal{S} = Sp(2g,\mathbb{R}) \times_{Sp(2g-2,\mathbb{R})} \mathfrak{D}_{g-1}$ and since the subgroups $SL(2,\mathbb{R})$ and $Sp(2g-2,\mathbb{R})$ commute, the right action of $SL(2,\mathbb{R})$ on $Sp(2g,\mathbb{R})$ descend to an action of $SL(2,\mathbb{R})$ on $\mathcal{S}$. The action of the unitary subgroup $U(1)$ of $Sp(2,\mathbb{R})$ is just the standard circle action on the fibres of the sphere bundle $\mathcal{S} \to \mathfrak{D}_g$ given by multiplication with complex numbers of absolute value one. The connection $\mathcal{Z}_0 = \mathcal{T} \oplus \mathcal{Y}_0$ induces a real analytic splitting

$$T\mathcal{S} = T\mathcal{U} \oplus \mathcal{Z} = T\mathcal{U} \oplus \mathcal{T} \oplus \mathcal{Y}$$

where $T\mathcal{U}$ denotes the tangent bundles of the fibres of the fibration $\mathcal{S} \to \Omega$, the *horizontal bundle* $\mathcal{Z}$ is the image of $\mathcal{Z}_0 \times T\mathfrak{D}_{g-1}$ under the projection $Sp(2g,\mathbb{R}) \times \mathfrak{D}_{g-1} \to \mathcal{S}$ and as before, $\mathcal{T}$ is the tangent bundle of the orbits of the $SL(2,\mathbb{R})$-action.

Lemma A.7. *The right action of $SL(2,\mathbb{R})$ on $\mathcal{S}$ projects to the standard action of $SL(2,\mathbb{R})$ on Ω.*

Proof. This follows as before from naturality and bi-invariance of the Killing form. ☐

The group $Sp(2g,\mathbb{Z})$ acts properly discontinuously from the left on the bundle $\mathcal{S} \to \Omega$ as a group of real analytic bundle automorphisms. In particular, it preserves the real analytic splitting of the tangent bundle of $\mathcal{S}$ into the tangent bundle of the leaves of the foliation $\mathcal{U}$ and the complementary bundle. Thus this splitting descends to an $SL(2,\mathbb{R})$-invariant real analytic splitting of the tangent bundle of the quotient. This quotient is just the sphere bundle of the quotient vector bundle (in the orbifold sense) over the locally symmetric space

$$\mathcal{A}_g = Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/U(g).$$

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