TYPICAL AND ATYPICAL PROPERTIES OF PERIODIC TEICHMÜLLER GEODESICS

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Abstract. Consider a component $\mathcal{Q}$ of a stratum in the moduli space of area one abelian differentials on a surface of genus g. Call a property $\mathcal{P}$ for periodic orbits of the Teichmüller flow on $\mathcal{Q}$ *typical* if the growth rate of orbits with property $\mathcal{P}$ is maximal. Typical are: The logarithms of the eigenvalues of the symplectic matrix defined by the orbit are arbitrarily close to the Lyapunov exponents of $\mathcal{Q}$, and its trace field is a totally real splitting field of degree g over $\mathbb{Q}$. If $g \geq 3$ then periodic orbits whose $SL(2,\mathbb{R})$-orbit closure equals $\mathcal{Q}$ are typical. We also show that $\mathcal{Q}$ contains only finitely many algebraically primitive Teichmüller curves, and only finitely many affine invariant submanifolds of rank $\ell \geq 2$.

1. Introduction

The mapping class group $\mathrm{Mod}(S)$ of a closed surface S of genus $g \geq 2$ acts by precomposition of marking on the Teichmüller space $\mathcal{T}(S)$ of marked complex structures on S. The action is properly discontinuous, with quotient the moduli space $\mathcal{M}_g$ of complex structures on S.

The goal of this paper is to describe properties of this action which are invariant under conjugation and hold true for conjugacy classes of mapping classes which are typical in the following sense.

The moduli space of area one abelian differentials on S decomposes into *strata* of differentials with zeros of given multiplicities. There is a natural $SL(2,\mathbb{R})$-action on any connected component $\mathcal{Q}$ of this moduli space. The action of the diagonal subgroup is the *Teichmüller flow* Φ^t.

Let Γ be the set of all periodic orbits for Φ^t on $\mathcal{Q}$. The length of a periodic orbit $\gamma \in \Gamma$ is denoted by $\ell(\gamma)$. Let $h > 0$ be the entropy of the unique Φ^t-invariant Borel probability measure on $\mathcal{Q}$ in the Lebesgue measure class [M82, V86]. As an application of [EMR12] (see also [EM11, H11]) we showed in [H13] that

$$\sharp\{\gamma \in \Gamma \mid \ell(\gamma) \leq R\} \sim \frac{e^{hR}}{hR}.$$

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Call a subset $\mathcal{A}$ of Γ *typical* if

$$\sharp\{\gamma \in \mathcal{A} \mid \ell(\gamma) \leq R\} \sim \frac{e^{hR}}{hR}.$$

Thus a subset of Γ is typical if its growth rate is maximal. The intersection of two typical subsets is typical.

A periodic orbit $\gamma \in \Gamma$ for Φ^t determines the conjugacy class of a pseudo-Anosov mapping class. Each mapping class acts on $H_1(S, \mathbb{Z})$. This defines a natural surjective [FM12] homomorphism

$$\Psi : \mathrm{Mod}(S) \to Sp(2g, \mathbb{Z}).$$

Thus a periodic orbit γ of Φ^t determines the conjugacy class $[A(\gamma)]$ of a matrix $A(\gamma) \in Sp(2g, \mathbb{Z})$.

Let

$$1 = \kappa_1 > \kappa_2 > \cdots > \kappa_g > 0$$

be the positive Lyapunov exponents of the Kontsevich Zorich cocycle with respect to the normalized Lebesgue measure on $\mathcal{Q}$. The fact that there are no multiplicities in the Lyapunov spectrum was shown in [AV07]. For $\gamma \in \Gamma$ let $\hat{\alpha}_i(\gamma)$ be the logarithm of the absolute value of the i-th eigenvalue of the matrix $A(\gamma)$ ordered in decreasing order and write $\alpha_i(\gamma) = \hat{\alpha}_i(\gamma)/\ell(\gamma)$. As $A(\gamma)$ is symplectic, with real leading eigenvalue $e^{\ell(\gamma)}$, we have

$$1 = \alpha_1(\gamma) \geq \cdots \geq \alpha_g(\gamma) \geq 0 \geq -\alpha_g(\gamma) \geq \cdots \geq -\alpha_1(\gamma) = -1.$$

As eigenvalues of matrices are invariant under conjugation, for $-g \leq i \leq g$ we obtain in this way a function $\alpha_i : \Gamma \to [-1, 1]$.

The characteristic polynomial of a symplectic matrix $A \in Sp(2g, \mathbb{Z})$ is a reciprocal polynomial of degree $2g$ with integral coefficients. Its roots define a number field K of degree at most $2g$ over $\mathbb{Q}$ which is a quadratic extension of the so-called *trace field* of A. The Galois group of K is isomorphic to a subgroup of the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^g \rtimes \Sigma_g$ where Σ_g is the symmetric group in g variables (see [VV02] for details). The field K and the Galois group only depend on the conjugacy class of A.

For $\gamma \in \Gamma$ let $G(\gamma)$ be the Galois group of the number field defined by the conjugacy class $[A(\gamma)]$. We show

Theorem 1. (1) *For $\epsilon > 0$ the set $\{\gamma \in \Gamma \mid |\alpha_i(\gamma) - \kappa_i| < \epsilon\}$ $(1 \leq i \leq g)$ is typical.*
(2) *The set of all $\gamma \in \Gamma$ such that the trace field of $[A(\gamma)]$ is totally real, of degree g over $\mathbb{Q}$, and $G(\gamma) = (\mathbb{Z}/2\mathbb{Z})^g \rtimes \Sigma_g$ is typical.*

The proof of Theorem 1 uses a result on the Zariski closure of the image under the map Ψ of pseudo-Anosov mapping classes obtained from the first return map of the Teichmüller flow on $\mathcal{Q}$ to a small contractible flow box in $\mathcal{Q}$. Although our viewpoint is a bit different, this discussion can be translated into properties of the *Rauzy-Veech group* of $\mathcal{Q}$ and yields the following result which was conjectured by Zorich [Z99].

Corollary 1. *The Rauzy-Veech group of any component of a stratum is a Zariski dense subgroup of* $Sp(2g,\mathbb{Z})$.

For hyperelliptic strata, Avila, Matheus and Yoccoz [AMY16] showed that the Rauzy-Veech group is a subgroup of $Sp(2g,\mathbb{Z})$ of finite index. In Proposition 3.4 we observe that for strata with at least one simple zero, the Rauzy Veech group coincides with $Sp(2g,\mathbb{Z})$. Theorem 6.3 is a more precise version of Corollary 1 which is valid for *affine invariant manifolds* as well.

By the groundbreaking work of Eskin, Mirzakhani and Mohammadi [EMM15], such affine invariant manifolds are precisely the closures of orbits for the $SL(2,\mathbb{R})$-action on $\mathcal{Q}$. Examples of non-trivial orbit closures are arithmetic *Teichmüller curves*. They arise from branched covers of the torus, and they are dense in any stratum of abelian differentials. Other examples of orbit closures different from entire components of strata can be constructed using more general branched coverings.

The *rank* of an affine invariant manifold $\mathcal{M}$ is defined by

$$\mathrm{rk}(\mathcal{M}) = \frac{1}{2}\dim(pT\mathcal{M})$$

where p is the projection of period coordinates into absolute cohomology [W15]. Teichmüller curves are affine invariant manifolds of rank one, and the rank of a component of a stratum equals g.

We establish a finiteness result for affine invariant submanifolds of rank at least two which is independently due to Eskin, Filip and Wright [EFW17].

Theorem 2. *Let* $g \geq 2$ *and let* $\mathcal{Q}$ *be a component of a stratum in the moduli space of abelian differentials. For every* $2 \leq \ell \leq g$, *there are only finitely many proper affine invariant submanifolds in* $\mathcal{Q}$ *of rank* ℓ.

As a corollary, we obtain

Corollary 2. *Let* $\mathcal{Q}$ *be any component of a stratum in genus* $g \geq 3$. *Then the set of all* $\gamma \in \Gamma$ *whose* $SL(2,\mathbb{R})$*-orbit closure equals* $\mathcal{Q}$ *is typical.*

For $g = 2$, Corollary 2 is false in a very strong sense. Namely, McMullen [McM03a] showed that in this case, the orbit closure of any periodic orbit is an affine invariant manifold of rank one. If the trace field K of the orbit is quadratic, then K defines a Hilbert modular surface in the moduli space of principally polarized abelian varieties which contains the image of the orbit closure under the Torelli map. Such a Hilbert modular surface is a quotient of $\mathbf{H}^2 \times \mathbf{H}^2$ by the lattice $PSL(2,\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of algebraic integers in K. This insight is the starting point of a complete classification of orbit closures in genus 2 [McM03b].

In higher genus, Apisa [Ap15] classified all orbit closures in hyperelliptic components of strata. For other components of strata, a classification of orbit closures is not available. However, there is substantial recent progress towards a geometric understanding of orbit closures. In particular, Mirzakhani and Wright [MW16] showed that all affine invariant manifolds of maximal rank either are components of strata or are contained in the hyperelliptic locus. We refer to the work [LNW15] of Lanneau, Nguyen and Wright for an excellent recent overview of what is known and for a structural result for rank one affine invariant manifolds.

Teichmüller curves are affine invariant manifolds of dimension 2. To each such Teichmüller curve, there is associated a *trace field* which is an algebraic number field of degree at most g over $\mathbb{Q}$. This trace field coincides with the trace field of every periodic orbit contained in the curve [KS00]. The Teichmüller curve is called *algebraically primitive* if the degree of its trace field equals g.

The stratum $\mathcal{H}(2)$ of abelian differentials with a single zero on a surface of genus 2 contains infinitely many algebraically primitive Teichmüller curves [McM03b]. Recently, Bainbridge, Habegger and Möller [BHM14] showed finiteness of algebraically primitive Teichmüller curves in any stratum in genus 3. Finiteness of algebraically primitive Teichmüller curves in strata of differentials with a single zero for surfaces of prime genus $g \geq 3$ was established in [MW15]. Our final result generalizes this to every stratum in every genus $g \geq 3$, with a different proof. A stronger finiteness result is contained in [EFW17].

Theorem 3. *Any component $\mathcal{Q}$ of a stratum in genus $g \geq 3$ contains only finitely many algebraically primitive Teichmüller curves.*

Plan of the paper: In Section 2 we establish the first part of Theorem 1 as a fairly easy consequence of the results in [H13].

Section 3 introduces the idea of local monodromy groups and their Zarisky closures and uses it to show Corollary 1. In Section 4, this result together with group sieving and the first part of Theorem 1 leads to the second part of Theorem 1 and to Corollary 1.

Section 5 contains some properties of the absolute period foliation of an affine invariant manifold. In Section 6 we look at the local monodromy group of an affine invariant manifold and show that it is Zariski dense in the symplectic group of rank corresponding to the rank of the manifold. We then compare in Section 7 the Chern connection on the Hodge bundle to the Gauss Manin connection which leads to the proof of the first part of Theorem 2 in Secction 8. Section 9 uses information on the absolute period foliation to complete the proof of Theorem 2. The proofs of Theorem 2 and Theorem 3 are contained in Section 8.

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2. Lyapunov exponents

In this section we consider any component $\mathcal{Q}$ of a stratum of area one abelian differentials. The results in this section are equally valid for any affine invariant submanifold for the $SL(2,\mathbb{R})$-action [EMM15], in particular they hold true for all components of strata of quadratic differentials.

The *Teichmüller flow* Φ^t acts on $\mathcal{Q}$ preserving a Borel probability measure λ in the Lebesgue measure class, the so-called *Masur Veech measure*. Let $h > 0$ be the entropy of Φ^t with respect to the measure λ.

Denote by $\mathcal{H} = \mathcal{Q} \times H^1(S,\mathbb{R}) \to \mathcal{Q}$ the trivial vector bundle over $\mathcal{Q}$ with fibre $H^1(S,\mathbb{R})$. The *Kontsevich Zorich cocycle* over $\mathcal{Q}$ is an extension of the Teichmüller flow Φ^t on $\mathcal{Q}$ to a flow on $\mathcal{H}$ which can roughly be described as follows. Consider a component $\tilde{\mathcal{Q}}$ of the preimage of $\mathcal{Q}$ in the Teichmüller space of marked area one abelian differentials. Then $\mathcal{Q}$ is the quotient of $\tilde{\mathcal{Q}}$ under the action of the stabilizer $\mathrm{Stab}(\tilde{\mathcal{Q}})$ of $\tilde{\mathcal{Q}}$ in the mapping class group $\mathrm{Mod}(S)$. For $q \in \tilde{\mathcal{Q}}$ and large T, the differential $\Phi^T q$ can be brought back to a fixed fundamental domain for the action of $\mathrm{Stab}(\tilde{\mathcal{Q}})$ by an element $\varphi \in \mathrm{Stab}(\tilde{\mathcal{Q}})$. The action of φ on the first cohomology group $H^1(S,\mathbb{R})$ is essentially the Kontsevich Zorich cocycle.

The Kontsevich Zorich cocycle can also be described using the *Gauss Manin connection* on $\mathcal{H}$. The Gauss Manin connection is a flat connection on $\mathcal{H}$ defined by local trivializations which identify nearby integral cohomology classes. Parallel transport for the Gauss Manin connection defines a lift of the Teichmüller flow Φ^t to a flow Θ^t on $\mathcal{H}$ which preserves the symplectic structure on $\mathcal{H}$ defined by the algebraic intersection form on $H^1(S,\mathbb{R})$.

There are some technical difficulties due to nontrivial point stabilizers for the action of the mapping class group on Teichmüller space. To avoid dealing with this issue (although this can be done with some amount of care) we define the *good subset* $\mathcal{Q}_{\rm good}$ of $\mathcal{Q}$ to be the set of all points $q \in \mathcal{Q}$ with the following property. Let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of marked abelian differentials and let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of q; then an element of $\mathrm{Mod}(S)$ which fixes $\tilde{q}$ acts as the identity on $\tilde{\mathcal{Q}}$ (compare [H13] for more information on this technical condition). The good subset is open, dense and Φ^t-invariant [H13].

The Kontsevich Zorich cocycle is bounded, with values in the symplectic group $Sp(2g,\mathbb{R})$, and therefore its *Lyapunov exponents* for the invariant measure λ are defined. These exponents measure the asymptotic growth rate of vectors along orbits of Φ^t which are typical for λ. Since the Gauss Manin connection preserves the symplectic structure on $\mathcal{H}$, these exponents are invariant under multiplication with -1. Let $1 = \kappa_1 > \cdots > \kappa_g > 0$ be the largest g Lyapunov exponents of the Teichmüller flow on $\mathcal{Q}$. That these exponents are all positive and pairwise distinct was shown in [AV07]. For more general *affine invariant manifolds*, the analogue statement need not hold true. We refer to [Au15] for a discussion and examples.

Let

$$\Gamma \subset \mathcal{Q}$$

be the countable collection of all periodic orbits for Φ^t contained in $\mathcal{Q}$. Denote by $\ell(\gamma)$ the period of $\gamma \in \Gamma$. The orbit $\gamma \in \Gamma$ determines a conjugacy class in $\mathrm{Mod}(S)$ of pseudo-Anosov elements. Let $\varphi \in \mathrm{Mod}(S)$ be an element in this conjugacy class; then $A(\gamma) = \Psi(\varphi) \in \mathrm{Sp}(2g, \mathbb{Z})$ is determined by γ up to conjugation. Furthermore, the largest absolute value of an eigenvalue of $A(\gamma)$ equals $e^{\ell(\gamma)}$. More precisely, the matrix $A(\gamma)$ is Perron Frobenius, with leading eigenvalue $e^{\ell(\gamma)}$, and the eigenspace for the eigenvalue $e^{\ell(\gamma)}$ is spanned by the real cohomology class defined by intersection with the attracting measured geodesic lamination of φ.

If we define $1 = \alpha_1(\gamma) > \cdots \geq \alpha_g(\gamma) \geq 0$ to be the quotients by $\ell(\gamma)$ of the logarithms of the g largest absolute values of the eigenvalues of the matrix $A(\gamma)$, ordered in decreasing order and counted with multiplicities, then the numbers $\alpha_i(\gamma)$ only depend on γ but not on any choices made.

Let $\epsilon > 0$. For $\gamma \in \Gamma$ define $\chi_\epsilon(\gamma) = 1$ if $|\alpha_i(\gamma) - \kappa_i| < \epsilon$ for every $i \in \{1, \ldots, g\}$ and define $\chi_\epsilon(\gamma) = 0$ otherwise.

For $R_1 < R_2$ let $\Gamma(R_1, R_2) \subset \Gamma$ be the set of all periodic orbits of prime period contained in the interval (R_1, R_2). For an open or closed subset V of $\mathcal{Q}$ denote by $\chi(V)$ the characteristic function of V and define

$$H(V, R_1, R_2) = \sum_{\gamma \in \Gamma(R_1, R_2)} \int_\gamma \chi(V) \quad \text{and}$$
$$H_\epsilon(V, R_1, R_2) = \sum_{\gamma \in \Gamma(R_1, R_2)} \int_\gamma \chi(V)\chi_\epsilon(\gamma).$$

Clearly we have

$$H_\delta(V, R_1, R_2) \leq H_\epsilon(V, R_1, R_2) \leq H(V, R_1, R_2)$$

for all $\epsilon > \delta > 0$.

Call a point $q \in \mathcal{Q}$ *birecurrent* if it is contained in its own α- and ω limit set. By the Poincaré recurrence theorem, the set of birecurrent points in $\mathcal{Q}$ has full Lebesgue measure. In [H13] (Corollary 4.8 and Proposition 5.4) we showed

Proposition 2.1. *For every good birecurrent point $q \in \mathcal{Q}_{\mathrm{good}}$, for every neighborhood U of q in $\mathcal{Q}$ and for every $\delta > 0$ there is an open neighborhood $V \subset U$ of q in $\mathcal{Q}$ and a number $t_0 > 0$ such that*

$$H(V, R - t_0, R + t_0)e^{-hR} \in (2t_0\lambda(V)(1 - \delta), 2t_0\lambda(V)(1 + \delta))$$

for all sufficiently large $R > 0$.

The proof of Proposition 2.1 is based on a more technical result which will be used several times in the sequel. Lemma 2.2 below combines Lemma 4.7 and Proposition 5.4 of [H13]. For its formulation, we say that a closed curve η in $\mathcal{Q}_{\mathrm{good}}$ defines the conjugacy class of a pseudo-Anosov mapping class $\varphi \in \mathrm{Mod}(S)$ if the following holds true. Let $\tilde{\eta}$ be a lift of η to an arc in the Teichmüller space of abelian differentials, parametrized one some interval $[0, a] \subset \mathbb{R}$; then $\tilde{\eta}(a) = \psi(\tilde{\eta}(0))$ for a mapping class ψ which is conjugate to φ. This definition does not depend on any choices made.

Lemma 2.2. *Let $q \in \mathcal{Q}_{\rm good}$ be a good birecurrent point and let $\delta > 0$. Then every neighborhood U of q contains an open and contractible neighborhood $V \subset \mathcal{Q}_{\rm good}$ of q, such that there exists a nested sequence of neighborhoods $Z_0 \subset Z_1 \subset Z_2 \subset V$ of q, with Z_i closed and Z_i contained in the interior of Z_{i+1}, and there is a number $R_0 > 0$ with the following properties.*

Let $z \in Z_0$ and let $R > R_0$ be such that $\Phi^R z \in Z_0$. Let $\hat{E}$ be the connected component containing z of the intersection $\Phi^t V \cap V$.

a) The length of the connected subsegment of the orbit $\cup_{t\in\mathbb{R}}\Phi^t z \cap Z_0$ containing $\Phi^R z$ equals $2t_0$.
b) $\lambda(Z_0) > (1-\delta)\lambda(V)$, and the Lebesgue measure of the intersection $\Phi^R Z_1 \cap Z_2 \cap \hat{E}$ is contained in the interval
$$[e^{-hR}\lambda(V)(1-\delta), e^{-hR}\lambda(V)(1+\delta)].$$
c) Connect $\Phi^R z$ to z by an arc in V and let η be the concatenation of the orbit segment $\cup_{0\leq t\leq R}\Phi^t z$ with this arc. We call η a characteristic curve *of the orbit segment $\cup_{t\in[0,R]}\Phi^t z$. There is a unique periodic orbit γ for Φ^t of length at most $R+\delta$ which intersects $\Phi^R Z_2 \cap Z_2 \cap \hat{E}$. The curve η and the orbit γ define the same conjugacy class in* $\mathrm{Mod}(S)$.

Note that in the above statement, we slightly adjusted the choice of the sets Z_i compared to the terminology in [H13] for clarity of exposition.

We use Lemma 2.2 to show

Proposition 2.3. *For every birecurrent point $q \in \mathcal{Q}_{\rm good}$, for every neighborhood U of q in $\mathcal{Q}$ and for every $\delta > 0$ there is an open neighborhood $V \subset U$ of q in $\mathcal{Q}$ and a number $t_0 > 0$ with the properties stated in Proposition 2.1 such that for every $\epsilon > 0$ we have*
$$\liminf_{R\to\infty} H_\epsilon(V, R-t_0, R+t_0)e^{-hR} \geq 2t_0\lambda(V)(1-\delta).$$

Proof. Let $\| \ \|$ be the *Hodge norm* on the bundle $\mathcal{H} \to \mathcal{Q}$ (see [ABEM12] for definitions and the most important properties). Denote as before by Θ^t the lift of the Teichmüller flow to a flow on $\mathcal{H}$ defined by parallel transport for the Gauss Manin connection. Recall that Θ^t preserves the symplectic structure on $\mathcal{H}$, but in general it does not preserve the Hodge norm. For $z \in \mathcal{Q}$ let $\mathcal{H}_z$ be the fibre of $\mathcal{H}$ at z. For $1 \leq i \leq g$ and for $t > 0$ let
$$\zeta_i(t,z)$$
be the minimum of the operator norms of the restriction of $\Theta^t(z)$ to a symplectic subspace of $\mathcal{H}_z$ of real dimension $2(g-i+1)$. Define
$$\kappa_i(t,z) = \frac{1}{t}\log\zeta_i(t,z).$$

Let $\epsilon > 0, \delta > 0$ and let U be a neighborhood of a birecurrent point $q \in \mathcal{Q}_{\rm good}$. Since the Kontsevich Zorich cocycle is locally constant (or, equivalently, the Gauss Manin connection is flat), we can find a collection of nested neighborhoods $Z_0 \subset Z_1 \subset Z_2 \subset V \subset U$ with the properties in Lemma 2.2 and such that furthermore, with the notations from the lemma, if $z\in Z_0, R>R_0$ and if $\Phi^R z\in Z_0$ then the periodic orbit γ for Φ^t determined by the characteristic curve η of the orbit segment $\cup_{t\in[0,R]}\Phi^t z$ satisfies

$$|\kappa_i(R,z)-\alpha_i(\gamma)|\leq \epsilon/2. \tag{1}$$

Namely, for a sufficiently small contractible neighborhood V of q in $\mathcal{Q}_{\rm good}$, the trivialization of $\mathcal{H}|V$ defined by the Gauss Manin connection almost preserves the Hodge norm. Then the estimate (1) holds true if we replace $\alpha_i(\gamma)$ be the i-th absolute value in decreasing order of an eigenvalue of the symplectic transformation A_η of $\mathcal{H}_z$ which is defined by parallel transport for the Gauss Manin connection along a characteristic curve η for the orbit segment $\cup_{t\in[0,R]}\Phi^t z$. But by property (c) in Lemma 2.2, the characteristic curve η defines the same conjugacy class of a pseudo-Anosov mapping class as γ. This means that the numbers $\alpha_i(\gamma)$ are precisely the absolute values of the eigenvalues of the transformation A_η. Thus the estimate for the characteristic curve η and the transformation A_η implies the estimate for γ.

By Oseledec's theorem and ergodicity, there is number $R(\epsilon)>R_0$ and a Borel subset B of Z_0 of measure $\lambda(B)>\lambda(Z_0)(1-\delta)$ with the following property. Let $u\in B$ and let $R>R(\epsilon)$; then $|\kappa_i(R,u)-\kappa_i|\leq\epsilon/2$.

Since the Lebesgue measure is mixing for the Teichmüller flow, there is a number $R_1>R(\epsilon)$ such that

$$\lambda(\Phi^R B\cap B)\geq \lambda(B)^2(1-\delta)\geq\lambda(Z_0)^2(1-\delta)^3\geq\lambda(V)^2(1-\delta)^5$$

for all $R\geq R_1$. By the estimate (b) in Lemma 2.2, this implies that the number of components of the intersection $\Phi^R V\cap V$ containing points in $\Phi^R B\cap B$ is at least $e^{hR}\lambda(V)(1-\delta)^6$. By property (c) in Lemma 2.2, for each such component there is a periodic orbit of Φ^t passing through $Z_2\subset V$. The estimate (1) together with the definition of B yields that each such orbit γ satisfies $\chi_\epsilon(\gamma)=1$. Thus by (a) of Lemma 2.2, each such component of intersection $\Phi^R V\cap V$ contributes $2t_0$ to the value $H_\epsilon(V,R-t_0,R+t_0)$. Together this shows the proposition. ☐

As a corollary, we obtain the first part of Theorem 1. We formulate it more generally for strata of abelian or quadratic differentials. As before, κ_i denotes the i-th Lyapunov exponent of the Kontsevich Zorich cocycle.

Corollary 2.4. *For $\epsilon>0$, the set $\{\gamma\in\Gamma\mid|\alpha_i(\gamma)-\kappa_i|<\epsilon\}$ $(1\leq i\leq g)$ is typical.*

Proof. In [H13] the following is shown. As $R\to\infty$, the measures

$$\mu_R=e^{-hR}\sum_{\gamma\in\Gamma,\ell(\gamma)\leq R}\delta(\gamma)$$

converge weakly to the Lebesgue measure on $\mathcal{Q}$. The Lebesgue measure of $\mathcal{Q}-\mathcal{Q}_{\rm good}$ vanishes, so it suffices to study the Lebesgue measure on $\mathcal{Q}_{\rm good}$.

By [EM11, EMR12, H11], there is no escape of mass: We have

$$\sharp\{\gamma\in\Gamma\mid\ell(\gamma)\leq R\}\sim\frac{e^{hR}}{hR}. \tag{2}$$

In fact, there is a compact subset K of $\mathcal{Q}$ such that the growth rate of all periodic orbits which do not intersect K is strictly smaller than h.

Let $\epsilon > 0$. By Proposition 2.1 and Proposition 2.3, the measures

$$\mu_R = e^{-hR} \sum_{\gamma \in \Gamma, \ell(\gamma) \leq R} \chi_\epsilon(\gamma)\delta(\gamma)$$

also converge weakly to the Lebesgue measure on $\mathcal{Q}$. Since there is no escape of mass, as in [H13] it now follows from (2) that periodic orbits γ with $\chi_\epsilon(\gamma) > 0$ are typical. □

3. Local Zariski density: The Zorich conjecture

In this section we prove the Zorich conjecture which is stated as Corollary 1 in the introduction. Throughout the section, we denote by $\mathcal{Q}$ a component of a stratum in the moduli space of area one abelian differentials on S, and by $\tilde{\mathcal{Q}}$ a component of its preimage in the Teichmüller space of abelian differentials. As in Section 2, we denote by $\mathcal{Q}_{\rm good}$ the open dense Φ^t-invariant subset of good points. All results of this section are also true for components of quadratic differentials, however the proof requires some more details which we postpone to forthcoming work.

A periodic orbit of Φ^t on $\mathcal{Q}$ determines a conjugacy class in $Sp(2g,\mathbb{Z})$. However, it will be convenient to look at actual elements of $Sp(2g,\mathbb{Z})$ rather than at conjugacy classes.

Choose a birecurrent point $q \in \mathcal{Q}_{\rm good}$. Let $\delta > 0$ and let $Z_0 \subset Z_1 \subset Z_2 \subset V \subset \mathcal{Q}_{\rm good}$ be a nested family of neighborhoods of q in $\mathcal{Q}_{\rm good}$ as in Lemma 2.2.

For sufficiently large $R_0 > 0$ let $z \in Z_0$ and let $T > R_0$ be such that $\Phi^T z \in Z_0$. A characteristic curve of this orbit segment determines uniquely a periodic orbit γ of Φ^t which intersects Z_2 in an arc of length $2t_0$. There may be more than one such intersection arc, but there is a unique arc determined by the component of the intersection $Z_2 \cap \Phi^T Z_2$ containing the point z as described in (c) of Lemma 2.2. Choose the midpoint of this intersection arc as a basepoint for γ and as an initial point for a parametrization of γ.

Let Γ_0 be the set of all parametrized periodic orbits of this form for points $z \in Z_0$ with $\Phi^T z \in Z_0$. By Lemma 2.2, the map which associates to a component of $\Phi^T V \cap V$ containing points in $\Phi^T Z_0 \cap Z_0$ the corresponding parametrized periodic orbit in Γ_0 is a bijection.

Fix once and for all a lift $\tilde{V}$ of the contractible set V to $\tilde{\mathcal{Q}}$. A periodic orbit γ which intersects Z_2 in an arc of length $2t_0$ lifts to a subarc of a flow line of the Teichmüller flow on $\tilde{\mathcal{Q}}$ with starting point in $\tilde{V}$. The endpoints of this arc are identified by a pseudo-Anosov element $\Omega(\gamma) \in \mathrm{Mod}(S)$.

The following *shadowing property* is a variation of Lemma 2.2, using the same notations. Versions of this lemma are familiar in hyperbolic dynamics.

Proposition 3.1. *For $\gamma_1,\dots,\gamma_k \in \Gamma_0$, there is a point $z \in Z_2$, and there are numbers $0 < t_1 < \cdots < t_k$ with the following properties.*

(1) $\Phi^{t_i} z \in Z_2$.
(2) *For each $i \leq k$, a characteristic curve ζ_i of the orbit segment $\{\Phi^t z \mid t_{i-1} \leq t \leq t_i\}$ defines the same conjugacy class in* $\mathrm{Mod}(S)$ *as the periodic orbit γ_i.*
(3) *There is a parametrized periodic orbit γ for Φ^t with initial point in Z_2 which defines the same conjugacy class in* $\mathrm{Mod}(S)$ *as a characteristic curve ζ of the orbit segment $\{\Phi^t z \mid 0 \leq t \leq t_k\}$.*
(4) $\Omega(\gamma) = \Omega(\gamma_k) \circ \cdots \circ \Omega(\gamma_1)$.

Proof. In the case that the arcs γ_i are contained in a fixed compact invariant subset K for Φ^t and that the set V is chosen small in dependence of K, the lemma is identical with the slight weakening of Theorem 4.3 of [H10]. That the statement holds true in the form presented here is immediate from the construction of the set Z_2 in [H13] and Proposition 5.4 of [H13] which is based on an extension of Theorem 4.3 of [H10] to arbitrary orbits of the Teichmüller flow which recur to Z_2. $\square$

As a consequence, the subsemigroup $\Omega(\Gamma_0)$ of $\mathrm{Mod}(S)$ generated by $\{\Omega(\gamma) \mid \gamma \in \Gamma_0\}$ consists of pseudo-Anosov elements whose corresponding periodic orbits are contained in the stratum $\mathcal{Q}$ and pass through the set Z_2. This can be viewed as a version of Rauzy-Veech induction as used in [AV07, AMY16] which is valid for strata of quadratic differentials as well, or as a version of symbolic dynamics for the Teichmüller flow on strata as in [H11, H16].

Our next goal is to obtain information on the image of this subsemigroup under the homomorphism $\Psi : \mathrm{Mod}(S) \to SL(2g,\mathbb{Z})$. For this we choose an odd prime p and study the image of $\Psi\Omega(\Gamma_0)$ under the natural reduction map

$$\Lambda_p : Sp(2g,\mathbb{Z}) \to Sp(2g, F_p)$$

where F_p is the field with p elements. Denote by ι the symplectic form on F_p^{2g} which is preserved by $Sp(2g, F_p)$.

The following lemma relies on the results in [Hl08]. For its formulation, define a *transvection* in $Sp(2g, F_p)$ to be a map $A \in Sp(2g, F_p)$ which fixes a subspace of F_p^{2g} of codimension one and has determinant one (see [Hl08]). Any map of the form

$$\alpha \to \alpha + \iota(\alpha,\beta)\beta$$

for some $0 \neq \beta \in F_p^{2g}$ (here as before, ι is the symplectic form) is a transvection. We call this map a *transvection by β*.

Lemma 3.2. *Let $p \geq 3$ be an odd prime and let $G < Sp(2g, F_p)$ be a subgroup generated by $2g$ transvections by the elements of a set $\mathcal{E} = \{e_1,\dots,e_{2g}\} \subset F_p^{2g}$ which spans F_p^{2g}. Assume that there is no nontrivial partition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ so that $\iota(e_{i_1}, e_{i_2}) = 0$ for all $e_{i_j} \in \mathcal{E}_j$. Then $G = Sp(2g, F_p)$.*

Proof. For each i write $A_i(x) = x + \iota(x, e_i)e_i$. Let $G < Sp(2g, F_p)$ be the subgroup generated by the transvections $A_1, \dots, A_{2g}$. Since the vectors $e_1, \dots, e_{2g}$ span F_p^{2g}, the intersection of the invariant subspaces of the transvections A_i $(i \leq 2g)$ is trivial.

We claim that the standard representation of G on F_p^{2g} is irreducible. Namely, assume to the contrary that there is an invariant proper linear subspace $W \subset F_p^{2g}$. Let $0 \neq w \in W$; then there is at least one i so that $\iota(w, e_i) \neq 0$. By invariance, we have $w + \iota(w, e_i)e_i \in W$ and hence $e_i \in W$ since F_p is a field.

As a consequence, W is spanned by some of the e_i, say by $e_{i_1}, \dots, e_{i_k}$, and if j is such that $\iota(e_{i_s}, e_j) \neq 0$ for some $s \leq k$ then $e_j \in W$. However, this implies that $W = F_p^{2g}$ by the assumption on the set $\mathcal{E} = \{e_i\}$.

To summarize, G is an irreducible subgroup of $Sp(2g, F_p)$ generated by transvections (where irreducible means that the standard representation of G on F_p^{2g} is irreducible). Furthermore, as p is an odd prime by assumption, the order of each of these transvections is not divisible by 2. Theorem 3.1 of [Hl08] now yields that $G = Sp(2g, F_p)$ which is what we wanted to show. $\square$

Remark 3.3. By Proposition 6.5 of [FM12], Lemma 3.2 is not true for $p = 2$.

The following proposition is the main step towards the proof of Corollary 1 and is of independent interest. A slightly weaker version for affine invariant manifolds will be established in Section 6.

Proposition 3.4.
(1) *For an odd prime $p \geq 3$ the subgroup of $Sp(2g, F_p)$ generated by $\{\Lambda_p \Psi(\Omega(\gamma)) \mid \gamma \in \Gamma_0\}$ equals the entire group $Sp(2g, F_p)$.*
(2) *If $\mathcal{Q}$ is a stratum of abelian differentials with at least one simple zero then the subgroup of* $\mathrm{Mod}(S)$ *generated by $\{\Omega(\gamma) \mid \gamma \in \Gamma_0\}$ equals the entire mapping class group.*

Proof. We begin with showing the second part of the proposition.

The mapping class group is generated by finitely many Dehn twists about simple closed curves. A specific generating system are the so-called *Humphries generators* (see p.112 of [FM12]). These generators are Dehn twists about the set of simple closed curves $a_1, \dots, a_g, c_1, \dots, c_{g-1}, m_1, m_2$ shown in the Figure 1.

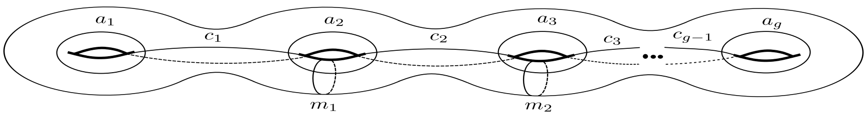

Figure 1

Let $\mathcal{Q}$ be a stratum of abelian differentials with at least one simple zero. By [KZ03], $\mathcal{Q}$ is connected. For a simple closed curve c denote by T_c the positive Dehn twist about c.

For the second part of the proposition it suffices to show that for each of the simple closed curves c shown in Figure 1, there is $\sigma(c) \in \{1,-1\}$ so that the subgroup of $\mathrm{Mod}(S)$ generated by $\Omega(\gamma)$ $(\gamma \in \Gamma_0)$ contains $T_c^{\sigma(c)}$.

By [H16], to each component $\mathcal{Q}$ of a stratum there is associated a collection $\mathcal{T}(\mathcal{Q})$ of *large train tracks* which parametrize the component in a sense described in [H16]. In particular, if $\tau \in \mathcal{T}(\mathcal{Q})$ and if $\varphi \in \mathrm{Mod}(S)$ is a pseudo-Anosov mapping class with *train track expansion* τ (this means that the train track $\varphi(\tau)$ is *carried* by τ, a property which we denote by $\varphi(\tau) \prec \tau$ in the sequel), then the periodic orbit of the Teichmüller flow corresponding to the conjugacy class of φ is contained in the closure of $\mathcal{Q}$. This statement can be viewed as a version of Rauzy Veech induction.

If $\mathcal{U}$ is a component of a stratum and if $\mathcal{U}$ is contained in the closure of $\mathcal{Q}$, then any train track associated to $\mathcal{U}$ can be obtained from some train track associated to $\mathcal{Q}$ by removing some branches [H16]. Furthermore, if we call a train track τ *orientable* if there exists a consistent orientation of the branches of τ (here consistent means that the orientation is compatible at the switches), then orientable train tracks correspond to strata of abelian differentials.

Let now $\mathcal{U}$ be the stratum of abelian differentials with one simple zero and one zero of order $2g-3$. We claim that there is a train track τ for $\mathcal{U}$ with the following property. For each of the Humphries generators T_c of $\mathrm{Mod}(S)$ there exists $\sigma(c) \in \{1,-1\}$ such that we have $T_c^{\sigma(c)}(\tau) \prec \tau$, ie the train track $T_c^{\sigma(c)}(\tau)$ is *carried* by τ.

Let as before ι be the intersection form on $H_1(S,\mathbb{Z})$. Denote by $[c]$ the homology class of an oriented simple closed curve c. Orient the curves in Figure 1 in such a way that for each i we have $\iota([c_i],[a_i]) = 1 = \iota([c_i],[a_{i+1}])$ and $\iota([a_2],[m_1]) = -1, \iota([a_3],[m_2]) = 1$. For example, we can find such an orientation so that the curves a_{2i-1} in Figure 1 are oriented counter-clockwise, and the curves a_{2i} clockwise.

Construct from this oriented curve system a train track η by replacing each intersection of oriented curves by a large branch as shown in Figure 2. Informally,

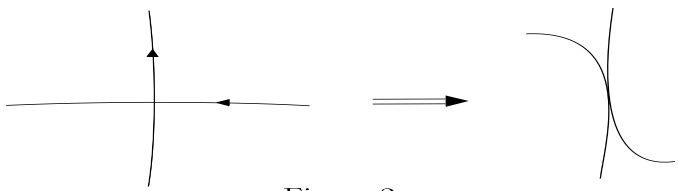

Figure 2

this amounts to following the oriented curves a_i, c_i in the direction prescribed by their orientation. It is easy to check that η is orientable.

Each of the curves a_i, c_i, m_i is embedded in η, and the image of η under either the positive or the negative Dehn twist about any of these curves is carried by η. Namely, the curves are embedded as a subgraph which either consists of a single large branch and a single small branch (in the case of the curves a_1, a_g, m_1, m_2) or of two large branches and two small branches as shown in Figure 3 below, or of three large branches and three small branches (for the curves a_2, a_3). For a simple closed curve c as shown in Figure 3, the train track obtained from τ by a single positive Dehn twist about c is obtained from two *splits* at the two large branches shown in the figure.

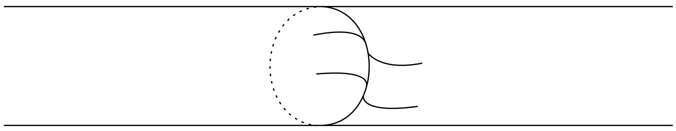

Figure 3

The train track η has two complementary components, one of them is a four-gon. It is not hard to see that η is *large* in the sense of [H16] (ie it is birecurrent, and it carries a large geodesic lamination of the same combinatorial type as η). Thus η is associated to the stratum $\mathcal{U}$ of differentials with one simple zero and one zero of order $2g-3$ [H16]. By construction, it has the properties required in the above claim.

A stratum $\mathcal{Q}$ of abelian differentials with a simple zero contains $\mathcal{U}$ in its closure [KZ03]. There is a train track τ for $\mathcal{Q}$ so that η can be obtained from τ by removing some branches [H16], i.e. η is a subtrack of τ. We can choose τ in such a way that for each of the Humphries generators T_c, we have $T_c^{\sigma(c)}\tau \prec \tau$. Thus τ is a train track as required in the above claim (see again [H16] for details of this construction).

Choose a pseudo-Anosov mapping class φ which admits τ as a *train track expansion*. This means that $\varphi\tau \prec \tau$. Assume that φ maps every branch of τ *onto* τ. This will guarantee that the periodic orbit γ defined by φ is contained in $\mathcal{Q}$ rather than in the boundary of $\mathcal{Q}$, see [H16]. Then for each of the Humphries generators $T_c^{\sigma(c)} \in \mathrm{Mod}(S)$, the composition $\varphi \circ T_c^{\sigma(c)}$ satisfies $(\varphi \circ T_c^{\sigma(c)})\tau \prec \tau$, moreover $\varphi \circ T_c^{\sigma(c)}$ maps every branch of τ *onto* τ. But this just means that $\varphi \circ T_c^{\sigma(c)}$ is a pseudo-Anosov mapping class which admits τ as a train track expansion. In particular, $\varphi \circ T_c^{\sigma(c)}$ determines a periodic orbit in $\mathcal{Q}$. We may assume without loss of generality that this orbit is contained in $\mathcal{Q}_{\mathrm{good}}$.

We next show that for every periodic orbit γ in $\mathcal{Q}_{\mathrm{good}}$ defined by the conjugacy class of a pseudo-Anosov mapping class φ with train track expansion τ as above and for every neighborhood U_0 of $\gamma(0)$ (see the proof of Proposition 2.3), the subgroup of $\mathrm{Mod}(S)$ generated by the parametrized orbits in the set Γ_0 contains each of the Humphries generators $T_c^{\sigma(c)}$ and hence this group is the entire mapping class group.

Namely, write $\beta = T_c^{\sigma(c)}$ for one of these Dehn twists. By the above discussion, for each k the mapping class $\varphi^k \circ \beta \circ \varphi^k$ is pseudo-Anosov, with train track expansion τ, and it defines a periodic orbit $\gamma_{\varphi^k\circ\beta\circ\varphi^k}$ in $\mathcal{Q}$. As $k \to \infty$, the normalized Φ^t-invariant measures supported on the periodic orbits $\gamma_{\varphi^k\circ\beta\circ\varphi^k}$ converge weakly to the normalized Lebesgue measure on γ. Namely, the vertical projective measured laminations μ_k, μ of abelian differentials $q_k, q \in \mathcal{Q}$ which generated the periodic orbit $\gamma_{\varphi^k \circ \beta \circ \varphi^k}, \gamma$ are all carried by τ, and $\mu_k \to \mu$ in the space of projective transverse measures on τ. By the same reasoning, the horizontal projective measured geodesic laminations of q_k converge to the projective measured geodesic lamination of q.

Thus for sufficiently large k the periodic orbit $\gamma_{\varphi^k \circ \beta \circ \varphi^k}$ passes through the set $Z_0 \subset U_0$ used for the construction of the set Γ_0 and hence it defines an element of Γ_0. Then $\beta = \varphi^{-k} \circ (\varphi^k \circ \beta \circ \varphi^k) \circ \varphi^{-k}$ is contained in the group generated by $\{\Omega(\gamma) \mid \gamma \in \Gamma_0\}$ as claimed. As a consequence, the second part of the proposition holds true for any choice of a neighborhood U_0 of $\gamma(0)$.

To show the second part of the proposition for an arbitrary open neighborhood U in $\mathcal{Q}$ of a birecurrent point $q \in \mathcal{Q}_{\rm good}$, recall that by ergodicity of the Teichmüller flow and the Anosov closing lemma established in [H13], "generic" periodic orbits for Φ^t passing through U become equidistributed for the Lebesgue measure (see [H13] for a detailed discussion). In particular, they pass through the set $Z_0 \subset U_0$ chosen as above.

Now use the argument for the set $Z_0 \subset U_0$ as follows. Let Γ_1 be the set of periodic orbits constructed from q and the open neighborhood U of q. Let $\gamma \in \Gamma_1$ be a generic periodic orbit which passes through the set Z_0. Let $T > 0$ be such that $\gamma(T) \in Z_0$. Let $\hat{\gamma}$ be the reparametrization of γ which satisfies $\hat{\gamma}(0) = \gamma(T)$. Apply the above construction to $\hat{\gamma}$ and the pseudo-Anosov mapping class $\hat{\varphi}$ defined by the parametrized orbit γ. We conclude that for a sufficiently large k, a reparametrization of the periodic orbit corresponding to $\varphi^k \circ \beta \circ \varphi^k$ is contained in Γ_1. By the reasoning used for the set U_0, we obtain the second part of the proposition for periodic orbits passing through U.

The first part of this proof also immediately implies the first part of the proposition for strata of abelian differentials with at least one simple zero. We are left with showing the first part of the proposition for arbitrary strata $\mathcal{Q}$ of abelian differentials.

Let c be an oriented non-separating simple closed curve on S which defines the homology class $[c]$. The action on homology of the positive Dehn twist T_c about c equals

$$T_c(\alpha) = \alpha + \iota(\alpha, [c])[c] \tag{3}$$

(Proposition 6.3 of [FM12]). In other words, the Dehn twist T_c acts on $H_1(S, \mathbb{Z})$ as a transvection by $[c]$.

By the main result of [KZ03], for $g \geq 4$ the stratum $\mathcal{H}(2g-2)$ of abelian differentials with a single zero consists of three connected components. One of these components is hyperelliptic, the other two components are distinguished by the parity of the spin structure they define. The stratum $\mathcal{H}(4)$ consists of two components; one component is hyperelliptic, the second component has odd spin structure. The stratum $\mathcal{H}(2)$ is connected.

Any component $\mathcal{Q}$ of a stratum with more than one zero contains a component of $\mathcal{H}(2g-2)$ in its closure. Thus following the reasoning in the first part of this proof, it suffices to find for each of the components $\mathcal{V}$ of $\mathcal{H}(2g-2)$ a train track ζ associated to $\mathcal{V}$ with the following property. Let $p \geq 3$ be an odd prime. Then the subgroup of $Sp(2g, F_p)$ which is generated by those transvections in $Sp(2g, F_p)$ which are images under $\Lambda_p \circ \Psi$ of Dehn twists about embedded curves c in ζ with $T_c^{\sigma(c)}\zeta \prec \zeta$ ($\sigma \in \{-1, 1\}$) is all of $Sp(2g, F_p)$.

Let again η be the train track constructed in the beginning of this proof from the Humphries generators of $\mathrm{Mod}(S)$. Let ζ be the train track obtained from η by removing the small branch contained in the curve m_2. This train track is orientable, filling and birecurrent. If $g \geq 3$ then ζ is not invariant under a hyperelliptic involution. Hence it corresponds to one of the two non-hyperelliptic components of $\mathcal{H}(2g-2)$. These components are distinguished by the parity of the spin structure they define. The formula in the proof of Proposition 4.9 of [H16] calculates this parity (see also the formulas in [KZ03]). It is odd if g is odd, and even if g is even.

For each curve $c \in \{c_i, a_i, m_1\}$ we have $T_c^{\sigma(c)}\zeta \prec \zeta$. Now for any choice of orientations of the curves in $\{c_i, a_i, m_1\}$, the homology classes $\{[c_i], [a_i], [m_1]\}$ are a basis of $H_1(S, \mathbb{Z})$. Moreover, any two curves intersect in at most one point and their union is a connected subset of S. This implies that the transvections by the homology classes of these curves satisfy the assumptions in Lemma 3.2. As a consequence, for each odd prime $p \geq 3$ these transvections generate $Sp(2g, F_p)$. This is what we wanted to show.

The other two components of $\mathcal{H}(2g-2)$ are treated in the same way. A train track for the hyperelliptic component of $\mathcal{H}(2g-2)$ can be constructed from the following curve system. Remove the curves m_1, m_2 from the Humphries generators and add a simple closed curve c which intersects a_1 in a single point and does not intersect any other of the curves shown in Figure 1. The orientation of the curve c can be chosen in such a way that all the properties used above hold true. The resulting set of curves is clearly invariant under the hyperelliptic involution. The same argument as for the non-hyperelliptic components discussed above yields the case of the hyperelliptic component. In particular, we established the proposition for $g = 3$ [KZ03].

We are left with finding for $g \geq 4$ a train track for the second non-hyperelliptic component of $\mathcal{H}(2g-2)$ with parity of the spin structure opposite to the parity of g. Thus let $\mathcal{Q}$ be the component of abelian differentials in genus $g \geq 4$ with a single zero with parity of the spin structure $g+1$ mod 2. By Lemma 14 of [KZ03], there are differentials $q \in \mathcal{Q}$ which can be obtained from an abelian differential with a single zero in genus $g-1$ by "bubbling a handle". This can be translated into train tracks as follows. Start with a train track η on a surface S' of genus $g-1$ constructed above for the non-hyperelliptic component with a single zero in genus $g-1$ whose spin structure equals $g-1$ mod 2. Attach to S' a cylinder C by removing from $S'-\eta$ two small disks. There is no ambiguity here since $S'-\eta$ is connected. Following the strategy in [H16] we extend the train track η by adding first an embedded simple closed curve c defining the core curve of C. Attach two small branches b_1, b_2 to c, one at each side of c, such that after adding these branches, c consists of a single large branch and a single small branch. Connect the branch b_1 to the curve a_1 and connect the branch b_2 to the curve a_2 (notations are as in Figure 1) using the above orientation rule. It is now easy to check that the resulting train track τ is large and defines the stratum of differentials with parity of spin structure $g-1$ mod 2. Furthermore, τ carries a system of embedded simple closed curves a_i, b_i defining a basis for $H_1(S,\mathbb{Z})$ with the properties stated in Lemma 3.2. For each i there is a train track α_i, β_i carried by η and a choice of a sign $\sigma(a_i), \sigma(b_i)$ such that $T_{a_i}^{\sigma(a_i)}\alpha_i \prec \alpha_i, T_{b_i}^{\sigma(b_i)}\beta_i \prec \beta_i$. The reasoning in the beginning of this proof can now be applied to suitably chosen pseudo-Anosov mapping classes φ_i, ζ_i with train track expansion τ so that $\varphi_i\tau \prec \alpha_i, \zeta_i\tau \prec \beta_i$. This completes the proof of the proposition. □

The first part of the following corollary establishes Zorich's conjecture [Z99] (we leave the easy translation into the language of Rauzy induction to the reader). For its formulation, call a component $\mathcal{Q}$ of a stratum *locally Zarisky dense* if the following holds true. Let U be any open subset of $\mathcal{Q}_{\rm good}$ and let $\Omega(\Gamma_0)$ be the sub-semigroup of ${\rm Mod}(S)$ generated in the sense discussed above by the periodic orbits Γ_0 for Φ^t passing through U. We require that the sub-semigroup $\Psi\Omega(\Gamma_0)$ of $Sp(2g,\mathbb{Z})$ is Zariski dense in $Sp(2g,\mathbb{R})$.

Corollary 3.5. (1) *A component $\mathcal{Q}$ of a stratum is locally Zariski dense.*
(2) *If $\mathcal{Q}$ is a stratum of abelian differentials with a simple zero then the preimage of $\mathcal{Q}$ in the Teichmüller space of area one abelian differentials is connected.*

Proof. The first part of the corollary follows from the first part of Proposition 3.4 and the well known fact that a subgroup of $Sp(2g,\mathbb{Z})$ which surjects onto $Sp(2g,F_p)$ for all odd primes $p\geq 3$ is Zariski dense in $Sp(2g,\mathbb{R})$ (see [Lu99] for more and for references).

Now let $\mathcal{Q}$ be a stratum of abelian or quadratic differentials with a simple zero and let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of abelian differentials. By the second part of Proposition 3.4, the stabilizer of $\tilde{\mathcal{Q}}$ in the mapping class group equals the entire mapping class group. As the components of the preimage of $\mathcal{Q}$ are permuted by the mapping class group, the second part of the corollary follows. □

Remark 3.6. More generally, one can ask about the orbifold fundamental group of a component $\mathcal{Q}$ of a stratum of abelian or quadratic differentials. If $\mathcal{Q}$ is a stratum with $k\geq 2$ zeros then the zero forgetful map maps this orbifold fundamental group into the mapping class group ${\rm Mod}(S)$ of S. Proposition 3.4 shows that for strata with a simple zero, this map is onto. For some strata in genus 3, the fundamental group has been identified with tools from algebraic geometry in [LM14].

For most components of strata, we do not even know the image of the orbifold fundamental group in $Sp(2g,\mathbb{Z})$ besides surjecting onto $Sp(2g,F_p)$ for all odd primes p, see Proposition 3.4. For hyperelliptic components, this image has been determined in [AMY16]. One finds that the group is precisely the subgroup stabilized by the hyperelliptic involution, in particular it is of finite index in $Sp(2g,\mathbb{Z})$. We conjecture that the latter property holds true for all components of all strata.

4. Galois groups

In this section we consider again an arbitrary component $\mathcal{Q}$ of a stratum of abelian differentials. We continue to use the assumptions and notations from section 2 and Section 3. Recall in particular the construction of the set Γ_0 of parametrized periodic orbits in $\mathcal{Q}_{\rm good}$ defined by a small neighborhood of a point $q \in \mathcal{Q}_{\rm good}$ which is birecurrent under the Teichmüller flow. We showed in Section 3 that this set determines a sub-semigroup $\Omega(\Gamma_0)$ of $\mathrm{Mod}(S)$ consisting of pseudo-Anosov elements whose image under the homomorphism Ψ is Zariski dense $Sp(2g,\mathbb{R})$. More precisely, by the first part of Proposition 3.4, for every odd prime p the image $G_p = \Lambda_p(\Psi\Omega(\Gamma_0))$ of the semi-group $\Psi(\Omega(\Gamma_0))$ generates the entire group $Sp(2g,F_p)$. Here as before, $\Lambda_p : Sp(2g,\mathbb{Z}) \to Sp(2g,F_p)$ denotes reduction modulo p.

Now $Sp(2g,F_p)$ is a finite group and therefore for every $A \in Sp(2g,F_p)$ there is some $\ell \geq 1$ such that $A^\ell = A^{-1}$. As a consequence, for all $x,y \in G_p$ we have $xy^{-1} \in G_p$ as well and hence $G_p < Sp(2g,F_p)$ is a group. Then G_p equals the group generated by $\Lambda_p(\Psi\Omega(\Gamma_0))$ and thus $G_p = Sp(2g,F_p)$.

Our next goal is to makes this statement quantitative. To this end denote for a periodic orbit γ for Φ^t by δ_γ the Φ^t-invariant measure supported on γ whose total mass equals the period $\ell(\gamma)$ of γ. Recall that periodic orbits in the set Γ_0 are parametrized, so a single unparametrized periodic orbit may give rise to many different elements of Γ_0.

Let as before p be an odd prime and let $N(p)$ be the number of elements of $Sp(2g,F_p)$. The following proposition holds true for any finite group G of order N with the property that there is a homomorphism $\rho : \mathrm{Mod}(S) \to G$ whose restriction to the semigroup $\Omega(\Gamma_0)$ is surjective.

Proposition 4.1. *Let $B \in Sp(2g,F_p)$ be arbitrary, let p be an odd prime and define*

$$\mathcal{B}(R,B) = \{\gamma \in \Gamma_0 \mid \ell(\gamma) \leq R, \Lambda_p \circ \Psi \circ \Omega(\gamma) = B\}.$$

Then as $R \to \infty$,

$$\sharp\mathcal{B}(R,B) \sim \frac{e^{hR}\lambda(Z_0)}{2ht_0N(p)}$$

independent of B up to a multiplicative error which only depends on Z_0 and which can be arranged to be arbitrarily close to one.

Proof. We show first that there is a number $a > 0$ such that

$$\sharp\mathcal{B}(R,B) \geq ae^{hR}$$

for all $B \in Sp(2g,F_p)$ and for all sufficiently large R.

To this end let U be any neighborhood of a birecurrent point $q \in \mathcal{Q}_{\rm good}$. Let $R_0 > 0, \delta > 0$ and let $Z_0 \subset Z_1 \subset Z_2 \subset V \subset U$ be as in Lemma 2.2. Let $\mathcal{G}$ be the sub-semigroup of $\mathrm{Mod}(S)$ generated by $\{\Omega(\gamma) \mid \gamma \in \Gamma_0\}$. By Lemma 3.1, this semigroup consists of pseudo-Anosov elements. Furthermore, each $\rho \in \mathcal{G}$ is represented by a parametrized periodic orbit ζ for Φ^t which intersects the set Z_2 in a segment of length $2t_0$ containing $\zeta(0)$ as its midpoint. Vice versa, every periodic orbit which passes through Z_0 admits a parametrization so that the corresponding element of $\mathrm{Mod}(S)$ is contained in $\mathcal{G}$. The sub-semigroup $\Psi(\mathcal{G})$ of $Sp(2g,\mathbb{Z})$ is mapped by Λ_p onto the finite group $Sp(2g,F_p)$.

Since $Sp(2g,F_p)$ is a *finite* group and the above argument applies to every neighborhood U of q in $\mathcal{Q}_{\mathrm{good}}$, in particular to $U=Z_0$, there is a number $\hat{R}>R_0$ with the following property. Let $A\in Sp(2g,F_p)$ be arbitrary. Then there is some $z\in Z_0$ with $\Phi^t z\in Z_0$ for some $t\in(R_0,\hat{R})$ which determines a parametrized periodic orbit $\gamma(A)\in\Gamma_0$ with $\Lambda_p\Psi(\Omega(\gamma(A)))=A$.

Write $Z=Z_0$. Let $v\in Z$ be such that $\Phi^T v\in Z,\Phi^{T+U}v\in Z$ for some $T,U>R_0$. By Lemma 2.2 and Lemma 3.1, the pseudo-orbits $\{\Phi^t v\mid 0\le t\le T\}$ and $\{\Phi^t v\mid T\le t\le T+U\}$ determine two parametrized periodic orbits γ_1,γ_2 for Φ^t which define elements $\Lambda_p\Psi(\Omega(\gamma_1)),\Lambda_p\Psi(\Omega(\gamma_2))\in Sp(2g,F_p)$. Let $\gamma=\gamma_2\hat{\circ}\gamma_1$ be the periodic orbit for $\{\Phi^t v\mid 0\le t\le T+U\}$. The notation $\gamma=\gamma_2\hat{\circ}\gamma_1$ indicates that the element γ of $\mathrm{Mod}(S)$ defined by γ is the product of the elements of $\mathrm{Mod}(S)$ defined by γ_1 and γ_2. We have

$$\Lambda_p\Psi(\Omega(\gamma_2\hat{\circ}\gamma_1))=\Lambda_p\Psi(\Omega(\gamma_2))\circ\Lambda_p\Psi(\Omega(\gamma_1)).$$

Fix an element $B\in Sp(2g,F_p)$. If $A\in Sp(2g,F_p)$ is arbitrary and if $\zeta\in\Gamma_0$ is such that $\Lambda_p\Psi(\Omega(\zeta))=A$ then $\Lambda_p\Psi(\Omega(\gamma(BA^{-1})\hat{\circ}\zeta))=B$. In particular, by Proposition 2.1, for sufficiently large $R>R_0$ the number of parametrized periodic orbits $\gamma\in\Gamma_0$ with $\ell(\gamma)\le R+\hat{R}$ and $\Lambda_p\Psi(\Omega(\gamma))=B$ is not smaller than the number of orbits in Γ_0 of length at most R.

For large enough R, the number of orbits in Γ_0 of length at most R can be estimated as in [H13]. Namely, by Proposition 2.1 and the choice of the set Z, for the fixed number $\delta>0$ used in the construction of Z and for large R, the volume of Z with respect to the sum of the measures supported on periodic orbits of Φ^t of length in the interval $[R-t_0,R+t_0]$ is contained in the interval

$$[2t_0e^{hR}\lambda(Z)(1-\delta),2t_0e^{hR}\lambda(Z)(1+\delta)].$$

A periodic orbit intersects the set Z in arcs of length $2t_0$ [H13]. As each such intersection component defines an element of Γ_0 (recall that Γ_0 consists of parametrized orbits), we conclude that for large enough R the number of elements of Γ_0 of length contained in $[R-t_0,R+t_0]$ roughly equals $e^{hR}\lambda(Z)$. Summation yields that for sufficiently large k the number of elements of Γ_0 of length at most $2kt_0$ roughly equals

$$\sum_{i=1}^{k}e^{h(2i-1)t_0}\lambda(Z)\sim e^{2hkt_0}\lambda(Z)/2ht_0.$$

As a consequence, there is a number $a>0$ not depending on B such that up to passing to a subsequence, the measures

$$he^{-hR}\sum_{\gamma\in\mathcal{B}(R,B)}\delta_\gamma\chi_Z[\gamma(-t_0),\gamma(t_0)]$$

converge to a measure $\hat{\lambda}_B$ on Z of total mass contained in $[a\lambda(Z)(1-\delta), \lambda(Z)(1+\delta)]$. Here as before, δ_γ is the Φ^t-invariant measure supported on γ of total mass $\ell(\gamma)$, and χ_Z is the characteristic function of Z.

It is immediate from the construction that $\hat{\lambda}_B$ is the restriction to Z of a Φ^t-invariant Borel measure λ_B on $\mathcal{Q}$. By the main result of [H13] (see also Proposition 2.1), this measure is contained in the measure class of the Lebesgue measure λ. Thus by ergodicity of λ under the Teichmüller flow, we have $\lambda_B = c(B)\lambda$ for a number $c(B) \in [a, 1]$ and such that $\sum_B c(B) = 1$.

Our goal is to show that $c(B)$ is independent of B. To this end recall that the Lebesgue measure λ is mixing of all orders [M82]. In particular, for large enough numbers $R, S > 0$ we have

$$\lambda(Z \cap \Phi^R Z \cap \Phi^{R+S}(Z)) \sim \lambda(Z)^3$$

and therefore $\lambda_B(Z \cap \Phi^R Z \cap \Phi^{R+S} Z) \sim c(B)\lambda(Z)^3$.

For $R > 0, T > 0$ and $B \in Sp(2g, F_p)$ let $\Gamma(R, T, B, Z)$ be the set of all parametrized periodic orbits $\gamma \in \Gamma_0$ for Φ^t with the following properties.

(1) The length of γ is contained in the interval $[R + T - t_0, R + T + t_0]$.
(2) γ is determined by some $v \in Z$ and a return time to Z which is close to $R+T$. Moreover, there is a number $U \in [R-t_0, R+t_0]$ such that $\Phi^U v \in Z$.
(3) $\Lambda_p \Psi(\Omega(\gamma)) = B$.

Define similarly a set $\Gamma(R + T, B, Z)$ containing all orbits with properties (1) and (3) above. It follows from the above discussion (compare [H13]) that

$$\sharp\Gamma(R + T, B, Z) \sim c(B)\lambda(Z)e^{h(R+T)}/2ht_0$$

for large enough R and similarly

$$\sharp\Gamma(R, T, B, Z) \sim c(B)\lambda(Z)^2 e^{h(R+T)}/(2ht_0)^2.$$

Each orbit $\gamma \in \Gamma(R, T, B, Z)$ can be represented in the form $\gamma = \gamma_2 \hat{\circ} \gamma_1$ for some $\gamma_1 \in \Gamma(R, A, Z)$ and some $\gamma_2 \in \Gamma(T, BA^{-1}, Z)$. Moreover, if $\gamma_1 \in \Gamma(R, A, Z)$ for some $A \in Sp(2g, F_p)$ then for any $\gamma_2 \in \Gamma(T, BA^{-1}, Z)$ we have $\gamma_2 \hat{\circ} \gamma_1 \in \Gamma(R, T, B, Z)$.

As a consequence, for an arbitrarily chosen $\epsilon > 0$ and for sufficiently large $R > 0$, as $T \to \infty$ we observe that

$$\begin{aligned}\sharp\Gamma(R, T, B, Z) &= \sum_{A \in Sp(2g, F_p)} \sharp\Gamma(T, BA^{-1}, Z)\sharp\Gamma(R, A, Z) \\ &\sim \frac{1}{(2ht_0)^2}\lambda(Z)^2 e^{h(R+T)} \sum_{A \in Sp(2g, F_p)} c(BA^{-1})c(A).\end{aligned}$$

Now let $B \in Sp(2g, F_p)$ be such that $c(B) = \min\{c(A) \mid A\}$. Such an element exists since $Sp(2g, F_p)$ is finite. Since $\sum_A c(BA^{-1}) = 1$ we have $\sharp\Gamma(R, T, B, Z) \sim c(B)\lambda(Z)^2 e^{h(R+T)}/(2ht_0)^2$ only if $c(A) = c(B) = \frac{1}{N(p)}$ for all A. The proposition follows. ∎

Remark 4.2. Proposition 4.1 can be viewed as an equidistribution result for conjugacy classes of elements in $Sp(2g, F_p)$ defined by periodic orbits of the Teichmüller flow which parallels the familiar equidistribution of random walks on finite connected graphs. The main difficulty lies in the fact that periodic orbits represent conjugacy classes in the mapping class group rather than actual elements, moreover we look at periodic orbits in strata rather than at all periodic orbits.

Now we are ready to complete the proof of the second part of Theorem 1. To this end recall that the characteristic polynomial of a symplectic matrix $A \in Sp(2g, \mathbb{Z})$ is reciprocal of degree $2g$. The roots of such a polynomial come in pairs. The Galois group of the number field defined by the polynomial is a subgroup of the semidirect product

$$(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$$

where S_g is the symmetric group in g elements (see [VV02] for a nice account on this classical fact), and S_g acts on $(\mathbb{Z}/2\mathbb{Z})^g$ by permutation of the factors.

In the sequel we call the Galois group of the field defined by the characteristic polynomial of a matrix $A \in Sp(2g, \mathbb{Z})$ simply the Galois group of A. It only depends on the conjugacy class of A. We say that the Galois group of A is *full* if it coincides with $(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$.

Having a full Galois group makes also sense for an element in $Sp(2g, F_p)$. We use this as in [R08] as follows.

Let $p \geq 5$ be a prime and let $N(p)$ be the number of elements of $Sp(2g, F_p)$. By Proposition 4.1, for large enough R and every $B \in Sp(2g, F_p)$, the number of orbits $\gamma \in \Gamma_0$ of length at most R with $\Lambda_p \circ \Psi \circ \Omega(\gamma) = B$ roughly equals $\frac{e^{hR}}{N(p)}\lambda(Z_0)/2ht_0$. On the other hand, if we denote by $R_p(2g)$ the subset of $Sp(2g, F_p)$ of elements with reducible characteristic polynomial then

$$\frac{|R_p(2g)|}{N(p)} < 1 - \frac{1}{3g}$$

(see Theorem 6.2 of [R08] for a reference to this classical result of Borel).

We follow the proof of Theorem 6.2 of [R08]. Namely, let $p_1, \dots, p_k$ be k distinct primes, and let $K = p_1 \cdots p_k$. Then the reduction $\Lambda_K(A)$ modulo K of any element $A \in Sp(2g, \mathbb{Z})$ is defined, and we have

$$\Lambda_K(A) = \Lambda_{p_1}(A) \times \cdots \times \Lambda_{p_k}(A).$$

It follows from the discussion preceding Proposition 4.1 that reduction mod K defines a surjection of the semigroup $\Psi\Omega(\Gamma_0)$ onto the finite group $\Lambda_{p_1}(A) \times \cdots \times \Lambda_{p_k}(A) = \Lambda_K(A)$. Now if $A \in Sp(2g, \mathbb{Z})$ has a reducible characteristic polynomial then the same holds true for $\Lambda_{p_i}(A)$ for all i. The proportion of the number of elements in $\Lambda_{p_1}(A) \times \cdots \times \Lambda_{p_k}(A)$ with this property is at most $(1 - \frac{1}{3g})^k$.

By Proposition 4.1 (taking into account the comment preceding the proposition), the above estimate implies that for large enough R, the proportion of all orbits γ of length at most R with the property that the Galois group of the characteristic polynomial of $A(\gamma)$ is *not* full is at most of the order of $(1 - \frac{1}{3g})^k$. As $k \to \infty$, we conclude that the Galois group of a typical periodic orbit for Φ^t is full. Thus we have shown

Corollary 4.3. *Let $\mathcal{Q}$ be a component of the stratum of abelian or quadratic differentials. The set of all $\gamma \in \Gamma$ such that the trace field of $[A(\gamma)]$ is of degree g over $\mathbb{Q}$, and $G(\gamma) = (\mathbb{Z}/2\mathbb{Z})^g \rtimes \Sigma_g$ is typical.*

Remark 4.4. The only property used in the proof of Corollary 4.3 which is not available for components of strata of quadratic differentials is the first part of Proposition 3.4. However, it is not hard to establish this part for strata of quadratic differentials, with the same proof (and some extra combinatorial discussion). Thus we could use this to show the analogue of Corollary 4.3 for components of strata of quadratic differentials. This leads to stating that for a typical such orbit, the trace field of the corresponding symplectic matrix A has maximal Galois group and hence either is totally real or completely imaginary. Furthermore, if this field is totally real then the roots of the characteristic polynomial of A are pairwise distinct. In spite of Corollary 2.4 and in contrast to the case of random walks on $\mathrm{Mod}(S)$ (see [R08] for details), this does not imply however that the Lyapunov exponents of the Kontsevich Zorich cocycle over components of strata of quadratic differentials are pairwise distinct.

Let $\omega \in \tilde{\mathcal{Q}}$ be a lift of a point on a typical periodic orbit γ for Φ^t. The periods of ω define an abelian subgroup $\Lambda = \omega(H_1(S, \mathbb{Z}))$ of $\mathbb{C}$ of rank two. Let $e_1, e_2 \in \Lambda$ be two points which are linearly independent over $\mathbb{R}$. Let K be the smallest subfield of $\mathbb{R}$ such that every element of Λ can be written as $ae_1 + be_2$, with $a, b \in K$; then $\Lambda \otimes_K K = K^2$. If we write $T = \Psi(A(\gamma)) + \Psi(A(\gamma))^{-1}$, then the field K also is the field of the characteristic polynomial of T. We call K the *trace field* of γ (see the appendix of [KS00] for more details).

Definition 4.5. The periodic orbit γ is called *algebraically primitive* if the trace field K of γ is a totally real number field of degree g over $\mathbb{Q}$, with maximal Galois group.

The following corollary completes the proof of the second part of Theorem 1.

Corollary 4.6. *Algebraically primitive periodic orbits for Φ^t are typical.*

Proof. Since the Lyapunov spectrum of $\mathcal{Q}$ is simple [AV07], the first part of Theorem 1 implies that for a typical periodic orbit γ, the absolute values of the eigenvalues of $A(\gamma)$ are pairwise distinct and hence all eigenvalues are real. Thus by the discussion following Proposition 4.1, we only have to show that for a symplectic matrix $A \in Sp(2g, \mathbb{R})$ with $2g$ distinct real eigenvalues r_i, r_i^{-1} $(i \leq g, r_i > 1)$ the field defined by $A + A^{-1}$ is totally real. However, this is immediate from the fact that the roots of the polynomial defining the trace field are of the form $r_i + r_i^{-1}$ where r_i are the roots of the characteristic polynomial of A. □

5. The local structure of affine invariant manifolds

In this section we begin the investigation of affine invariant manifolds. Our first goal is to gain some understanding of their local structure. Most or perhaps all of the statements in this section are known to the experts but hard to find in the literature.

We begin with introducing the geometric setup which will be used throughout the remainder of this paper.

A point in *Siegel upper half-space* $\mathfrak{D}_g = \mathrm{Sp}(2g,\mathbb{R})/U(g)$ is a principally polarized abelian variety of dimension g. Here as usual, $U(g)$ denotes the unitary group of rank g. There is a natural rank g holomorphic vector bundle $\tilde{\mathcal{V}} \to \mathfrak{D}_g$ whose fibre over y is just the complex vector space defining y. The polarization and the complex structure define a Hermitean metric h on $\tilde{\mathcal{V}}$. The group $Sp(2g,\mathbb{R})$ acts from the left on the bundle $\tilde{\mathcal{V}}$ as a group of bundle automorphisms preserving the polarization and the complex structure, and hence this action preserves the Hermitean metric. Thus the bundle $\tilde{\mathcal{V}}$ projects to a holomorphic Hermitean (orbifold) vector bundle

$$\mathcal{V} \to Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/U(g) = \mathcal{A}_g.$$

Let $\mathcal{M}_g$ be the moduli space of closed Riemann surfaces of genus g. The *Hodge bundle*

$$\Pi : \mathcal{H} \to \mathcal{M}_g$$

is the pullback of the holomorphic bundle $\mathcal{V} \to \mathcal{A}_g$ under the *Torelli map*

$$\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g = Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/U(g).$$

As the Torelli map is holomorphic, $\mathcal{H}$ is a g-dimensional holomorphic Hermitean vector bundle on $\mathcal{M}_g$ (in the orbifold sense). Its fibre over $x \in \mathcal{M}_g$ can be identified with the vector space of holomorphic one-forms on x. The Hermitean inner product on $\mathcal{H}$ is given by

$$(\omega,\zeta) = \frac{i}{2}\int \omega \wedge \overline{\zeta}.$$

With this interpretation, the sphere bundle in $\mathcal{H}$ for the inner product $(,)$ is just the moduli space of area one abelian differentials.

Each point $x \in \mathcal{M}_g$ determines a complex structure J_x on the first real cohomology $H^1(S,\mathbb{R})$. Namely, every cohomology class $\alpha \in H^1(S,\mathbb{R})$ can be represented by a unique harmonic one-form for the complex structure x, and this one-form is the real part of a unique holomorphic one-form ω on x. The imaginary part of this holomorphic one-form is a harmonic one-form which represents the cohomology class $J_x\alpha$. The complex structure J_x is *compatible* with the intersection form ι, i.e. we have

$$\iota(J_x\alpha, J_x\beta) = \iota(\alpha,\beta) \text{ for all } \alpha,\beta \in H^1(S,\mathbb{R}).$$

With this interpretation, the flat vector bundle over $\mathcal{M}_g$ whose fibre at any point equals the complex cohomology group $H^1(S,\mathbb{C}) = H^1(S,\mathbb{R})\otimes\mathbb{C}$ can be decomposed as

$$H^1(S,\mathbb{C}) = \mathcal{H} \oplus \overline{\mathcal{H}}$$

where the *holomorphic* bundle $\mathcal{H} = \{\alpha + iJ\alpha \mid \alpha \in H^1(S,\mathbb{R})\}$ admits a natural identification with the bundle of holomorphic one-forms on $\mathcal{M}_g$, i.e. $\mathcal{H}$ is just the Hodge bundle over $\mathcal{M}_g$.

As a real vector bundle, the bundle $\mathcal{H}$ is isomorphic to the bundle with fibre $H^1(S,\mathbb{R})$. Via the identification $\mathcal{H} \sim H^1(S,\mathbb{R})$ as real vector bundles, we can view the Gauss Manin connection as a flat connection on $\mathcal{H}$. This bundle $\mathcal{H}$ is equipped with a complex structure J which varies real analytically with the basepoint.

Denote by $\mathcal{H}_+ \subset \mathcal{H}$ the complement of the zero section in the Hodge bundle $\mathcal{H}$. This is a complex orbifold. The pull-back

$$\Pi^*\mathcal{H} \to \mathcal{H}_+$$

to $\mathcal{H}_+$ of the Hodge bundle on $\mathcal{M}_g$ is a holomorphic vector bundle on $\mathcal{H}_+$. The pull-back of the Gauss-Manin connection is a flat connection on $\Pi^*\mathcal{H}$. In the sequel we simply write $\mathcal{H}$ for the bundle $\Pi^*\mathcal{H}$ whenever what is meant is clear from the context. This is precisely the notation we used in the previous sections, where we also used the Gauss Manin connection on $\Pi^*\mathcal{H}$.

Let $\mathcal{Q}_+ \subset \mathcal{H}_+$ be a component of a stratum of abelian differentials. We use the notation $\mathcal{Q}_+$ to indicate that unlike in the previous sections, we do not normalize the area of an abelian differential. Then $\mathcal{Q}_+$ is a complex suborbifold of $\mathcal{H}_+$. Period coordinates for $\mathcal{Q}_+$ define the complex structure.

The component $GL^+(2,\mathbb{R})$ of the identity of the full linear group $GL(2,\mathbb{R})$ acts on $\mathcal{H}$, and this action preserves $\mathcal{Q}_+$. An *affine invariant manifold* $\mathcal{C}_+$ in $\mathcal{Q}_+$ is the closure in $\mathcal{Q}_+$ of an orbit of the $GL^+(2,\mathbb{R})$-action. Such a manifold is complex affine in period coordinates [EMM15]. In particular, $\mathcal{C}_+ \subset \mathcal{Q}_+$ is a complex suborbifold. Period coordinates determine a projection

$$p : T\mathcal{C}_+ \to \mathcal{H} \oplus \overline{\mathcal{H}} = H^1(S,\mathbb{C})$$

to absolute periods (see [W14] for a clear exposition). The image $p(T\mathcal{C}_+)$ is a flat subbundle of the flat complex vector bundle $\mathcal{H} \oplus \overline{\mathcal{H}}|_{\mathcal{C}_+} = H^1(S,\mathbb{C})|_{\mathcal{C}_+}$. The map p is compatible with respect to the complex structure on $\mathcal{C}_+$ defined by period coordinates and the constant complex structure on $H^1(S,\mathbb{C})$.

By the main result of [F16], there is a *holomorphic* subbundle $\mathcal{Z}$ of $\mathcal{H}|_{\mathcal{C}_+}$ such that

$$p(T\mathcal{C}_+) = \mathcal{Z} \oplus \overline{\mathcal{Z}}.$$

We call $\mathcal{Z}$ the *absolute holomorphic tangent bundle* of $\mathcal{C}_+$. In particular, the bundle $p(T\mathcal{C}_+)$ is invariant under the complex structure.

As a real vector bundle, $\mathcal{Z}$ is isomorphic to $p(T\mathcal{C}_+) \cap H^1(S,\mathbb{R})$. This bundle is invariant under the compatible complex structure, in particular $\mathcal{Z} \subset H^1(S,\mathbb{R})|_{\mathcal{C}_+}$ is symplectic. Moreover, the bundle is flat, i.e. it is invariant under the restriction of the Gauss Manin connection to $\mathcal{C}_+$ [F16].

Define the *rank* of the affine invariant manifold $\mathcal{C}_+$ as

$$\mathrm{rk}(\mathcal{C}_+) = \frac{1}{2}\dim_{\mathbb{C}} p(T\mathcal{C}_+) = \dim_{\mathbb{C}}\mathcal{Z}.$$

With this definition, components of strata are affine invariant manifolds of rank g.

Every component $\mathcal{Q}$ of a stratum in the bundle of area one abelian differentials which consists of differentials with at least two zeros admits a foliation $\mathcal{AP}(\mathcal{Q})$ whose leaves locally consist of differentials with the same absolute periods. This foliation is called the *absolute period foliation* (we adopt this terminology from [McM13], other authors call it the relative period foliation). The leaves of this foliation admit a complex affine structure (see e.g. [McM13]).

If $\mathcal{C} \subset \mathcal{Q}$ is an affine invariant manifold whose dimension is strictly bigger than its rank then $\mathcal{C}$ intersects the leaves of the absolute period foliation of $\mathcal{Q}$ nontrivially. This fact alone does not imply that $\mathcal{C} \cap \mathcal{AP}(\mathcal{Q})$ is a foliation of $\mathcal{C}$. Our next goal is to establish some structural results on $\mathcal{C} \cap \mathcal{AP}(\mathcal{Q})$.

We begin with collecting some more detailed information on the absolute period foliation of the stratum $\mathcal{Q}$. Its tangent bundle $T\mathcal{AP}(\mathcal{Q})$ has an explicit description via so-called *Schiffer variations* [McM13].

Let first ω be an abelian differential with a simple zero p. There are four horizontal separatrices at p for the flat metric defined by ω. In a complex coordinate z near p so that $\omega = (z/2)dz$, the horizontal separatrices are the four rays contained in the real or the imaginary axis. The restriction of ω to these rays defines an orientation on the rays. With respect to this orientation, the two rays contained in the real axis are outgoing from p, while the rays contained in the imaginary axis are incoming. The *Schiffer variation* with weight one at p is the tangent at ω of a deformation obtained by cutting S open along the vertical axis at p and refold so that the singular point p slides backwards along the incoming rays in the imaginary axis. We refer to [McM13] for a more detailed description.

If ω has a zero of order $n \geq 2$ at p then the Schiffer variation at p is defined as follows (see p.1235 of [McM13]). Choose a coordinate z near p so that $\omega = (z^n/2)dz$ in this coordinate. This choice of coordinate is unique up to multiplication with $e^{\ell 2\pi i/(n+1)}$ for some $\ell \leq n$. There are $n+1$ horizontal separatrices at p for the flat metric defined by ω whose orientations point towards p. For small $u > 0$ cut the surface S open along the initial subsegments of length $2u$ of these $n+1$ segments. The result is a $2n+2$-gon which we refold as in the case of a simple zero. As before, we call the tangent at ω of this deformation the Schiffer variation with weight one at p.

Now let $\mathcal{Q}$ be a stratum of abelian differentials consisting of differentials with $k \geq 1$ zeros. By passing to a finite cover we may assume that the zeros are *numbered.* For $\omega \in \mathcal{Q}$ let $Z(\omega)$ be the set of numbered zeros of ω. Let moreover $V(\omega) \sim \mathbb{C}^k$ be the complex vector space freely generated by the set $Z(\omega)$. Then the tangent space $T\mathcal{AP}(\mathcal{D})$ of the absolute period foliation of $\mathcal{D}$ at ω is naturally isomorphic to the hyperplane in $V(\omega)$ of all points whose coordinates sum up to zero [McM13, H15], i.e. of points with zero mean.

More explicitly, let $\mathfrak{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ be any k-tuple of *real* numbers with $\sum_i a_i = 0$. Then $\mathfrak{a}$ defines a smooth vector field $X_\mathfrak{a}$ on $\mathcal{Q}$ as follows. For each $\omega \in \mathcal{D}$, the value of $X_\mathfrak{a}$ at ω is the Schiffer variation for the tuple $(a_1, \dots, a_k)$ of speed parameters at the numbered zeros of ω. Thus $X_\mathfrak{a}$ is tangent to the absolute period foliation. The real vector space of dimension $k-1$ spanned by these vector fields is the tangent bundle of the *real rel foliation* $\mathcal{R}$ which is the intersection of the absolute period foliation with the *strong unstable foliation* W^{su} of $\mathcal{Q}$. The leaf of this foliation through $q \in \mathcal{Q}$ locally consists of all differentials with the same horizontal measured geodesic lamination as q. We refer again to [H15] for references. Denote by $\Lambda^t_{\mathfrak{a}}$ the flow defined by the Schiffer variation $X_{\mathfrak{a}}$ for the weight $\mathfrak{a}$. The flow lines for this flow are contained in the leaves of $\mathcal{R}$.

Similarly, we define the *imaginary rel foliation* of $\mathcal{Q}$ to be the intersection of the absolute period foliation with the *strong stable foliation* W^{ss} of $\mathcal{Q}$. The leaf of the foliation W^{ss} through q locally consists of all differentials with the same vertical measured geodesic lamination as q. Exchanging the roles of the horizontal and the vertical foliation in the definition of the Schiffer variations identifies the tangent bundle of the imaginary rel foliation of $\mathcal{Q}$ with the purely imaginary weight vectors of zero mean on the numbered zeros of the differentials in $\mathcal{Q}$. As the tangent bundle of the absolute period foliation is spanned by its intersection with the tangent bundle of the strong stable and the strong unstable foliation, mapping a real weight vector to its multiple with $i = \sqrt{-1}$ defines a natural almost complex structure J on $T\mathcal{AP}(\mathcal{Q})$. This almost complex structure is in fact integrable [McM13].

The Teichmüller flow Φ^t preserves the absolute period foliation. The following is Lemma 2.2 of [H15].

Lemma 5.1. *$d\Phi^t X_{\mathfrak{a}} = e^t X_{\mathfrak{a}}$ and $d\Phi^t X_{i\mathfrak{a}} = e^{-t} X_{i\mathfrak{a}}$ for every $\mathfrak{a} \in \mathbb{R}^k$ with zero mean.*

We observe next that an affine invariant submanifold $\mathcal{C}$ of $\mathcal{Q}$ intersects the absolute period foliation of $\mathcal{Q}$ in a real analytic foliation $\mathcal{AP}(\mathcal{C})$ with complex affine leaves. For the formulation, we denote by $\hat{\mathcal{Q}}$ a finite cover of $\mathcal{Q}$ on which the zeros of the differentials are numbered.

Lemma 5.2. *Let $\mathcal{C}$ be an affine invariant submanifold of $\hat{\mathcal{Q}}$ with $r = \dim(\mathcal{C}_+) - 2\mathrm{rk}(\mathcal{C}) > 0$. Then $\mathcal{C}$ intersects the real rel foliation in a real analytic foliation of real dimension r. Furthermore, if $q \in \mathcal{C}$ and $\mathfrak{a} \in \mathbb{R}^k$ are such that $X_{\mathfrak{a}}(q) \in T\mathcal{AP}(\mathcal{C})$ then $X_{\mathfrak{a}}(z) \in T\mathcal{AP}(\mathcal{C})$, $X_{i\mathfrak{a}}(z) = JX_{\mathfrak{a}}(z) \in T\mathcal{AP}(\mathcal{C})$ for every $z \in \mathcal{C}$.*

Proof. Let $\mathcal{C} \subset \hat{\mathcal{Q}}$ be an affine invariant manifold. For the purpose of the lemma, we may assume that

$$r = \dim(\mathcal{C}_+) - 2\mathrm{rk}(\mathcal{C}) > 0.$$

Then for each $q \in \mathcal{C}$ there is a vector $X \in T_q\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to $\mathcal{C}$. By invariance of $\mathcal{C}$ under the Teichmüller flow, we have $d\Phi^t(X) \in T\mathcal{AP}(\hat{\mathcal{Q}}) \cap T\mathcal{C}$ for all t.

A vector $X \in T\mathcal{C} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ decomposes as $X = X^u + X^s$ where $X^u \in T\mathcal{AP}(\hat{\mathcal{Q}})$ is real (and hence tangent to the strong unstable foliation) and X^s is imaginary (and hence tangent to the strong stable foliation). We claim that we can find a vector $Y \in T\mathcal{C} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which either is tangent to the strong unstable or to the strong stable foliation. To this end we may assume that $X^u \neq 0$. Since this is an open condition and since the Teichmüller flow on $\mathcal{C}$ is topologically transitive, we may furthermore assume that the Φ^t-orbit of the footpoint q of X is dense in $\mathcal{C}$. Then there is a sequence $t_i \to \infty$ such that $\Phi^{t_i}(q) \to q$.

Choose any smooth norm $\| \|$ on $T\hat{\mathcal{Q}}$. As $X \neq 0$, up to passing to a subsequence,

$$d\Phi^{t_i}(X)/\|d\Phi^{t_i}(X)\|$$

converges to a vector $Y \in T_q\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to the strong unstable foliation. As the bundle $T\mathcal{C} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ is a smooth $d\Phi^t$-invariant subbundle of the restriction of the tangent bundle of $\hat{\mathcal{Q}}$ to $\mathcal{C}$, we have $Y \in T\mathcal{C} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which is what we wanted to show.

Using Lemma 5.1 and density of the Φ^t-orbit of q, if $0 \neq \mathfrak{a} \in \mathbb{R}^k$ is a vector of zero mean such that $Y = X_{\mathfrak{a}}(q)$ then $X_{\mathfrak{a}}(u) \in T\mathcal{C}$ for all $u \in \mathcal{C}$. As a consequence, $\mathcal{C}$ is invariant under the flow $\Lambda^t_{\mathfrak{a}}$.

By invariance of $T\mathcal{C}_+$ under the complex structure J, if $r = 1$ then

$$T\mathcal{C} \cap T\mathcal{AP}(\hat{\mathcal{Q}}) = \mathbb{R}X_{\mathfrak{a}} \oplus J\mathbb{R}X_{\mathfrak{a}}$$

and we are done. Otherwise there is a tangent vector $Y \in T\mathcal{C} \cap T\mathcal{AP}(\hat{\mathcal{Q}}) - \mathbb{C}X_{\mathfrak{a}}$. Apply the above argument to Y, perhaps via replacing the Teichmüller flow by its inverse. In finitely many such steps we conclude that there is a smooth subbundle $\mathcal{B}$ of $T\mathcal{C} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to the strong unstable foliation (i.e. real for the real structure), of rank r, and such that $T\mathcal{C} \cap T\mathcal{AP}(\hat{\mathcal{Q}}) = \mathbb{C}\mathcal{B}$. Moreover, if $\omega \in \mathcal{C}$ and if $\mathfrak{a} \in \mathbb{R}^k$ is such that $X_{\mathfrak{a}}(\omega) \in \mathcal{B}$ then $X_{\mathfrak{a}}(q) \in \mathcal{B}$ for every $q \in \mathcal{C}$. As a consequence, $\mathcal{C}$ is invariant under the flow $\Lambda^t_{\mathfrak{a}}$. The same argument applied to the imaginary subbundle $i\mathcal{B}$ of $T\mathcal{C} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ yields the statement of the lemma. □

Recall the definition of the foliation W^{ss}, W^{su} of $\mathcal{Q}$. As a corollary, we obtain

Corollary 5.3. *Let $\mathcal{C}$ be an affine invariant manifold with $\dim_{\mathbb{C}}(\mathcal{C}_+) = r$. Then $\mathcal{C} \cap W^i$ is a smooth foliation of $\mathcal{C}$ into leaves of real dimension $r - 1$.*

Proof. We show the statement of the corollary for the foliation W^{su}, the statement for the foliation W^{ss} is completely analogous.

By Lemma 5.2, the intersection of $\mathcal{C}$ with a leaf of W^{su} is foliated into leaves of the real Rel foliation of real dimension $r - 2\mathrm{rk}(\mathcal{C}_+) = m$.

The image of the projection $p : T\mathcal{C}_+ \to H^1(S, \mathbb{C})$ is a flat complex subbundle of the flat bundle $H^1(S, \mathbb{C})|\mathcal{C}_+$ which is invariant under multiplication with i. This shows that for a differential $z \in \mathcal{C}_+$ near q, the set of all differentials in $\mathcal{C}_+$ whose absolute periods coincide with the absolute periods of z and whose imaginary parts coincide with the imaginary part of z is a submanifold of $\mathcal{C}_+$ of dimension m. The intersection of $\mathcal{C}_+$ with the leaf of the strong unstable foliation is the union of these submanifolds over all points with the property that the imaginary part of the absolute period coincides with the imaginary part of the absolute period of q. From this the corollary follows. □

6. Local Zariski density for affine invariant manifolds

The goal of this section is to prove a weaker analogue of the first part of Proposition 3.4 for affine invariant manifolds. Throughout this section we assume that $g \geq 3$. We use the assumptions and notations from Section 5.

Let $\mathcal{Q}_+ \subset \mathcal{H}_+ \mathcal{C}_+$ Recall from Section 5 that the image of the projection $p : T\mathcal{C}_+ \to H^1(S,\mathbb{C})$ to absolute periods is a flat subbundle $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ of $H_1(S,\mathbb{C})|\mathcal{C}_+$ which is invariant under the complex structure. We denote by $\ell \geq 1$ its complex dimension. Then $p(T\mathcal{C}_+) \cap H^1(S,\mathbb{R})$ is a symplectic subspace of dimension 2ℓ which can be identified with the holomorphic bundle $\mathcal{Z}$ as a real vector bundle.

Definition 6.1. The *monodromy group* of the affine invariant invariant manifold $\mathcal{C}_+$ of rank ℓ is the subgroup of $Sp(2\ell,\mathbb{R})$ which is the monodromy of the bundle $\mathcal{Z}$ for the restriction of the Gauss Manin connection.

Example 6.2. If $\mathcal{C}_+$ is a Teichmüller curve then the monodromy group of $\mathcal{C}_+$ is just the Veech group of $\mathcal{C}_+$. Thus this monodromy group is a lattice in $Sp(2,\mathbb{R}) = SL(2,\mathbb{R})$, in particular it is Zariski dense in $SL(2,\mathbb{R})$.

The monodromy group of a component of a stratum is a subgroup of $Sp(2g,\mathbb{Z})$.

The goal of this section is to show

Theorem 6.3. *The monodromy group of an affine invariant manifold $\mathcal{C}_+$ of rank ℓ is a Zariski dense subgroup of $Sp(2\ell,\mathbb{R})$.*

We begin with summarizing a result of Wright [W15]. He introduced the following two deformations of a translation surface (X,ω) (i.e. a Riemann surface X equipped with a holomorphic one-form ω).

Recall that the *horocycle flow* is defined as part of the $SL(2,\mathbb{R})$-action,

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \subset SL(2,\mathbb{R}),$$

and the *vertical stretch* is defined by

$$a_t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \subset GL^+(2,\mathbb{R}).$$

For a collection $\mathcal{Y}$ of horizontal cylinders on a translation surface X, define the *cylinder shear* $u_t^{\mathcal{Y}}(X)$ to be the translation surface obtained by applying the horocycle flow to the cylinders in $\mathcal{Y}$ but not to the rest of X. Similarly, the *cylinder stretch* $a_t^{\mathcal{Y}}(X)$ is obtained by applying the vertical stretch only to the cylinders in $\mathcal{Y}$.

The following lemma is a fairly easy consequence of the work of Wright [W15]. For its formulation, a translation surface (X,ω) is called *horizontally periodic* if it is a union of horizontal cylinders.

Lemma 6.4. *Let $\mathcal{C}_+$ be an affine invariant manifold of rank ℓ. Suppose that $(X,\omega)\in\mathcal{C}_+$ is horizontally periodic, and that there is a decomposition of X into ℓ collections $\mathcal{Y}_1,\dots,\mathcal{Y}_\ell$ of horizontal cylinders so that for each i, the cylinder shear $u_t^{\mathcal{Y}_i}(X)$ remains in $\mathcal{C}_+$. Then for each i, the moduli of the cylinders in the collection $\mathcal{Y}_i$ are rationally dependent.*

Proof. For each i, the collection $\mathcal{Y}_i$ consists of $r_i\geq 1$ cylinders. By assumption, the cylinder shear $u_t^{\mathcal{Y}_i}(X)$ remains entirely in $\mathcal{C}_+$. A local version of Lemma 3.1 of [W15], applied to this cylinder shear rather than to the full horocycle flow (note that the proof of this local version is identical to the proof given in Section 3 of [W15]) shows the following. If the moduli $m_1^i,\dots,m_{r_i}^i$ of the cylinders in $\mathcal{Y}_i$ are not rationally dependent then there is a proper subcollection $\mathcal{V}$ of $\mathcal{Y}_i$ consisting of $1\leq s<r_i$ cylinders so that the cylinder shear $u_t^{\mathcal{V}}$ for this subcollection is contained in $\mathcal{C}_+$.

But then there are at least $\ell+1$ pairwise distinct collections of horizontal cylinders in (X,ω) with the property that the cylinder shear of X for each of these collections is contained in $\mathcal{C}_+$ (see Section 3 of [W15] for details). This violates Theorem 1.10 of [W15] and yields that indeed, for fixed i the moduli m_j^i $(1\leq j\leq r_i)$ of the cylinders in $\mathcal{Y}_i$ are rationally dependent. □

Define a *piecewise affine automorphism* of a translation surface (X,ω) to be a continuous self-map $F:X\to X$ with the property that there exists a decomposition $X=\cup_i X_i$ into finitely many components with geodesic boundary which is preserved by F and such that the restriction of F to each of these components is affine. In contrast to an affine automorphism of (X,ω), we allow that the restriction of F to some of the components X_i equals the identity.

The main consequence of Lemma 6.4 we are going to use is the following

Corollary 6.5. *Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell\geq 2$. Then there is a horizontally periodic surface $(X,\omega)\in\mathcal{C}_+$ which admits an abelian group D of rank ℓ of piecewise affine automorphisms. The group D acts on S as a group of Dehn-multitwists, and it acts on $H^1(S,\mathbb{R})$ as a group of transvections of rank ℓ.*

Proof. By Theorem 1.10 of [W15] and its proof (more precisely the results in Section 8 of [W15]), the affine invariant manifold $\mathcal{C}_+$ contains a horizontally periodic surface (X,ω) which admits a decomposition into ℓ cylinder families $\mathcal{Y}_1,\dots,\mathcal{Y}_\ell$ with the properties stated in Lemma 6.4. Moreover, for each i and each t the image of X under the vertical stretch $a_t^{\mathcal{Y}_i}(X)$ is contained in $\mathcal{C}_+$. These vertical stretches commute.

The vertical stretch $a_t^{\mathcal{Y}_i}$ changes the heights of the horizontal cylinders in the family $\mathcal{Y}_i$ while keeping their circumferences fixed. The image translation surfaces are horizontally periodic. Using Lemma 6.4, this implies that we can find $t_1,\dots,t_\ell\in\mathbb{R}$ so that the modulus of *every* horizontal cylinder in

$$Z=a_{t_1}^{\mathcal{Y}_1}\cdots a_{t_\ell}^{\mathcal{Y}_\ell}(X)$$

is rational.

Using again the results in Section 8 of [W15], the affine invariant manifold $\mathcal{C}_+$ contains the images of the translation surface Z under the cylinder shears $u_t^{\mathcal{Y}_i}(Z)$ where by abuse of notation, we denote again by $\mathcal{Y}_i$ the cylinder family on Z which is the image of the horizontal cylinder family $\mathcal{Y}_i$ on (X,ω). As the moduli of all cylinders in the family $\mathcal{Y}_i$ are rational, these cylinder shears are eventually periodic. This means that for each i there exists some number $r_i > 0$ such that for some fixed marking of Z, the surface $u_{r_i}^{\mathcal{Y}_i}(Z)$ is the image of Z by a Dehn multitwist T_i about the core curves of the cylinders in $\mathcal{Y}_i$.

Since the core curves of the horizontal cylinders in Z are pairwise disjoint, the Dehn multitwists T_i commute. Therefore these multitwists generate an abelian group of rank ℓ of piecewise affine automorphisms of Z. The multitwist T_i acts as a transvection on $H_1(S,\mathbb{R})$ by some multiple of the homology class defined by the family $\mathcal{Y}_i$. The proposition now follows from another application of Theorem 1.10 of [W15]: The rank of the subspace of $H_1(S,\mathbb{R})$ spanned by the homology classes of the cylinder families $\mathcal{Y}_i$ $(i \leq \ell)$ equals ℓ. □

The following result is the analogue of the first part of Corollary 3.5 for affine invariant manifolds. For its formulation, we define as in Section 3 an affine invariant manifold $\mathcal{C}_+$ of rank ℓ to be *locally Zariski dense* if for every open contractible subset U of $\mathcal{C}_+$ the subsemigroup of $Sp(2\ell,\mathbb{R})$ generated by the monodromy of those periodic orbits for Φ^t in $\mathcal{C}_+$ which pass through U is Zariski dense in $Sp(2\ell,\mathbb{R})$. Note that this makes sense because the bundle $\mathcal{Z} \to \mathcal{C}_+$ is flat and therefore admits a family of canonical symplectic trivializations over the contractible set U. Replacing one of these trivializations by another one changes the local monodromy group by a conjugation.

Proposition 6.6. *An affine invariant manifold is locally Zariski dense.*

Proof. Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 1$, and let $\mathcal{C} \subset \mathcal{C}_+$ be its subset of differentials of area one. We use an argument similar to the proof of Proposition 3.4 (compare also the proof of Theorem 5.1 of [W14] for an argument along these lines). Following the reasoning in the proof of Proposition 3.4, it suffices to show the existence of a single birecurrent point $q \in \mathcal{C}$ with the following property. For every open neighborhood U of q, the subgroup of $SL(2\ell,\mathbb{R})$ generated by the monodromoy of those periodic orbits for Φ^t in $\mathcal{C}$ which pass through U is Zariski dense in $SL(2\ell,\mathbb{R})$.

To this end choose a translation surface $(X,\omega) \in \mathcal{C}$ with the properties stated in Corollary 6.5. Denote by H the abelian group of rank ℓ of Dehn multitwists which is contained in the group of piecewise affine automorphisms of X.

Choose a contractible neighborhood U of ω in $\mathcal{C}$ which is small enough that the bundle $\mathcal{Z}|U$ is trivial as a symplectic vector bundle, with a real analytic trivialization. Let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of area one abelian differentials and let $\tilde{U}$ be a lift of U to $\tilde{\mathcal{C}}$. We may assume that the canonical projection $\tilde{U} \to U$ is a homeomorphism.

By Corollary 5.3, every point $q \in \mathcal{C}$ admits a neighborhood with a *local product structure*. This means the following. Let Σ be the set of zeros of a differential in $\mathcal{Q}$. By possibly decreasing the size of U we may assume that there are two disjoint compact subsets D,K of the set of (marked) projective measured geodesic laminations on S, viewed as points in $H^1(S,\Sigma)$ via intersection, with the following property. The sets D,K are homeomorphic to closed balls of dimension $h=\dim_{\mathbb{C}}(\mathcal{C}_+)-1$. Furthermore, there is a continuous map $\omega: D\times K\to \tilde U$ such that for any pair $(\xi,\nu)\in D\times K$, the vertical projective measured geodesic lamination of $\omega(\xi,\nu)$ equals ξ and its horizontal projective measured geodesic lamination equals ν. Moreover, there is some $\epsilon>0$ such that

$$\tilde V=\cup_{-\epsilon\leq t\leq \epsilon}\cup_{(\mu,\nu)\in D\times K}\Phi^t\omega(\mu,\nu)$$

is a compact subset of $\tilde U$ whose dense interior contains the lift $\tilde\omega$ of ω to $\tilde U$.

The cylinder shears of the translation surface (X,ω) which are used to construct the Dehn multi-twists T_i preserve the horizontal projective measured lamination of ω, but they modify the vertical projective measured lamination. These cylinder shears define ℓ smooth paths c_i in $\mathcal{C}$ which lift to smooth paths $\tilde c_i$ in $\tilde{\mathcal{C}}$ beginning at the preimage $\tilde\omega$ of ω in U and connecting $\tilde\omega$ to $T_i\tilde\omega$.

Fix $i\leq \ell$ and write $T=T_i$ and $\tilde c=\tilde c_i$ for simplicity. Recall that the set D is homeomorphic to a compact ball of dimension h, and the same holds true for TD. Furthermore, $D\cap TD$ contains the projective class μ of the horizontal projective measured lamination of $\tilde\omega$. We claim that there is an open neighborhood E of μ in D which is contained in $D\cap TD$.

To this end cover the compact path $\tilde c$ by finitely many open subsets U_i ($i=0,\dots,k$) of $\tilde{\mathcal{C}}$ whose closures $\overline{U_i}$ have a product structure as described above. These product structures are defined by compacts sets D_i,K_i in the space of projective measured geodesic laminations. For each i, the set D_i is homeomorphic to an h-dimensional compact ball and coincides with the set of all vertical projective measured laminations of all points in $\overline{U_i}$.

Let i,j be such that $U_i\cap U_j\cap\tilde c\neq\emptyset$. Then $D_i\cap D_j$ contains the projective class μ of the horizontal measured geodesic lamination of $\tilde\omega$. Now $U_i\cap U_j$ is open and therefore locally near a point $x\in U_i\cap U_j\cap\tilde c$, period coordinates define a local product structure on a neighborhood E of x of the form described above, determined by a pair of subsets of the space of projective measured geodesic laminations which are homeomorphic to compact balls of dimension h. These sets contain all projective classes of vertical and horizontal measured geodesic laminations, respectively, of points in E. As $E\subset U_i\cap U_j$, this implies that $D_i\cap D_j$ contains a compact neighborhood of μ in D, and by induction, the same holds true for $\cap_i D_i$. In particular, there is a compact neighborhood $R\subset D\cap TD$ of μ which is homeomorphic to a compact ball of dimension h. Similarly, by making K smaller we may assume that the sets K and TK are disjoint.

Let $\tilde V=\cup_{-\epsilon\leq t\leq\epsilon}\cup_{(\xi,\nu)\in R\times K}\Phi^t\omega(\xi,\nu)\subset\tilde U$. Let $V\subset U$ be the projection of $\tilde V$ into $\mathcal{C}$. Choose a periodic orbit γ for Φ^t which is generated by an abelian differential q contained in the interior of V. Let $\tilde q$ be the preimage of q in $\tilde V$. We may assume that the vertical projective measured lamination ζ_1 of $\tilde q$ as well as its image under T are contained in the interior of R. We use the birecurrent point q as a starting point for the construction of a set Γ_0 of periodic orbits passing through V as in

Proposition 3.1. Denote by $\Omega(\Gamma_0)$ the corresponding sub-semigroup of $\mathrm{Mod}(S)$ and let $G < \mathrm{Mod}(S)$ be the subgroup generated by $\Omega(\Gamma_0)$.

We claim that G contains the Dehn multitwist $T = T_i$. Since G is a group, this is the case if we can find a pseudo-Anosov element $\varphi \in \Gamma_0$ such that $T\varphi \in \Gamma_0$.

We establish this fact using a fixed point argument for the action of $\mathrm{Mod}(S)$ on the sphere of projective measured geodesic laminations which is motivated by the argument in the proof of Proposition 5.4 of [H13]. Let φ be the pseudo Anosov mapping class which satisfies $\varphi(\tilde{q}) = \Phi^\tau(\tilde{q})$ where $\tau > 0$ is the period of γ. The projective measured geodesic lamination $\zeta_1 \in R$ is the attracting fixed point for the action of the map φ on the space of projective measured geodesic laminations of S. As φ preserves the component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ containing $\tilde{q}$, by possibly replacing φ by a large power we may assume that $\varphi(D)$ is contained in the interior of R and that the same holds true for $T\varphi(D)$. Recall to this end that $T\zeta_1$ is contained in the interior of R by assumption. Similarly, we may assume that $\varphi^{-1}T^{-1}K$ is contained in the interior of K.

To summarize, the mapping class $\psi = T \circ \varphi$ maps the compact ball D into its interior. Therefore it has an attracting fixed point in the interior of D. Similarly, it has a repelling fixed point in the interior of K. As ψ is pseudo-Anosov, it acts on the space of projective measured geodesic laminations with north-south dynamics. As a consequence, the fixed points of ψ in D, K, respectively, are the attracting and repelling measured geodesic laminations of ψ, and the periodic orbit defined by ψ is contained in $\mathcal{C}$.

As this argument is valid for each $i \leq \ell$ and we can choose the pseudo-Anosov mapping class φ such that its attracting measured geodesic lamination is arbitrarily close to μ, we conclude that for each i there is a pseudo-Anosov mapping class $\varphi_i \in \Omega(\Gamma_0)$ such that $T_i\varphi \in \Omega(\Gamma_0)$ for all i. Since G is a group, we deduce that $T_i \in G$ for all i and hence $H < G$ as claimed.

The subgroup $\Psi(H) < Sp(2\ell, \mathbb{R})$ is an abelian group generated by ℓ transvections. The intersection of their fixed sets intersects $\mathcal{Z} = p(T\mathcal{C}_+) \cap H^1(S, \mathbb{R})$ in a Lagrangian linear subspace A_1 of $\mathcal{Z}$.

Recall that the bundle $\mathcal{Z}|U$ is equipped with a fixed trivialization. Let again $\tilde{U}$ be a lift of U to $\tilde{\mathcal{C}}$. For $\tilde{q} \in \tilde{U}$, the real part $\mathrm{Re}(\tilde{q})$ of $\tilde{q}$ is a harmonic one-form for the complex structure underlying $\tilde{q}$ which defines a cohomology class $[\mathrm{Re}(\tilde{q})] \in \mathcal{Z} \subset H^1(S, \mathbb{R})$. As $\tilde{q}$ varies in $\tilde{U}$ these cohomology classes vary through an open subset of $\mathcal{Z} \sim \mathbb{R}^{2\ell}$.

Denote by $[c_i] \in H_1(S, \mathbb{Z})$ the homology class of the oriented weighted multicurve c_i which determines the Dehn multi-twist T_i. The evaluation of the real part of the marked abelian differential $\tilde{\omega}$ on each of the classes $[c_i]$ is positive. As this is an open condition, by decreasing the size of U we may assume that $\langle [\mathrm{Re}(\tilde{q})], [c_i] \rangle \neq 0$ for all i and all $\tilde{q} \in \tilde{U}$ where $\langle , \rangle$ is the natural pairing $H^1(S, \mathbb{R}) \times H_1(S, \mathbb{R}) \to \mathbb{R}$. Now periodic orbits for Φ^t are dense in $\mathcal{C}$ and therefore we can find a periodic point $z \in U$ with preimage $\tilde{z} \in \tilde{U}$. By assumption on $\tilde{U}$ we have $\langle [\mathrm{Re}(\tilde{z})], [c_i] \rangle \neq 0$ for all i.

Let $\varphi \in \mathrm{Mod}(S)$ be a pseudo Anosov element whose cotangent line passes through $\tilde{z}$. There is a number $\kappa > 1$ such that $\varphi^*\mathrm{Re}(\tilde{z}) = \kappa\mathrm{Re}(\tilde{z})$, moreover κ is the Perron Frobenius eigenvalue for the action of φ on $H_1(S,\mathbb{R})$. By invariance of the natural pairing $\langle , \rangle$ under φ, as $k \to \infty$ the homology classes $\varphi^k([c_i])$ converge up to rescaling to a class $u \in H_1(S,\mathbb{R})$ whose contraction with the intersection form defines $\pm[\mathrm{Re}(\tilde{z})]$, viewed as a linear functional on $H_1(S,\mathbb{R})$. By this we mean that $\iota(u,a) = \langle \pm[\mathrm{Re}(\tilde{z})], a\rangle$ for all $a \in H_1(S,\mathbb{R})$.

As a consequence, for sufficiently large j and all i,ℓ we have $\iota([\varphi^j c_i],[c_\ell]) \neq 0$. It now follows from the arguments in the beginning of this proof and equivariance that the group generated by $\Psi\Omega(\Gamma_0)$ contains a subgroup generated by at least $\ell+1$ transvections with integral homology classes which are independent over $\mathbb{R}$. These are the images under Ψ of the Dehn multitwists $T_i = T_{c_i}$ $(i \leq \ell)$ and images under Ψ of the Dehn multitwists $\varphi^j T_i \varphi^{-j} = T_{\varphi^j c_i}$. Moreover, $\iota([\varphi^j c_i],[c_u]) \neq 0$ for all i,u.

Let $A_2 \subset A_1$ be the common fixed set in $\mathcal{Z}$ of the transvections which are the images of the multitwists $T_i, \varphi^j T_u \varphi^{-j}$. Then A_2 is a linear subspace of A_1, and for large enough j its codimension in A_1 is $s \geq 1$. Let $i_1,\dots,i_s \subset \{1,\dots,\ell\}$ be such that the homology classes $[c_i],[\varphi^j c_{i_u}] \in H_1(S,\mathbb{Z})$ $(i \leq \ell, u \leq s)$ are independent over $\mathbb{R}$ and that the common fixed set in $\mathcal{Z}$ of the transvections defined by the corresponding Dehn multitwists is A_2. Using again the fact that the set of real parts of differentials in $\tilde{U}$ define an open subset of $p(T\mathcal{C}_+) \cap H^1(S,\mathbb{R})$, we can find some $\bar{z} \in \tilde{U}$ and some $a \in A_2$ so that $\langle [\mathrm{Re}(\bar{z})], a\rangle > 0$. As before, we may assume that $\bar{z}$ is the preimage of a periodic point. Argue now as in the previous paragraph and find a multitwist β in the group generated by $\Omega(\Gamma_0)$ so that the common fixed set of the subgroup generated by $\Psi(\beta)$ and A_2 has codimension at least one in A_2.

Repeat this construction. In at most ℓ steps we find integral homology classes $a_1,\dots,a_\ell,a_{\ell+1},\dots,a_{2\ell} \in H_1(S,\mathbb{Z})$ (where for $i \leq \ell$ the class a_i is the class of the weighted multicurve which determines T_i) with the following properties.

(1) Let $E \subset H_1(S,\mathbb{R})$ be the real vector space spanned by the classes a_i. Then the dimension of E equals 2ℓ. Each element $a \in E$ defines a linear functional on $H^1(S,\mathbb{R})$ by evaluation, and the restriction to $\mathcal{Z}$ of this linear subspace of $H^1(S,\mathbb{R})^*$ is non-degenerate. In particular, E is a symplectic subspace of $H_1(S,\mathbb{R})$.
(2) $\iota(a_j,a_i) \neq 0$ for all $i \leq \ell$, $j \geq \ell+1$.
(3) For each i the transvection $b \to b + \iota(b,a_i)a_i$ is contained in the group generated by $\Psi(\Omega(\Gamma_0))$.

By the choice of the homology classes a_i, the $(2\ell,2\ell)$-matrix $(\iota(a_i,a_j))$ whose (i,j)-entry is the intersection $\iota(a_i,a_j)$ is integral and of maximal rank. Choose a prime $p \geq 5$ so that each of the entries of $(\iota(a_i,a_j))$ is prime to p. All but finitely many primes will do. Then the reduction mod p of the matrix $(\iota(a_i,a_j))$ is of maximal rank as well. In particular, if F_p denotes the field with p elements then the reductions mod p of the homology classes a_i span a 2ℓ-dimensional symplectic subspace E_p of $H_1(S,F_p)$.

Let $L < \mathrm{Sp}(E)$ be the subgroup of the symplectic group of E which is generated by the transvections with the elements a_i. Its reduction L_p mod p acts on E_p as a group of symplectic transformations. Lemma 3.2 shows that $L_p = Sp(2\ell, F_p)$. Note that property (2) above guarantees that all conditions in Lemma 3.2 are fulfilled. Then L is a Zariski dense subgroup of the group of symplectic automorphisms of E [Lu99]. By duality and the discussion in the proof of Proposition 3.4, this just implies that $\mathcal{C}_+$ is locally Zariski dense. □

Corollary 6.7. *Let $\mathcal{C}$ be the hyperplane of area one differentials in an affine invariant manifold $\mathcal{C}_+$ of rank $\ell \geq 1$. Then for every open subset U of $\mathcal{C}$ there exists a periodic orbit γ for Φ^t through U with the following properties.*

(1) *The eigenvalues of the matrix $A = \Psi(\Omega(\gamma))|\mathcal{Z}$ are real and pairwise distinct.*
(2) *No product of two eigenvalues of A is an eigenvalue.*

Proof. By Proposition 6.6, for every small open contractible subset U of $\mathcal{C}$, the image under Ψ of the subsemigroup $\Psi(\Omega(\Gamma_0))$ defined as in Proposition 3.1 by the monodromy along periodic orbits through U is Zariski dense in $Sp(2\ell, \mathbb{R})$. The statement of the corollary is now an immediate consequence of the main result of [Be97]. □

Recall that for an affine invariant manifold $\mathcal{C}$ of rank $\ell \leq g$ the projected tangent space $p(T\mathcal{C})$ can be identified with the complexification of a symplectic subspace $\mathbb{R}^{2\ell}$ of $\mathbb{R}^{2g} = H^1(S, \mathbb{R})$. The stabilizer of this subspace is a subgroup G of $Sp(2g, \mathbb{R})$ which is isomorphic to $Sp(2\ell, \mathbb{R}) \times Sp(2(g-\ell), \mathbb{R})$.

Let $\Pi : G \to G_1 = Sp(2\ell, \mathbb{R})$ be the natural projection. Proposition 6.6 shows that $\Pi(G \cap Sp(2g, \mathbb{Z}))$ is a Zariski dense subgroup of G_1. The following consequence of this fact was communicated to me by Yves Benoist. Although it is not used for the proofs of the results stated in the introduction, we include it here since it relates affine invariant manifolds to proper subvarieties of $\mathcal{A}_g$.

Proposition 6.8. *If $\Pi(G \cap Sp(2g, \mathbb{Z}))$ is Zariski dense in $Sp(2\ell, \mathbb{R})$ then either $\Pi(G \cap Sp(2g, \mathbb{Z}))$ is a lattice in $Sp(2\ell, \mathbb{R})$ or dense.*

Proof. Using the above notations, write $G_{\mathbb{Z}} = Sp(2g, \mathbb{Z}) \cap G$ and let $F < Sp(2\ell, \mathbb{R})$ be the Zariski closure of $\Pi(G_{\mathbb{Z}})$.

The group F is defined over $\mathbb{Q}$. Namely, the set of polynomials P which vanish on $G_{\mathbb{Z}}$ is invariant under the Galois action. As a consequence, either $F_{\mathbb{Z}} = G_{\mathbb{Z}}$ is a lattice in F, or there is a nontrivial character on F defined over $\mathbb{Q}$.

Assume for contradiction that there exists a nontrivial character on F defined over $\mathbb{Q}$. Define

$$F^0 = \cap\{\ker(\chi) \mid \chi \text{ is a character on } F \text{ defined over } \mathbb{Q}\}.$$

Then $F^0 = F$ since up to multiplication with an integer, the evaluation on $G_{\mathbb{Z}}$ of a nontrivial character χ defined over $\mathbb{Q}$ has to be integral in $\mathbb{C}^*$ which is impossible. This contradiction yields that $F_{\mathbb{Z}}$ is a lattice in F.

The group $G_1 = Sp(2\ell, \mathbb{R})$ is simple, and $\Delta = \Pi(G_{\mathbb{Z}}) < G_1$ is Zariski dense. Then $\Delta < G_1$ either is discrete or dense. We have to show that if Δ is discrete then Δ is a lattice.

Thus assume that Δ is discrete. Consider the surjective homomorphism $\varphi : F \to G_1$. Its kernel K is a locally compact group which intersects the lattice $F_{\mathbb{Z}}$ in a discrete subgroup. The exact sequence

$$1 \to K \to F \to G_1 \to 1$$

induces a sequence

$$K/K \cap F_{\mathbb{Z}} \to F/F_{\mathbb{Z}} \to G_1/\varphi(F_{\mathbb{Z}}).$$

Now the Haar measure on F can locally be represented as a product of the Haar measure on the orbits of K and the quotient Haar measure. If the volume of $G_1/\varphi(F_{\mathbb{Z}})$ is infinite then this shows that the volume of $F/F_{\mathbb{Z}}$ has to be infinite. But $F_{\mathbb{Z}}$ is a lattice in F which is a contradiction. □

7. Connections on the Hodge bundle

The main goal of this section is to analyze differential geometric properties of the projected tangent bundle of an affine invariant manifold and use this is to establish a first rigidity result geared towards Theorem 2 from the introduction. We always assume that $g \geq 3$.

Recall from Section 6 that the Hodge bundle on the moduli space $\mathcal{M}_g$ of curves is the pull-back under the Torelli map of the Hermitean holomorphic (orbifold) vector bundle $\mathcal{V} \to \mathcal{A}_g$. The pull-back $\Pi^*\mathcal{H} \to \mathcal{H}_+$ to $\mathcal{H}_+$ of the Hodge bundle on $\mathcal{M}_g$ is a holomorphic vector bundle on $\mathcal{H}_+$. The Hermitean structure on $\mathcal{H}$ defines a Hermitean structure on $\Pi^*\mathcal{H}$. The bundle $\Pi^*\mathcal{H}$ naturally splits as a direct sum

$$\Pi^*\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$$

of complex vector bundles. Here the fibre of $\mathcal{T}$ over a point q is just the $\mathbb{C}$-span of q, and the fibre of $\mathcal{L}$ is the orthogonal complement of $\mathcal{T}$ for the natural Hermitean metric, or, equivalently, the orthogonal complement of $\mathcal{T}$ for the symplectic form. The complex line bundle $\mathcal{T}$ is holomorphic. Via identification of $\mathcal{L}$ with the quotient bundle $\Pi^*\mathcal{H}/\mathcal{T}$, we may assume that $\mathcal{L}$ is holomorphic. Its complex dimension equals $g - 1 \geq 2$.

The group $GL^+(2, \mathbb{R})$ acts on $\Pi^*\mathcal{H} \to \mathcal{H}_+$ as a group of bundle automorphisms. For each point $q \in \mathcal{H}_+$, the symplectic subspace of $H^1(S, \mathbb{R})$ spanned by the real and imaginary part of q is locally constant along the orbits of the $GL^+(2, \mathbb{R})$-action and hence its symplectic complement is locally constant as well. Thus

Lemma 7.1. *The restriction of the bundle $\mathcal{L}$ to the orbits of the $GL^+(2, \mathbb{R})$-action is flat.*

The space $\mathcal{H}_+$ is foliated into the orbits of the $GL^+(2, \mathbb{R})$-action. By Lemma 7.1, the Gauss-Manin connection on the flat bundle $\mathcal{H} \to \mathcal{H}_+$ restricts to a flat leafwise connection ∇^{GM} on the bundle $\mathcal{L} \to \mathcal{H}_+$. Here a leafwise connection is a connection whose covariant derivative is only defined for vectors tangent to the foliation. As the Gauss Manin connection is real analytic in period coordinates, the $GL^+(2,\mathbb{R})$-action on $\mathcal{H}_+$ is real analytic and the splitting $\Pi^*\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$ is real analytic, the leafwise connection is real analytic. This means that it is defined by a connection matrix which is real analytic in real analytic coordinates. The leafwise connection preserves the symplectic structure of $\mathcal{L}$, but there is no information on the complex structure.

For each $k \leq g-2$, the leafwise connection ∇^{GM} extends to a flat leafwise connection on the bundle $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$ whose fibre at q is the $2k$-th exterior power of the fibre of $\mathcal{L}$ at q, viewed as a real vector space.

The Hermitean holomorphic vector bundle $\Pi^*\mathcal{H}$ admits a unique *Chern connection* ∇ (see e.g. [GH78]). The Chern connection defines parallel transport of the fibres of $\mathcal{H}$ along smooth curves in $\mathcal{H}_+$. As the Chern connection is Hermitean, parallel transport preserves the metric. Viewing again $\Pi^*\mathcal{H}$ as the real vector bundle $H^1(S,\mathbb{R})$ equipped with the complex structure J, the complex structure J is parallel for ∇. In particular, parallel transport preserves J and the Hermitean metric. Since the $GL^+(2,\mathbb{R})$-orbits on $\mathcal{H}_+$ are holomorphic suborbifolds of $\mathcal{H}_+$ and the restriction of the bundle $\mathcal{T}$ to each leaf of the foliation is just the tangent bundle of the quotient $GL^+(2,\mathbb{R})/(\mathbb{R}^+ \times S^1) = \mathbf{H}^2$, by naturality the restriction of the Chern connection to the leaves of the foliation of $\mathcal{H}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action preserves the decomposition $\Pi^*\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$.

For every $k \leq g-2$, the complex structure J on $\mathcal{L}$ extends to an automorphism of the real tensor bundle $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$. The restriction of the connection ∇ to the orbits of the $GL^+(2,\mathbb{R})$-action extends to a connection on $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$ which commutes with this automorphism.

The Hermitean metric which determines the Chern connection is defined by the polarization and the complex structure. These data are real analytic in period coordinates (recall that the Torelli map is holomorphic) and consequently the connection matrix for the Chern connection in period coordinates is real analytic (see [GH78]).

To summarize, for every $k \leq g-2$, both the Chern connection and the Gauss Manin connection restrict to leafwise connections of the restriction of the bundle $\wedge^{2k}_{\mathbb{R}}\mathcal{L} \to \mathcal{H}_+$ to the orbits of the $GL^+(2,\mathbb{R})$-action, and these leafwise connections depend in a real analytic fashion on period coordinates.

The tangent bundle $\mathcal{F}$ of the foliation of $\mathcal{H}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action is a real analytic subbundle of $T\mathcal{H}_+$. By the above discussion, $\nabla - \nabla^{GM}$ defines a real analytic tensor field

$$\Xi \in \Omega(\mathcal{F}^* \otimes \mathcal{L}^* \otimes \mathcal{L})$$

where we denote by $\Omega(\mathcal{F}^* \otimes \mathcal{L}^* \otimes \mathcal{L})$ the vector space of real analytic sections of the real analytic vector bundle $\mathcal{F}^* \otimes \mathcal{L}^* \otimes \mathcal{L}$. For every $k \leq g-2$ the tensor field

$$\Xi^k \in \Omega(\mathcal{F}^* \otimes \wedge^{2k}_{\mathbb{R}}\mathcal{L}^* \otimes \wedge^{2k}_{\mathbb{R}}\mathcal{L})$$

defined as the action of $\nabla - \nabla^{GM}$ on $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$ is real analytic as well. If $\mathcal{C}_+ \subset \mathcal{H}_+$ is an affine invariant manifold of rank $2 \leq \ell \leq g-1$, with absolute holomorphic tangent bundle $\mathcal{Z}$, then the restriction of the tensor field $\Xi^{\ell-1}$ to $\mathcal{C}_+$ preserves the

J-invariant section of $\wedge_{\mathbb{R}}^{2\ell-2}\mathcal{L}$ which is defined by $p(T\mathcal{C}_+)\cap H^1(S,\mathbb{R})$. This section associates to a point q the exterior product of a normalized oriented basis of the (real) vector space $p(T\mathcal{C}_+)\cap\mathcal{L}$.

The next proposition is the key step towards Theorem 2.

Proposition 7.2. *Let $\mathcal{C}_+\subset\mathcal{Q}_+$ be an affine invariant manifold of rank $\ell\geq 3$. Then one of the following two possibilities holds true.*

1. *There are finitely many proper affine invariant submanifolds of $\mathcal{C}_+$ which contain every affine invariant submanifold of $\mathcal{C}_+$ of rank $2\leq k\leq \ell-1$.*
2. *Up to passing to a finite cover, the restriction of the bundle $\mathcal{L}\cap\mathcal{Z}$ to an open dense $GL^+(2,\mathbb{R})$-invariant subset of $\mathcal{C}_+$ admits a non-trivial $GL^+(2,\mathbb{R})$-invariant real analytic splitting $\mathcal{L}\cap\mathcal{Z}=\mathcal{E}_1\oplus\mathcal{E}_2$ into two complex subbundles whose restrictions to each orbit of the $GL^+(2,\mathbb{R})$-action are flat.*

Proof. Let $\mathcal{Q}_+\subset\mathcal{H}_+$ be a component of a stratum and let $\mathcal{C}_+\subset\mathcal{Q}_+$ be an affine invariant manifold of rank $\ell\geq 3$, with absolute holomorphic tangent bundle $\mathcal{Z}\to\mathcal{C}_+$. As before, there is a splitting

$$\mathcal{Z}=\mathcal{T}\oplus(\mathcal{L}\cap\mathcal{Z}).$$

The bundle

$$\mathcal{W}=\mathcal{L}\cap\mathcal{Z}\to\mathcal{C}_+$$

is holomorphic. It also can be viewed as a real analytic real vector bundle with a real analytic complex structure J (which is just a real analytic section of the real analytic endomorphism bundle of $\mathcal{W}$ with $J^2=-\mathrm{Id}$).

For $1\leq k\leq \ell-2$ denote by $\mathrm{Gr}(2k)\to\mathcal{C}_+$ the fibre bundle whose fibre over q is the Grassmannian of oriented $2k$-dimensional real subspaces of $\mathcal{W}_q$. This is a real analytic fibre bundle with compact fibre. It contains a real analytic subbundle $\mathcal{P}(k)\to\mathcal{C}_+$ whose fibre over q is the Grassmannian of *complex* k-dimensional subspaces of $\mathcal{W}_q$ (for the complex structure J).

The Hermitean metric on $\mathcal{W}$ naturally extends to a real analytic Riemannian metric on $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$. The bundle $\mathrm{Gr}(2k)$ can be identified with the set of pure vectors in the sphere subbundle of $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$ for this metric. Namely, an oriented $2k$-dimensional real linear subspace E of $\mathcal{W}_q$ defines uniquely a pure vector in $\wedge_{\mathbb{R}}^{2k}\mathcal{W}_q$ of norm one which is just the exterior product of an orthonormal basis of E with respect to the inner product on $\mathcal{W}_q$. The points in $\mathcal{P}(k)$ correspond precisely to those pure vectors which are invariant under the extension of J to an automorphism of $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$.

From now on, we work on the real analytic hyperplane $\mathcal{C}\subset\mathcal{C}_+$ of abelian differentials in $\mathcal{C}$ of area one, and we replace the action of $GL^+(2,\mathbb{R})$ by the action of $SL(2,\mathbb{R})$. The tangent bundle $\mathcal{F}$ of the orbits of the $SL(2,\mathbb{R})$-action is naturally trivialized by the generator X of the Teichmüller flow, the generator Y of the horocycle flow, and the generator Z of the circle group of rotations. Let B^k (or C^k,D^k) be the contraction of the tensor field Ξ^k with the vector fields X,Y,Z. Since these vector fields are real analytic, B^k (or C^k,D^k) is a real analytic section of the endomorphism bundle $\mathrm{End}(\wedge_{\mathbb{R}}^{2k}(\mathcal{W}))$ of $\wedge_{\mathbb{R}}^{2k}(\mathcal{W})$.

For $1 \leq k \leq \ell - 2$ and $q \in \mathcal{C}$ let

$$\rho^k(q,0) \subset \mathcal{P}(k)_q$$

be the set of all k-dimensional complex linear subspaces L of $\mathcal{W}_q$ with $B^k L = 0 = C^k L = D^k L$ (here we view as before a k-dimensional complex subspace of $\mathcal{W}_q$ as a pure J-invariant vector in $\wedge^{2k}_{\mathbb{R}} \mathcal{W}_q$). Since the contractions B^k, C^k, D^k of the tensor field Ξ^k are linear, the set $\rho^k(q,0)$ can be identified with the set of all J-invariant pure vectors which are contained in some (perhaps trivial) linear subspace of $\wedge^k_{\mathbb{R}}(\mathcal{W}_q)$.

Define a *real analytic subset* of $\mathcal{P}(k)$ to be the intersection of the zero sets of a finite or countable number of real analytic functions on $\mathcal{P}(k)$. Recall that this is well defined since $\mathcal{P}(k)$ has a natural real analytic structure. We allow such functions to be constant zero, i.e. we do not exclude that such a set coincides with $\mathcal{P}(k)$. The real analytic set is *proper* if it does not coincide with $\mathcal{P}(k)$. Then there is at least one defining function which is not identically zero and hence the set is closed and nowhere dense.

Since the tensor fields Ξ^k and the vector fields X, Y, Z are real analytic, $\cup_q \rho^k(q,0)$ is a real analytic subset of $\mathcal{P}(k)$, defined as the common zero set of three real analytic functions.

For $t \in \mathbb{R}$ define

$$\rho_1^k(q,t) \subset \mathrm{Gr}(2k)_q$$

to be the preimage of $\rho^k(u_t q, 0)$ under parallel transport for the Gauss Manin connection along the flow line $s \to u_s q$ ($s \in [0,t]$) of the horocycle flow. By the previous paragraph and the fact that parallel transport is real analytic, $\cup_q \rho_1^k(q,t)$ is a real analytic subset of $\mathrm{Gr}(2k)_q$ and hence the same holds true for

$$\xi_1^k = \cap_{t\in\mathbb{R}} \cup_q \rho_1^k(q,t) \subset \mathcal{P}(k)$$

(take the intersections for all $t \in \mathbb{Q}$).

By construction, the set ξ_1^k is invariant under the extension of the horocycle flow by parallel transport with respect to the Gauss Manin connection of the fibres of the bundle $\mathcal{P}(k) \to \mathcal{C}$. It also is invariant under parallel transport with respect to the Chern connection: Namely, by definition, if $Z \in \xi_1^k(q)$ and if $Z(t)$ is the parallel transport of $Z = Z(0)$ for the Gauss Manin connection along the orbit of the horocycle flow through q, then the covariant derivative of the section $t \to Z(t)$ for the Chern connection vanishes.

Similarly, for $t \in \mathbb{R}$ define

$$\rho_2^k(q,t) \subset \mathrm{Gr}(2k)_q$$

to be the preimage of $\xi_1^k(\Phi^t q)$ under parallel transport for the Gauss Manin connection along the flow line $s \to \Phi^s q$ ($s \in [0,t]$) of the Teichmüller flow and let

$$\xi_2^k = \cap_{t\in\mathbb{R}} \cup_q \rho_2^k(q,t) \subset \mathcal{P}(k).$$

Then ξ_2^k is invariant under the extension of the Teichmüller flow by parallel transport both for the Gauss Manin connection and the Chern connection. As the Teichmüller flow maps orbits of the horocycle flow to orbits of the horocycle flow with the parametrization multiplied by a constant, using Lemma 7.1 it follows that the set ξ_2^k consists of points whose parallel transports along the orbits of the upper triangular subgroup of $SL(2,\mathbb{R})$ for both the Chern connection and the Gauss Manin connection coincide.

Repeat this construction with the circle group of rotations to find a set

$$\xi^k \subset \mathcal{P}(k).$$

Then ξ^k is a real analytic subset of $\mathcal{P}(k)$ which is invariant under the action of $SL(2,\mathbb{R})$ defined by parallel transport for the Gauss Manin connection.

If $\mathcal{U} \subset \mathcal{C}$ is a proper affine invariant manifold of rank $2 \leq k+1 < \ell$ then it follows from the discussion preceding this proof (see [F16]) that for every $q \in \mathcal{U}$ the projected tangent space $p(T\mathcal{U})$ defines a point in $\xi^k(q) \subset \mathcal{P}(k)_q$. In particular, we have $\xi^k \neq \emptyset$.

Let $\pi : \mathcal{P}(k) \to \mathcal{C}$ be the natural projection and let

$$\mathcal{A} = \pi(\xi^k).$$

The fibres of π are compact and hence π is closed. Therefore $\mathcal{A}$ is a closed $SL(2,\mathbb{R})$-invariant subset of $\mathcal{C}$ which contains all affine invariant submanifolds of $\mathcal{C}$ of rank $k+1$. Our goal is to show that either $\mathcal{A}$ is nowhere dense in $\mathcal{C}$, or the second property in the statement of the proposition is fullfilled.

To this end assume that $\mathcal{A}$ contains an open subset of $\mathcal{C}$. By invariance and ergodicity, $\mathcal{A}$ contains an open dense invariant set. On the other hand, $\mathcal{A}$ is closed and hence we have $\mathcal{A} = \mathcal{C}$. This is equivalent to stating that $\xi^k(q) \subset \mathcal{P}(k)_q$ is a non-empty compact set for all $q \in \mathcal{C}$. In other words, for every $q \in \mathcal{C}$ there is a line in $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$ spanned by a pure vector Z which is an eigenvector for the extension of the complex structure J and which is contained in the kernel of B^k, C^k, D^k. Moreover, the same holds true for the parallel transport with respect to the Gauss Manin connection of this eigenline along the orbits of the $SL(2,\mathbb{R})$-action (by Lemma 7.1, this makes sense).

With respect to a real analytic local trivialization of the bundle $\mathcal{P}(k)$ over an open set $V \subset \mathcal{C}$, the set ξ^k is of the form $(q, \xi^k(q))$ where $\xi^k(q)$ is a real analytic subset of the compact projective variety of k-dimensional complex linear subspaces of $\mathbb{C}^{\ell-1}$ depending in a real analytic fashion on q. More precisely, $\xi^k(q)$ can be identfied with the space of all J-invariant pure vectors which are contained in some linear subspace R_q of $\wedge_{\mathbb{R}}^{2k}(\mathcal{W}_q)$. As a consequence, there is an open subset V of $\mathcal{C}$ so that the restriction of ξ^k to V is a fibre bundle over V (take an open set where the dimension of R_q is minimal). By invariance and ergodicity, there is an open dense $SL(2,\mathbb{R})$-invariant connected subset V of $\mathcal{C}$ such that $\xi^k|V$ is a fibre bundle.

We claim that for $q \in V$, the set $\xi^k(q)$ consists of finitely many points. To this end choose a periodic orbit $\gamma \subset V$ for the Teichmüller flow so that for $q \in \gamma$, the restriction A to $\mathbb{R}^{2\ell} = \mathcal{Z}_q$ of the transformation $\Psi(\Omega(\gamma)) \in Sp(2\ell,\mathbb{R})$ has 2ℓ distinct real eigenvalues. Such an orbit exists by Corollary 6.7. Note that A is the return map for parallel transport of $\mathcal{Z}$ along γ with respect to the Gauss Manin connection. The map A preserves the decomposition $\mathcal{Z}_q = \mathcal{T}_q \oplus \mathcal{W}_q$.

We are looking for k-dimensional complex linear subspaces of $\mathcal{W}_q$ whose images under iteration of A are complex. Recall that the complex structure and the symplectic structure define an inner product $\langle , \rangle$ on $\wedge^{2k}\mathcal{W}_q$. Thus the set $\xi^k(q)$ consists of pure vectors $Y \in \wedge^{2k}_{\mathbb{R}}\mathcal{W}_q$ whose norms are preserved by the symplectic linear map A. As the eigenvalues of A are all real and positive, of multiplicity one, the eigenvalues for the action of A on $\wedge^{2k}\mathcal{W}_q$ are all real and positive. More precisely, the eigenvalues for this action are precisely the products of $2k$ eigenvalues of the linear map A. In particular, these eigenvalues are all real and positive, and the action of A on $\wedge^{2k}_{\mathbb{R}}\mathcal{W}_q$ is diagonalizable over $\mathbb{R}$. As a consequence, a vector $Y \in \wedge^{2k}_{\mathbb{R}}\mathcal{W}_q$ whose norm is preserved by the action of A is an eigenvector for the eigenvalue one.

We showed so far that a pure vector $Y \in \xi^k(q)$ corresponds to a fixed point for the action of A on the set $\mathcal{P}(k)_q$ of all k-dimensional complex subspaces of $\mathcal{W}_q$, viewed as a compact subset of the Grassmannian $\mathrm{Gr}(2k)_q$ of $2k$-dimensional subspaces of $\mathcal{W}_q$. The fixed points for the action of A on $\mathrm{Gr}(2k)_q$ are precisely the k-dimensional oriented linear subspaces which are direct sums of eigenspaces of A. Thus the number of such subspaces is finite and hence $\xi^k(q)$ is a finite set.

By the choice of V, the cardinality of the set $\xi^k(q)$ is locally constant and hence constant on V since V is connected. As the dependence of $\xi^k(q)$ on $q \in V$ is real analytic, any choice of a point in $\xi^k(q)$ defines locally near q an analytic section of $\mathcal{P}(k)$ and hence a real analytic J-invariant subbundle of $\mathcal{W}$. If this local section is invariant under parallel transport along the orbits of the $SL(2,\mathbb{R})$-action (this is equivalent to triviality of the monodromy), then it defines a real analytic J-invariant subbundle of $\mathcal{W}|V$. Otherwise as $\xi^k(q) \subset \mathcal{P}(k)_q$ consists of finitely many points and is invariant under parallel transport along the orbits of the $SL(2,\mathbb{R})$-action, this parallel transport acts as a finite group of permutations on the finite set $\xi^k(q)$. Thus we can pass to a finite cover of $\mathcal{C}$ so that on the covering space, using the same notation, the induced subbundle of $\mathcal{W}$ is invariant under parallel transport.

In other words, up to passing to a finite cover, ξ^k defines a real analytic complex $SL(2,\mathbb{R})$-invariant k-dimensional vector bundle over the open dense invariant subset V of $\mathcal{C}$. By invariance of both the complex and the symplectic structure under parallel transport along the orbits of the $SL(2,\mathbb{R})$-action, ξ^k then defines a splitting of the bundle $\mathcal{L}|V$ as predicted in the second part of the proposition. □

Remark 7.3. By Proposition 6.6 (see also [W14]), a real analytic splitting of the bundle $\mathcal{L}$ as stated in the second part of Proposition 7.2 can not be flat, i.e. invariant under the Gauss Manin connection. However, Proposition 6.6 does not rule out a real analytic splitting which is not flat.

8. Invariant splittings of the lifted Hodge bundle

In this section we use information on the moduli space of principally polarized abelian varieties to rule out the second case in Proposition 7.2. We continue to use all assumptions and notations from Section 7.

Recall the splitting $\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$ of the lifted Hodge bundle. The leaves of the absolute period foliation consist of differentials whose real and imaginary part define fixed classes in $H^1(S,\mathbb{R})$. Thus by the fact that the Hodge bundle is flat, the restriction of the bundle $\mathcal{T}$ to each leaf of the absolute period foliation is flat and hence the same holds true for its symplectic complement $\mathcal{L}$. This is equivalent to stating that parallel transport of the Hodge bundle with respect to the Gauss Manin connection along curves contained in the absolute period foliation preserves the subbundle $\mathcal{L}$. Equivalently, the restriction of the Gauss Manin connection to each such leaf defines a leafwise flat connection on $\mathcal{L}$.

Let $\mathcal{C}$ be an affine invariant manifold with absolute holomorphic tangent bundle $\mathcal{Z}$. Assume that there is an open dense $SL(2,\mathbb{R})$-invariant subset V of $\mathcal{C}$ and there is an $SL(2,\mathbb{R})$-invariant real analytic splitting $\mathcal{L}\cap\mathcal{Z}|V=\mathcal{E}_1\oplus\mathcal{E}_2$ into complex orthogonal subbundles as in the second part of Proposition 7.2. The restriction of the splitting to an orbit of the $SL(2,\mathbb{R})$-action is invariant under the Gauss Manin connection. Equivalently, the restrictions of the bundles $\mathcal{E}_i$ to the orbits of the $SL(2,\mathbb{R})$-action are parallel.

Recall from Section 5 that if $\dim(\mathcal{C}_+)>2\mathrm{rk}(\mathcal{C}_+)$ then the absolute period foliation $\mathcal{AP}(\mathcal{C})$ of $\mathcal{C}$ is defined.

Lemma 8.1. *The restriction of the bundle $\mathcal{E}_i\to V$ to a leaf of $\mathcal{AP}(\mathcal{C})$ is invariant under the Gauss Manin connection.*

Proof. We may assume that the dimension of $\mathcal{AP}(\mathcal{C})$ is positive. Furthermore, by passing to a finite cover we assume that the zeros of the differentials in $\mathcal{C}$ are numbered.

The splitting $\mathcal{Z}|V=\mathcal{T}\oplus\mathcal{E}_1\oplus\mathcal{E}_2$ can be used to project the Gauss Manin connection ∇^{GM} on $p(T\mathcal{C})=\mathcal{Z}$ to a real analytic connection $\hat\nabla$ along the leaves of the absolute period foliation which preserves this decomposition. More precisely, given a local vector field $X\subset T\mathcal{AP}(\mathcal{C})$ and a section Y of $\mathcal{E}_i$, define

$$\hat\nabla_X Y=P_i(\nabla^{GM}_X Y)$$

where

$$P_i:\mathcal{E}_1\oplus\mathcal{E}_2\to\mathcal{E}_i$$

is the natural projection. Recall that this makes sense since the Gauss Manin connection restricted to a leaf of $\mathcal{AP}(\mathcal{C})$ preserves the bundle $\mathcal{L}\cap\mathcal{Z}$.

Let $\mathfrak{a}\in\mathbb{C}^k$ be such that the vector field $X_{\mathfrak{a}}$ is tangent to $\mathcal{C}$. The existence of such a vector $\mathfrak{a}$ was shown in Lemma 5.2. For every $q\in V\subset\mathcal{C}$ and every $Y\in\mathcal{E}_1(q)$ we can extend Y by parallel transport for $\hat\nabla$ along the flow line of the flow $\Lambda^t_{\mathfrak{a}}$ generated by $X_{\mathfrak{a}}$. Let us denote this extension by $\hat Y$; then

$$\beta(X_{\mathfrak{a}},Y)=\frac{\nabla^{GM}}{dt}\hat Y(\Lambda^t_{\mathfrak{a}}(q))|_{t=0}\in\mathcal{E}_2(q)$$

only depends on $X_{\mathfrak{a}}$ and Y, moreover this dependence is linear in both variables. In this way we obtain a real analytic tensor field

$$\beta\in\Omega(T\mathcal{AP}(\mathcal{C})^*\otimes\mathcal{E}_1^*\otimes\mathcal{E}_2).$$

Here as before, $\Omega(T\mathcal{AP}(\mathcal{C})^*\otimes\mathcal{E}_1^*\otimes\mathcal{E}_2)$ is the vector space of real analytic sections of the bundle $T\mathcal{AP}(\mathcal{C})^*\otimes\mathcal{E}_1^*\otimes\mathcal{E}_2$. The splitting $\mathcal{L}\cap\mathcal{Z}=\mathcal{E}_1\oplus\mathcal{E}_2$ is parallel along the leaves of the absolute period foliation if and only if β vanishes identically.

The Teichmüller flow Φ^t acts on the bundle $\mathcal{L}$ by parallel transport with respect to the Gauss Manin connection, and this action preserves the bundles $\mathcal{E}_i$ $(i=1,2)$ and the absolute period foliation of $\mathcal{C}$. Thus by definition, the tensor field β is equivariant under the action of Φ^t.

Assume to the contrary that β does not vanish identically. As β is real analytic and bilinear, there is then an open subset U of $V \subset \mathcal{C}$ and either a real or a purely imaginary vector $\mathfrak{a} \in \mathbb{C}^k$ such that $X_{\mathfrak{a}}$ is tangent to $\mathcal{C}$ and that moreover the contraction of β with $X_{\mathfrak{a}}$ does not vanish on U.

Assume that $\mathfrak{a} \in \mathbb{R}^k$ is real, the case of a purely imaginary vector is treated in the same way; then $d\Phi^t X_{\mathfrak{a}} = e^t X_{\mathfrak{a}}$ by Lemma 5.1. Let now $\gamma \subset \mathcal{C}$ be a periodic orbit with the properties stated in Corollary 6.7 which passes through U. Let $q \in \gamma \cap U$. The eigenvalues of the matrix $A = \Psi(\Omega(\gamma))|\mathcal{Z}_q$ (where we identify $\mathcal{Z}_q$ with the symplectic subspace of $H^1(S,\mathbb{R})$ it defines) are real, positive and of multiplicity one. The largest eigenvalue of A equals $e^{\ell(\gamma)}$ where $\ell(\gamma)$ is the length of the orbit γ.

Write $\mathcal{W} = \mathcal{Z} \cap \mathcal{L}$; then $\mathcal{W}_q$ is a sum of eigenspaces for A (viewed as a transformation of $\mathcal{Z}_q$) for eigenvalues strictly smaller than $e^{\ell(\gamma)}$. Furthermore, the decomposition $\mathcal{W}_q = \mathcal{E}_1(q) \oplus \mathcal{E}_2(q)$ $(i=1,2)$ is invariant under A, and $\mathcal{E}_i(q)$ is a direct sum of eigenspaces for A.

Denote by $||^{GM}$ parallel transport for the Gauss Manin connection. By equivariance of the tensor field β under the flow Φ^t, for $Y \in \mathcal{E}_1(q)$ we have
$$\beta(d\Phi^{\ell(\gamma)} X_{\mathfrak{a}}, ||_\gamma^{GM} Y) = ||_\gamma^{GM} \beta(X_{\mathfrak{a}}, Y) \in \mathcal{E}_2(q).$$

Recall that $\mathcal{E}_i(q)$ is a direct sum of eigenspaces for A. By the above discussion, if $Y \in \mathcal{E}_1(p)$ is an eigenvector of A for the eigenvalue $a > 0$, then
$$\beta(d\Phi^{\ell(\gamma)} X_{\mathfrak{a}}, AY) = e^{\ell(\gamma)} a \beta(X_{\mathfrak{a}}, Y) = A\beta(X_{\mathfrak{a}}, Y) \in \mathcal{E}_2(p).$$
In other words, the contraction B of β with $X_{\mathfrak{a}}$ maps an eigenspace of $\mathcal{E}_1(p)$ for the eigenvalue a to an eigenspace for the eigenvalue $e^{\ell(\gamma)}a$.

But $e^{\ell(\gamma)}$ is an eigenvalue of A and by the choice of γ, no product of two eigenvalues of A is an eigenvalue. Therefore B has to be the zero map. This contradicts the assumption that the contraction of β with $X_{\mathfrak{a}}$ does not vanish at q and hence that the map B is non-trivial.

As a consequence, the parallel transport for $\hat{\nabla}$ of a vector $Y \in \mathcal{E}_1$ along a path which is entirely contained in a leaf of the absolute period foliation of $V \subset \mathcal{C}$ coincides with parallel transport with respect to the Gauss Manin connection. Equivalently, the restriction of the Gauss Manin connection to a leaf of the absolute period foliation preserves the splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$. This is what we wanted to show. □

Remark 8.2. It is an interesting question whether it is possible to deduce from Lemma 8.1, Proposition 6.8 and Moore's theorem for the action of $SL(2,\mathbb{R})$ on $Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})$ that a splitting as in the second part of Proposition 6.6 does not exist. The main difficulty is that we do not know whether there is a leaf of the foliation $\mathcal{F}$ of the bundle $\mathcal{S}$ as described in the appendix which intersects the image of the period map in more than one component.

We saw so far that an invariant splitting of the Hodge bundle over an affine invariant manifold $\mathcal{C}$ as predicted by the second part of Proposition 7.2 has to be flat along the leaves of the absolute period foliation. If this foliation is non-trivial and if we knew that it has dense leaves (this is an open question, see however [H15]), we could try to show that such a splitting has to be flat globally. As such tools are not available, we use the curvature of the projection of the Gauss Manin connection to the invariant subbundle to arrive at a contradiction similar to the reasoning in the proof of Lemma 8.1. To compute this curvature we use the fact that the Gauss Manin connection is the pull-back of the standard flat connection of the vector bundle $\mathcal{V} \to \mathcal{A}_g$ under the Torelli map. As detailed in the appendix, this connection can be controlled with standard methods from differential geometry on locally symmetric spaces.

Proposition 8.3. *Let $\mathcal{Z}$ be the absolute holomorphic tangent bundle of an affine invariant manifold $\mathcal{C}$ of rank at least three. Then there is no open dense $SL(2,\mathbb{R})$-invariant subset V of $\mathcal{C}$ such that the bundle $\mathcal{W}|V = \mathcal{Z} \cap \mathcal{L}|V$ admits an $SL(2,\mathbb{R})$-invariant real analytic direct decomposition into a sum of two complex vector bundles whose restrictions to each $SL(2,\mathbb{R})$-orbit are flat.*

Proof. We argue by contradiction and we assume that an open dense invariant set V and such a splitting $\mathcal{W}|V = \mathcal{E}_1 \oplus \mathcal{E}_2$ exists. By Lemma 8.1, we know that this splitting is invariant under the restriction of the Gauss Manin connection to the leaves of the absolute period foliation of $\mathcal{C}$. Furthermore, it naturally induces an invariant splitting of the bundle $\mathcal{W} \oplus \overline{\mathcal{W}} \subset p(T\mathcal{C})$ into two subbundles which are complex for the flat complex structure on $H^1(S,\mathbb{C}) = H^1(S,\mathbb{R}) \otimes \mathbb{C}$.

Let $\mathcal{T}(S)$ be the Teichmüller space of the surface S and let $\mathcal{I}_g < \mathrm{Mod}(S)$ be the Torelli group. The group $\mathcal{I}_g$ acts properly and freely from the right on $\mathcal{T}(S)$, with quotient the *Torelli space* $\mathcal{T}(S)/\mathcal{I}_g$. The *period map* F maps the bundle $\mathcal{D}$ of area one abelian differential over $\mathcal{T}(S)/\mathcal{I}_g$ into the sphere subbundle $\mathcal{S}$ of the tautological vector bundle $\mathcal{V}$ over the Siegel upper half-space $\mathfrak{D}_g$ (see the beginning of Section 6 and the appendix for the notations). The map $F : \mathcal{D} \to \mathcal{S}$ is equivariant with respect to the standard action of $SL(2,\mathbb{R})$ on $\mathcal{D}$ and the right action of $SL(2,\mathbb{R})$ on $\mathcal{S}$ (which commutes with the left action of $Sp(2g,\mathbb{R})$). We refer to the appendix for more details on these facts.

By Lemma A.5, the composition of the map F with the equivariant projection

$$\Pi : \mathcal{S} = Sp(2g,\mathbb{R}) \times_{U(g)} S^{2g-1} \to \Omega = Sp(2g,\mathbb{R})/Sp(2g-2,\mathbb{R})$$

is equivariant for the standard $SL(2,\mathbb{R})$-actions as well.

Apply the map $\Pi \circ F$ to a component $\hat{\mathcal{C}}$ of the preimage of $\mathcal{C}$ in $\mathcal{D}$. More precisely, the absolute holomorphic tangent bundle of $\mathcal{C}$ determines a symplectic subspace $\mathbb{R}^{2\ell}$ of $(\mathbb{R}^{2g},\omega) = H^1(S,\mathbb{R})$. We denote by $\mathbb{C}^{2\ell}$ its complexification. The composition of the map $\Pi \circ F$ with symplectic orthogonal projection then defines a map

$$\Xi : \hat{\mathcal{C}} \to \Omega_\ell = \{x + iy \in \mathbb{C}^{2\ell} \mid \omega(x,y) = 1\}.$$

By naturality of this construction, the Gauss Manin connection on the bundle $p(T\mathcal{C})$ with fibre $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ is just the pull-back via Ξ of the natural flat connection on $T\mathcal{O}$. Using the notations from the appendix, the leafwise connection ∇^{GM} on the bundle $\mathcal{L}$ is the pull-back of the connection $\nabla^{\mathcal{R}}$ on the symplectic complement $\mathcal{R} \subset T\Omega_\ell$ of the tangent bundle of the orbits of the $SL(2,\mathbb{R})$-action.

By Lemma 8.1, the splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is invariant under the $SL(2,\mathbb{R})$-action and parallel with respect to the restriction of the Gauss Manin connection to the leaves of the absolute period foliation. Using again Lemma A.5 in the appendix, this means that the splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is the pull-back by Ξ of a real analytic splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ of the bundle $\mathcal{R}$ into a sum of two complex vector bundles.

We claim that the curvature form Θ for the connection $\nabla^{\mathcal{R}}$ preserves this decomposition on the image of the map Ξ. To this end let $\gamma \subset \mathcal{C}$ be a periodic orbit for Φ^t with the properties as in Corollary 6.7. Then $\Xi(\gamma)$ is an orbit in Ω_ℓ for the action of the diagonal subgroup of $SL(2,\mathbb{R})$. This orbit is periodic under the action of an element $A \in Sp(2\ell,\mathbb{R})$ (which is the restriction of an element of $Sp(2g,\mathbb{Z})$ stabilizing the subspace $\mathbb{R}^{2\ell}$) whose eigenvalues are all real, of multiplicity one, and such that no product of two eigenvalues is an eigenvalue.

By the assumption of the splitting of $\mathcal{W}$ and Lemma A.5, the local splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is invariant under the action of $SL(2,\mathbb{R})$. As a consequence, the complex subspaces $\mathcal{R}_i$ are sums of eigenspaces for A, containing with an eigenspace for the eigenvalue $a > 1$ the eigenspace for a^{-1}. As in the proof of Lemma 8.1, we conclude that the connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ preserves the splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ along $\Xi(\gamma)$.

Namely, let $\nabla^{\mathcal{R}_1}$ be the projection of the connection $\nabla^{\mathcal{R}}$ to a connection on $\mathcal{R}_1$. Then $\nabla^{\mathcal{R}} - \nabla^{\mathcal{R}_1}$ is a real analytic tensor field $\beta \in \Omega(T^*\Omega_\ell \otimes \mathcal{R}_1^* \otimes \mathcal{R}_2)$. Since $\mathcal{R}_1$ and $\nabla^{\mathcal{R}}$ are invariant under the action of the diagonal flow $\Psi^t \subset SL(2,\mathbb{R})$ (we use the notation Ψ^t here to indicate that we are looking at a flow on the space Ω_ℓ), this tensor field is equivariant under the action of Ψ^t. As in the proof of Lemma 8.1, as no product of two eigenvalues of the matrix A is an eigenvalue, this implies that the restriction of β to $\Xi(\gamma)$ vanishes.

By Corollary 6.7, the set of points $q \in V \subset \mathcal{C}$ which are contained in a periodic orbit with the above properties is dense in V. Hence its image under the restriction of the map Ξ to a small contractible open subset of V is a dense subset of a nonempty open subset E of Ω_ℓ where the splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is defined. As a consequence, the real analytic tensor field β vanishes on E and hence the splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is flat on E.

Using the terminology from the appendix, this shows that the curvature form of the connection $\nabla^{\mathcal{R}}$ splits $\mathcal{R}$ as a complex vector bundle contradicting Lemma A.2. This shows the proposition. We refer to Theorem 5.1 of [W14] for a similar result using related arguments. $\square$

Remark 8.4. The reasoning in the proof of Lemma 8.1 and Proposition 8.3 also implies that the Lyapunov filtration for the action of the Teichmüller flow on a stratum of abelian differentials is not smooth (or, less restrictive, is not of the class

C^2). As we use covariant differentiation in our argument, mere continuity of the filtration can not be ruled out immediately.

Corollary 8.5. (1) *Let $\mathcal{Q}$ be a component of a stratum; then for every $2 \leq \ell \leq g-1$ there are finitely many affine invariant submanifolds of $\mathcal{Q}$ of rank ℓ which contain every affine invariant submanifold of rank ℓ.*
(2) *The smallest stratum of differentials with a single zero contains only finitely many affine invariant submanifolds of rank at least two.*

Proof. Let $\mathcal{C}$ be an affine invariant manifold of rank $\ell \geq 3$. By Proposition 7.2 and Proposition 8.3, there are finitely many proper affine invariant submanifolds of $\mathcal{C}$ which contain every affine invariant submanifold of $\mathcal{C}$ of rank $2 \leq k \leq \ell - 1$.

An application of this fact to a component $\mathcal{Q}$ of a stratum shows that for $2 \leq \ell \leq g-1$, there are finitely many proper affine invariant submanifolds $\mathcal{C}_1, \dots, \mathcal{C}_m$ of $\mathcal{Q}$ which contain every affine invariant submanifold of $\mathcal{Q}$ of rank ℓ. The dimension of $\mathcal{C}_i$ is strictly smaller than the dimension of $\mathcal{Q}$.

By reordering we may assume that there is some $u \leq m$ such that for all $i \leq u$ the rank $\mathrm{rk}(\mathcal{C}_i)$ of $\mathcal{C}_i$ is bigger than ℓ and that for $i > u$ the rank $\mathrm{rk}(\mathcal{C}_i)$ of $\mathcal{C}_i$ is at most ℓ. Apply the first paragraph of this proof to each of the affine invariant manifolds $\mathcal{C}_i$ ($i \leq u$). We conclude that for each i there are finitely many proper affine invariant submanifolds of $\mathcal{C}_i$ of rank $r \in [\ell, \mathrm{rk}(\mathcal{C}_i))$ which contain every affine invariant submanifold of $\mathcal{C}_i$ of rank ℓ. The dimension of each of these submanifolds is strictly smaller than the dimension of $\mathcal{C}_i$. In finitely many such steps, each applied to all affine invariant submanifolds of rank stricltly bigger than ℓ found in the previous step, we deduce the statement of the first part of the corollary.

Now let $\mathcal{H}(2g-2)$ be a stratum of differentials with a single zero. Period coordinates for $\mathcal{H}(2g-2)$ are given by absolute periods, and the dimension of an affine invariant manifold $\mathcal{C} \subset \mathcal{H}(2g-2)$ of rank ℓ equals 2ℓ. Thus $\mathcal{C}$ does not contain any proper affine invariant submanifold of rank ℓ.

By Proposition 7.2 and the first part of this proof, there are finitely many proper affine invariant submanifolds $\mathcal{C}_1, \dots, \mathcal{C}_s$ of $\mathcal{H}(2g-2)$ which contain every affine invariant submanifold of $\mathcal{H}(2g-2)$ of rank at most $g-1$. In particular, there are only finitely many such manifolds of rank $g-1$.

To show finiteness of affine invariant manifolds of any rank $2 \leq \ell \leq g-1$, apply Proposition 7.2 and the first part of this proof to each of the finitely many affine invariant manifolds constructed in some previous step and proceed by inverse induction on the rank. □

Remark 8.6. The proof of the second part of Corollary 8.5 immediately extends to the following statement. An affine invariant manifold $\mathcal{C}$ with trivial absolute period foliation contains only finitely many affine invariant manifolds of rank at least two.

9. Nested affine invariant submanifolds of the same rank

The goal of this section is to analyze affine invariant submanifolds of affine invariant manifolds $\mathcal{C}$ of the same rank and to complete the proof of Theorem 2. Our strategy is a variation of the strategy used in Section 8. Namely, we reduce the finiteness statement to the nonexistence of a certain real analytic $SL(2,\mathbb{R})$-invariant splitting of the tangent bundle of $\mathcal{C}$.

We continue to use the notations from the previous sections. In particular, we denote by $\mathcal{AP}(\mathcal{C})$ the absolute period foliation of an affine invariant manifold $\mathcal{C}$. We always assume that this foliation is not trivial. Moreover, we always assume that the zeros of a differential $q \in \mathcal{C}$ are numbered; this may require to replace $\mathcal{C}$ by a finite cover.

Recall that the Hodge bundle $\mathcal{H} \to \mathcal{M}_g$ is a holomorphic vector bundle in the orbifold sense. We denote by J its complex structure. Using the notation from the appendix, let $\mathcal{V} \to Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/U(g)$ be the tautological vector bundle over the moduli space $\mathcal{A}_g$ of principally polarized abelian differentials. The period map $F : \mathcal{H} \to \mathcal{V}$ is holomorphic.

For an affine invariant manifold $\mathcal{C}$ contained in a component $\mathcal{Q}$ of a stratum of the moduli space of area one abelian differentials denote as before by $\mathcal{C}_+ \subset \mathcal{H}_+ \subset \mathcal{H}$ the extension of $\mathcal{C}$ be scaling; then $\mathcal{C}_+$ is a complex suborbifold of $\mathcal{H}_+$. The foliation of $\mathcal{C}$ into the orbits of the $GL^+(2,\mathbb{R})$-action is holomorphic.

We use these facts in the proof of the following analogue of Proposition 7.2. Lemma A.4 in the appendix discusses splittings in the sphere subbundle $\mathcal{S}$ of the tautological vector bundle $\mathcal{V}$ over the moduli space $\mathcal{A}_g$ of principally polarized abelian differentials which puts this proposition into the framework of homogenous spaces.

Proposition 9.1. *Let $\mathcal{C}$ be an affine invariant manifold of rank $\ell \geq 2$. Then one of the following two possibilities holds true.*

1. *There are at most finitely many proper affine invariant submanifolds of $\mathcal{C}$ of rank ℓ.*
2. *Up to passing to a finite cover, the tangent bundle $T\mathcal{C}$ of $\mathcal{C}$ admits a nontrivial Φ^t-invariant real analytic splitting $T\mathcal{C} = \mathcal{A} \oplus \mathcal{V}$ where $\mathcal{A}$ is a complex subbundle of $T\mathcal{AP}(\mathcal{C})$ and where $\mathcal{V}$ contains the tangent bundle of the orbits of the $SL(2,\mathbb{R})$-action.*

Proof. By the main results of [EMM15], it suffices to show that the second possibility in the proposition holds true under the following assumption: There is a number $k \geq 1$, and there is an open subset V of $\mathcal{C}$ such that the set of all affine invariant submanifolds of $\mathcal{C}$ of complex codimension k whose rank coincides with the rank of $\mathcal{C}$ is dense in V. We therefore suppose from now on that this is the case.

Let as before J be the complex structure on $\mathcal{C}$. Let $\mathcal{N}$ be a J-invariant $SL(2,\mathbb{R})$-invariant real analytic subbundle of $T\mathcal{C}$ which is complementary to the tangent bundle of the orbits of the $SL(2,\mathbb{R})$-action and invariant under the natural complex structure J. Such a bundle exists because the leaves of the foliation into the orbits of the action of $GL^+(2,\mathbb{R})$ of the natural extension $\mathcal{C}_+$ of $\mathcal{C}$ by scaling are both complex and affine. In fact, using the notations from Section 7, if we write $\mathcal{W}=\mathcal{L}\cap\mathcal{Z}$ then we can take $\mathcal{N}=p^{-1}(\mathcal{W}\oplus\overline{\mathcal{W}})$.

Let $\mathcal{P}\to\mathcal{C}$ be the real analytic fibre bundle whose fibre at a point $q\in\mathcal{C}$ equals the Grassmannian of all *complex* subspaces of $\mathcal{N}_q$ of complex codimension k. This is a real analytic subbundle of the fibre bundle whose fibre at q equals the Grassmannian of all oriented real linear subspaces of codimension $2k$ in $\mathcal{N}_q$. The bundle $\mathcal{P}$ admits a natural $SL(2,\mathbb{R})$-invariant stratification $\mathcal{P}=\cup_{i=0}^k\mathcal{P}_i$ where $\mathcal{P}_i$ consists of all subspaces which intersect $T\mathcal{AP}(\mathcal{C})$ in a subspace of complex codimension $k-i$ (note that some of these strata may be empty).

The flow Φ^t acts on $\mathcal{C}$ as a one-parameter group of real analytic transformations, and its differential preserves the bundle $\mathcal{N}$. For $q\in\mathcal{C}$ and $t\in\mathbb{R}$ let $\rho(q,t)$ be the image of $\mathcal{P}(\Phi^t q)$ under the map $d\Phi^{-t}$. Then

$$\mathcal{R}_\infty=\cap_t\cup_q\rho(q,t)$$

is a (possibly empty) real analytic subvariety of $\mathcal{P}$. This subvariety is invariant under the action of Φ^t.

The tangent bundle of an affine invariant submanifold $\mathcal{D}$ of $\mathcal{C}$ intersects the complex vector bundle $\mathcal{N}$ in a complex subbundle $\mathcal{N}\cap T\mathcal{D}|\mathcal{D}$. This subbundle is invariant under the action of the flow Φ^t. Thus if $q\in\mathcal{C}$ is contained in an affine invariant submanifold of $\mathcal{C}$ of the same rank as $\mathcal{C}$ and of complex codimension k, then $\mathcal{R}_\infty\cap\mathcal{P}_0(q)\neq\emptyset$. In particular, by the assumption on V, the subvariety $\mathcal{R}_\infty$ of $\mathcal{P}$ is not empty, and the restriction of the canonical projection $\pi:\mathcal{P}\to V$ to $\mathcal{R}_\infty$ is surjective.

Following the reasoning in the proof of Proposition 7.2, by restricting to a smaller open Φ^t-invariant set $U\subset V$ we may assume that $\mathcal{R}_\infty\to U$ is a real analytic fibre bundle. In particular, the fibre is not empty and compact.

Now $\mathcal{P}_0\subset\mathcal{P}$ is an open Φ^t-invariant subset of $\mathcal{P}$, and $\mathcal{R}_\infty\cap\mathcal{P}_0(q)\neq\emptyset$ for a dense set of points q of U. Therefore up to perhaps decreasing the set U, we may assume that the intersection of $\mathcal{R}_\infty$ with $\mathcal{P}_0(q)$ is not empty for every $q\in U$.

To summarize, the restriction of $\mathcal{R}_\infty$ to the preimage of U under the projection $\mathcal{P}\to U$ is a fibre bundle with compact fibre which contains a non-empty open subbundle $\mathcal{E}=\mathcal{R}_\infty\cap\mathcal{P}_0$. This subbundle is invariant under the natural action of the Teichmüller flow Φ^t.

Each point $q\in\mathcal{E}$ is a complex subspace of $\mathcal{N}_q$ of complex codimension k which intersects $T\mathcal{AP}(\mathcal{C})$ in a subspace of complex codimension k. Thus for $q\in U$ we can define

$$E(q)=\cap_{z\in\mathcal{E}(q)}z\subset\mathcal{N}_q\subset T\mathcal{C}_q.$$

Then $E(q)$ is a (possibly trivial) complex linear subspace of $T\mathcal{C}_q$. As $\mathcal{E}\to U$ is an open subset of a real analytic fibre bundle, by possibly replacing the set U by a proper open dense Φ^t-invariant subset we may assume that for $q\in U$, the dimension of $E(q)$ is minimal (perhaps zero). In particular, if this dimension is positive, then $\cup_{q\in U}E(q)$ is a real analytic complex subbundle of $\mathcal{N}|U$.

Our next goal is to show that indeed, for $q \in U$ the complex dimension of $E(q)$ is not smaller than $\ell - 1 \geq 1$. To this end let $\gamma \subset U$ be a periodic orbit with the properties stated in Corollary 6.7. For $q \in \gamma$, the return map $d\Phi^{\ell(\gamma)}(q)$ acts on $\mathcal{N}_q$ as a semisimple linear map A, with all eigenvalues real and positive, and all eigenvalues but the largest $e^{\ell(\gamma)}$ and the smallest $e^{-\ell(\gamma)}$ simple over $\mathbb{C}$ (i.e. eigenspaces for eigenvalues different from $e^{\ell(\gamma)}$ and $e^{-\ell(\gamma)}$ are complex subspaces of complex dimension one). As $\mathcal{N}_q$ is complementary to the tangent space of the orbits of the $SL(2,\mathbb{R})$-action, the eigenspaces for the eigenvalues $e^{\ell(\gamma)}, e^{-\ell(\gamma)}$ are contained in $T\mathcal{AP}(\mathcal{C})$. Moreover, if $a > 0$ is an eigenvalue of A, then the same holds true for a^{-1}.

By definition, a point $z \in \mathcal{E}(q)$ is a complex subspace of $T\mathcal{C}_q$ of complex codimension k which is complementary to some k-dimensional complex subspace of $T\mathcal{AP}(\mathcal{C})_q$, and its image under the map A is complex as well. We claim that such a subspace has to contain the sum of the eigenspaces for A with respect to the eigenvalues of absolute value different from $e^{\ell(\gamma)}, e^{-\ell(\gamma)}$.

To this end recall that the fibre $\mathcal{P}(q)$ of the bundle $\mathcal{P}$ at q is a closed subset of the Grassmann manifold of all oriented linear subspaces of $T\mathcal{C}_q$ of real codimension $2k$. Thus if $z \in \mathcal{E}(q)$, then any limit of a subsequence of the sequence $A^i z$ $(i \to \pm\infty)$ is complex. Such a limit y is a fixed point for the action of A on $\mathcal{P}(q)$ and hence it is a direct sum of subspaces of eigenspaces. Furthermore it is complex and hence symplectic. Therefore y contains with an eigenspace for the eigenvalue $1 < a < e^{\ell(\gamma)}$ (which is of dimension one by assumption) also the eigenspace for the eigenvalue a^{-1}.

By the same reasoning, the dimension of the intersection of y with the eigenspace for the eigenvalue $e^{\ell(\gamma)}$ coincides with the dimension of the intersection with the eigenspace for the eigenvalue $e^{-\ell(\gamma)}$. Furthermore, as $y \in \mathcal{E} \subset \mathcal{P}_0$ by assumption and as $T\mathcal{AP}(\mathcal{C})_q$ equals the direct sum of the eigenspaces for action of A on $\mathcal{N}_p$ with respect to the eigenvalues $e^{\ell(\gamma)}, e^{-\ell(\gamma)}$, we conclude that the real dimension of this intersection equals $\dim_{\mathbb{C}}(\mathcal{AP}(\mathcal{C}) - k$. For reasons of dimension, this implies that y contains the sum of the eigenspaces for the eigenvalues different from $e^{\ell(\gamma)}, e^{-\ell(\gamma)}$.

Now this discussion is valid for each $y \in \mathcal{E}_q$ and therefore for each $q \in \gamma$, the sum of these eigenspaces is contained in $\cap E(q)$. In particular, we have $\dim_{\mathbb{C}} E(q) \geq \ell - 1$ and hence this dimension is contained in the interval $[\ell - 1, \dim(\mathcal{C}) - k - 1]$, moreover $\mathcal{N}_q = T\mathcal{AP}(\mathcal{C})_q + E(q)$.

As $E(q)$ depends in a real analytic fashion on $q \in U$, the assignment $q \to E(q)$ is a real analytic Φ^t-invariant subbundle of $T\mathcal{C}$. This bundle can be used to construct a real analytic $SL(2,\mathbb{R})$-invariant splitting of $T\mathcal{C}$ as claimed in the second part of the proposition.

Namely, let $q \in U$ and let $\mathfrak{a}_1, \dots, \mathfrak{a}_{m-1} \in \mathbb{R}^m$ be linearly independent with zero mean (here m is the number of zeros of a differential in $\mathcal{C}$) such that for some $u \leq m - 1$ the vector fields $X_{\mathfrak{a}_1}, \dots, X_{\mathfrak{a}_u}$ are tangent to $\mathcal{C}$ and such that moreover their complex span is a linear subspace of $T\mathcal{AP}(\mathcal{C})$ complementary to $E(q)$. By invariance and Lemma 5.2, these vector fields span a subbundle of $T\mathcal{AP}(\mathcal{C})$ which is complementary to the bundle $\cup_q E(q)$. This is what we wanted to show. $\square$

By Proposition 9.1, if the affine invariant manifold $\mathcal{C}$ contains infinitely many affine invariant manifolds of the same rank as $\mathcal{C}$, then there exists an open dense Φ^t-invariant subset U of $\mathcal{C}$, and there is a Φ^t-invariant real analytic splitting $T\mathcal{C}|U = \mathcal{A}\oplus\mathcal{V}$ as a sum of complex vector bundles with the following additional property. The bundle $\mathcal{A}$ is a subbundle of $T\mathcal{AP}(\mathcal{C})$, and $\mathcal{V}$ contains the tangent bundle of the orbits of $SL(2,\mathbb{R})$-action. The following lemma shows that such a splitting does not exist.

Lemma 9.2. *Let $\mathcal{C}$ be an affine invariant manifold; then there is no nontrivial Φ^t-invariant real analytic splitting $T\mathcal{C} = \mathcal{A}\oplus\mathcal{V}$ on an open dense invariant subset of $\mathcal{C}$ with the following property. $\mathcal{A}$ is a complex subbundle of $T\mathcal{AP}(\mathcal{C})$, and the complex bundle $\mathcal{V}$ contains the tangent bundle of the orbits of the $SL(2,\mathbb{R})$-action.*

Proof. We proceed as in the proof of Lemma 8.1 and Proposition 8.3. Namely, assume to the contrary that there is an open dense invariant set $U\subset\mathcal{C}$, and there is a Φ^t-invariant splitting $T\mathcal{C}|U = \mathcal{A}\oplus\mathcal{V}$ as in the statement of the lemma.

An affine invariant manifold is locally defined by real linear equations in period coordinates (see [W14]). The affine structure of $\mathcal{C}$ defines a flat connection $\nabla^{\mathcal{C}}$ on $T\mathcal{C}$ which is invariant under affine transformations. In particular, this connection is invariant under the $SL(2,\mathbb{R})$-action. The bundle $\mathcal{A}\subset T\mathcal{AP}(\mathcal{C})$ is flat, i.e. invariant under parallel transport for $\nabla^{\mathcal{C}}$. Namely, it is globally trivialized by globally defined vector fields $X_{\mathfrak{a}_i}$ where $\mathfrak{a}_i\in\mathbb{C}^m$ (compare the proof of Proposition 9.1).

Our goal is to show that the bundle $\mathcal{V}$ is flat as well. To this end recall that there is a real analytic complex subbundle $\mathcal{N}\subset T\mathcal{C}$ which is invariant under the $SL(2,\mathbb{R})$-action and transverse to the tangent bundle $\mathcal{T}$ of the foliation of $\mathcal{C}$ into the orbits of the $SL(2,\mathbb{R})$-action. The restriction of the bundle $\mathcal{N}$ to the leaves of the absolute period foliation is flat. Using the splitting $T\mathcal{C} = \mathcal{N}\oplus\mathcal{T}$, project the flat connection $\nabla^{\mathcal{C}}$ on $T\mathcal{C}$ to a connection $\nabla^{\mathcal{N}}$ on $\mathcal{N}$. The bundle $\mathcal{N}|U$ admits a real analytic Φ^t-invariant decomposition $\mathcal{N} = \mathcal{W}\oplus\mathcal{A}$ where $\mathcal{W} = \mathcal{N}\cap\mathcal{V}$. Use this decomposition of $\mathcal{N}$ to project the connection $\nabla^{\mathcal{N}}$ to a connection $\nabla^{\mathcal{W}}$ on $\mathcal{W}$ (compare the proof of Lemma 8.1). This projection determines a Φ^t-equivariant real analytic tensor field

$$\beta\in\Omega(T^*\mathcal{C}\otimes\mathcal{W}^*\otimes\mathcal{A})$$

(compare the discussion in Section 8).

We aim at showing that β vanishes identically, and as before, for this it suffices to show that β vanishes at any point $q\in\mathcal{C}$ which is contained in a periodic orbit γ for Φ^t with the properties stated in Corollary 6.7. For $q\in\gamma$, the return map $A = d\Phi^{\ell(\gamma)}(q)$ acts on $\mathcal{N}_q$ as a semisimple automorphism, with all eigenvalues real and distinct from one. Moreover, no product of two eigenvalues is an eigenvalue.

Denote by $\mathcal{R}^{su}$ the intersection of $T\mathcal{AP}(\mathcal{C})$ with the tangent bundle of the strong unstable foliation, and let similarly $\mathcal{R}^{ss}$ be the intersection of $T\mathcal{AP}(\mathcal{C})$ with the tangent bundle of the strong stable foliation. Then the eigenspace for the restriction of the return map A to $\mathcal{N}_q$ for the eigenvalue $e^{-\ell(\gamma)}$ equals the subspace $\mathcal{R}^{ss}_q$, and the eigenspace for the eigenvalue $e^{\ell(\gamma)}$ is the subspace $\mathcal{R}^{su}_q$.

The bundle $\mathcal{A}$ decomposes as $\mathcal{A} = \mathcal{A}^{su} \oplus \mathcal{A}^{ss}$ where $\mathcal{A}^{su} = \mathcal{A} \cap \mathcal{R}^{su}$ and $\mathcal{A}^{ss} = \mathcal{A} \cap \mathcal{R}^{ss}$. We use this to show that β vanishes at q. Namely, since the decomposition is invariant under the action of the Teichmüller flow, all connections are invariant and hence the contraction of β with the generator $X \in T\mathcal{C}$ of Φ^t vanishes everywhere.

By equivariance and the reasoning in the proof of Lemma 8.1, this implies that if β does not vanish at q then up to replacing $\Phi^{\ell(\gamma)}$ by $\Phi^{-\ell(\gamma)}$, we may assume that there are vectors $Y \in \mathcal{N}_q \subset T\mathcal{C}_q, Z \in \mathcal{W}_q$ so that the product of the growth rates of Y and Z under $\Phi^{\ell(\gamma)}$ equals $e^{\ell(\gamma)}$. However, by the choice of γ and the fact that the eigenvalues of the restriction of A to $\mathcal{N}_q$ are off the unit circle, there are no two nontrivial vectors with this property.

As a consequence, the tensor field β vanishes identically, and the bundle $\mathcal{V}$ is flat (compare the discussion in Section 8). This implies that $T\mathcal{C}$ splits as a direct sum of two flat subbundles. By Theorem 5.1 of [W14], such a splitting does not exist. The lemma follows. $\square$

As an immediate consequence of Proposition 7.2 and Lemma 9.2, we obtain

Corollary 9.3. *An affine invariant manifold $\mathcal{C}$ of rank at least two contains only finitely many affine invariant submanifolds of the same rank.*

Theorem 2 from the introduction is now an immediate consequence of Proposition 8.5 and Corollary 9.3.

10. Algebraically primitive Teichmüller curves

A point in the moduli space of abelian differentials can be viewed as a *translation surface* (X, ω), where X denotes a Riemann surface of genus g. If the *Veech group* of such a translation surface contains a pseudo-Anosov element, then the *holonomy field* of (X, ω) coincides with the trace field of the pseudo-Anosov element (see e.g. the appendix of [KS00]- we do not need any additional information on this field). By Lemma 2.10 of [LNW15], if $\mathcal{C}$ is a rank one affine invariant manifold then for all $(X, \omega) \in \mathcal{C}$, the holonomy field of (X, ω) equals the *field of definition* of $\mathcal{C}$ [W14].

For the proof of Corollary 2 we have a closer look at rank one affine invariant manifolds $\mathcal{C}$ whose field of definition $\mathfrak{k}$ is of degree g over $\mathbb{Q}$. Then $\mathfrak{k}$ is a totally real [F16] number field of degree g, with ring of integers $\mathcal{O}_{\mathfrak{k}}$. Via the g field embeddings $\mathfrak{k} \to \mathbb{R}$, the group $SL(2, \mathcal{O}_{\mathfrak{k}})$ embeds into $G = SL(2, \mathbb{R}) \times \cdots \times SL(2, \mathbb{R}) < Sp(2g, \mathbb{R})$ and in fact, $SL(2, \mathcal{O}_{\mathfrak{k}})$ is a lattice in G. The trace field of every periodic orbit γ in $\mathcal{C}$ equals $\mathfrak{k}$ and hence the image of the corresponding pseudo-Anosov element $\Omega(\gamma)$ under the homomorphism $\Psi : \mathrm{Mod}(S) \to Sp(2g, \mathbb{R})$ is contained in $SL(2, \mathcal{O}_{\mathfrak{k}})$.

The following observation is immediate from Lemma 6.6 and [G12]. For its formulation, define the *extended local monodromy group* of an open contractible subset U of $\mathcal{C}$ to be the subgroup of $SL(2, \mathcal{O}_{\mathfrak{k}})$ which is generated by the monodromy of those (parametrized) periodic orbits for Φ^t in $\mathcal{C}$ which pass through U. Compare with Proposition 6.8.

Lemma 10.1. *For a rank one affine invariant manifold $\mathcal{C}$ whose field of definition is of degree g over $\mathbb{Q}$, any extended local monodromoy group is Zariski dense in $SL(2,\mathbb{R})\times\cdots\times SL(2,\mathbb{R})$.*

Proof. By Proposition 6.6, the projection of the extended local monodromy group of an open set $U\subset\mathcal{C}$ to the first factor $SL(2,\mathbb{R})$ of $G=SL(2,\mathbb{R})\times\cdots\times SL(2,\mathbb{R})$ is Zariski dense in $SL(2,\mathbb{R})$ and hence it is non-elementary. Moreover, by definition and [KS00, LNW15], the invariant trace field of the extended local monodromy group equals $\mathfrak{k}$ (compare [G12] for the definition of the invariant trace field). Thus by Corollary 2.2 of [G12], the extended local monodromy group of U is Zariski dense in G. □

In the statement of the next corollary, the affine invariant manifold $\mathcal{B}_+$ may be a component of a stratum. As before, we put a lower index $+$ whenever we do not normalize the area of a holomorphic differential.

Corollary 10.2. *Let $\mathcal{C}_+$ be a rank one affine invariant manifold whose field of definition is of degree g over $\mathbb{Q}$. Assume that $\mathcal{C}_+$ is properly contained in an affine invariant manifold $\mathcal{B}_+$ of rank at least three. Let $\mathcal{Z}\to\mathcal{B}_+$ be the absolute holomorphic tangent bundle of $\mathcal{B}_+$; then $\mathcal{Z}|\mathcal{C}_+$ splits as a sum of holomorphic line bundles which are invariant under both the Chern connection and the Gauss Manin connection.*

Proof. In the case that the rank of $\mathcal{B}_+$ equals g (and hence $\mathcal{Z}=\mathcal{H}$), the statement is immediate from Theorem 1.5 of [W14]. Thus assume that the rank of $\mathcal{B}_+$ is at most $g-1$.

Since $\mathcal{C}_+\subset\mathcal{B}_+$, the restriction of $\mathcal{H}$ to $\mathcal{C}_+$ has two splittings which are invariant under the extended local monodromy of $\mathcal{C}_+$. The first splitting is the splitting into g line bundles obtained from the different field embeddings of the field of definition of $\mathcal{C}_+$ into $\mathbb{R}$ (see Theorem 1.5 of [W14]). The second splitting is the splitting into the absolute holomorphic tangent bundle $\mathcal{Z}$ of $\mathcal{B}_+$ (which is a holomorphic subbundle of $\mathcal{H}|_{\mathcal{B}_+}$ whose complex rank equals the rank of $\mathcal{B}_+$) and its symplectic complement. Since by Lemma 10.1 the extended local monodromy group of $\mathcal{C}_+$ is Zariski dense in $SL(2,\mathbb{R})\times\cdots\times SL(2,\mathbb{R})$, the bundle $\mathcal{Z}|\mathcal{C}_+$ is a sum of invariant line bundles. □

Corollary 10.3. *For a component $\mathcal{Q}_+$ of a stratum in genus $g\geq 3$, all affine invariant submanifolds of rank one whose fields of definition are of degree g over $\mathbb{Q}$ are contained in a finite collection of affine invariant submanifolds of rank at most two.*

Proof. Let $\mathfrak{C}$ be the collection of all rank one affine invariant submanifolds of $\mathcal{Q}$ whose field of definition is a number field of degree g over $\mathbb{Q}$. For each $\mathcal{C}_+\in\mathfrak{C}$, the restriction of the bundle $\mathcal{L}$ to $\mathcal{C}_+$ splits as a sum of flat holomorphic line bundles which are invariant under the Gauss-Manin connection. Thus by Proposition 7.2 and its proof, all but finitely many elements of $\mathfrak{C}$ are submanifolds of a finite set of affine invariant submanifolds of $\mathcal{Q}$ of rank at most $g-1$.

Now if $\mathcal{B}_+$ is an affine invariant manifold of rank contained in $[3, g-1]$ which contains some $\mathcal{C}_+ \in \mathfrak{C}$ then by Corollary 10.2, the above reasoning can be applied to $\mathcal{B}_+$. In finitely many steps we find finitely many proper affine invariant manifolds $\mathcal{C}_1, \dots, \mathcal{C}_k \subset \mathcal{Q}_+$ of rank at most two which contain every $\mathcal{C}_+ \in \mathfrak{C}$. This is the statement of the corollary. □

Now we are ready to complete the proof of Corollary 2.

Corollary 10.4. *For $g \geq 3$ the $SL(2,\mathbb{R})$-orbit closure of a typical periodic orbit in any component of a stratum is the entire stratum.*

Proof. Let $\mathcal{Q}$ be a component of a stratum and let $U \subset \mathcal{Q}$ be a non-empty open set. Then a typical periodic orbit for Φ^t passes through U [H13]. Thus by Theorem 2 (see also [MW15, MW16]), the $SL(2,\mathbb{R})$-orbit closure of a typical periodic orbit either equals the entire stratum, or it is an affine invariant manifold of rank one.

By the second part of Theorem 1, the trace field of a typical perodic orbit γ is totally real and of degree g over $\mathbb{Q}$. If the rank of the $SL(2,\mathbb{R})$-orbit closure $\mathcal{C}$ of γ equals one then this trace field is the field of definition of $\mathcal{C}$ [LNW15]. Thus the corollary follows from Corollary 10.3. □

We complete this work with the proof of Theorem 3. We begin with

Proposition 10.5. *Let $g \geq 3$ and let $\mathcal{B}_+ \subset \mathcal{Q}_+$ be a rank two affine invariant manifold. Then the union of all algebraically primitive Teichmüller curves which are contained in $\mathcal{B}_+$ is nowhere dense in $\mathcal{B}_+$.*

Proof. Let $\mathcal{B}_+ \subset \mathcal{Q}_+$ be a rank two affine invariant manifold. We argue by contradiction, and we assume that the closure of the union of all algebraically primitive Teichmüller curves $\mathcal{C}_+ \subset \mathcal{B}_+$ contains some open subset V of $\mathcal{B}_+$.

Let $\mathcal{Z} \to \mathcal{B}_+$ be the absolute holomorphic tangent bundle of $\mathcal{B}_+$. Let U be a small contractible subset of V so that there is a trivialization of the Hodge bundle over U defined by the Gauss Manin connection. The extended local monodromy group of U preserves $\mathcal{Z}$. Let $\mathcal{C}_i \subset \mathcal{B}_+$ be a sequence of algebraically primitive Teichmüller curves which pass through U and whose closures contain a compact subset of U with non-empty interior W.

Let $\Pi : \mathcal{Q}_+ \to \mathcal{M}_g$ be the canonical projection and let $\mathcal{I}_g : \mathcal{M}_g \to \mathcal{A}_g$ be the Torelli map. The image under Π of the curve $\mathcal{C}_i$ is an algebraic curve which admits a *modular embedding.* Namely, by the main result of [Mo06], there is a totally real number field K_i of degree g over $\mathbb{Q}$, there is an order $\mathfrak{o}_{K_i}$ in K_i, and there is an embedding

$$SL(2, \mathfrak{o}_{K_i}) \to SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R}) \to Sp(2g,\mathbb{R})$$

which maps $SL(2,\mathfrak{o}_{K_i})$ into $Sp(2g,\mathbb{Z})$ and such that the image of $\mathcal{C}_i$ under the Torelli map is contained in the Hilbert modular variety $H(\mathfrak{o}_{K_i})$. This Hilbert modular variety is the quotient of $\mathbf{H}^2 \times \cdots \times \mathbf{H}^2$ under the lattice $SL(2,\mathfrak{o}_{K_i})$ in a Lie subgroup G_i of $Sp(2g,\mathbb{R})$ which is isomorphic to $SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$.

We claim that $G_i = G_j = G$ for all i. Namely, assume otherwise. Then there are algebraically primitive Teichmüller curves $\mathcal{C}_i, \mathcal{C}_j$ which intersect U and for which the groups G_i, G_j are distinct. By Lemma 10.1, the extended local monodromy groups of $\mathcal{C}_i \cap U$ and $\mathcal{C}_j \cap U$ are Zariski dense in G_i, G_j. Therefore the Zariski closure in $Sp(2g, \mathbb{R})$ of the extended local monodromy group of $U \subset \mathcal{B}_+$ contains $G_i \cup G_j$. But as $G_i \neq G_j$, a subgroup of $Sp(2g, \mathbb{R})$ which contains $G_i \cup G_j$ can not preserve the subspace $\mathcal{Z}$. This is a contractiction and implies that indeed, $G_i = G_j = G$ for all i.

Write $SL(2, \mathfrak{o}) = SL(2, \mathfrak{o}_{K_i})$. The Hilbert modular variety $H(\mathfrak{o}) = H(\mathfrak{o}_{K_i}) \subset \mathcal{A}_g$ consists of abelian varieties with real multiplication with the field $K = K_i$. The image of $\mathcal{C}_i$ under the map $\mathcal{I}_g \circ \Pi$ is contained in $H(\mathfrak{o})$. As a consequence, the set of points in $\mathcal{B}_+$ which are mapped by the composition of the foot-point projection $\Pi : \mathcal{B}_+ \to \mathcal{M}_g$ with the Torelli map $\mathcal{I}_g$ into $H(\mathfrak{o})$ contains a dense subset of the open set W. But $H(\mathfrak{o})$ is a complex submanifold of $\mathcal{A}_g$ and this composition map is holomorphic and therefore the image of $\mathcal{B}_+$ is contained in $H(\mathfrak{o})$.

We showed so far that each point in $\mathcal{B}_+$ is an abelian differential whose Jacobian has real multiplication with K. Now a point on an algebraically primitive Teichmüller curve is mapped to an eigenform for real multiplication [Mo06] and hence the closure of the set of differentials in $\mathcal{B}_+$ which are mapped to eigenforms for this real multiplication contains an open set. This implies as before that each point in $\mathcal{B}_+$ corresponds to such an eigenform and hence $\mathcal{B}_+$ is a rank one affine invariant manifold, contrary to our assumption. The proposition follows $\square$

Proof of Theorem 3:

Let $\mathcal{Q}$ be a component of a stratum in genus $g \geq 3$. By Corollary 10.3, there are finitely many affine invariant submanifolds $\mathcal{B}_1, \dots, \mathcal{B}_k$ of rank two which contain all but finitely many algebraically primitive Teichmüller curves.

Let $\mathcal{B}_i$ be such an affine invariant manifold of rank two. Assume that its dimension equals r for some $r \geq 4$. By Proposition 10.5, the closure of the union of all algebraically primitive Teichmüller curves which are contained in $\mathcal{B}_i$ is nowhere dense in $\mathcal{B}_i$. As this closure is invariant under the action of $GL(2, \mathbb{R})$, it consists of a finite union of affine invariant manifolds. The dimension of each of these invariant submanifolds is at most $r - 1$.

If there are submanifolds of rank two in this collection then we can repeat this argument with each of these finitely many submanifolds. By inverse induction on the dimension, this yields that all but finitely many algebraically primitive Teichmüller curves are contained in one of finitely many affine invariant manifolds of rank one. The field of definition of such a manifold coincides with the field of definition of the Teichmüller curve, in particular it is of degree g [LNW15].

By the main result of [LNW15], a rank one affine invariant manifold with field of definition of degree g over $\mathbb{Q}$ only contains finitely many Teichmüller curves. Thus the number of algebraically primitive Teichmüller curves in $\mathcal{Q}$ is finite as promised.

Appendix A. Structure of the homogeneous space $Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})$

In this appendix we collect some geometric properties of the Siegel upper half-space $\mathfrak{D}_g = Sp(2g,\mathbb{R})/U(g)$ and its quotient $\mathcal{A}_g = Sp(2g,\mathbb{Z})\backslash\mathfrak{D}_g$ which are either directly or indirectly used in the proofs of our main results.

The tautological vector bundle

$$\mathcal{V} \to \mathfrak{D}_g$$

over the Hermitean symmetric space $\mathfrak{D}_g = Sp(2g,\mathbb{R})/U(g)$ is obtained as follows.

Via the right action of the unitary group $U(g)$, the symplectic group $Sp(2g,\mathbb{R})$ is an $U(g)$-principal bundle over $\mathfrak{D}_g$. The bundle $\mathcal{V}$ is the associated vector bundle

$$\mathcal{V} = Sp(2g,\mathbb{R}) \times_{U(g)} \mathbb{C}^g$$

where $U(g)$ acts from the right by $(x,y,\alpha) \to (x\alpha,\alpha^{-1}y)$. The bundle $\mathcal{V}$ is holomorphic, and it is equipped with a hermitean metric obtained from an $U(g)$-invariant hermitean inner product on $\mathbb{C}^g$. As $U(g)$ acts transitively on the unit sphere in $\mathbb{C}^g$, with isotropy group $U(g-1)$, the associated sphere bundle

$$\mathcal{S} = Sp(2g,\mathbb{R}) \times_{U(g)} S^{2g-1}$$

in $\mathcal{V} \to \mathfrak{D}_g$ can naturally be identified with the homogeneous space

$$\mathcal{S} = Sp(2g,\mathbb{R})/U(g-1)$$

(Proposition 5.5 of [KN63]).

The group $Sp(2g-2,\mathbb{R})$ is the isometry group of Siegel upper half-space

$$\mathfrak{D}_{g-1} = Sp(2g-2,\mathbb{R})/U(g-1).$$

Since the action of $Sp(2g-2,\mathbb{R})$ on $\mathfrak{D}_{g-1}$ is transitive, with isotropy group $U(g-1)$, the bundle $\mathcal{S} = Sp(2g,\mathbb{R})/U(g-1) \to \mathfrak{D}_g$ can also be identified with the associated bundle

$$\mathcal{S} = Sp(2g,\mathbb{R}) \times_{Sp(2g-2,\mathbb{R})} \mathfrak{D}_{g-1}$$

where $Sp(2g-2,\mathbb{R})$ act via

$$(g,x)h = (gh, h^{-1}(x)).$$

The first factor projection then defines a projection

$$\Pi : \mathcal{S} \to Sp(2g,\mathbb{R})/Sp(2g-2,\mathbb{R}).$$

Let $\omega = \sum_i dx_i \wedge dy_i$ be the standard symplectic form on $\mathbb{R}^{2g}$. The standard representation of $Sp(2g,\mathbb{R})$ on $(\mathbb{R}^{2g},\omega)$ naturally extends to an action of $Sp(2g,\mathbb{R})$ on $\mathbb{R}^{2g}\otimes\mathbb{C} = \mathbb{C}^{2g}$. The open subset

$$\mathcal{O} = \{x+iy \mid x,y \in \mathbb{R}^{2g}, \omega(x,y) > 0\} \subset \mathbb{C}^{2g}$$

is $Sp(2g,\mathbb{R})$-invariant. It contains the invariant hypersurface

$$\Omega = \{x+iy \in \mathbb{C}^{2g} \mid \omega(x,y) = 1\}.$$

We claim that Ω can naturally and $Sp(2g,\mathbb{R})$-equivariantly be identified with the homogeneous space

$$Sp(2g,\mathbb{R})/Sp(2g-2,\mathbb{R}).$$

To this end just observe that the diagonal action of the group $Sp(2g,\mathbb{R})$ on Ω is transitive. The stabilizer in $Sp(2g,\mathbb{R})$ of a point $x+iy\in\Omega$ is isomorphic to a standard embedded

$$\mathrm{Id}\times Sp(2g-2,\mathbb{R}) < Sp(2,\mathbb{R})\times Sp(2g-2,\mathbb{R}) < Sp(2g,\mathbb{R}).$$

The group $SL(2,\mathbb{R})$ acts from the right on Ω as follows. The real and imaginary part of a point $x+iy\in\Omega$ define the basis of a two-dimensional symplectic subspace of $\mathbb{R}^{2g}$. The group $SL(2,\mathbb{R})$ acts by basis transformation on this subspace, preserving the symplectic form. The $SL(2,\mathbb{R})$-orbit through a point $x+iy\in\Omega$ preserves pointwise the symplectic complement of the subspace V of $\mathbb{R}^{2g}$ spanned by x and y as well as the complexification of V.

As the left action of $Sp(2g,\mathbb{R})$ on $\mathcal{O}$ is the restriction of a linear action on $\mathbb{C}^{2g}$, the tangent bundle of $\mathcal{O}$ admits an $Sp(2g,\mathbb{R})$-invariant flat connection. Vector fields which are tangent to the orbits of a one-parameter group of translations are globally parallel. This connection restricts to a flat connection ∇^{GM} on $T\mathcal{O}|\Omega$.

The bundle $T\mathcal{O}|\Omega$ splits a sum

$$T\mathcal{O}|\Omega = T\Omega\oplus\mathbb{R}$$

where the trivial line bundle $\mathbb{R}$ is the tangent bundle of the orbits of the one-parameter group of deformations $((x+iy),t)\to e^tx+ie^ty$ transverse to Ω. This splitting is not flat, i.e. it is not invariant under the connection ∇^{GM}.

The tangent bundle $T\Omega$ of Ω has a natural splitting

$$T\Omega = \mathcal{R}\oplus\mathcal{T}$$

where for a point $x+iy\in\Omega$ we have

$$\mathcal{R}_{x+iy} = \{u+iv \mid \omega(u,x)=\omega(u,y)=\omega(v,x)=\omega(v,y)=0\}$$

and where $\mathcal{T}$ is tangent to the orbits of the right action of $SL(2,\mathbb{R})$. Moreover, we have $\mathcal{R}=\mathcal{E}\otimes\mathbb{C}$ where $\mathcal{E}_{x+iy}=\{u\in\mathbb{R}^{2g}\mid \omega(u,y)=\omega(u,x)=0\}$.

By definition of the action of $Sp(2g,\mathbb{R})$, the bundles $\mathcal{R}$ and $\mathcal{T}$ are invariant under both the left action of $Sp(2g,\mathbb{R})$ and the right action of $SL(2,\mathbb{R})$. Thus the flat left $Sp(2g,\mathbb{R})$-invariant connection ∇^{GM} on $T\mathcal{O}|\Omega$ projects to a left $Sp(2g,\mathbb{R})$-invariant right $SL(2,\mathbb{R})$-invariant connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ defined as follows. Let

$$P: T\mathcal{O}|\Omega = \mathcal{R}\oplus\mathcal{T}\oplus\mathbb{R}\to\mathcal{R}$$

be the canonical projection, and for $X\in T\Omega$ and a local section Y of $\mathcal{R}$ define $\nabla^{\mathcal{R}}_XY = P\nabla^{GM}_X(Y)$.

We summarize this as follows.

Lemma A.1. *The flat $Sp(2g,\mathbb{R})$-invariant connection on $T\mathcal{O}$ projects to a connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ which is invariant under both the left $Sp(2g,\mathbb{R})$ action and the right $SL(2,\mathbb{R})$ action.*

The curvature of the connection $\nabla^{\mathcal{R}}$ is a two-form on Ω with values in the Lie algebra $\mathfrak{sp}(2g-2,\mathbb{R})$ of $Sp(2g-2,\mathbb{R})$, acting as an algebra of transformation on $\mathcal{R}$. The restriction of this two-form to the tangent bundle of the orbits of the $SL(2,\mathbb{R})$-action vanishes. Moreover, the two-form is equivariant with respect to the left action of $Sp(2g,\mathbb{R})$ and the right action of $SL(2,\mathbb{R})$.

We say that the curvature form Θ for a connection ∇ on a complex vector bundle $E \to M$ *splits* E *as a complex vector bundle* if there is a nontrivial Θ-invariant decomposition $E = E_1 \oplus E_2$ as a Whitney sum of two complex vector bundles. This means that for any $x \in M$ and any two vectors $Y, Z \in T_xM$ the map $\Theta(Y,Z)$ preserves the decomposition $E = E_1 \oplus E_2$.

Since Ω is not locally affine, the curvature form of the connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ does not vanish identically. Furthermore, the stabilizer in $Sp(2g,\mathbb{R})$ of a point $z \in \Omega$ can be identified with the subgroup $Sp(2g-2,\mathbb{R})$, which act on the fibre of $\mathcal{R}$ at z via the standard representation of $Sp(2g-2,\mathbb{R})$ on $\mathbb{C}^{2g-2}$, viewed as the complexification of the standard representation on $\mathbb{R}^{2g-2}$. Since the standard representation of $Sp(2g-2,\mathbb{R})$ on the complex vector space $\mathbb{C}^{2g-2}$ is irreducible, by equivariance, we have

Lemma A.2. *The curvature form of $\nabla^{\mathcal{R}}$ does not split $\mathcal{R}$ as a complex vector bundle.*

The complement of the zero section $\mathcal{V}_0 \subset \mathcal{V}$ of the bundle $\mathcal{V} \to \mathcal{D}_g$ is a complex manifold. The fibration $\mathcal{S} \to \Omega$ extends to a holomorphic fibration $\mathcal{V}_0 \to \mathcal{O}$ of complex manifolds. The fibres of the fibration define a foliation $\mathcal{F}$ of $\mathcal{V}_0$.

The following is immediate from the definition of the complex structure on $\mathcal{V}_0$ and on $\mathcal{O} \subset \mathbb{C}^{2g}$.

Lemma A.3. *The foliation $\mathcal{F}$ is holomorphic. A fibre is biholomorphic to $\mathfrak{D}_{g-1}$.*

Let for the moment G be an arbitrary Lie group. A *G-connection* for a G-principal bundle $X \to Y$ is given by a $\mathrm{Ad}(G)$-invariant subbundle of the tangent bundle of X transverse to the tangent bundle of the fibres. Such a bundle is called *horizontal*.

The group $Sp(2g,\mathbb{R})$ is an $Sp(2g-2,\mathbb{R})$-principal bundle over Ω. In the statement of the following Lemma, the type $(2g, 2g-1)$ stems from the fact that Ω is a hypersurface in the manifold $\mathcal{O}$ with invariant indefinite metric of type $(2g,2g)$.

Lemma A.4. *The $Sp(2g-2,\mathbb{R})$-principal bundle $Sp(2g,\mathbb{R}) \to \Omega$ admits a natural real analytic $Sp(2g-2,\mathbb{R})$-connection which is invariant under the left action of $Sp(2g,\mathbb{R})$ and the right action of $SL(2,\mathbb{R})$. The horizontal bundle $\mathcal{Z}_0$ contains the tangent bundle $\mathcal{T}$ of the orbits of the $SL(2,\mathbb{R})$-action, and it admits an $SL(2,\mathbb{R})$-invariant $Sp(2g,\mathbb{R})$-invariant pseudo-Riemannian metric h of type $(2g, 2g-1)$. The h-orthogonal complement $\mathcal{Y}_0$ of $\mathcal{T}$ in $\mathcal{Z}_0$ is a real analytic $SL(2,\mathbb{R})$-invariant $Sp(2g,\mathbb{R})$invariant bundle.*

Proof. The fibre containing the identity induces an embedding of Lie algebras

$$\mathfrak{sp}(2g-2,\mathbb{R}) \to \mathfrak{sp}(2g,\mathbb{R}).$$

The restriction of the Killing form B of $\mathfrak{sp}(2g,\mathbb{R})$ to the Lie algebra $\mathfrak{sp}(2g-2,\mathbb{R})$ is non-degenerate. Thus the B-orthogonal complement $\mathfrak{z}$ of $\mathfrak{sp}(2g-2,\mathbb{R})$ is a linear subspace of $\mathfrak{sp}(2g,\mathbb{R})$ which is complementary to $\mathfrak{sp}(2g-2,\mathbb{R})$ and invariant under the restriction of the adjoint representation Ad of $Sp(2g,\mathbb{R})$ to $Sp(2g-2,\mathbb{R})$. The restriction to $\mathfrak{z}$ of the Killing form is a non-degenerate bilinear form of type $(2g, 2g-1)$.

The group $Sp(2g,\mathbb{R})$ acts by left translation on itself, and this commutes with the right action of $Sp(2g-2,\mathbb{R})$. Hence this action defines an action by automorphisms of the principal bundle.

Define a $\mathfrak{sp}(2g-2,\mathbb{R})$-valued one-form θ on $Sp(2g,\mathbb{R})$ by requiring that $\theta(e)$ equals the canonical projection

$$T_eSp(2g,\mathbb{R}) = \mathfrak{z} \oplus \mathfrak{sp}(2g-2,\mathbb{R}) \to \mathfrak{sp}(2g-2,\mathbb{R})$$

and

$$\theta(g) = \theta \circ dg^{-1}.$$

Then for every $h \in Sp(2g-2,\mathbb{R})$ we have

$$\theta(gh) = \mathrm{Ad}(h^{-1}) \circ \theta(g)$$

and hence this defines an $Sp(2g,\mathbb{R})$-invariant connection on the $Sp(2g-2,\mathbb{R})$-principal bundle $Sp(2g,\mathbb{R}) \to \Omega$. Denote by $\mathcal{Z}_0$ the horizontal bundle. It is invariant under the left action of $Sp(2g,\mathbb{R})$ and the right action of $Sp(2g-2,\mathbb{R})$, and it is equipped with an invariant pseudo-Riemannian metric of type $(2g, 2g-1)$,

Now $\mathfrak{sp}(2,\mathbb{R}) \subset \mathfrak{z}$, and hence the tangent bundle for the right action of $Sp(2,\mathbb{R})$ is contained in the horizontal bundle $\mathcal{Z}_0$. Thus the subbundle $\mathcal{Y}_0$ of $\mathcal{Z}_0$ defined by the B-orthogonal complement $\mathfrak{y}$ in $\mathfrak{z}$ of the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$ is invariant as well. The lemma follows. □

Since $\mathcal{S} = Sp(2g,\mathbb{R}) \times_{Sp(2g-2,\mathbb{R})} \mathfrak{D}_{g-1}$ and since the subgroups $SL(2,\mathbb{R})$ and $Sp(2g-2,\mathbb{R})$ commute, the right action of $SL(2,\mathbb{R})$ on $Sp(2g,\mathbb{R})$ descend to an action of $SL(2,\mathbb{R})$ on $\mathcal{S}$. The action of the unitary subgroup $U(1)$ of $Sp(2,\mathbb{R})$ is just the standard circle action on the fibres of the sphere bundle $\mathcal{S} \to \mathfrak{D}_g$ given by multiplication with complex numbers of absolute value one. The connection $\mathcal{Z}_0 = \mathcal{T} \oplus \mathcal{Y}_0$ induces a real analytic splitting

$$T\mathcal{S} = T\mathcal{F} \oplus \mathcal{Z} = T\mathcal{F} \oplus \mathcal{T} \oplus \mathcal{Y}$$

where $T\mathcal{F}$ denotes the tangent bundles of the fibres of the fibration $\mathcal{S} \to \Omega$, the *horizontal bundle* $\mathcal{Z}$ is the image of $\mathcal{Z}_0 \times T\mathfrak{D}_{g-1}$ under the projection $Sp(2g,\mathbb{R}) \times \mathfrak{D}_{g-1} \to \mathcal{S}$ and as before, $\mathcal{T}$ is the tangent bundle of the orbits of the $SL(2,\mathbb{R})$-action.

We also note

Lemma A.5. *The right action of $SL(2,\mathbb{R})$ on $\mathcal{S}$ projects to the standard action of $SL(2,\mathbb{R})$ on Ω.*

Proof. This follows as before from naturality and bi-invariance of the Killing form. □

The group $Sp(2g,\mathbb{Z})$ acts properly discontinuously from the left on the bundle $\mathcal{S}\to\Omega$ as a group of real analytic bundle automorphisms. In particular, it preserves the real analytic splitting of the tangent bundle of $\mathcal{S}$ into the tangent bundle of the leaves of the foliation $\mathcal{F}$ and the complementary bundle. Thus this splitting descends to an $SL(2,\mathbb{R})$-invariant real analytic splitting of the tangent bundle of the quotient. This quotient is just the sphere bundle of the quotient vector bundle (in the orbifold sense) over the locally symmetric space

$$\mathcal{A}_g = Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/U(g).$$

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