# TYPICAL PROPERTIES OF PERIODIC TEICHMÜLLER GEODESICS: STRETCH FACTORS

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Abstract. Consider a component $\mathcal{Q}$ of a stratum in the moduli space of area one abelian or quadratic differentials on a surface of genus $g$. Call a property $\mathcal{P}$ for periodic orbits of the Teichmüller flow on $\mathcal{Q}$ *typical* if the growth rate of orbits with property $\mathcal{P}$ is maximal. We show that the following property is typical. If $\mathcal{Q}$ is a stratum of abelian differentials, then the trace field of the symplectic matrix defined by the orbit is a totally real splitting field of degree $g$ over $\mathbb{Q}$. If $\mathcal{Q}$ is a component of a stratum of quadratic differentials with $k \geq 0$ zeros of odd order then the stretch factor of a typical orbit is of degree $2g-2+k$ over $\mathbb{Q}$.

## Contents



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## 1. Introduction

The *mapping class group* $\mathrm{Mod}(S)$ of a closed surface $S$ of genus $g \geq 2$ acts by precomposition of marking on the *Teichmüller space* $\mathcal{T}(S)$ of marked complex structures on $S$. The action is properly discontinuous, with quotient the *moduli space* $\mathcal{M}_g$ of complex structures on $S$.

The fibre over a Riemann surface $x \in \mathcal{M}_g$ of the *Hodge bundle* $\mathcal{H} \to \mathcal{M}_g$ equals the vector space of holomorphic one-forms on $x$. The Hodge bundle is a holomorphic vector bundle of complex dimension $g$ (in the orbifold sense) which decomposes into *strata* of differentials with zeros of given multiplicities. There is a natural $SL(2,\mathbb{R})$-action on $\mathcal{H}$ preserving its sphere subbundle of area one abelian differentials on $S$ as well as any connected component of a stratum. The action of the diagonal subgroup is called the *Teichmüller flow* $\Phi^t$.

The cotangent bundle of moduli space can naturally be identified with the bundle of holomorphic quadratic differentials over $\mathcal{M}_g$. It also admits a natural $SL(2,\mathbb{R})$-action preserving the strata of differentials with zeros of given multiplicities and the sphere bundle of area one quadratic differentials. The action of the diagonal subgroup is again called the Teichmüller flow $\Phi^t$.

Let $\mathcal{Q}$ be a component of stratum of area one abelian or quadratic differentials and let $\Gamma$ be the set of all periodic orbits for $\Phi^t$ in $\mathcal{Q}$. The length of a periodic orbit $\gamma \in \Gamma$ is denoted by $\ell(\gamma)$. Let $m \geq 1$ be the number of singular points of the differentials in $\mathcal{Q}$ and let $h = 2g-1+m$ if $\mathcal{Q}$ is a component of abelian differentials, and let $h = 2g-2+m$ otherwise. As an application of [EMR12] (see also [EM11]) we showed in [H13] that

$$\sharp\{\gamma \in \Gamma \mid \ell(\gamma) \leq R\}\frac{hR}{e^{hR}} \to 1 \quad (R \to \infty).$$

Call a subset $\mathcal{A}$ of $\Gamma$ *typical* if

$$\sharp\{\gamma \in \mathcal{A} \mid \ell(\gamma) \leq R\}\frac{hR}{e^{hR}} \to 1 \quad (R \to \infty).$$

Thus a subset of $\Gamma$ is typical if its growth rate is maximal. The intersection of two typical subsets of $\Gamma$ is typical.

A periodic orbit $\gamma \in \Gamma$ for $\Phi^t$ determines the conjugacy class of a pseudo-Anosov mapping class. The mapping class group acts on the first integral cohomology group

$H^1(S,\mathbb{Z})$ of $S$ preserving the intersection form $\iota$ on $H^1(S,\mathbb{Z})$. This action defines a natural surjective [FM12] homomorphism

$$\Psi : \mathrm{Mod}(S) \to Sp(2g,\mathbb{Z}).$$

Thus a periodic orbit $\gamma \in \Gamma$ determines the conjugacy class $[A(\gamma)]$ of a matrix $A(\gamma) \in Sp(2g,\mathbb{Z})$.

The characteristic polynomial of a symplectic matrix $A \in Sp(2g,\mathbb{Z})$ is a reciprocal polynomial of degree $2g$ with integral coefficients. Its roots define a number field $\mathfrak{k}$ of degree at most $2g$ over $\mathbb{Q}$ which is a quadratic extension of the so-called *trace field* of $A$. The field $\mathfrak{k}$ only depends on the conjugacy class of $A$. We show

**Theorem 1.** *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials. The set of all $\gamma \in \Gamma$ such that the field of $[A(\gamma)]$ is of degree $2g$ over $\mathbb{Q}$, separable and totally real is typical.*

Theorem 1 can be used to analyze *stretch factors* of pseudo-Anosov elements $\varphi \in \mathrm{Mod}(S)$. Here the stretch factor of $\varphi$ is the unique number $\lambda > 1$ such that there exists a measured foliation $\xi$ on $S$ with $\varphi(\xi) = \lambda\xi$, and it only depends on the conjugacy class of $\varphi$. In the case that $\varphi$ fixes a pair of oriented projective measured foliations, this stretch factor is just the leading eigenvalue for the action of $\varphi$ on $H^1(S,\mathbb{R})$. Theorem 1 then states that for a typical pseudo Anosov conjugacy class preserving a pair of oriented projective measured foliations, the stretch factor is an algebraic integer of degree $2g$ over $\mathbb{Q}$.

The maximal degree over $\mathbb{Q}$ of the stretch factor for arbitrary pseudo-Anosov elements is known to be $6g-6$. This was claimed by Thurston in [Th88] and was verified in [St15]. The article [St15] shows more precisely that a number $d$ is the algebraic degree of the stretch factor of a pseudo-Anosov mapping class if and only if either $d$ is at most $3g-3$, or $d$ is even and at most $6g-6$. We show

**Theorem 2.** *Let $\mathcal{D}$ a component of a stratum of area one quadratic differentials consisting of differentials with $k \geq 0$ zeros of odd order. Then the algebraic degree of the stretch factor of a pseudo-Anosov conjugacy class defined by a typical periodic orbit in $\mathcal{D}$ equals $2g-2+k$.*

Unlike in Theorem 1, we do not show that the extension of $\mathbb{Q}$ determined by a typical stretch factor is totally real. The proof of this fact in the case of a component of a stratum of abelian differentials uses simplicity of the Lyapunov spectrum for the Kontsevich Zorich cocycle [AV07] which is not available at the moment for components of strata of quadratic differentials. We will address this question in forthcoming work.

As an easy corollary, we obtain

**Corollary.** *For every $g \geq 2$ and every even number $2m \leq 6g-6$, there are infinitely many distinct conjugacy classes of pseudo-Anosov mapping classes whose stretch factors are algebraic intergers of degree $2m$ over $\mathbb{Q}$.*

The proofs of these results use a result of independent interest which we explain next.

Period coordinates on a component $\mathcal{Q}$ of a stratum of abelian differentials with singular set $\Sigma \subset S$ are obtained by integration of a holomorphic one-form $q \in \mathcal{Q}$ over a basis of the relative homology group $H_1(S,\Sigma;\mathbb{Z})$. Thus a tangent vector of $\mathcal{Q}$ defines a point in $H_1(S,\Sigma;\mathbb{C})^*$.

By the groundbreaking work of Eskin, Mirzakhani and Mohammadi [EMM15], the orbit closures of the $SL(2,\mathbb{R})$-action on the moduli space of abelian differentials are precisely the so-called *affine invariant manifolds*. Such manifolds are cut out by linear equations in period coordinates. The *rank* of an affine invariant manifold $\mathcal{C}$ is defined by

$$\mathrm{rk}(\mathcal{C}) = \frac{1}{2}\dim_{\mathbb{C}}(pT\mathcal{C})$$

where $p$ is the projection of $H_1(S,\Sigma;\mathbb{C})^*$ into $H_1(S;\mathbb{C})^* = H^1(S;\mathbb{C})$ [W14]. The rank of a component of a stratum equals $g$.

Recall that the $\alpha$*-limit set* of an orbit $\{\Phi^t x\}$ of the Teichmüller flow on a component $\mathcal{Q}$ of abelian or quadratic differentials is the set of points $y \in \mathcal{Q}$ for which there exists a sequence $t_i \to \infty$ so that $\Phi^{t_i}x \to y$. Similarly, the $\omega$*-limit set* is defined to be the set of accumulation points of the backward orbit $t \to \Phi^{-t}x$. A point $x \in \mathcal{Q}$ is *birecurrent* if it is contained in its own $\alpha$- and $\omega$-limit set.

The Hodge bundle $\mathcal{H}$ is equipped with a natural flat connection, the so-called *Gauss-Manin connection*. This connection defines a trivialization of $\mathcal{H}$ over any contractible subset of moduli space not containing any singular points, and this trivialization is unique up to conjugation. If $U$ is a contractible subset of an affine invariant manifold $\mathcal{C}$, then there is a natural trivialization of the pullback $\Pi^*\mathcal{H}$ of the Hodge bundle $\mathcal{H}$ over $U$, defined by the pullback connection. If we fix such a trivialization, then we can identify the fibres of $\Pi^*\mathcal{H}$ over $U$. Thus for any fixed basepoint $x \in U$, we can study the monodromy of the pullback connection at $x$ and relate it to the monodromy along periodic orbits of $\Phi^t$ passing through $U$.

More precisely, if we denote by $\mathcal{Z}_{\mathbb{R}}$ the projection of the tangent bundle of $\mathcal{C}$ to $H^1(S,\mathbb{R}) \subset H^1(S,\mathbb{C})$ then $\mathcal{Z}_{\mathbb{R}}$ is a flat symplectic [AEM12, F16] subbundle of the restriction of $\Pi^*\mathcal{H}$ to $\mathcal{C}$, and we can study the subsemigroup of $Sp(\mathcal{Z}_{\mathbb{R}},\mathbb{R})$ generated by the return maps of periodic orbits $\gamma$ for $\Phi^t$ passing through $U$.

**Definition.** The monodromy of the affine invariant manifold $\mathcal{C}$ is *locally Zariski dense* if for every birecurrent point $q \in \mathcal{C}$ and every neighborhood $U$ of $q$ in $\mathcal{C}$, the subsemigroup of $Sp(\mathcal{Z}_{\mathbb{R}},\mathbb{R})$ generated by the monodromy maps of parametrized periodic orbits for $\Phi^t$ beginning at a point in $U$ is Zariski dense in $Sp(\mathcal{Z}_{\mathbb{R}},\mathbb{R})$.

The following result is used in the proof of the main theorems.

**Theorem 3.** *The monodromy of any affine invariant manifold is locally Zariski dense.*

For components $Q$ of strata of abelian differentials, there is an easy translation of Theorem 3 into the language of Rauzy induction and the so-called *Rauzy-Veech group* of $\mathcal{Q}$ which yields a solution to a conjecture of Zorich (Conjecture 5 of [Z99]). However, Rauzy induction plays no role in our approach, and we leave this translation to other authors. The Rauzy-Veech group of a component of a stratum of abelian differentials has independently been studied by Avila, Matheus and Yoccoz [AMY16] and Gutierrez-Romo [GR17], with different methods. They show that it is an explicit finite index subgroup of $Sp(2g, \mathbb{Z})$. In Proposition 4.14 which was shown to us by Yves Benoist we give a version of this result for all affine invariant manifolds.

**Strategy of the proofs and organization of the article:** The proofs of the above results use tools from hyperbolic and non-uniform hyperbolic dynamics and algebraic groups. Many of the arguments and results are valid in much larger context, for example for the geodesic flow on a closed rank one manifold of non-positive curvature. With the exception of the proof of Theorem 3, we do not use any methods or results from the theory of flat surfaces.

The basic strategy for the proof of Theorem 1 is motivated by work of Rivin [R08] who showed the following. Let $\mu$ be a symmetric probability measure on $Sp(2g, \mathbb{Z})$ whose finite support generates $Sp(2g, \mathbb{Z})$. Then $\mu$ generates a random walk on $Sp(2g, \mathbb{Z})$. As the step length tends to infinity, the probability that the characteristic polynomial of a random element is reducible tends to zero.

Rivin's argument consists in studying for a prime $p \geq 5$ the projection of the random walk to the finite simple group $Sp(2g, F_p)$ where $F_p$ is the field with $p$ elements. Since this group is finite, this projected random walk equidistributes. By a counting result due to Borel, a definite proportion of the elements of $Sp(2g, F_p)$ which is independent of $p$ has an irreducible characteristic polynomial. Since the mod $p$ reduction of a reducible polynomial with coefficients in $\mathbb{Z}$ is reducible, this implies that as the step length of the walk tends to infinity, a definitive proportion of the random matrices in $Sp(2g, \mathbb{Z})$ have irreducible characteristic polynomials. An application of this argument to varying primes then yields Rivin's result.

A natural route towards Theorem 1 is to replace the equidistribution result obtained from the projection of a random walk on $\mathrm{Mod}(S)$ to $Sp(2g, \mathbb{Z})$ by equidistribution based on lattice counting. There is by now a vast literature on lattice counting in Teichmüller space, see [ABEM12] for an example. However, an argument along this line only seems applicable to study the principal stratum in the moduli space of quadratic differentials. Instead we work directly with the dynamics of the Teichmüller flow on an affine invariant manifold. Our approach has three partially independent parts.

The first part addresses the issue that periodic orbits for the Teichmüller flow correspond to conjugacy classes of pseudo-Anosov mapping classes rather than to actual group elements. To mimic lattice counting we choose a suitable contractible flow box $V$ for the Teichmüller flow on an affine invariant manifold and lift flow lines through this box to the Teichmüller space of abelian differentials. Using a strong shadowing result reminiscent of hyperbolic dynamics we associate to an orbit segment beginning and ending in a suitable open subset $Y$ of $V$ a pseudo-Anosov element in Mod$(S)$ in such a way that concatenation of orbit segments translates into multiplication of group elements. This construction is carried out in Section 3 and is based on earlier results in [H13, H18].

The image of the resulting subsemigroup of Mod$(S)$ under the homomorphism $\Psi$ defines a subsemigroup of a symplectic group. For a component of a stratum, this symplectic group is the group $Sp(2g,\mathbb{Z})$, but for an affine invariant manifold $\mathcal{C}$, it is the group $Sp(\mathcal{Z}_{\mathbb{R}},\mathbb{R})$ introduced above. The second part of our approach consists in establishing Theorem 3 which is the main algebraic result of this article. Its proof is contained in Section 4 and builds on results of Wright [W15] on horizontally periodic translation surfaces in affine invariant manifolds. We also use the results from Section 3 and tools from the theory of algebraic groups developed in the context of strong approximation.

The third and most involved part of this work is an equidistribution result for a homomorphism of Mod$(S)$ onto a finite group $G$ which is contained in Section 5. In our application, the group $G$ is just one of the groups $Sp(2g,F_p)$. For such a homomorphism we construct a cocyle over the Teichmüller flow on a component of a stratum of abelian or quadratic differentials with values in $G$, and we prove equidistribution for this cocycle with respect to the Masur Veech measure. This part of the article is independent of Section 4.

The only information on the Teichmüller flow we use is quantitative non-uniform hyperbolicity in the sense of [H13, H18] and the fact that periodic orbits equidistribute for the Masur Veech measure. The results in Section 5 are valid in much broader context, for example they hold true for the geodesic flow on a rank one manifold of non-positive curvature equipped with the measure of maximal entropy.

An application of the results from Sections 3-5 completes the proof of Theorem 1 and Theorem 2 in Section 6. In the introductory Section 2 we introduce the Hodge bundle and the Gauss Manin connection. We then establish some basic properties of affine invariant manifolds.

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## 2. The geometry of affine invariant manifolds

The goal of this section is to collect some geometric and dynamical properties of affine invariant manifolds which are used throughout this article.

2.1. **The Hodge bundle.** Let $\mathcal{M}_g$ be the moduli space of closed Riemann surfaces of genus $g$. This is the quotient of *Teichmüller space* $\mathcal{T}(S)$ under the action of the mapping class group $\mathrm{Mod}(S)$. The *Hodge bundle* $\mathcal{H} \to \mathcal{M}_g$ is a holomorphic vector bundle over $\mathcal{M}_g$ (in the orbifold sense). As a real vector bundle, it has the following description. The action of the mapping class group $\mathrm{Mod}(S)$ on the first real cohomology group $H^1(S,\mathbb{R})$ defines a homomorphism

$$\Psi : \mathrm{Mod}(S) \to Sp(2g,\mathbb{Z}).$$

The Hodge bundle is then the flat orbifold vector bundle

$$\Pi : \mathcal{H} = \mathcal{T}(S) \times_{\mathrm{Mod}(S)} H^1(S,\mathbb{R}) \to \mathcal{M}_g \tag{1}$$

for the standard left action of $\mathrm{Mod}(S)$ on Teichmüller space $\mathcal{T}(S)$ and the right action of $\mathrm{Mod}(S)$ on $H^1(S,\mathbb{R})$ via $\Psi$. This description determines a flat connection on $\mathcal{H}$ which is called the *Gauss Manin* connection.

As the Hodge bundle $\mathcal{H}$ is a holomorphic vector bundle over the complex orbifold $\mathcal{M}_g$, it is a complex orbifold in its own right and the same holds true for the complement $\mathcal{H}_+ \subset \mathcal{H}$ of the zero section in $\mathcal{H}$. The pull-back

$$\Pi^*\mathcal{H} \to \mathcal{H}_+$$

to $\mathcal{H}_+$ of $\mathcal{H}$ is a holomorphic vector bundle on $\mathcal{H}_+$. The pull-back of the Gauss-Manin connection is a flat connection on $\Pi^*\mathcal{H}$ which we call again the Gauss Manin connection.

2.2. **Affine invariant manifolds.** Let $\mathcal{Q} \subset \mathcal{H}$ be a component of a stratum of area one abelian differentials. Define the *good subset* $\mathcal{Q}_{\rm good}$ of $\mathcal{Q}$ to be the set of all points $q \in \mathcal{Q}$ with the following property. Let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of *marked* abelian differentials and let $\tilde q \in \tilde{\mathcal{Q}}$ be a lift of $q$; then an element of $\mathrm{Mod}(S)$ which fixes $\tilde q$ acts as the identity on $\tilde{\mathcal{Q}}$ (compare [H13] for more information on this technical condition). Then $\mathcal{Q}_{\rm good}$ is precisely the subset of $\mathcal{Q}$ of manifold points. Lemma 4.5 of [H13] shows that the good subset $\mathcal{Q}_{\rm good}$ of $\mathcal{Q}$ is open, dense and $\Phi^t$-invariant, furthermore it is invariant under scaling.

**Definition 2.1.** A closed curve $\eta : [0,a] \to \mathcal{Q}_{\rm good}$ *defines the conjugacy class of a pseudo-Anosov mapping class* $\varphi \in \mathrm{Mod}(S)$ if the following holds true. Let $\tilde\eta : [0,a] \to \tilde{\mathcal{Q}}$ be a lift of $\eta$ to an arc in the Teichmüller space of abelian differentials. Then $\psi\tilde\eta(a) = \tilde\eta(0)$ for a unique $\psi \in \mathrm{Mod}(S)$, and we require that $\psi$ is conjugate to $\varphi$.

As any two lifts of an arc in $\mathcal{Q}_{\rm good}$ to the Teichmüller space of abelian differentials are translates of each other by some element in the mapping class group, the property captured in Definition 2.1 does not depend on any choices made.

Using Definition 2.1, the above discussion easily leads to the following statement (here parallel transport means parallel transport with respect to the Gauss Manin connection).

**Lemma 2.2.** *Let $\eta \subset \mathcal{Q}_{\rm good}$ be a closed curve which defines the conjugacy class of a pseudo-Anosov mapping class $\varphi \in {\rm Mod}(S)$. Then the characteristic polynomial of the holonomy map obtained by parallel transport of the bundle $\Pi^*\mathcal{H}$ along $\eta$ coincides with the characteristic polynomial of the map $\Psi \circ \varphi \in Sp(2g,\mathbb{Z})$.*

*Proof.* Since the Gauss Manin connection is flat, parallel transport along a closed based loop in $\mathcal{Q}_{\rm good}$ is invariant under homotopy of the based loop in $\mathcal{Q}_{\rm good}$ and hence the holonomy along such a based loop is an invariant of its homotopy class. Furthermore, moving the basepoint, i.e. changing the loop with a free homotopy, results in conjugation of the holonomy map.

Now the characteristic polynomial of an element $A \in Sp(2g,\mathbb{Z})$ is invariant under conjugation and hence the characteristic polynomial of the holonomy of an unparametrized loop in $\mathcal{Q}_{\rm good}$ is defined. For a loop $\eta : [0,a] \to \mathcal{Q}_{\rm good}$ which defines the conjugacy class of a pseudo-Anosov element $\varphi$, this polynomial can be computed as follows.

Choose any lift $\tilde\eta$ of $\eta$ to the Teichmüller space of area one abelian differentials. By the definition of the Gauss Manin connection, the characteristic polynomial of the holonomy map along $\eta$ is the characteristic polynomial of $\Psi \circ \zeta$ where $\zeta \in {\rm Mod}(S)$ maps the endpoint $\tilde\eta(a)$ of $\tilde\eta$ back to $\tilde\eta(0)$. As $\zeta$ is conjugate to $\varphi$ and hence $\Psi \circ \zeta$ is conjugate to $\Psi \circ \varphi$, the lemma follows. □

Let $\mathcal{Q}_+$ be a component of a stratum of (non-normalized) abelian differentials on the surface $S$ with fixed number and multiplicities of zeros. We use the notation $\mathcal{Q}_+$ if we are looking at differentials whose area may be different from one. Denote by $\Sigma \subset S$ the set of zeros of a differential in $\mathcal{Q}_+$.

*Period coordinates* for $\mathcal{Q}_+$ are defined by integration of a differential $q$ over a basis of $H_1(S,\Sigma;\mathbb{Z})$. These coordinates take values in $H_1(S,\Sigma;\mathbb{R})^* \otimes_{\mathbb{R}} \mathbb{C}$ and induce an affine structure on $\mathcal{Q}_+$.

An *affine invariant manifold* $\mathcal{C}_+$ in $\mathcal{Q}_+$ is the closure in $\mathcal{Q}_+$ of an orbit of the $GL^+(2,\mathbb{R})$-action. Such an affine invariant manifold is complex affine in period coordinates [EMM15]. In particular, $\mathcal{C}_+ \subset \mathcal{Q}_+$ is a complex suborbifold. Period coordinates determine a projection

$$p : T\mathcal{C}_+ \to \Pi^*\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}|\mathcal{C}_+$$

to absolute periods (see [W14] for a clear exposition). The image $p(T\mathcal{C}_+)$ is flat, i.e. it is invariant under the restriction of the Gauss Manin connection to a connection on $\Pi^*\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}|_{\mathcal{C}_+}$.

By the main result of [F16], there is a *holomorphic* subbundle $\mathcal{Z}$ of $\Pi^*\mathcal{H}|_{\mathcal{C}_+}$ such that

$$p(T\mathcal{C}_+) = \mathcal{Z} \oplus \overline{\mathcal{Z}}.$$

We call $\mathcal{Z}$ the *absolute holomorphic tangent bundle* of $\mathcal{C}_+$. As a consequence, the bundle $p(T\mathcal{C}_+)$ is invariant under the complex structure on $\Pi^*\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}$ obtained by extension of scalars.

As a real vector bundle, $\mathcal{Z}$ is isomorphic to $p(T\mathcal{C}_+)\cap\Pi^*\mathcal{H}|\mathcal{C}_+$. Since $\mathcal{Z}$ is complex, the bundle $p(T\mathcal{C}_+)\cap\Pi^*\mathcal{H}\to\mathcal{C}_+$ is symplectic [AEM12]. Moreover, it is flat, i.e. it is invariant under the Gauss Manin connection.

Define the *rank* of the affine invariant manifold $\mathcal{C}_+$ as [W14]

$$\mathrm{rk}(\mathcal{C}_+)=\frac{1}{2}\mathrm{dim}_{\mathbb{C}}\,p(T\mathcal{C}_+)=\mathrm{dim}_{\mathbb{C}}\mathcal{Z}.$$

With this definition, components of strata are affine invariant manifolds of rank $g$.

## 3. Non-uniform hyperbolic dynamics of the Teichmüller flow

The goal of this section is to give an account on non-uniform hyperbolicity of the dynamics of the Teichmüller flow $\Phi^t$ on components of strata and on affine invariant manifolds. We establish a strengthening of a shadowing result for the Teichmüller flow on components of strata from [H13] (see also [H18]) which is moreover also valid for the Teichmüller flow on affine invariant manifolds. The slogan is that an ordered sequence of orbit segment in an affine invariant manifold with prescribed transitions through a finite collection of small relatively compact flow boxes is shadowed by a periodic orbit contained in the affine invariant manifold with the same transitions. We also introduce the idea of encoding iterated transitions through a flow box by a semigroup.

3.1. **Product structures and the Hodge distance.** In this subsection we introduce local product structures for affine invariant manifolds $\mathcal{C}$ and the Hodge distance on strong stable and strong unstable manifolds. These will be used to obtain effective control on non-uniform hyperbolicity of the Teichmüller flow on $\mathcal{C}$.

An affine invariant manifold $\mathcal{C}_+\subset\mathcal{H}_+$ is described in period coordinates as the set of solutions of a system of linear equations [EMM15]. Here as before, we write $\mathcal{C}_+$ if we consider differentials whose area is not necessarily one. In particular, each manifold point of $\mathcal{C}_+$ has a neighborhood $U$ which is mapped by period coordinates homeomorphically onto an open subset $V$ of an affine subspace of $H_1(S,\Sigma;\mathbb{R})^*\otimes_{\mathbb{R}}\mathbb{C}$ where $\Sigma$ is the set of zeros of the differentials in the stratum containing $\mathcal{C}_+$. This affine subspace is invariant under the complex structure induced from the complex structure on $H_1(S,\Sigma;\mathbb{R})^*\otimes_{\mathbb{R}}\mathbb{C}$ [F16].

To avoid technical difficulties, we will concentrate on the manifold points in an affine invariant manifold $\mathcal{C}_+$. To this end recall from Subsection 2.2 the definition of the set $\mathcal{Q}_{\rm good}$ of good points of a component $\mathcal{Q}$ of area one abelian differentials. This notion extends in an obvious way to the notion of a set of good points in $\mathcal{C}_+$. Denote by $\mathcal{C}_{+,\rm good}$ the good subset of $\mathcal{C}_+$ and by $\mathcal{C}_{\rm good}$ the intersection of $\mathcal{C}_{+,\rm good}$ with the hypersurface $\mathcal{C}$ of area one differentials. As before, the good subset $\mathcal{C}_{\rm good}$ of $\mathcal{C}$ is precisely the set of manifold points. The proof of the easy Lemma 4.5 of [H13] is equally valid for affine invariant manifolds and shows that the set $\mathcal{C}_{\rm good}\subset\mathcal{C}$ of good points is open, dense and invariant under the Teichmüller flow.

In period coordinates, a local leaf of the *strong unstable foliation* $W^{su}$ through a point $w\in H_1(S,\Sigma;\mathbb{R})^*\otimes_{\mathbb{R}}\mathbb{C}$ consists of all differentials whose imaginary parts coincide with the imaginary part of $w$, and the local leaf of the *strong stable foliation* consists of all differentials whose real parts coincide with the real part of $w$. As $\mathcal{C}_+$ is complex affine in period coordinates, we obtain

**Lemma 3.1.** *Let $\mathcal{C}_+$ be an affine invariant manifold. Then $\mathcal{C}_{\rm good} \cap W^i$ is a smooth foliation of $\mathcal{C}_{\rm good}$ into leaves of real dimension $\dim_{\mathbb{C}}(\mathcal{C}_+) - 1$ $(i = ss, su)$.*

Lemma 3.1 implies that for every affine invariant manifold $\mathcal{C}$, every point $q \in \mathcal{C}_{\rm good}$ has a neighborhood *with a product structure*. We next define a set with a product structure formally. The definition we give is a bit less restrictive than other of its versions, but it is convenient for the purpose of this section.

The real and imaginary part, respectively, of an abelian differential $\omega$ defines a *measured foliation* on $S$. This is a smooth oriented foliation of $S - \Sigma$, equipped with a transverse invariant measure. The tangent bundle of the foliation is the subbundle of $T(S - \Sigma)$ annihilated by the real or imaginary part of $\omega$, respectively. The space of *marked projective measured foliations* $\mathcal{PMF}$ on $S$ is equipped with a natural topology so that it is homeomorphic to a sphere of dimension $6g - 7$. Period coordinates for the component $\mathcal{Q}$ of a stratum containing $\mathcal{C}$ show that nearby differentials in $\mathcal{Q}$ whose real parts define the same class in $H_1(S, \Sigma; \mathbb{R})^*$ determine the same marked measured foliations on $S$ (up to equivalence as in the definition of $\mathcal{PMF}$).

**Definition 3.2.** Let $\mathcal{C}$ be an affine invariant manifold and let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials. A subset $\tilde{V}$ of $\tilde{\mathcal{C}}$ *admits a product structure* if there are two disjoint compact subsets $D, K$ of the set of (marked) projective measured foliations on $S$, viewed as projective classes of points in $H_1(S, \Sigma; \mathbb{R})^*$ via integration of the transverse measure along arcs with endpoints in $\Sigma$, with the following properties.

(1) The sets $D, K$ are homeomorphic to closed balls of dimension
$$m = \dim_{\mathbb{C}}(\mathcal{C}_+) - 1.$$
(2) There is a continuous map
$$\Lambda : D \times K \to \tilde{V}$$
such that for any pair $(\xi, \nu) \in D \times K$, the horizontal projective measured foliation of $\Lambda(\xi, \nu)$ equals $\xi$, and its vertical projective measured foliation equals $\nu$.
(3) There is some $\epsilon > 0$ such that
$$\tilde{V} = \cup_{-\epsilon \leq t \leq \epsilon} \cup_{(\xi,\nu) \in D \times K} \Phi^t \Lambda(\xi, \nu).$$

A closed contractible set $V \subset \mathcal{C}_{\rm good}$ with dense interior *admits a product structure* if some (and hence any) component $\tilde{V}$ of $V$ of the preimage of $V$ in the Teichmüller space of abelian differentials has a product structure.

We say that an open subset $U$ of $\mathcal{C}_{\rm good}$ *has a product structure* if its closure has a product structure in the sense of Definition 3.2. We refer to Section 3.1 of [H13] for a detailed description of this construction for strata. The requirement (1) in

Definition 3.2 is made for convenience of exposition; we will occasionally talk about a set with a product structure which only has properties (2) and (3) above.

The following observation is immediate from the definition.

**Lemma 3.3.** *Let $U \subset \mathcal{C}_{\rm good}$ be an open or closed set with a product structure as in Definition 3.2. Then each component of the intersection of $U$ with an orbit of the Teichmüller flow is an arc of length $2\epsilon$.*

*Proof.* Let $V \subset \mathcal{C}_{\rm good}$ be a set with a product structure, and let $\tilde V$ be a component of the preimage of $V$ in the Teichmüller space of marked abelian differentials. As $V$ is contained in $\mathcal{C}_{\rm good}$ and is contractible, a component of the intersection of $V$ with an orbit of the Teichmüller flow lifts to a component of the intersection of $\tilde V$ with an orbit of the Teichmüller flow. The lemma is now immediate from the definition and the fact that the Teichmüller flow preserves the projective class of the horizontal and vertical measured foliation, respectively. □

Let $\tilde V$ be as in (3) of Definition 3.2. For each $\tilde z \in \tilde V$, the product structure determines a closed *local strong unstable manifold*

$$W^{su}_{\rm loc}(\tilde z)$$

containing $\tilde z$ which is homeomorphic to a closed ball of dimension $m$. This set consists of all points whose marked horizontal measured foliation coincides with the marked horizontal measured foliation of $\tilde z$, and whose marked vertical projective measured foliation is contained in the set $K$. Similarly we obtain a *local strong stable manifold* $W^{ss}_{\rm loc}(\tilde z)$ by exchanging the roles of the horizontal and the vertical measured foliations. The sets $W^i_{\rm loc}(\tilde z)$ $(i = ss, su)$ need not be contained in $\tilde V$, but every $\tilde y \in W^i_{\rm loc}(\tilde z)$ can be moved into $\tilde V$ with a small translate along the flow line of $\Phi^t$ through $\tilde y$. For $z \in V$ we let $W^i_{\rm loc}(z)$ be the projection to $\mathcal{C}$ of $W^i_{\rm loc}(\tilde z)$ where $\tilde z \in \tilde V$ is the preimage of $z$ $(i = ss, su)$. Note that these sets are contained in $\mathcal{C}_{\rm good}$ by invariance of $\mathcal{C}_{\rm good}$ under the Teichmüller flow.

**Example 3.4.** Let $\mathcal{Q}$ be a component of a stratum of abelian or quadratic differentials. Let $q \in \mathcal{Q}_{\rm good}$ and let $A^{su}$ be a neighborhood of $q$ in $W^{su}_{\rm loc}(q)$. Then for a sufficiently small neighborhood $A^{ss}$ of $q$ in $W^{ss}_{\rm loc}(q)$ and every $z \in A^{ss}$ there exists a *holonomy homeomorphism*

$$\Xi_z : A^{su} \to \Xi_z(A^{su}) \subset W^{su}_{\rm loc}(z)$$

with $\Xi_z(q) = z$ determined by the requirement that $\Xi_z(u) \in \cup_{-\epsilon\leq t\leq\epsilon}\Phi^t W^{ss}_{\rm loc}(u)$ for some small $\epsilon > 0$. The holonomy homeomorphisms $\Xi_z$ are smooth and depend smoothly on $z$.

Define $V(A^{ss}, A^{su}) = \cup_{z\in A^{ss}}\Xi_z A^{su}$ and

$$V(A^{ss}, A^{su}, t_0) = \cup_{-t_0\leq t\leq t_0}\Phi^t V(A^{ss}, A^{su}).$$

If we choose $A^i$ to be a sufficiently small ball neighborhood of $q$ in $W^i_{\rm loc}(q)$ and $t_0$ sufficiently small, then $V(A^{ss}, A^{su}, t_0)$ is a neighborhood of $q$ with a product structure in the sense of Definition 3.2.

The tangent bundle of the strong stable or strong unstable foliation of a component $\mathcal{Q}$ of a stratum can be equipped with the so-called *modified Hodge norm* which induces a *Hodge distance* $d_H$ on the leaves of the foliation of a stratum of abelian differentials.

The following result is the first part of Theorem 8.12 of [ABEM12].

**Theorem 3.5.** *There exists a number $c_H > 0$ not depending on choices such that for every $q \in \mathcal{Q}$, any $q' \in W^{ss}_{\rm loc}(q)$ and all $t > 0$ we have*

$$d_H(\Phi^t q, \Phi^t q') \leq c_H d_H(q, q').$$

The following is Theorem 2 of [H18]. In its formulation, $B^i(q,r)$ denotes the ball of radius $r$ about $q$ for the Hodge distance on the local leaf $W^i_{\rm loc}(q)$ of the foliation $W^i$ through $q$. The balls $B^i(u, r_0)$ are not required to be contained in the set $U$ $(i = ss, su)$.

**Theorem 3.6.** *Let $q \in \mathcal{Q}_{\rm good}$ be a birecurrent point. Then there is a number $r_0 = r_0(q) > 0$, and there is a neighborhood $U$ of $q$ in $\mathcal{Q}_{\rm good}$ with the following property. Let $z \in U$ be birecurrent; then for every $a > 0$ there is a number $T(z,a) > 0$ so that for all $T > T(z,a)$, we have $\Phi^T B^{ss}(z, r_0) \subset B^{ss}(\Phi^T(z), a)$ and $\Phi^T B^{su}(z,a) \supset B^{su}(\Phi^T(z), r_0)$.*

3.2. **Shadowing and Anosov closing.** The goal of this subsection is to establish a strong version of shadowing and Anosov closing for the Teichmüller flow on affine invariant manifolds which is familiar in hyperbolic dynamics. We begin with some basic definitions.

**Definition 3.7.** Let $\mathcal{Y} = \{Y_i \mid i \in \mathcal{I}\}$ be a non-empty finite collection of open relatively compact subsets of an affine invariant manifold $\mathcal{C}$. For some $n > 0$, an $(n, \mathcal{Y})$-*pseudo-orbit* for the Teichmüller flow $\Phi^t$ on $\mathcal{C}$ consists of a sequence of points $q_0, q_1, \dots, q_m \in \mathcal{C}$ and a sequence of numbers $t_0, \dots, t_{m-1} \in [n, \infty)$ with the following property. For every $1 \leq j \leq m$, there exists some $\kappa(j) \in \mathcal{I}$ such that $\Phi^{t_{j-1}} q_{j-1}, q_j \in Y_{\kappa(j)}$. The pseudo-orbit is called *periodic* if $q_m = q_0$.

Although we describe a pseudo-orbit by a sequence of pairs $(q_i, t_i) \in \mathcal{C} \times (0, \infty)$, we view a pseudo-orbit as a finite ordered collection of compact orbit segments such that the endpoint of the $i-1$-th segment is close to the starting point of the $i$-th segment. With this interpretation, the shadowing property [Bw73] for hyperbolic flows on a compact Riemannian manifold states that for sufficiently large $n$ and sufficiently small $\epsilon$, if $\mathcal{Y}_\epsilon$ is the collection of all open balls of radius $\epsilon$ then an $(n, \mathcal{Y}_\epsilon)$-pseudo-orbit is fellow-traveled by an orbit with prescribed precision: For every number $\sigma > 0$, there are $n > 0, \epsilon > 0$ such that for any $(n, \mathcal{Y}_\epsilon)$-pseudo-orbit $\eta$, there exists an orbit segment whose Hausdorff distance to $\eta$ is less than $\sigma$. In the case that the pseudo-orbit is periodic, this orbit segment can be chosen to be a periodic orbit. The point here is that there is no upper bound on the number of orbit segments contained in the pseudo-orbit.

For the Teichmüller flow on components of strata or, more generally, on affine invariant manifolds, we can not expect that the shadowing property for all small balls holds true. However, Proposition 3.8 below establishes a local version of shadowing for periodic pseudo-orbits on affine invariant manifolds.

For its formulation, assume that the open relatively compact subsets $Y_i$ $(i \in \mathcal{I})$ of the affine invariant manifold $\mathcal{C}$ have disjoint closure. Assume furthermore that the closure of each $Y_i$ is contained in an open relatively compact contractible subset $V_i$ of $\mathcal{C}_{\rm good}$ and that the sets $V_i$ are pairwise disjoint. Now suppose we are given a periodic $(n, \mathcal{Y})$-pseudo-orbit, specified by points $q_0, q_1, \dots, q_m = q_0 \in \mathcal{C}$, numbers $t_0, \dots, t_{m-1} \in [n, \infty)$ and indices $\kappa(j) \in \mathcal{I}$. Connect $\Phi^{t_{j-1}} q_{j-1}$ to $q_j$ by an arc $\alpha_j$ in $V_{\kappa(j)}$. The concatenation of the orbit segments connecting $q_{j-1}$ to $\Phi^{t_{j-1}} q_{j-1}$ with the arcs $\alpha_j$ defines a closed curve $\eta$ in $\mathcal{C}$ which we call a $\mathcal{V}$-*characteristic curve* of the pseudo-orbit, where $\mathcal{V} = \{V_i \mid i \in \mathcal{I}\}$. Note that such a characteristic curve is by no means unique, but as the sets $V_j$ are contractible and have been selected a priori, any other such curve can be obtained by a collection of small deformations in a fixed contractible subset of $\mathcal{C}_{\rm good}$.

By the definition of the set $\mathcal{C}_{\rm good}$, every path $\eta : [0, a] \to \mathcal{C}_{\rm good}$ admits a lift $\tilde{\eta}$ to the Teichmüller space of marked abelian differentials, and such a lift is unique up to translation by an element of $\mathrm{Mod}(S)$. Now assume that the path $\eta$ is closed. Then the endpoints of each lift $\tilde{\eta}$ of $\eta$ are identified by a unique element of $\mathrm{Mod}(S)$, and two distinct lifts give rise to conjugate elements in $\mathrm{Mod}(S)$ as made precise in Lemma 2.2.

The following is the version of the shadowing properties we need. For its formulation, recall from [M82, V86] that for every component $\mathcal{Q}$ of a stratum of abelian or quadratic differentials, there is a $\Phi^t$-invariant Borel probability measure $\lambda$ on $\mathcal{Q}$ in the Lebesgue measure class, the so-called normalized *Masur Veech measure*.

**Proposition 3.8.** *Let $\mathcal{C}$ be an affine invariant manifold, let $q_1, \dots, q_k \in \mathcal{C}_{\rm good}$ be birecurrent points and for each $j$ let $U_j$ be a neighborhood of $q_j$ in $\mathcal{C}_{\rm good}$. Then there are open relative compact neighborhoods $Y_j \subset V_j \subset U_j$ of $q_j$, where $V_j$ contractible with a product structure, and there is a number $R_0 > 0$ with the following property. Let $\mathcal{Y} = \{Y_j \mid j\}$, let $\mathcal{V} = \{V_j \mid j\}$ and let $\eta$ be a $\mathcal{V}$-characteristic curve of a periodic $(R_0, \mathcal{Y})$-pseudo-orbit, given by points $y_0, \dots, y_{m-1}$ and numbers $t_i > R_0$ such that $\Phi^{t_{i-1}} y_{i-1}, y_i \in Y_{\kappa(i)}$ for some $\kappa(i) \in \{1, \dots, m\}$. Then there is a periodic orbit $\gamma \subset \mathcal{C}_{\rm good}$ for $\Phi^t$ which passes through each of the sets $V_{\kappa(i)}$ at times close to $\sum_{s \leq i-1} t_s$ and which defines the same conjugacy class in $\mathrm{Mod}(S)$ as $\eta$. If $\mathcal{C} = \mathcal{Q}$ is a component of a stratum then for any number $\delta > 0$ and any finite set of birecurrent points $\Sigma \subset \cup_j V_j$, the sets $Y_j \subset V_j$ can be chosen so that $\lambda(Y_j) \geq \lambda(V_j)(1 - \delta)$ and that $\Sigma \subset \cup_j Y_j$.*

*Proof.* The proof is divided into three steps. In the first step, we construct the neighborhoods $Y_j \subset V_j \subset U_j$ of the points $q_j$ and determine the number $R > 0$ whose existence is stated in the proposition. These sets have some additional properties used to obtain the dynamical control we need.

In the second step we consider the element $\varphi \in \mathrm{Mod}(S)$ determined by a $\mathcal{V}$-characteristic curve of a periodic $(R, \mathcal{Y})$-pseudo orbit, and we show that it is pseudo-Anosov. In particular, it determines a periodic orbit for the Teichmüller flow in the moduli space of abelian differentials. We then use a fixed point argument to show that this orbit is contained in $\mathcal{C}$ and has the properties stated in the proposition.

The last step contains the measure control for components of strata which is the last part of the proposition.

*Step 1.*

Using the notation from the proposition, for each $j \leq k$ choose a closed contractible neighborhood $V_j \subset U_j$ of $q_j$ with a product structure which furthermore has the properties stated in Theorem 3.6. Recall that such a product structure is determined by a choice $\tilde{V}_j$ of a component of the preimage of $V_j$ in the Teichmüller space of marked abelian differentials, of two closed disjoint subsets $D_j, K_j$ of the space of projective measured foliations which are homeomorphic to closed balls of dimension $d = \dim_{\mathbb{C}}(\mathcal{C}_+) - 1$, an embedding

$$\Lambda_j : D_j \times K_j \to \tilde{V}_j$$

and a number $\epsilon_j > 0$ with the properties stated in Definition 3.2.

For $\tilde{z} \in \tilde{V}_j$ denote by $W^{ss}_{\rm loc}(\tilde{z})$ the local strong stable manifold of $\tilde{z}$, and let similarly $W^{su}_{\rm loc}(\tilde{z})$ be the local strong unstable manifold. We require that the projections into $\mathcal{C}$ of the union of all these local strong stable and strong unstable manifolds are contained in a fixed contractible subset of $U_j$. Note that this is not automatic as some of these local manifolds may not be contained in $V_j$, but it can be achieved by making $V_j$ smaller if necessary. For $z \in V_j$ we denote by $W^i_{\rm loc}(z)$ the projection to $\mathcal{C}$ of the set $W^i_{\rm loc}(\tilde{z})$ where $\tilde{z}$ is the preimage of $z$ in $\tilde{V}_j$; this does not depend on the choice of the component $\tilde{V}_j$. By perhaps decreasing the size of $V_j$ we may assume that $W^i_{\rm loc}(\tilde{z}) \subset B^i(\tilde{z}, r_0)$ for all $\tilde{z} \in V_j$, where $r_0 > 0$ is as in Theorem 3.6.

Recall from Example 3.4 that for two points $\tilde{z}, \tilde{u} \in \tilde{V}_j$ there is a holonomy map

$$\Xi(\tilde{u}, \tilde{z}) : W^{su}_{\rm loc}(\tilde{u}) \to W^{su}_{\rm loc}(\tilde{z}).$$

For each $\tilde{v} \in W^{su}_{\rm loc}(\tilde{u})$, the point $\Xi(\tilde{u}, \tilde{z})(\tilde{v})$ is the unique point in $W^{su}_{\rm loc}(\tilde{z})$ whose marked vertical projective measured foliation coincides with the marked vertical projective measured foliation of $\tilde{v}$.

The holonomy maps $\Xi(\tilde{u}, \tilde{z})$ are smooth and depend smoothly on $\tilde{u}, \tilde{z}$. In particular, they are bilipschitz for the Hodge distance $d_H$. Furthermore, if $\tilde{z} \in W^{su}_{\rm loc}(\tilde{u})$ then $\Xi(\tilde{u}, \tilde{z}) = \mathrm{Id}$. Thus by perhaps decreasing the size of the sets $V_j$ we may assume that the bilipschitz constants for these holonomy maps are at most 2.

Choose a compact neighborhood $Z_j \subset V_j$ of $q_j$ with a product structure which is contained in the interior of $V_j$. For $z \in Z_j$ let $W^i_{{\rm loc}, Z_j}(z)$ $(i = su, ss)$ be the local strong stable and strong unstable manifold for $Z_j$. By continuity and compactness, there exists a number $r > 0$ such that for any $z \in Z_j$, the $d_H$-distance between the set $W^i_{{\rm loc}, Z_j}(z)$ and the boundary of $W^i_{\rm loc}(z)$ is at least $r$.

By Theorem 3.5 and Theorem 3.6 and the choice of the sets $Z_j$, we can find a contractible neighborhood $Y_j \subset Z_j$ of $q_j$ with a product structure and a number $T_j > 0$ with the following property. If $z \in Y_j$ and if $T > T_j$ then

$$\begin{aligned} d_H(\Phi^T z', \Phi^T z'') &\leq \frac{r}{4} \text{ for all } z', z'' \in W^{ss}_{\text{loc}}(z) \text{ and} \\ d_H(\Phi^{-T} z', \Phi^{-T} z'') &\leq \frac{r}{4} \text{ for all } z', z'' \in W^{su}_{\text{loc}}(z). \end{aligned} \tag{2}$$

Namely, choose $T_j > 0$ so that the estimate (2) is satisfied for $q_j$ and the constant $r/8c_H$ instead of $r/4$. Such a number exists by Theorem 3.6 and the choice of the sets $V_j$. By continuity, the estimate with $r/4c_H$ then holds true for this number $T_j$ and for all points $z$ in a neighborhood $Y_j$ of $q_j$ which can be chosen to be contractible, with a product structure. By Theorem 3.5, the estimate (2) then holds true for all $T \geq T_j$ and for all $z \in Y_j$. Define $\mathcal{Y} = \{Y_j\}, \mathcal{V} = \{V_j\}$ and let $R = \max_j T_j$.

*Step 2.*

Using the notations from Step 1, let $\eta$ be a $\mathcal{V}$-characteristic curve of a periodic $(R, \mathcal{Y})$-pseudo-orbit. By definition, $\eta$ is determined by points $y_i \in Y_{\kappa(i)}$, numbers $t_i > R$ $(0 \leq i \leq m-1)$ and arcs in the contractible sets $V_{\kappa(i)}$. Parametrize $\eta$ in such a way that for each orbit segment, the parametrization coincides with the parametrization as a flow line of the Teichmüller flow and that $\eta(\sum_{i<\ell} t_i + \ell) = y_\ell$ (i.e. the connecting arcs $\alpha_j$ are parametrized on a unit interval). For simplicity of notation, assume that $\eta(0) \in Y_1$. Let $T = \sum_j t_j + m > 0$ be such that $\eta(T) = \eta(0)$.

Let as before $\mathcal{Q}$ be the component of the stratum containing $\mathcal{C}$ and let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of marked abelian differentials. Let $\tilde{\mathcal{C}} \subset \tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{C}$. We assume that these components are chosen in such a way that they contain the set $\tilde{V}_1$. Let $\tilde{\eta}$ be a lift of $\eta$ to $\tilde{\mathcal{C}}$ which begins at $\tilde{\eta}(0) = \tilde{y}_0 \in \tilde{V}_1$. Then there is an element $\varphi \in \text{Mod}(S)$ which maps the endpoint $\tilde{\eta}(T)$ of $\tilde{\eta}$ back to $\tilde{y}_0$. As any element of $\text{Mod}(S)$ either stabilizes $\tilde{\mathcal{C}}$ or maps $\tilde{\mathcal{C}}$ to a disjoint component of the preimage of $\mathcal{C}$, we know that $\varphi \in \text{Stab}(\tilde{\mathcal{C}})$.

By Lemma 5.1 of [H13] (and after perhaps increasing the number $R > 0$), the mapping class $\varphi$ is pseudo-Anosov (see also the bottom of p.523 of [H13]). Our goal is to show that it defines a periodic orbit $\gamma$ in $\mathcal{C}$ with the properties stated in the proposition. Note that this is not implied by the fact that $\varphi \in \text{Stab}(\tilde{\mathcal{C}})$. To this end we use a variation of the argument in the proof of Proposition 5.4 of [H13].

Let $\tilde{\gamma} \subset \tilde{\mathcal{Q}}$ be the cotangent line of the axis in Teichmüller space of the pseudo-Anosov element $\varphi$. The curve $\tilde{\gamma}$ is a $\varphi$-invariant orbit of the Teichmüller flow in $\tilde{\mathcal{Q}}$ which projects to the periodic orbit $\gamma$. The (biinfinite) lift $\tilde{\eta}$ of the characteristic curve $\eta$ is contained in a uniformly bounded neighborhood of $\tilde{\gamma}$.

The pseudo-Anosov element $\varphi$ acts with north-south dynamics on the Thurston sphere $\mathcal{PMF}$ of projective measured foliations of the surface $S$. This means that $\varphi$ has precisely two fixed points in $\mathcal{PMF}$, one is attracting, the other repelling. Furthermore, if $\tilde{u} \in \tilde{\gamma}$ is arbitrary, then the vertical projective measured foliation $\nu$

of $\tilde{u}$ equals the attracting fixed point of $\varphi$, and the horizontal projective measured foliation $\xi$ of $\tilde{u}$ equals the repelling fixed point of $\varphi$.

It now suffices to verify that with the above notation, we have $\xi \in D_1, \nu \in K_1$. Namely, every flow line of the Teichmüller flow in the Teichmüller space of abelian differentials which is defined by a differential with horizonal measured foliation in $D_1$ and vertical measured foliation in $K_1$ passes through the set $\tilde{V}_1$, in particular it is entirely contained in $\tilde{\mathcal{C}}$ by invariance of $\tilde{\mathcal{C}}$. Thus if $\xi \in D_1, \nu \in K_1$ then the periodic orbit $\gamma$ is contained in $\mathcal{C}$, and it passes through the set $V_1$. As the initial point of the periodic pseudo-orbit was arbitrarily chosen among the starting points of the orbit segments which determine the pseudo-orbit, the periodic orbit $\gamma$ passes through each of the sets $V_{\kappa(i)}$, and the crossing times fulfill the estimate stated in the proposition. Thus $\gamma$ has all the properties stated in the proposition.

Using the argument on p.524 of [H13], we show that indeed $\nu \in K_1$. To this end we claim that

$$\Phi^{-t_0} W^{su}_{\rm loc}(\tilde{\eta}(t_0)) \subset W^{su}_{\rm loc}(\tilde{y}_0).$$

Namely, since $t_0 > R$ and since $\eta(t_0) \in Y_{\kappa(1)}$, the estimate (2) shows that the $d_H$-diameter of $A = \Phi^{-t_0} W^{su}_{\rm loc}(\tilde{\eta}(t_0))$ is at most $r/4$. On the other hand, the set $A$ contains the point $\tilde{\eta}(0) = \tilde{y}_0 \in \tilde{Y}_1 \subset \tilde{V}_1$. Now by assumption, the Hodge distance between $\tilde{y}_0$ and the boundary of $W^{su}_{\rm loc}(\tilde{y}_0)$ is at least $r$ and hence indeed $\Phi^{-t_0} W^{su}_{\rm loc}(\tilde{\eta}(t_0)) \subset W^{su}_{\rm loc}(\tilde{z}_0)$. In particular, if we denote by $K_{\kappa(1)} \subset \mathcal{PMF}$ the closed set of all vertical projective measured foliations for points in the component $\tilde{V}_{\kappa(1)}$ of the preimage of $V_{\kappa(1)}$ containing $\tilde{\eta}(t_0)$, then we have $K_{\kappa(1)} \subset K_1$.

The above reasoning can be iterated: For $s \geq 1$ let $K_{\kappa(s)}$ be the set of all projective measured foliations of all marked abelian differentials which are contained in the component $\tilde{V}_{\kappa(s)}$ of the preimage of $V_{\kappa(s)}$ containing $\tilde{\eta}(\sum_{j<s} t_j + s)$. We show by induction on $s$ that for any $s \geq 1$, the set $K_{\kappa(s)}$ is entirely contained in $K_1 = K_{\kappa(0)}$. The case $s = 1$ was discussed in the previous paragraph, so let us assume that this holds true for all $s < s_0$ for some $s_0 \geq 2$. Replacing the starting point $y_0$ of the periodic pseudo-orbit by $y_1$, we conclude from the induction hypothesis that $K_{\kappa(s_0)} \subset K_{\kappa(1)}$. However, we showed above that $K_{\kappa(1)} \subset K_1$. This yields the induction step.

To summarize, for each $t > 0$ the vertical projective measured foliation of $\tilde{\eta}(t)$ is contained in the compact set $K_0$. Now the attracting fixed point of $\varphi$ is the limit as $t \to \infty$ of the vertical projective measured foliation of $\tilde{\eta}(t)$. Namely, the path $\tilde{\eta}$ is invariant under the pseudo-Anosov element $\varphi$. Since $\varphi$ acts with north-south dynamics on $\mathcal{PMF}$, any non-constant orbit on $\mathcal{PMF}$ under forward iteration of $\varphi$ converges to the attracting fixed point of $\varphi$. Thus this attracting fixed point of $\varphi$ is indeed contained in the compact set $K_1$.

Reversing the direction of the flow $\Phi^t$ and replacing $\varphi$ by $\varphi^{-1}$, the same argument applies to the repelling fixed point of $\varphi$ and shows that this repelling fixed point is contained in $D_1 = D_{\kappa(0)}$. In particular, the periodic orbit of $\Phi^t$ defined by $\varphi$ is contained in $\mathcal{C}$, and it passes through $V_1$. As remarked earlier, this suffices for the proof of the main part of the proposition.

*Step 3.*

Consider now a component $\mathcal{Q}$ of stratum of abelian or quadratic differentials, equipped with Masur Veech measure $\lambda$. We have to show that for any given $\delta > 0$ the sets $Y_j$ can be chosen in such a way that $\lambda(Y_j) > \lambda(V_j)(1-\delta)$. To this end note that we may choose the sets $V_j$ as in the beginning of this proof in such a way that the Lebesgue measure of their boundaries vanish. In a second step, we choose the sets $Z_j$ in such a way that they satisfy $\lambda(Z_j) \geq (1-\delta/2)\lambda(V_j)$. Let $r > 0$ be sufficiently small that the modified Hodge distance of every point $z \in Z_j$ to the boundary of $W^i_{loc}(z)$ is at least $r$.

By our choices and Theorem 3.6, for every birecurrent point $z \in Z_j$ there exists a number $T = T(z) > 0$ such that the estimates (2) above hold true provided that $t \geq T(z)$, with $r/4$ replaced by $r/8$. As $\Phi^t$ is smooth and as birecurrent points in $Z_j$ have full Masur Veech measure, we can find a number $T_j > 0$ such that $T_j \geq T(z)$ for a subset $Y_j'$ of $Z_j$ of measure at least $(1-\delta)\lambda(V_j)$. Then the corresponding estimate for $r/4$ holds true for an open neighborhood $Y_j$ of $Y_j'$. Let $R = \max\{T_j \mid j\}$.

Now the proof of the Anosov closing property only used the estimate (2) beyond some standard properties of the Teichmüller flow. This yields the proposition with the additional volume control on the nested sets $Y_j \subset V_j$. Furthermore, by Theorem 3.6 and as the sets $V_j$ are open, we may also construct the sets $Y_j$ in such a way that $\cup_j Y_j$ contains any prescribed finite subset $\Sigma \subset \cup_j V_j$ consisting of birecurrent points. This finishes the proof. □

**Remark 3.9.** Let $\mathcal{C}$ be an affine invariant manifold, contained in a component $\mathcal{Q}$ of a stratum, and let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials. If $\varphi \in \mathrm{Mod}(S)$ defines a periodic orbit of the Teichmüller flow on $\mathcal{C}$, then $\varphi$ is a pseudo-Anosov mapping class which is conjugate to an element of $\mathrm{Stab}(\tilde{\mathcal{C}})$. However, it is not true that any pseudo-Anosov mapping class in $\mathrm{Stab}(\tilde{\mathcal{C}})$ determines a periodic orbit for $\Phi^t$ contained in the closure of $\mathcal{C}$. An example of this situation is the case that $\mathcal{C}$ equals a non-principal stratum of abelian differentials with at least one simple zero. In this case the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials is connected [H14] and hence the stabilizer of this preimage equals the entire mapping class group. However, the set of periodic orbits for the Teichmüller flow contained in the closure of $\mathcal{C}$ is a proper subset of the set of all periodic orbits.

In the case of a single birecurrent point $q$ on an affine invariant manifold $\mathcal{C}$, Proposition 3.8 predicts for every contractible neighborhood $U$ of $q$ a nested set of neighborhoods $Y \subset V \subset U$ of $q$ and a number $R > 0$ with the following property. For every $y \in Y$ and $T > R$ so that $\Phi^T y \in Y$, there is a periodic orbit passing through $V$ of period close to $T$ which defines the same conjugacy class in $\mathrm{Mod}(S)$ as a characteristic curve of the periodic $(R, Y)$-pseudo-orbit $(y, T)$.

3.3. **Semigroups defined by recurring orbits.** The goal of this subsection is to establish a parametrized version of Proposition 3.8. This is needed to associate to a periodic orbit of $\Phi^t$ on an affine invariant manifold $\mathcal{C}$ which passes through an a priori chosen subset of $\mathcal{C}$ an element of $\mathrm{Mod}(S)$ rather than a conjugacy class in such a way that adjunction of orbit segments in a pseudo-orbit corresponds to multiplication of group elements.

To this end let again $q \in \mathcal{C}_{\rm good}$ be a good birecurrent point. Let $U \subset \mathcal{C}_{\rm good}$ be a neighborhood of $q$ and let $Y \subset V \subset U$ be a nested family of neighborhoods of $q$ in $\mathcal{C}_{\rm good}$ as in Proposition 3.8. We may assume that $V$ is contractible and has a product structure, that $Y$ consists of a finite union of contractible sets with product structures and that any connected component of the intersection with $Y$ or $V$ of an orbit segment of the Teichmüller flow is an arc of fixed length $2t_0$.

For $R_0 > 0$ as in Proposition 3.8 let $y \in Y$ and let $T > R_0$ be such that $\Phi^T y \in Y$. A characteristic curve of this orbit segment determines uniquely a periodic orbit $\gamma$ of $\Phi^t$ which intersects $V$ in an arc of length $2t_0$. There may be more than one such intersection arc, but there is a unique arc determined by the requirement that the parametrized periodic orbit starting at a point in this arc uniformly fellow-travels the pseudo-orbit issuing from $y$. Choose the midpoint of this intersection arc as a basepoint for $\gamma$ and as an initial point for a unit speed parametrization of $\gamma$.

Let $\Gamma_0$ be the set of all parametrized periodic orbits of this form for points $y \in Y$ with $\Phi^T y \in Y$ $(T > R_0)$. There is a bijection between such periodic orbits and subsets of $\Phi^T V \cap V$ containing points in $\Phi^T Y \cap Y$. With some care, these subsets can be chosen to be components of $\Phi^T V \cap V$ [H13], but we will not need this somewhat technical fact in the sequel.

Fix once and for all a lift $\tilde{V}$ of the contractible set $V$ to a component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ in the Teichmüller space of marked abelian differentials. A parametrized periodic orbit $\gamma$ which starts in $V$ lifts to a subarc of a flow line of the Teichmüller flow on $\tilde{\mathcal{C}}$ with starting point in $\tilde{V}$. The endpoint of this arc is mapped to its starting point by a pseudo-Anosov element $\Omega(\gamma) \in \mathrm{Mod}(S)$. The conjugacy class of $\Omega(\gamma)$ is uniquely determined by $\gamma$, and the element $\Omega(\gamma)$ only depends on the choice of $\tilde{V}$ (and the component of $\gamma \cap V$ as explained above). Thus a characteristic curve of a sufficiently long orbit segment beginning and ending in $Y$ determines an element in $\mathrm{Mod}(S)$.

The following proposition is a parametrized version of shadowing as established in Proposition 3.8.

**Proposition 3.10.** *For $\gamma_1, \dots, \gamma_m \in \Gamma_0$, there is a point $z \in V$, and there are numbers $0 < t_1 < \cdots < t_m$ with the following properties.*

(1) $\Phi^{t_i} z \in V$.
(2) *For each $i \le m$, a $V$-characteristic curve of the orbit segment $\{\Phi^t z \mid t_{i-1} \le t \le t_i\}$ defines the element $\Omega(\gamma_i)$ in* $\mathrm{Mod}(S)$.
(3) *A $V$-characteristic curve of the orbit segment $\{\Phi^t z \mid 0 \le t \le t_m\}$ determines a parametrized periodic orbit $\gamma$ for $\Phi^t$ with initial point in $V$, and $\Omega(\gamma) = \Omega(\gamma_k) \circ \cdots \circ \Omega(\gamma_1)$.*

*Proof.* The proposition is a fairly immediate consequence of Proposition 3.8 and the definitions.

Namely, recall that an orbit $\gamma \in \Gamma_0$ is constructed from a point $y \in Y$ and a number $s(\gamma, y) > R_0$ so that $\Phi^{s(\gamma,y)} \in Y$. The orbit $\gamma$ then is the unique periodic orbit determined by the characteristic curve of the pseudo-orbit $(y, s(\gamma, y))$.

Now let $\gamma_1, \dots, \gamma_m \in \Gamma_0$, and for each $i \leq m$ let $(y_i, s_i)$ be as in the previous paragraph for $\gamma_i$. By Proposition 3.8, there exists a parametrized periodic orbit $\gamma \in \mathcal{C}$ beginning at a point $z \in V$ which passes through $V$ at times $t_i$ close to $\sum_{\ell < i-1} s_\ell$ and which defines the same conjugacy class in $\mathrm{Mod}(S)$ as the concatenation of the pseudo-orbits $(y_1, s_1), \dots, (y_m, s_m)$. But this just means that for each $i$ a $V$-characteristic curve of the orbit segment $\cup_{t \in [t_{i-1}, t_i]} \Phi^t z$ defines the element $\Omega(\gamma_i)$ in $\mathrm{Mod}(S)$. It is now immediate from the construction that $\gamma$ can be parametrized in such a way that the properties in the proposition are fulfilled. □

As a consequence, the subsemigroup $\langle \Omega(\Gamma_0) \rangle$ of $\mathrm{Mod}(S)$ generated by $\{\Omega(\gamma) \mid \gamma \in \Gamma_0\}$ consists of pseudo-Anosov elements whose corresponding periodic orbits are contained in the affine invariant manifold $\mathcal{C}$ and pass through the set $V$. This can be viewed as a version of Rauzy-Veech induction as used in [AV07, AMY16] which is valid for all affine invariant manifolds, in particular for strata of quadratic differentials, or as a version of symbolic dynamics for the Teichmüller flow on strata.

## 4. Local Zariski density for affine invariant manifolds

The goal of this section is to prove Theorem 3. Throughout this section we assume that $g \geq 2$, and we use the assumptions and notations from Section 2.

Let $\mathcal{Q}_+ \subset \mathcal{H}_+$ be a component of a stratum and let $\mathcal{C}_+ \subset \mathcal{Q}_+$ be an affine invariant manifold. Recall from Section 2 that the image of the projection $p : T\mathcal{C}_+ \to \Pi^*\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}|\mathcal{C}_+$ to absolute periods is a flat subbundle of $\Pi^*\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}|\mathcal{C}_+$ which is invariant under both the complex structure defined by enlargement of coefficients (the tensor product) as well as the complex structure of the Hodge bundle. We denote by $2\ell \geq 2$ its complex dimension. Then $p(T\mathcal{C}_+) \cap \Pi^*\mathcal{H}|\mathcal{C}_+$ is a flat bundle $\mathcal{Z} = \mathcal{Z}_{\mathbb{R}}$ whose fibre is a symplectic subspace of the fibre of $\Pi^*\mathcal{H}$ (recall that the fibre of $\Pi^*\mathcal{H}$ can be identified with $H^1(S, \mathbb{R})$) of real dimension $2\ell$. As before, by a flat subbundle of the bundle $\Pi^*\mathcal{H}|\mathcal{C}_+$ we mean a bundle which is invariant under the restriction of the Gauss Manin connection. We call $\mathcal{Z}$ the *absolute real tangent bundle* of $\mathcal{C}_+$. The Gauss Manin connection restricts to a flat connection on $\mathcal{Z}$.

The monodromy of the restriction of the Gauss Manin connection to $\mathcal{Z}$ is defined as the subgroup of $GL(2\ell, \mathbb{R})$ which is generated by parallel transport along loops in $\mathcal{C}_+$ based at some fixed point $p$. As the Gauss Manin connection is symplectic, this monodromy group is a subgroup of $Sp(2\ell, \mathbb{R})$. Its conjugacy class does not depend on any choices made.

**Definition 4.1.** The *monodromy group* of the affine invariant manifold $\mathcal{C}_+$ of rank $\ell$ is the subgroup of $Sp(2\ell, \mathbb{R})$ which is the monodromy of the absolute real tangent bundle $\mathcal{Z}$ of $\mathcal{C}_+$ for the restriction of the Gauss Manin connection.

A geometric description of the monodromy group of $\mathcal{C}_+$ is as follows. Observe first that the monodromy coincides with the monodromy of the restriction of the bundle $\mathcal{Z}$ to the intersection $\mathcal{C}$ of $\mathcal{C}_+$ with the moduli space of area one abelian differentials. Let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials. The stabilizer $\mathrm{Stab}(\tilde{\mathcal{C}})$ of $\tilde{\mathcal{C}}$ in the mapping class group maps via the natural surjective homomorphism $\Psi : \mathrm{Mod}(S) \to Sp(2g, \mathbb{Z})$ to a subgroup of $Sp(2g, \mathbb{Z})$. There is a linear symplectic subspace $H \subset \mathbb{R}^{2g}$ of dimension $2\ell$ which is preserved by $\Psi(\mathrm{Stab}(\tilde{\mathcal{C}}))$. The monodromy group of $\mathcal{C}$ then is the projection of $\Psi(\mathrm{Stab}(\tilde{\mathcal{C}}))$ to the group of symplectic automorphisms of $H$. This description is immediate from the description of the Gauss Manin connection in Section 2.1.

**Example 4.2.** If $\mathcal{C}_+$ is a Teichmüller curve, then the monodromy group of $\mathcal{C}_+$ is just the Veech group of $\mathcal{C}_+$, acting on the two-dimensional symplectic subspace of $H^1(S, \mathbb{R})$ which is spanned by the real and imaginary part, respectively, of an abelian differential $\omega \in \mathcal{C}_+$. Thus this monodromy group is a lattice in $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$, in particular it is Zariski dense in $SL(2, \mathbb{R})$.

The proof of Theorem 3 is divided into two steps contained in two subsections. The first step establishes Zariski density of the (global) monodromy group, and in a second step, we extend this result to the local monodromy group.

4.1. **The monodromy group of affine invariant manifolds.** The goal of this subsection is to show that the monodromy group of any affine invariant manifold is Zariski dense in $Sp(2g, \mathbb{R})$. We will make use of the fact that an abelian differential on $S$ defines a singular euclidean metric on $S$ with cone points of cone angle a multiple of $2\pi$ at the zeros of the differential. This singular euclidean metric is given by a family of charts, defined on the complement of the zeros of the differential, with chart transitions being translations. As it is customary in the literature, if we view an abelian differential on $S$ as a singular euclidean metric, we refer to these data as a *translation surface*. We denote such a translation surface by $X$ or by a pair $(X, \omega)$ if we like to specify the abelian differential $\omega$ which defines the translation structure. Note that $\omega$ can be read off from the horizontal and vertical measured foliations of the translation surface.

We begin with evoking a result of Wright [W15]. He introduced the following two deformations of a translation surface $(X, \omega)$.

The *horocycle flow* is defined as part of the $SL(2, \mathbb{R})$-action,

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \subset SL(2, \mathbb{R}),$$

and the *vertical stretch* is defined by

$$a_t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \subset GL^+(2, \mathbb{R}).$$

For a collection $\mathcal{Y}$ of horizontal cylinders on a translation surface $X$ (i.e. cylinders foliated by leaves of the horizontal foliation), define the *cylinder shear* $u_t^{\mathcal{Y}}(X)$ to be the translation surface obtained by applying the horocycle flow to the cylinders in $\mathcal{Y}$ but not to the rest of $X$. Similarly, the *cylinder stretch* $a_t^{\mathcal{Y}}(X)$ is obtained by applying the vertical stretch only to the cylinders in $\mathcal{Y}$.

The following lemma is a consequence of the work of Wright [W15]. For its formulation, a translation surface $(X, \omega)$ is called *horizontally periodic* if it is a union of horizontal cylinders. Via the natural pairing $\langle , \rangle$ between first homology and first cohomology of $S$, a class in $H_1(S, \mathbb{R})$ defines an element of $H^1(S, \mathbb{R})^*$.

**Lemma 4.3.** *Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell$. Then there exists a horizontally periodic surface $(X, \omega) \in \mathcal{C}_+$ with the following properties. There is a decomposition of $X$ into $\ell + 1$ collections $\mathcal{Y}_1, \ldots, \mathcal{Y}_\ell, \mathcal{Y}_{\ell+1}$ of horizontal cylinder families. The family $\mathcal{Y}_{\ell+1}$ may be empty. The homology classes of the cylinder families $\mathcal{Y}_i$ $(i \leq \ell)$ span a subspace of the dual $pT\mathcal{C}_+^*$ of $pT\mathcal{C}_+$ of dimension $\ell$, and the moduli of all of the cylinders in each of the collections $\mathcal{Y}_i$ $(i \leq \ell)$ are rational. For each $i \leq \ell$, the cylinder shear $u_t^{\mathcal{Y}_i}(X)$ remains in $\mathcal{C}_+$.*

*Proof.* Let $(X, \omega) \in \mathcal{C}_+$ be a translation surface with the maximal number of parallel cylinders. We may assume that these cylinders are horizontal. Following the proof of Theorem 1.10 of [W15], $(X, \omega)$ is horizontally periodic, and the core curves of the horizontal cylinders span a subspace of the dual $pT\mathcal{C}_+^*$ of $pT\mathcal{C}_+$ of dimension $\ell$. No set of core curves of parallel cylinders on a translation surface $Y \in \mathcal{C}_+$ may span a subspace of $pT\mathcal{C}_+^*$ of dimension greater than $\ell$.

Following Definition 4.4 of [W15], call two homology classes in $H_1(S, \mathbb{R})$ $\mathcal{C}_+$-*collinear* if they have collinear images in $T\mathcal{C}_+^*$, i.e. if they are scalar multiples. By Definition 4.6 of [W15], two cylinders in $X$ are called $\mathcal{C}_+$-*parallel* if they are parallel at $X$ and at every nearby $X' \in \mathcal{C}_+$. Being $\mathcal{C}_+$-parallel is an equivalence relation on the set of cylinders.

Let $\mathcal{Z}_i$ $(i = 1, \ldots, k)$ be the set of equivalence classes of horizontal cylinders in $(X, \omega)$ for this equivalence relation. By the choice of $(X, \omega)$ and the results in Section 4 of [W15], we have $k = \ell$, i.e. the horizontal cylinders of $(X, \omega)$ group into precisely $\ell$ equivalence classes. Lemma 4.11 of [W15] shows that the cylinder shear of any of the $\mathcal{C}_+$-parallel cylinder families $\mathcal{Z}_i$ remains in $\mathcal{C}_+$.

Consider one of the families $\mathcal{Z}_i$. The cylinder shear for $\mathcal{Z}_i$ remains in $\mathcal{C}_+$. Corollary 3.4 of [W15] states that if the moduli of the cylinders in this family are not all rationally dependent, then there is a proper decomposition $\mathcal{Z}_i = \mathcal{A} \cup \mathcal{B}$ so that the cylinder shears for the families $\mathcal{A}, \mathcal{B}$ remain in $\mathcal{C}_+$. Thus we can subdivide the cylinder family $\mathcal{Z}_i = \cup_j \mathcal{Z}_i^j$ where $j \geq 1$, where the moduli of the cylinders in each of the families $\mathcal{Z}_i^j$ are rationally dependent and such that for each $j$, the cylinder shear $u_t^{\mathcal{Z}_i^j}(X)$ remains in $\mathcal{C}_+$.

By Theorem 5.1 of [W15], for all $i \leq \ell$ the vertical stretch of the cylinder family $\mathcal{Z}_i$ is contained in $\mathcal{C}_+$. This vertical stretch changes the moduli of the cylinders in the family $\mathcal{Z}_i$ while keeping the moduli of the cylinders in the family $\mathcal{Z}_j$ fixed for all $j \neq i$. If $A_1, A_2 \subset \mathcal{Z}_i$ are cylinders with rationally dependent moduli, then the moduli of their images under the vertical stretch are rationally dependent as well. As a consequence, by successively modifying $(X, \omega)$ with a sequence of vertical stretches of the cylinder families $\mathcal{Z}_i$ $(i = 1, \ldots, \ell)$ we can assure that in the image surface $(X', \omega')$ which is again horizontally periodic, the moduli of all cylinders in the cylinder families $\mathcal{Y}_1, \dots, \mathcal{Y}_\ell$ which are the images in $X'$ of the families $\mathcal{Z}_1^1, \dots, \mathcal{Z}_\ell^1$ are rational.

Let $\mathcal{Y}_{\ell+1} = X' - \cup_i \mathcal{Y}_i$. Then the surface $(X', \omega)$ and the cylinder families $\mathcal{Y}_i$ have the properties stated in the lemma. □

Define a *piecewise affine transformation* of a translation surface $(X, \omega)$ to be a continuous self-map $F : X \to X$ with the following property. There exists an $F$-invariant decomposition $X = \cup_i X_i$ into finitely many components with geodesic boundary for the singular euclidean metric, and the restriction of $F$ to each of these components is affine. In contrast to an affine automorphism of $(X, \omega)$, we allow that the restriction of $F$ to some of the components $X_i$ equals the identity. A cylinder shear of a collection $\mathcal{Y}$ of horizontal cylinders with non-empty complement is such a piecewise affine transformation. If the result of such a transformation is isometric to $(X, \omega)$ then we call the piecewise affine transformation a *piecewise affine automorphism* of $(X, \omega)$.

A *transvection* in a $2g$-dimensional symplectic vector space over a field $K$ is a map $A \in Sp(2g, K)$ which fixes a subspace of $K^{2g}$ of codimension one and has determinant one (see [Hl08]). Any map of the form

$$\alpha \to \alpha + \iota(\alpha, \beta)\beta$$

for some $0 \neq \beta \in K^{2g}$ (here as before, $\iota$ is the symplectic form) is a transvection. We call this map a *transvection by $\beta$*. The main consequence of Lemma 4.3 we are going to use is the following

**Corollary 4.4.** *Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 1$. Then there is a horizontally periodic surface $(X, \omega) \in \mathcal{C}_+$, and there is a free abelian group of rank $\ell$ of piecewise affine transformations of $(X, \omega)$ which preserves $\mathcal{C}_+$. This group of piecewise affine transformations contains a lattice $H$, i.e. a subgroup isomorphic to $\mathbb{Z}^\ell$, which acts on $(X, \omega)$ as a group of Dehn-multitwists, and it acts on $H_1(S, \mathbb{R})$ as a group of transvections of rank $\ell$. This action restricts to a group of linear automorphisms of $pT\mathcal{C}_+^*$ of rank $\ell$.*

*Proof.* Let $(X, \omega)$ be a translation surface as in Lemma 4.3. Let $\mathcal{Y}_i$ $(i \leq \ell)$ be one of the cylinder families whose existence was shown in Lemma 4.3. The moduli of all cylinders in the family are rational. Moreover, the cylinder shear $u_t^{\mathcal{Y}_i}(X)$ for this cylinder family remains in $\mathcal{C}$.

As all the moduli of the cylinders are rational, this cylinder shear is eventually periodic. This means that for each $i$ there exists some number $r_i > 0$ such that for some fixed marking of the surface $X$, the surface $u_{r_i}^{\mathcal{Y}_i}(X)$ is the image of $X$ by a Dehn multitwist $T_i$ about the core curves of the cylinders in $\mathcal{Y}_i$.

Since the core curves of the horizontal cylinders in $X$ are pairwise disjoint, the Dehn multitwists $T_i$ commute. Therefore these multitwists generate a free abelian group of rank $\ell$ of piecewise affine automorphisms of $X$. The multitwist $T_i$ acts as a transvection on $H_1(S, \mathbb{R})$ by a homology class of the form $\sum_s b_i^s \zeta_i^s$ where $b_i^s \in \mathbb{Z}$ and where $\zeta_i^s$ runs through the homology classes of the waist curves of the oriented cylinders in the family $\mathcal{Y}_i$.

Each of the homology classes $a_i = \sum_s b_i^s \zeta_i^s$ $(i \le \ell)$ induces a linear functional on the fibre of $T\mathcal{C}_+$ at $X$. The corollary now follows from the fact that by the choice of $(X, \omega)$, the rank of the subspace of $T\mathcal{C}_+^*$ spanned by these homology classes equals $\ell$. Then the subgroup of $\mathrm{Mod}(S)$ generated by the Dehn multitwists $T_i$ $(i = 1, \dots, \ell)$ acts on $H_1(S, \mathbb{R})$ as an abelian group of transvections of rank $\ell$. ☐

Our criterion for Zariski density relies on a result of Hall [Hl08]. For its formulation, for a prime $p \ge 2$ let $F_p$ be the field with $p$ elements. Then $Sp(2g, F_p)$ is a finite group. Therefore for every $A \in Sp(2g, F_p)$ there is some $\ell \ge 1$ such that $A^\ell = A^{-1}$. As a consequence, if $G < Sp(2g, F_p)$ is any subsemigroup then for all $x, y \in G$ we have $xy^{-1} \in G$ as well and hence $G < Sp(2g, F_p)$ is a group.

In the formulation of the following lemma, $\iota$ denotes the symplectic form on a symplectic vector space $F_p^{2\ell}$ over $F_p$ of rank $2\ell$.

**Lemma 4.5.** *Let $p \ge 3$ be an odd prime and let $G < Sp(2\ell, F_p)$ be a subgroup generated by $2\ell$ transvections by the elements of a set $\mathcal{E} = \{e_1, \dots, e_{2\ell}\} \subset F_p^{2\ell}$ which spans $F_p^{2\ell}$. Assume that there is no nontrivial partition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ so that $\iota(e_{i_1}, e_{i_2}) = 0$ for all $e_{i_j} \in \mathcal{E}_j$. Then $G = Sp(2\ell, F_p)$.*

*Proof.* For each $i$ write $A_i(x) = x + \iota(x, e_i)e_i$. Let $G < Sp(2\ell, F_p)$ be the subgroup generated by the transvections $A_1, \dots, A_{2\ell}$. Since the vectors $e_1, \dots, e_{2\ell}$ span $F_p^{2\ell}$, the intersection of the invariant subspaces of the transvections $A_i$ $(i \le 2\ell)$ is trivial.

We claim that the standard representation of $G$ on $F_p^{2\ell}$ is irreducible. Namely, assume to the contrary that there is an invariant proper linear subspace $W \subset F_p^{2\ell}$. Let $0 \ne w \in W$; then there is at least one $i$ so that $\iota(w, e_i) \ne 0$. By invariance, we have $w + \iota(w, e_i)e_i \in W$ and hence $e_i \in W$ since $F_p$ is a field.

As a consequence, $W$ is spanned by some of the $e_i$, say by $e_{i_1}, \dots, e_{i_k}$, and if $j$ is such that $\iota(e_{i_s}, e_j) \ne 0$ for some $s \le k$ then $e_j \in W$. However, this implies that $W = F_p^{2\ell}$ by the assumption on the set $\mathcal{E} = \{e_i\}$.

To summarize, $G$ is an irreducible subgroup of $Sp(2\ell, F_p)$ generated by transvections (where irreducible means that the standard representation of $G$ on $F_p^{2\ell}$ is irreducible). Furthermore, as $p$ is an odd prime by assumption, the order of each of these transvections is not divisible by 2. Theorem 3.1 of [Hl08] now yields that $G = Sp(2\ell, F_p)$ which is what we wanted to show. ☐

**Remark 4.6.** By Proposition 6.5 of [FM12], Lemma 4.5 is not true for $p = 2$.

We use Lemma 4.5 to establish a criterion for Zariski density of a subgroup of $Sp(2\ell, \mathbb{R})$ acting on a $2\ell$-dimensional symplectic subspace of $H^1(S, \mathbb{R})$. In its formulation, we use the standard pairing

$$\langle , \rangle : H^1(S, \mathbb{R}) \times H_1(S, \mathbb{R}) \to \mathbb{R}$$

between homology and cohomology to view a class in $H_1(S,\mathbb{R})$ as an element of $H^1(S,\mathbb{R})^*$. A symplectic automorphism of $H_1(S,\mathbb{R})$ induces a symplectic automorphism of $H^1(S,\mathbb{R})$. Recall also that the real part $\mathrm{Re}(\tilde q)$ and the imaginary part $\mathrm{Im}(\tilde q)$ of a marked abelian differential $\tilde q$ define a cohomology class $[\mathrm{Re}(\tilde q)], [\mathrm{Im}(\tilde q)] \in H^1(S,\mathbb{R})$.

For a symplectic subspace $E$ of $H^1(S,\mathbb{R})$ denote by $Sp(E^*)$ the group of symplectic automorphisms of its dual $E^*$. The image of $\mathrm{Mod}(S)$ under the homomorphism $\Psi$ is the integral symplectic group $Sp(2g,\mathbb{Z})$ and hence reduction of coefficients modulo a prime $p$ makes sense. By a weighted oriented simple multicurve $c$ on $S$ we mean a simple oriented multicurve with integral weights. For some fixed choice of a marking of $S$, such a weighted oriented simple multicurve then defines a homology class $[c] \in H_1(S,\mathbb{Z})$. In the formulation of the proposition below we use such a fixed choice of a marking for $S$.

**Proposition 4.7.** *Let $\mathcal{C}$ be an affine invariant manifold of rank $\ell$, let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials and let $\mathcal{Z} = p(T\tilde{\mathcal{C}}_+) \cap \Pi^*\mathcal{H}|\mathcal{C}_+$. Let $c_1,\dots,c_\ell$ be pairwise disjoint weighted oriented simple multicurves whose (marked) homology classes $[c_i]$ span a subspace of $\mathcal{Z}^*$ of rank $\ell$. Let $U \subset \mathcal{C}$ be an open contractible set and assume that there is a component $\tilde U$ of the preimage of $U$ in $\tilde{\mathcal{C}}$ such that $\langle [\mathrm{Re}(\tilde z)], [c_i]\rangle > 0$ for all $\tilde z \in \tilde U$, all $1 \le i \le \ell$. Let $\Omega(\Gamma_0) \subset \mathrm{Mod}(S)$ be the subsemigroup determined by a suitable pair of open contractible subsets $Y \subset V$ of $U$ and the lift $\tilde U$ of $U$ as in Proposition 3.10. Then the subgroup of $Sp(\mathcal{Z}^*)$ generated by $\Psi(\Omega(\Gamma_0))$ and the Dehn multitwists $T_{c_i}$ about the multicurves $c_i$ is Zariski dense in $Sp(\mathcal{Z}^*)$. If $\ell = g$ then for all but finitely many primes $p \ge 3$, this semigroup surjects onto $Sp(2g,F_p)$.*

*Proof.* Let $\mathcal{C}$ be an affine invariant manifold of rank $\ell$. Let $U \subset \mathcal{C}$ be an open contractible set with the properties stated in the proposition and let $\tilde U$ be a component of the preimage of $U$ in the Teichmüller space of marked abelian differentials. Via perhaps decreasing the size of $U$ we may assume that $\tilde U$ has a product structure, defined by disjoint compact balls $D, K$ of dimension $\dim_{\mathbb{C}}(\mathcal{C}_+) - 1$ in the sphere of projective measured foliations on $S$ as in Definition 3.2. The real parts $\mathrm{Re}(\tilde z)$ of the differentials $\tilde z \in \tilde U$ project to an open subset of the $2\ell$-dimensional subspace $\mathcal{Z}$ of $H^1(S,\mathbb{R})$ as defined in the proposition.

Let $Y \subset V \subset U$ be a pair of open subsets of $U$ as in Proposition 3.8 and use these sets and the components $\tilde Y \subset \tilde V \subset \tilde U$ of the preimages of $Y \subset V$ to construct the subsemigroup $\Omega(\Gamma_0)$ of $\mathrm{Mod}(S)$.

Let $c_1,\dots,c_\ell$ be pairwise disjoint simple oriented weighted multicurves. With respect to some fixed marking of $S$, used for the choice of the lift $\tilde U$ of $U$, assume that the homology classes $[c_i]$ of $c_i$ span a linear subspace $L$ of $\mathcal{Z}^*$ of dimension $\ell$. As the multicurves $c_i$ are pairwise disjoint, this subspace is isotropic. The projection which associates to a marked abelian differential $\tilde z \in \tilde U$ the cohomology class $[\mathrm{Re}(\tilde z)] \in H^1(S,\mathbb{R})$ of its real part $\mathrm{Re}(\tilde z)$ maps the open subset $\tilde Y$ of $\tilde{\mathcal{C}}$ to an open subset of the dual $L^*$ of $L$.

Let $\tilde{z} \in \tilde{Y}$ be the lift of a periodic point $z \in Y$ for $\Phi^t$; such a point exists by Proposition 3.8. Let $\varphi \in \Omega(\Gamma_0) < \mathrm{Mod}(S)$ be the pseudo-Anosov element which preserves the $\Phi^t$-orbit of $\tilde{z}$. Recall the assumption $\langle [\mathrm{Re}(\tilde{z})], [c_i] \rangle > 0$ for all $i$.

There is a number $\kappa > 1$ such that $\varphi^* \mathrm{Re}(\tilde{z}) = \kappa \mathrm{Re}(\tilde{z})$, moreover $\kappa$ is the Perron Frobenius eigenvalue for the action of $\varphi$ on $H^1(S, \mathbb{R})$. By invariance of the natural pairing $\langle , \rangle$ under $\varphi$, as $k \to \infty$ the homology classes $[\varphi^k c_i]$ converge up to rescaling to a class $u \in H_1(S, \mathbb{R})$ whose contraction with the intersection form $\iota$ defines $\pm[\mathrm{Re}(\tilde{z})]$, viewed as a linear functional on $H_1(S, \mathbb{R})$. By this we mean that $\iota(u, a) = \langle \pm[\mathrm{Re}(\tilde{z})], a \rangle$ for all $a \in H_1(S, \mathbb{R})$. As a consequence, for all sufficiently large $n > 0$ and all $i, j \leq \ell$ we have $\iota([\varphi^n c_i], [c_j]) \neq 0$.

Let $G < \mathrm{Mod}(S)$ be the group generated by the semigroup $\Omega(\Gamma_0)$ as well as the Dehn multitwists $T_i = T_{c_i}$ $(i \leq \ell)$. Then $G$ contains the multitwists $\varphi^n T_i \varphi^{-n} = T_{\varphi^n c_i}$ (see Fact 3.7 on p.73 of [FM12] for this equation).

Let $A_1 < \mathcal{Z}^*$ be the linear subspace of rank $\ell$ which is the common fixed set in $\mathcal{Z}^*$ for the transvections $\Psi(T_{c_i})$ of $\mathcal{Z}^*$ $(i = 1, \ldots, \ell)$. Then $A_1$ is a Lagrangian subspace of the symplectic vector space $\mathcal{Z}^*$. Let $A_2 \subset A_1$ be the common fixed set in $\mathcal{Z}^*$ of the transvections which are the images under the map $\Psi$ of all multitwists $T_i, \varphi^n T_j \varphi^{-n}$. Then $A_2$ is a linear subspace of $A_1$, and for large enough $n$ its codimension in $A_1$ is $s \geq 1$. Let $i_1, \ldots, i_s \subset \{1, \ldots, \ell\}$ be such that the homology classes $[c_j], [\varphi^n c_{i_p}] \in H_1(S, \mathbb{Z})$ $(j \leq \ell, p \leq s)$ are independent over $\mathbb{R}$ and that the common fixed set in $\mathcal{Z}^*$ of the transvections defined by the corresponding Dehn multitwists is $A_2$.

Since the set of real parts of differentials in $\tilde{Y}$ define an open subset of the symplectic vector space $\mathcal{Z}$, we can find some $\tilde{y} \in \tilde{Y}$ and some $a \in A_2$ so that $\langle [\mathrm{Re}(\tilde{y})], a \rangle \neq 0$. As this condition is open, we may assume as before that $\tilde{y}$ is the preimage of a periodic point of $Y$. Argue now as in the previous paragraph and find a multitwist $\beta$ in the subgroup $G$ of $\mathrm{Mod}(S)$ generated by $\Omega(\Gamma_0)$ so that the common fixed set of the subgroup generated by $\Psi(\beta)$ and $A_2$ has codimension at least one in $A_2$.

Repeat this construction. In at most $\ell$ steps we find integral homology classes $a_1, \ldots, a_\ell, a_{\ell+1}, \ldots, a_{2\ell} \in H_1(S, \mathbb{Z})$ (where for $i \leq \ell$ the class $a_i$ is the class $[c_i]$ of the oriented weighted multicurve $c_i$) with the following properties.

(1) Let $W \subset H_1(S, \mathbb{R})$ be the real vector space spanned by the classes $a_i$. The dimension of $W$ equals $2\ell$. Viewing $W$ as a linear subspace of $H^1(S, \mathbb{R})^*$, its restriction to $\mathcal{Z}$ is non-degenerate. In particular, $W$ is a symplectic subspace of $H_1(S, \mathbb{R})$.
(2) $\iota(a_j, a_i) \neq 0$ for all $i \leq \ell$, $j \geq \ell + 1$.
(3) For each $j$ the transvection $b \to b + \iota(b, a_j) a_j$ is contained in the group generated by $\Psi(\Omega(\Gamma_0))$ and the Dehn multitwists $\Psi(T_{c_i})$ $(i \leq \ell)$.

By the choice of the homology classes $a_i$, the $(2\ell, 2\ell)$-matrix $(\iota(a_i, a_j))$ whose $(i, j)$-entry is the homology intersection number $\iota(a_i, a_j)$ is integral and of maximal rank. Choose a prime $p \geq 5$ so that each of the entries of $(\iota(a_i, a_j))$ is prime to $p$. All but finitely many primes will do. Then the reduction mod $p$ of the matrix $(\iota(a_i, a_j))$

is of maximal rank as well. In particular, if $F_p$ denotes the field with $p$ elements then the reductions mod $p$ of the homology classes $a_i$ span a $2\ell$-dimensional symplectic subspace $W_p$ of $H_1(S, F_p)$.

Let $\Lambda < \mathrm{Sp}(W)$ be the subgroup of the symplectic group of $W$ which is generated by the transvections with the elements $a_i$. Its reduction $\Lambda_p$ mod $p$ acts on $W_p$ as a group of symplectic transformations. Lemma 4.5 shows that $W_p = Sp(2\ell, F_p)$. Note that property (2) above guarantees that all conditions in Lemma 4.5 are fulfilled. Then $\Lambda$ is a Zariski dense subgroup of the group of symplectic automorphisms of $W$ [Lu99]. By duality, this implies that the subgroup $G$ of $Sp(\mathcal{Z}^*)$ generated by $\Psi(T_{c_i})$ and $\Psi(\Omega(\Gamma_0))$ is Zariski dense in $Sp(\mathcal{Z}^*)$.

Now assume that $\ell = g$. The Dehn twists $T_{c_i}$ define elements of $Sp(2g, \mathbb{Z})$. All elements of $Sp(2g, \mathbb{R})$ constructed in the above way are integral, and the above proof shows that the subgroup of $Sp(2g, \mathbb{Z})$ constructed in the above way surjects onto $Sp(2g, F_p)$ for all but finitely many $p$. □

Let again $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 1$. Recall from Definition 4.1 the definition of the monodromy group of an affine invariant manifold $\mathcal{C}_+$ of rank $\ell$. We can now summarize the discussion in this section as follows.

**Corollary 4.8.** *For any affine invariant manifold $\mathcal{C}_+$ of rank $\ell$, the monodromy group of $\mathcal{C}_+$ is Zariski dense in $Sp(2\ell, \mathbb{R})$.*

*Proof.* Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 1$, and let $\mathcal{C} \subset \mathcal{C}_+$ be its subset of differentials of area one.

Choose a translation surface $(X, \omega) \in \mathcal{C}$ with the properties stated in Corollary 4.4. Denote by $H$ the free abelian group of rank $\ell$ of Dehn multitwists which is contained in the group of piecewise affine automorphisms of $X$ whose existence was shown in Corollary 4.4.

Choose a marking of the translation surface, i.e. a lift $\tilde{\omega}$ of $\omega$ to the Teichmüller space of abelian differentials. By construction, we have $\langle [\mathrm{Re}(\tilde{\omega}), [c_i] \rangle > 0$ for all $i$. As this is an open condition, we can find an open neighborhood $\tilde{U}$ of $\tilde{\omega}$ in the component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ such that this condition is fulfilled for all $\tilde{z} \in \tilde{U}$. We may assume that $\tilde{U}$ projects to a contractible subset $U$ of $\mathcal{C}$. The corollary now follows from Proposition 4.7. □

4.2. **The local monodromy group of affine invariant manifolds.** The goal of this subsection is to complete the proof of Theorem 3. We use the following

**Definition 4.9.** An affine invariant manifold $\mathcal{C}$ of rank $\ell$ is *locally Zariski dense* if for every open contractible subset $U$ of $\mathcal{C}_{\mathrm{good}}$ the subsemigroup of $Sp(2\ell, \mathbb{R})$ generated by the monodromy of those periodic orbits for $\Phi^t$ in $\mathcal{C}$ which pass through $U$ is Zariski dense in $Sp(2\ell, \mathbb{R})$.

Here as before, monodromy means monodromy of the restriction of the Gauss Manin connection to the bundle $\mathcal{Z} \to \mathcal{C}$, and this is computed with respect to a fixed trivialization of $\mathcal{Z}$ over $U$ which is parallel for the Gauss Manin connection. Replacing such a trivialization by another one changes the local monodromy group by a conjugation. We refer to subsection 2.2 for details.

We begin with reducing the statement of Theorem 3 to a statement on local Zariski density near a single point. For the next lemma, call a point $q \in \mathcal{C}$ *transitive* if its orbit under the Teichmüller flow is dense in $\mathcal{C}$. Transitive points are known to exist and are dense in $\mathcal{C}$ [EMM15].

**Lemma 4.10.** *An affine invariant manifold $\mathcal{C}$ of rank $\ell$ is locally Zariski dense if and only if there exists a transitive point $q \in \mathcal{C}_{\rm good}$ with the following property. For every open contractible neighborhood $U$ of $q$, the subgroup of $Sp(2\ell, \mathbb{R})$ generated by the monodromy of those periodic orbits for $\Phi^t$ in $\mathcal{C}$ which pass through $U$ is Zariski dense in $Sp(2\ell, \mathbb{R})$.*

*Proof.* That the condition stated in the lemma is necessary is obvious from the definition of local Zariski density. We have to show that it is also sufficient.

To this end let $q \in \mathcal{C}_{\rm good}$ be a transitive point as in the statement of the lemma. Let $U$ be any open subset of $\mathcal{C}$. Let $z \in U \cap \mathcal{C}_{\rm good}$ be an arbitrary transitive point; such a point exists since $\mathcal{C}_{\rm good}$ is an open and dense $\Phi^t$-invariant subset of $\mathcal{C}$ and the set of transitive points is dense. Write $U = U_z$ and let $U_q$ be a neighborhood of $q$.

By Proposition 3.8, we can find neighborhoods $Y_z \subset V_z \subset U_z$ of $z$, $Y_q \subset V_q \subset U_q$ of $q$ and a number $n > 0$ with the following properties. The sets $V_z, V_q$ are contractible. Write $\mathcal{Y} = \{Y_q, Y_z\}$ and let $u_0, u_1, u_2, u_3$ be a periodic $(n, \mathcal{Y})$-pseudo-orbit for $\Phi^t$, with $u_0 = u_3 \in Y_z$ and $u_1, u_2 \in Y_q$. There are numbers $t_i > n$ such that $\Phi^{t_i} u_i \in Y_{\kappa(i+1)}$ where $\kappa(i+1) = q$ for $i = 0, 1$ and $\kappa(i+1) = z$ otherwise. Such a pseudo-orbit exists since the Teichmüller flow on $\mathcal{C}$ is topologically transitive.

Let $\mathcal{V} = \{V_q, V_z\}$ and let $\eta$ be a $\mathcal{V}$-characteristic curve for this pseudo-orbit. Then $\eta$ determines a parametrized periodic orbit $\nu$ for $\Phi^t$ beginning in $V_z$, and this orbit passes through $V_q$.

Choose a component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials and let $\tilde{V}_z \subset \tilde{\mathcal{C}}$ be a component of the preimage of $V_z$. Let $\tilde{u}_0$ be the preimage of $u_0$ in $\tilde{V}_z$. For this fixed choice, the parametrized periodic orbit $\nu$ determines a pseudo-Anosov element $\Omega(\nu) \in {\rm Mod}(S)$ as follows. Let $\tilde{\eta}$ be the lift of the characteristic curve $\eta$ for the pseudo-orbit beginning at $\tilde{u}_0$. Then $\Omega(\nu)$ maps the endpoint of $\tilde{\eta}$ back to its starting point. Our goal is to show that the subsemigroup of $Sp(2\ell, \mathbb{R})$ generated by the elements $\Psi(\Omega(\nu))$ for parametrized periodic orbits $\nu$ of the above form is Zariski dense in $Sp(2\ell, \mathbb{R})$.

To this end let $\tilde{V}_q \subset \tilde{\mathcal{C}}$ be the component of the preimage of $V_q$ which contains $\Phi^{t_0} \tilde{u}_0$. If $\eta'$ is a characteristic curve of a pseudo-orbit defined by points $u_0, u_1', u_2, u_3 = u_0$, with $u_1' \in Y_q$, and times $t_0, t_1', t_2 > n$, and if $\nu'$ is the corresponding periodic orbit, then the element $\Omega(\nu')^{-1} \circ \Omega(\nu)$ (read from right to left) of $\mathrm{Mod}(S)$ maps the endpoint of the lift beginning in $\tilde{V}_z$ of the concatentation $(\eta')^{-1}\circ\eta$ back to its starting point $\tilde{u}_0$. Recall that this makes sense since $\eta,\eta'$ begin and end at the same point $u_0\in Y_z$.

Thus $\Psi(\Omega(\nu')^{-1}\circ\Omega(\nu))$ equals the holonomy for parallel transport with respect to the Gauss Manin connection of the following loop. Fix the point $u_0\in Y_z$ as a basepoint. The $(n,\mathcal{Y})$-pseudo-orbit given by the points $u_0,u_1,u_1$ and the times $t_0,t_1$ determine the homotopy class with fixed endpoints of an arc $\beta$ connecting $u_0$ to $u_1$, and there is an arc $\beta'$ for the $(n,\mathcal{Y})$-pseudo-orbit given by the points $u_0,u_1',u_1$ and the times $t_0,t_1'$. These arcs are constructed in such a way that they end at $u_1$. The holonomy of the concatenation of $\beta$ with the inverse of $\beta'$ equals the element $\Psi(\Omega(\nu')^{-1}\circ\Omega(\nu))$ (again read from right to left).

Now parallel transport along the distinguished orbit segment connecting $u_0$ to $u_1$ identifies the fibre of $\mathcal{Z}\subset\Pi^*\mathcal{H}$ at $u_0$ with the fibre of $\mathcal{Z}$ at $u_1$ as a symplectic vector space. This identification conjugates $\Psi(\Omega(\nu')^{-1}\circ\Omega(\nu))$ to $\Psi(\Omega(\xi')^{-1}\circ\Omega(\xi))$ where $\Omega(\xi),\Omega(\xi')$ are the elements of $\mathrm{Mod}(S)$ constructed in the same way from $\tilde{V}_q$ and from parametrized periodic orbits of $\Phi^t$ through $V_q$ determined by the one-segment periodic pseudo-orbits $(u_1,t_1)$ and $(u_1',t_1')$. Furthermore, the conjugating element does not depend on $\nu,\nu'$.

To complete the proof just note that a subsemigroup $G$ of $Sp(2\ell,\mathbb{R})$ is Zariski dense if and only if for any $h\in Sp(2\ell,\mathbb{R})$ the conjugate $hGh^{-1}$ is Zariski dense if and only if there exists an element $g\in G$ such that $g^{-1}G\subset Sp(2\ell,\mathbb{R})$ is not contained in any proper algebraic subvariety of $Sp(2\ell,\mathbb{R})$. Thus under the assumption of the lemma, the affine invariant manifold $\mathcal{C}$ is indeed locally Zariski dense. □

We need the following technical statement which is well known for components of strata. For its formulation, recall from Definition 3.2 the definition of a set with a product structure.

**Lemma 4.11.** *Let $\tilde{\mathcal{C}}$ be a component of the preimage in the Teichmüller space of abelian differentials of an affine invariant manifold $\mathcal{C}$. Let $\tilde{\alpha}:[0,1]\to\tilde{\mathcal{C}}$ be a smooth path which consists of differentials with the same horizontal projective measured foliation. Then there exists an open set $\tilde{V}\subset\tilde{\mathcal{C}}$ with a product structure which contains $\tilde{\alpha}$.*

*Proof.* Cover the compact path $\tilde{\alpha}$ by finitely many open subsets $W_i$ $(i=0,\dots,k)$ of $\tilde{\mathcal{C}}$ whose closures $\overline{W_i}$ have a product structure as in Definition 3.2. These product structures are defined by compacts balls $D_i,K_i\subset\mathcal{PMF}$, a map $\Lambda_i:D_i\times K_i\to\overline{W_i}$ and a number $\epsilon_i>0$. For each $i$, the set $D_i$ coincides with the set of all horizontal projective measured foliations of all points in $\overline{W_i}$. Let $\mathrm{int}(D_i)$ be the interior of $D_i$ (this is meant to be the interior of the $D_i$ viewed as an $m$-dimensional ball in $\mathcal{PMF}$ and not the interior of $D_i$ as a subset of $\mathcal{PMF}$). As the horizontal projective measured foliation of any point on $\tilde{\alpha}$ coincides with the horizontal projective measured foliation $\mu$ of $\tilde{\omega}=\tilde{\alpha}(0)$, we have $\mu\in\mathrm{int}(D_i)$ for each $i$.

Up to renumbering, we may assume that $W_i\cap W_{i+1}\cap\tilde{c}\neq\emptyset$ for all $i$. We now show by induction on $j\le k$ that the set $\cap_{i\le j}\mathrm{int}(D_i)$ is an open neighborhood of $\mu$ in each of the sets $D_i$ $(i\le j)$.

The case $j = 0$ is obvious, so assume that the claim is known for some $0 \leq j < k$. This means that $E_j = \cap_{i\leq j}\text{int}(D_i)$ is an open neighborhood of $\mu$ in each of the sets $\text{int}(D_i)$ for $i \leq j$. Note however that $E_j$ is not an open subset of $\mathcal{PMF}$.

Let $\overline{E}_j$ be the closure of $E_j$ in $D_j$. As $E_j$ is an open neighborhood of $\mu$ in $\text{int}(D_j)$, the subset $Z_j$ of $W_j$ with a product structure which is defined by $\overline{E}_j, K_j$ and the restriction of $\Lambda_j$ contains an open neighborhood of $\tilde{\alpha} \cap W_j$ (compare the remark after Definition 3.2). Thus by the assumption on the sets $W_u$, the intersection $Z_j \cap W_{j+1}$ contains an open neighborhood in $\tilde{\mathcal{C}}$ of $W_j \cap W_{j+1} \cap \tilde{\alpha}$. But this is only possible if $E_j \cap \text{int}(D_{j+1})$ is an open neighborhood of $\mu$ in both $E_j, D_{j+1}$. The induction step follows.

Let $E \subset E_k$ be a compact neighborhood of $\mu$ in $E_k$ which is homeomorphic to a closed ball of dimension $m$. Then $E$ is a compact neighborhood of $\mu$ in each of the sets $\text{int}(D_i)$, and by construction,

$$\tilde{\alpha} \subset \cup_i \cup_{-\epsilon_i \leq t \leq \epsilon_i} \Phi^t \Lambda_i(E \times K_i).$$

It now follows from the definition of a subset of $\tilde{\mathcal{C}}$ with a product structure that there is a neighborhood of $\tilde{\alpha}$ in $\tilde{\mathcal{C}}$ with a product structure which is of the form $\cup_{-\delta\leq t\leq\delta}\Phi^t\Lambda(E \times \cup_i K_i)$. Here for a point $(\xi, \eta) \in E \times K_i$, the point $\Lambda(\xi, \eta)$ is obtained from $\Lambda_i(\xi, \eta)$ by postcomposition with $\Phi^{\sigma_i(\xi,\eta)}$ where $\sigma_i : E \times K_i \to \mathbb{R}$ is a continuous function. The lemma follows. □

Now we are ready to show

**Theorem 4.12.** *An affine invariant manifold is locally Zariski dense.*

*Proof.* Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 1$, and let $\mathcal{C} \subset \mathcal{C}_+$ be its subset of differentials of area one. By Lemma 4.10, it suffices to show the existence of a single transitive point $q \in \mathcal{C}_{\text{good}}$ with the following property. For every open neighborhood $U$ of $q$, the subgroup of $Sp(2\ell, \mathbb{R})$ generated by the monodromies of those periodic orbits for $\Phi^t$ in $\mathcal{C}$ which pass through $U$ is Zariski dense in $Sp(2\ell, \mathbb{R})$.

Choose a translation surface $(X, \omega) \in \mathcal{C}$ with the properties stated in Corollary 4.4. Denote by $H$ the free abelian group of rank $\ell$ of Dehn multitwists which is contained in the group of piecewise affine automorphisms of $X$ whose existence was shown in Corollary 4.4.

Let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials and let $\tilde{\omega}$ be a preimage of $\omega$ in $\tilde{\mathcal{C}}$. Choose an open neighborhood $\tilde{U}$ of $\tilde{\omega}$ in $\tilde{\mathcal{C}}$ such that $\langle \text{Re}(\tilde{z}), [c_i]\rangle > 0$ for all $\tilde{z} \in \tilde{U}$ where $[c_i]$ are the homology classes of the marked weighted oriented multicurves which determine the Dehn multitwists $T_i$ generating the group $H$. We require that the projection $U$ of $\tilde{U}$ to $\mathcal{C}$ is contractible. Such a neighborhood always exists although the differential $\omega$ may not be contained in $\mathcal{C}_{\text{good}}$.

By Proposition 4.7, it suffices to find a birecurrent point $q \in U$ such that for each $i \leq \ell$ and every neighborhood $V \subset U$ of $q$, the Dehn multitwist $T_i$ is contained in the group generated by the periodic orbits of the Teichmüller flow through $V$ which is determined by the lift $\tilde{V}$ of $V$ contained in $\tilde{U}$.

The cylinder shears of the translation surface $(X,\omega)$ which are used to construct the Dehn multitwists $T_i$ generating the group $H$ preserve the horizontal projective measured foliation of $\omega$, but they deform the vertical projective measured foliation. These cylinder shears define $\ell$ smooth paths $\alpha_i$ $(i = 1,\dots,\ell)$ in $\mathcal{C}$ which lift to smooth paths $\tilde\alpha_i$ in $\tilde{\mathcal{C}}$ beginning at the preimage $\tilde\omega$ of $\omega$ in $\tilde U$ and connecting $\tilde\omega$ to $T_i\tilde\omega$.

By Lemma 4.11, for each $i \leq \ell$ there is a closed subset of $\tilde{\mathcal{C}}$ with a product structure as defined in Definition 3.2 whose interior contains the entire path $\tilde\alpha_i \subset \tilde{\mathcal{C}}$. Recall that such a neighborhood $\tilde A$ is determined by compact disjoint balls $D,K$ of dimension $m = \dim_{\mathbb{C}}(\mathcal{C}_+)-1$ in the Thurston sphere $\mathcal{PMF}$ of projective measured foliations, a number $\epsilon > 0$ and a map $\Lambda : D\times K \to \tilde{\mathcal{C}}$ with the properties stated in Definition 3.2 so that

$$\tilde A = \cup_{-\epsilon\leq t\leq\epsilon} \cup_{(\mu,\nu)\in D\times K} \Phi^t\Lambda(\mu,\nu).$$

Do this construction for all $j \leq \ell$ and as well for the maps $T_j^{-1}$. This results in a neighborhood $W$ of $\tilde\omega$ in $\tilde{\mathcal{C}}$ with a product structure with the following properties.

(1) There are compact disjoint sets $D,K$ in the Thurston sphere $\mathcal{PMF}$ of projective measured foliations, homeomorphic to closed balls of dimension $m$, there is a number $\epsilon > 0$ and there is a map $\Lambda : D\times K \to W$ with the properties stated in Definition 3.2 such that $W = \cup_{-\epsilon\leq t\leq\epsilon}\Phi^t\Lambda(D\times K)$.
(2) There exists a compact neighborhood $R$ of $\mu$ in $D$ homeomorphic to a closed ball of dimension $m$ so that $T_jR \subset D$ for all $j \leq \ell$.
(3) There is a compact neighborhood $B \subset K$ of the vertical projective measured foliation of $\tilde\omega$ such that $T_j^{-1}(B) \subset K$ for all $j$.

Let $A$ be the projection to $\mathcal{C}$ of the set

$$\tilde A = \cup_{-\epsilon\leq t\leq\epsilon}\Phi^t\Lambda(R\times B).$$

Then $A$ is a closed neighborhood of $\omega$. We may adjust $A$ in such a way that $A$ is contractible; this is always possible in spite of the fact that $\omega$ may not be contained in $\mathcal{C}_{\rm good}$ (see Section 2 of [H13] for a detailed discussion of this standard fact). Up to passing to a finite branched cover of $\mathcal{C}$, we then may assume that the absolute real tangent bundle $\mathcal{Z}$ of $\mathcal{C}$ admits a trivialization over $A$ which is parallel for the Gauss Manin connection. To this end recall that there is a finite branched cover of $\mathcal{M}_g$ which is the quotient of Teichmüller space by a torsion free subgroup of $\mathrm{Mod}(S)$ of finite index and recall the discussion in subsection 2.2.

We now show that for any birecurrent point $q$ contained in the interior of $A$ and every pair of open neighborhoods $Y \subset V$ of $q$ with the properties stated in Proposition 3.8, with corresponding set $\Gamma_0$ or periodic orbits for $\Phi^t$, the subgroup of $\mathrm{Mod}(S)$ generated by $\Omega(\Gamma_0)$ contains the Dehn multitwists $T_i$ $(i \leq \ell)$.

Thus let $q \in Y \subset V \subset A$ as above. By perhaps decreasing the size of $V$ we may assume that the component $\tilde Y \subset \tilde A$ of the preimage of $Y$ in $\tilde{\mathcal{C}}$ which is contained in $\tilde A$ equals the interior of the set

$$\overline{Y} = \cup_{-\delta'\leq t\leq\delta'}\Phi^t\Lambda(R'\times B')$$

for some closed balls $R' \subset R$, $B' \subset B$ and for a number $\delta' < \delta$. Using the neighborhoods $Y \subset V$ of $q$, construct a set $\Gamma_0$ of periodic orbits passing through $V$ as in Proposition 3.10. Denote by $\Omega(\Gamma_0)$ the corresponding subsemigroup of $\mathrm{Mod}(S)$ constructed with the above component $\tilde{V}$ of the preimage of $V$ and let $G < \mathrm{Mod}(S)$ be the subgroup generated by $\Omega(\Gamma_0)$.

Let $z \in Y$ be a periodic point for $\Phi^t$ as constructed in Proposition 3.10, and let $\gamma$ be the closed $\Phi^t$-orbit of $z$. Let $\tilde{z}$ be the lift of $z$ to $\tilde{A}$. The component $\tilde{\gamma}$ of the preimage of $\gamma$ which passes through $\tilde{z}$ is stabilized by a pseudo-Anosov mapping class $\varphi$. We claim that for every $j \leq \ell$ there is a number $k > 0$ such that for the Dehn multitwist $T = T_j$, we have $\varphi^k \circ T \circ \varphi^k \in \Omega(\Gamma_0)$. Since $G$ is a group, this implies that $T_j \in G$ for all $j$.

We establish the existence of numbers $k > 0, n > 0$ with the above property using a fixed point argument for the action of $\mathrm{Mod}(S)$ on the sphere of projective measured foliations which is motivated by the argument in the proof of Proposition 5.4 of [H13] (compare the proof of Proposition 3.8).

Let $\tau > 0$ be the period of $\gamma$; then $\varphi(\tilde{z}) = \Phi^{-\tau}(\tilde{z})$ (up to perhaps exchanging $\varphi$ and $\varphi^{-1}$). The horizontal projective measured foliation $\zeta \in R'$ of $\tilde{z}$ is the attracting fixed point for the action of the map $\varphi$ on the sphere $\mathcal{PMF}$ of projective measured foliations of $S$. As $\varphi$ preserves the component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ containing $\tilde{z}$ and acts with north-south dynamics on $\mathcal{PMF}$, there exists some large $k_0 > 0$ such that $\varphi^k(D)$ is contained in the interior of $R'$ for all $k \geq k_0$. Then $T\varphi^k(D)$ is contained in the interior of $D$ (recall to this end that $TR' \subset D$) and hence for any $k > k_0$, $\varphi^k \circ T \circ \varphi^k(D)$ is contained in the interior of $R'$. Since the attracting fixed point of $\varphi^{-1}$ is contained in the interior of $B'$, by perhaps increasing $k_0$ we also may assume that $\varphi^{-k}$ maps $K$ into the interior of $B'$ for all $k \geq k_0$.

As $\varphi$ is pseudo-Anosov, for large enough $k > k_0$ the mapping class $\varphi^k \circ T \circ \varphi^k$ is pseudo-Anosov (observe that for large $k$, this element acts with positive translation on the curve graph of $S$). Now $\varphi^k \circ T \circ \varphi^k(R')$ is contained in the interior of $R'$ and hence the attracting fixed point of $\varphi^k \circ T \circ \varphi^k$ is contained in the interior of $R'$.

The same argument shows that for sufficiently large $k$, the repelling fixed point of $\varphi^k \circ T \circ \varphi^k$ (which is the attracting fixed point of $\varphi^{-k} \circ T^{-1} \circ \varphi^{-k}$) is contained in the interior of $B'$. Namely, $T^{-1}(B') \subset K$ by construction and hence $\varphi^{-k} \circ T^{-1} \circ \varphi^{-k}$ maps $B'$ into its interior by the choice of $k$. In particular, we may assume that the periodic orbit of $\Phi^t$ defined by $\varphi^k \circ T \circ \varphi^k$ passes through $Y$.

As a consequence, the pseudo-Anosov elements $\varphi$ and $\varphi^k \circ T \circ \varphi^k$ are contained in the group $G$ and hence $G$ contains the multitwist $T = T_j$. As this argument is valid for each $j \leq \ell$, we deduce that the group $G$ contains each of the multi-twists $T_j$.

Theorem 4.12 now follows from Proposition 4.7 if we can make sure that for each $\tilde{z} \in \tilde{V}$ and each $i$ we have $\langle [\mathrm{Re}(\tilde{z}), [c_i] \rangle > 0$. But by construction, we have $\langle [\mathrm{Re}(\tilde{\omega}), [c_i] \rangle > 0$ for all $i$, and the set $D \subset \mathcal{PMF}$ in the definition of the neighborhood $W$ of $\tilde{\omega}$ as constructed above can be chosen to project to an arbitrarily small neighborhood of the projective class of $[\mathrm{Re}(\tilde{\omega})]$. Thus by continuity, we may choose the set $D$ as in (1) above in such a way that indeed, $\langle[\mathrm{Re}(\tilde{u}),[c_i]\rangle>0$ for all $i$ and all $\tilde{u}\in W$. Theorem 4.12 now follows from Proposition 4.7. □

For a prime $p$ let $\Lambda_p : Sp(2g,\mathbb{Z})\to Sp(2g,F_p)$ be reduction mod $p$. Recall from the remark before Lemma 4.5 that a sub-semigroup of a finite group $G$ is a subgroup of $G$. The proof of Theorem 4.12 and Proposition 4.7 shows the following version of Theorem 3 for affine invariant manifolds of rank $g$.

**Corollary 4.13.** *Let $\mathcal{C}$ be an affine invariant manifold of rank $g$. Then for all but finitely many primes $p\geq 3$, we have $\{\Lambda_p\Psi(\Omega(\gamma))\mid\gamma\in\Gamma_0\}=Sp(2g,F_p)$.*

Let again $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell\leq g$ and let $\tilde{\mathcal{C}}_+$ be a component of the preimage of $\mathcal{C}_+$ in the Teichmüller space of marked abelian differentials. Then the projected tangent space $p(T\mathcal{C}_+)$ can be identified with the complexification of a $2\ell$-dimensional symplectic subspace $V$ of $\mathbb{R}^{2g}=H^1(S,\mathbb{R})$. The stabilizer in $Sp(2g,\mathbb{R})$ of this subspace is the subgroup $G=Sp(V)\times Sp(V^\perp)$ of $Sp(2g,\mathbb{R})$ where $V^\perp$ is the orthogonal complement of $V$ with respect to the symplectic form. Thus the group $G$ is isomorphic to $Sp(2\ell,\mathbb{R})\times Sp(2(g-\ell),\mathbb{R})$.

Let $P:G\to Sp(V)=Sp(2\ell,\mathbb{R})$ be the natural projection. Theorem 4.12 shows that $P(G\cap Sp(2g,\mathbb{Z}))$ is a Zariski dense subgroup of $Sp(2\ell,\mathbb{R})$. The following consequence of this fact was communicated to me by Yves Benoist. Although it is not used in the sequel, we include it here since it relates affine invariant manifolds to proper subvarieties of $\mathcal{A}_g$.

**Proposition 4.14.** *If $P(G\cap Sp(2g,\mathbb{Z}))$ is Zariski dense in $Sp(2\ell,\mathbb{R})$ then either $P(G\cap Sp(2g,\mathbb{Z}))$ is a lattice in $Sp(2\ell,\mathbb{R})$ or dense.*

*Proof.* Using the above notations, write $G_{\mathbb{Z}}=Sp(2g,\mathbb{Z})\cap G$ and let $F<Sp(2\ell,\mathbb{R})$ be the Zariski closure of $P(G_{\mathbb{Z}})$.

The group $F$ is defined over $\mathbb{Q}$. Namely, the set of polynomials $P$ which vanish on $G_{\mathbb{Z}}$ is invariant under the Galois action. As a consequence, either $F_{\mathbb{Z}}=G_{\mathbb{Z}}$ is a lattice in $F$, or there is a nontrivial character on $F$ defined over $\mathbb{Q}$.

Assume for contradiction that there exists a nontrivial character on $F$ defined over $\mathbb{Q}$. Define

$$F^0=\cap\{\ker(\chi)\mid\chi\text{ is a character on }F\text{ defined over }\mathbb{Q}\}.$$

Then $F^0=F$ since up to multiplication with an integer, the evaluation on $G_{\mathbb{Z}}$ of a nontrivial character $\chi$ defined over $\mathbb{Q}$ has to be integral in $\mathbb{C}^*$ which is impossible. This contradiction yields that $F_{\mathbb{Z}}$ is a lattice in $F$.

The group $G_1=Sp(2\ell,\mathbb{R})$ is simple, and $\Delta=P(G_{\mathbb{Z}})<G_1$ is Zariski dense. Then $\Delta<G_1$ either is discrete or dense. We have to show that if $\Delta$ is discrete then $\Delta$ is a lattice.

Thus assume that $\Delta$ is discrete. Consider the surjective homomorphism $\varphi : F \to G_1$. Its kernel $K$ is a locally compact group which intersects the lattice $F_{\mathbb{Z}}$ in a discrete subgroup. The exact sequence

$$1 \to K \to F \to G_1 \to 1$$

induces a sequence

$$K/K \cap F_{\mathbb{Z}} \to F/F_{\mathbb{Z}} \to G_1/\varphi(F_{\mathbb{Z}}).$$

Now the Haar measure on $F$ can locally be represented as a product of the Haar measure on the orbits of $K$ and the quotient Haar measure. If the volume of $G_1/\varphi(F_{\mathbb{Z}})$ is infinite then this shows that the volume of $F/F_{\mathbb{Z}}$ has to be infinite. But $F_{\mathbb{Z}}$ is a lattice in $F$ which is a contradiction. □

**Remark 4.15.** If $\mathcal{C}$ is a Teichmüller curve, then the group $P(G \cap Sp(2g, \mathbb{Z}))$ is just the Veech group of $\mathcal{C}$, which is a lattice in $SL(2, \mathbb{R})$. The image under the Torelli map of the projection to moduli space of such a Teichmüller curve is a *Kobayashi geodesic* in the quotient of Siegel upper half-space $\mathfrak{D}_g$ by $Sp(2g, \mathbb{Z})$.

If $\mathcal{C}$ is *algebraically primitive*, then this Kobayashi geodesic is contained in a Hilbert modular variety defined by an order $\mathfrak{o}$ in the trace field of $\mathcal{C}$ [Mo06]. This Hilbert modular variety is the quotient of an embedded copy of $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$ in Siegel upper half-space $\mathfrak{D}_g$ by its stabilizer $SL(2, \mathfrak{o})$ in $Sp(2g, \mathbb{Z})$. The finite area Riemann surface $\Sigma$ obtained by projecting $\mathcal{C}$ to the moduli space of curves admits a *modular embedding* into $SL(2, \mathfrak{o}) \backslash \mathbb{H}^2 \times \cdots \times \mathbb{H}^2$ whose composition with the first factor projection $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2 \to \mathbb{H}^2$ is a finite area Riemann surface.

Proposition 4.14 suggests that we may expect a similar picture for affine invariant manifolds of higher rank.

## 5. Equidistribution for cocycles with values in finite groups

In this section we consider a component $\mathcal{Q}$ of a stratum of area one abelian or quadratic differentials. We continue to use the assumptions and notations from Section 2-4. Our goal is to establish an equidistribution result for a cocyle over the Teichmüller flow with values in a finite group $G$ constructed from a homomorphism $\rho : \mathrm{Mod}(S) \to G$.

5.1. **A cocycle for the Teichmüller flow.** Choose a Birkhoff regular (and hence transitive) point $q \in \mathcal{Q}_{\mathrm{good}}$ for the Masur Veech measure and a contractible neighborhood $V$ of $q$. We then can find an open neighborhood $Y \subset V$ of $q$ with the following property. If $y \in Y$ and $T > 0$ are such that $\Phi^T y \in Y$, then a $V$-characteristic curve for the periodic pseudo-orbit defined by $y, T$ determines a periodic orbit $\gamma$ passing through $V$. Note that we do not have to specify a number $R(\delta) > 0$ as a minimal return time to $Y$ so that Proposition 3.8 can be applied because we can choose $Y$ sufficiently small that any return time to $Y$ is bigger than the threshold from Proposition 3.8.

By perhaps decreasing $Y$ we may assume that $Y$ has a product structure of the form specified in Example 3.4. This means that there are open ball neighborhoods $A^i$ of $q$ in $W^i_{\rm loc}(q)$ $(i = ss, su)$, and there is a number $\beta > 0$ such that

$$Y = V(A^{ss}, A^{su}, \beta) = \cup_{-\beta<t<\beta}\Phi^t V(A^{ss}, A^{su}). \tag{3}$$

We may assume that the return time to $Y$ is bigger than $2\beta$ and that moreover the Masur Veech measure of the boundary of $Y$ vanishes.

Let $B$ be the closure of the set $\Phi^{-\beta}V(A^{ss}, A^{su})$. Note that $B$ is a compact transversal for the Teichmüller flow. We equip $B$ with the projection $\mu$ of the Masur Veech measure in $Y$. This is defined by

$$\mu(C) = \lambda\big(\cup_{0\leq t\leq 2\beta}\Phi^t C\big) \quad (C \subset B).$$

The measure of the boundary of $B$ vanishes.

For $y \in \mathcal{Q}$ let $T(y) \in (0, \infty]$ (or $S(y) \in [-\infty, 0)$) be the first positive (or the first negative) hitting time with $B$ of the $\Phi^t$-orbit of $y$. Since $B$ is closed, the functions $y \to T(y)$ and $y \to S(y)$ are measurable on $\mathcal{Q}$. By ergodicity of the Teichmüller flow for the Masur Veech measure, the set

$$\Omega = \{y \in B \mid T(\Phi^s y) < \infty, S(\Phi^s y) > -\infty \text{ for all } s \in \mathbb{R}\}$$

has full measure for $\mu$.

The *suspension* of the Teichmüller flow over $B$ is defined by the first return to $B$. This suspension is the Borel set

$$\mathcal{B} = \{(y, t) \in \Omega \times [0, \infty) \mid 0 \leq t \leq T(y) < \infty\}/\sim$$

where the equivalence relation $\sim$ identifies $(y, T(y))$ with $(\Phi^{T(y)}y, 0)$. In the sequel we identify $\mathcal{B}$ with the Borel set $\cup_{y\in\Omega}\cup_{0\leq t<T(y)}\Phi^t y \subset \mathcal{Q}$. This set is $\Phi^t$-invariant and has full Masur Veech measure.

We use this suspension to define a map $\theta : \mathcal{Q} \times [0, \infty) \to G$ as follows. Denote as before by $\Gamma_0$ the collection of all parametrized periodic orbits for $\Phi^t$ constructed from the pair $Y \subset V$. We use the convention from Proposition 3.10 that words in $\Omega(\Gamma_0)$ are read from left to right. Define $\theta(z, 0) = e$ (the identity in $G$) for all $z$, and $\theta(z, s) = e$ for all $s$ whenever $z \notin \mathcal{B}$. For $z \in \mathcal{B}$ assume by induction that $\theta(z, s)$ has been defined for some $s \geq 0$. Let $t \in (0, \infty)$ be the smallest positive number so that $\Phi^{s+t}z \in B$. Define $\theta(z, u) = \theta(z, s)$ for $s \leq u < s + t$, and let $\theta(z, s + t) = \theta(z, s)g$ where $g \in G$ is determined as follows. Since $\Phi^s z \in \mathcal{B}$ there is a largest number $u \leq s$ such that $\Phi^u z = y \in B$; then $\Phi^{s+t-u}y \in B$ and $T(y) = s + t - u$. Let $\gamma \in \Gamma_0$ be the parametrized periodic orbit for $\Phi^t$ defined by the periodic pseudo-orbit $\cup_{\beta\leq v\leq T(y)+\beta}\Phi^v y$; then $g = \rho(\Omega(\gamma))$.

The following is immediate from Proposition 3.10 and the definition of $\theta$.

**Lemma 5.1.** *$\theta(z, t)$ is a $G$-valued measurable cocycle for the Teichmüller flow.*

*Proof.* Since $B$ is a closed transversal for the Teichmüller flow and $\Omega \subset \mathcal{B}$ is measurable, the first return time $y \to T(y)$ is a measurable function on $\Omega$ and hence the same holds true for the function $\theta$.

The cocycle equality for $\theta$ means that $\theta(z,s+t)=\theta(z,s)\theta(\Phi^s z,t)$ for all $z$ and all $s,t\geq 0$. This is clear if the orbit segment connecting $\Phi^s v$ to $\Phi^{s+t}v$ does not cross through $B$.

Assume now that there is a single such crossing point in the interior of this segment. By definition, we then have $\theta(z,s+t)=\theta(z,s)g$ where $g$ is determined as follows. Let $y\in\Omega$ be such that $\Phi^s z=\Phi^u y$ for some $u\in[0,T(y))$ and let $\gamma$ be the periodic orbit for $\Phi^t$ determined by the pseudo-orbit $(y,T(y))$. Then $g=\rho(\Omega(\gamma))=\theta(\Phi^s z,t)$ and hence the cocycle equation follows from Proposition 3.10. ☐

By the cocycle equality, the image of $\theta$ is a subsemigroup of the finite group $G$ and hence a subgroup. In the sequel we assume that this group is all of $G$. Our goal is to show that the cocycle $\theta$ is equidistributed in a sense which is motivated by equidistribution of random walks on the finite group $G$.

We next summarize some result on random walks on the finite group $G$ needed in the sequel.

Let $\mathcal{P}(G)$ be the space of all probability measures on the finite group $G$ equipped with the $\ell^\infty$-norm. Then $\mathcal{P}(G)$ is a compact convex subset of a finite dimensional Banach space. For a number $\sigma\geq 0$ define $\mathcal{P}(G,\sigma)\subset\mathcal{P}(G)$ to be the subspace of all measures $\mu$ with $\min\{\mu(g)\mid g\in G\}\geq\sigma$.

The *convolution* of two probability measures $\nu,\mu$ on $G$ is defined by

$$\mu*\nu(g)=\sum_{h\in G}\mu(h)\nu(h^{-1}g). \tag{4}$$

A measure $\mu$ is *stationary* if $\mu*\mu=\mu$.

The next lemma quantifies the fact that iterated convolutions of a measure $\mu$ which gives positive mass to every element of $G$ converge to the equilibrium measure $\nu$ on $G$ defined by $\nu(g)=1/|G|$ for all $g\in G$.

**Lemma 5.2.** *Let $G$ be a finite group of order $N$ and let $\sigma>0$.*

(1) *If $\mu\in\mathcal{P}(G,\sigma)$ and $\nu\in\mathcal{P}(G)$ then $\mu*\nu\in\mathcal{P}(G,\sigma)$.*
(2) *For all $0<\sigma\leq\kappa<1/N$ there exists a number $\delta=\delta(\sigma,\kappa)>0$ with the following property. Let $\mu_1\in\mathcal{P}(G,\kappa),\mu_2\in\mathcal{P}(G,\sigma)$; then $\mu_2*\mu_1\in\mathcal{P}(G,\kappa+\delta)$.*

*Proof.* Convolution $*$ is a continuous convex bilinear map on the compact convex space of all probability measures on $G$.

Let $0\leq\sigma\leq\kappa<1/N$ and let $\mu_1\in\mathcal{P}(G,\kappa),\mu_2\in\mathcal{P}(G,\sigma)$. Then for every $g\in G$ we have

$$\mu_2*\mu_1(g)=\sum_h\mu_2(h)\mu_1(h^{-1}g)\geq\sum_h\kappa\mu_2(h)=\kappa.$$

Equality holds only if $\sigma=0$ and $\mu_2(h)=0$ for all $h$ with $\mu_1(h^{-1}g)>\kappa$.

Thus if $\sigma > 0$ then $\min\{\mu_2 * \mu_1(g) \mid g \in G\} > \kappa$ and hence $\mu_1 * \mu_2 \in \mathcal{P}(\sigma + \delta)$ for some $\delta > 0$ depending on $\mu_1, \mu_2$. The lemma now follows from continuity of the convolution and compactness of $\mathcal{P}(G, \sigma)$ and $\mathcal{P}(G, \kappa)$. □

We next characterize stationary measures. To this end we denote by $\mu^{(n)}$ the $n$-fold convolution of $\mu$ with itself.

**Lemma 5.3.** (1) *Let $\nu$ be a stationary measure on $G$. Then there exists a subgroup $H$ of $G$ such that $\nu(h) = 1/|H|$ for all $h \in H$, and $\nu(g) = 0$ for $g \in G - H$.*
(2) *Let $\mu \in \mathcal{P}(G)$ and let $H$ be the subgroup of $G$ generated by $\mathrm{supp}(\mu)$; then as $n \to \infty$, $\mu^{(n)}$ converges in $\mathcal{P}(G)$ to the stationary measure supported on $H$.*

*Proof.* Let $\nu$ be a stationary measure on $G$. We claim that $\{g \in G \mid \nu(g) > 0\} = H$ is a subgroup of $G$. Namely, if $g, h \in G$ and if $\nu(g), \nu(h) > 0$ then $\nu(gh) \geq \nu(g)\nu(g^{-1}gh) > 0$ which yields the claim.

Thus by perhaps replacing $G$ by $H$, it suffices to show that every positive stationary measure on $G$ is the equilibrium measure $\nu(g) = 1/|G|$ for all $g \in G$. However, this is immediate from Lemma 5.2.

This shows the first part of the lemma. The second part is equally well known, and its proof will be omitted. □

**Lemma 5.4.** *Let $\mu, \nu \in \mathcal{P}(G)$ and assume that $\mathrm{supp}(\mu) = H$, $\mathrm{supp}(\nu) = H'$ are subgroups of $G$. Then the subgroup of $G$ generated by the support of $\mu * \nu$ contains both $H, H'$.*

*Proof.* Assume that $\mathrm{supp}(\mu)$ is a subgroup $H$ of $G$, and $\mathrm{supp}(\nu)$ is a subgroup $H'$. In the case that $H = G$ (or $H' = G$), the statement of the lemma is immediate from the first part of Lemma 5.2. Thus assume that $H, H'$ are proper subgroups of $G$.

Let $g \in H'$; as $\mu(e) > 0$, we have $\mu * \nu(g) \geq \mu(e)\nu(g) > 0$ and similarly for $g \in H$. This shows the lemma. □

We use these statements for the proof of the following simple lemma which will be used in the proof of the main technical result in this section.

**Lemma 5.5.** *Let $\Upsilon \subset \mathcal{P}(G)$ be a closed subset with the following properties.*

- *For all $g \in G$, there exists a point $\xi_g \in \Upsilon$ with $\xi_g(g) > 0$.*
- *There exists a number $c > 0$, and for any $\xi, \eta \in \Upsilon$ there exists some $\zeta \in \Upsilon$ such that $\zeta(g) \geq c\xi * \eta(g)$ for all $g \in G$.*

*Then there exists some $\xi \in \Upsilon$ with $\xi(g) > 0$ for all $g \in G$.*

*Proof.* Let $\xi \in \Upsilon$ be such that $\min\{\xi(g) \mid g\}$ is maximal among all points in $\Upsilon$. If there is more than one point with this property then choose $\xi$ in such a way that the number of elements $g \in G$ for which the minimal value is atteint is minimal.

Our goal is to show that $\xi(g) > 0$ for all $g \in G$. We argue by contradiction and we assume that $\min\{\xi(g) \mid g \in G\} = 0$.

Let $H < G$ be the subgroup which is generated by the support of $\xi$. We first show that $\xi(h) > 0$ for all $h \in H$. Since $\xi$ vanishes on $G - H$, by the choice of $\xi$ to this end it suffices to find a measure $\eta \in \Upsilon$ so that $|\{h \in H \mid \eta(h) = 0\}| < |H_0|$. By the second part of Lemma 5.3, for this it suffices to show that for every $n \geq 0$ there exists a measure $\eta_n \in \Upsilon$ such that $\eta_n \geq c^n \xi$.

We establish this fact by induction on $n$. The case $n = 0$ is obvious, so assume that the statement holds true for all $k < n$ for some $n \geq 1$. Assume by induction that $\eta_{n-1} \in \Upsilon$ is such that $\eta_{n-1} \geq c^{n-1}\xi$. By the second property of the set $\Upsilon$ stated in the lemma, there exists a measure $\eta_n$ so that

$$\eta_n(g) \geq c\eta_{n-1} * \xi(g) = c \sum_h \eta_{n-1}(h)\xi(h^{-1}g) \geq c \sum_h c^{n-1}\xi^{(n)}(h)\xi(h^{-1}g) \geq c^n \xi^{(n)}(g)$$

which completes the induction step.

As a consequence, there exists a point $\nu \in \Upsilon$ with $\nu(h) > 0$ for all $h \in H$ and hence $\xi(h) > 0$ for all $h \in H$ by the definition of $\xi$.

On the other hand, if $H \neq G$ then by the above reasoning, applied to $\xi_g$ for some $g \in G - H$ we can find some $\zeta \in \Upsilon$ whose support is a subgroup $H'$ of $G$ which intersects $G - H$. It now follows from Lemma 5.4 and the above discussion that there exists a measure $\zeta \in \Upsilon$ whose support is a proper superset of the support of $\xi$. This contradicts once more the choice of $\xi$ and completes the proof of the lemma. □

5.2. **Volume control.** The goal of this subsection is to show that for any component $\mathcal{Q}$ of a stratum of abelian or quadratic differentials, the cocycle $\theta$ constructed in the previous subsection equidistributes with respect to the Masur Veech measure $\lambda$. By this we mean the following. Consider the set $Y$ defined in (3) which is used for the construction of $\theta$. Then for each $g \in G$ we have

$$\lim_{T \to \infty} \lambda\{z \in Y \mid \theta(z, T) = g\} = \lambda(Y)/|G|.$$

The idea of proof for this statement is to use the fact that the measure $\lambda$ is Bernoulli [M82] and make the idea precise that the cocycle $\theta$ has properties reminiscent of the Markov property which guarantees independence of the distribution of the random variable defined by $\theta$. For the volume control we need we have to partition a subset of $Y = V(A^{ss}, A^{su}, \beta)$ of full measure into sufficiently small sets of the form $V(C^{ss}, A^{su}, \beta)$ whose boundaries have measure zero. The choice of this partition depends on an a-priori chosen error term $\epsilon > 0$.

Recall that there are families $\lambda^{ss}, \lambda^{su}$ of conditional measures for $\lambda$ with the following properties. We have $d\lambda = d\lambda^{ss} \times d\lambda^{su} \times dt$, the measures $\lambda^{su}$ are the images of the measures $\lambda^{ss}$ under the *flip* $v \to -v$, and the transformation rule

$d\lambda^{ss} \circ \Phi^t = e^{-ht} d\lambda^{ss}$ is fulfilled where $h$ is the complex dimension of the stratum $\mathcal{Q}_+$ which consists of all non-trivial scalings of the points in $\mathcal{Q}$.

Denote as before by $B^{ss}(y,r)$ the ball of radius $r$ about $y$ for the modified Hodge distance in the local strong stable manifold of $y$. The following easy but technical observation is immediate from the construction of the Masur Veech measure from period coordinates. We refer to Section 2 of [H13] for a detailed discussion of this fact. As in Example 3.4, for $y \in A^{ss}$ denote by $\Xi_y : A^{su} \to \Xi_y(A^{su}) \subset W^{su}_{\mathrm{loc}}(y)$ the holonomy homeomorphism.

**Lemma 5.6.** *For every $\epsilon > 0$ there exists a number $r(\epsilon) > 0$ with the following property. Let $y \in A^{ss}$; then the Jacobian of the natural diffeomorphism*

$$B^{ss}(y, r(\epsilon)) \times \Xi_y A^{su} \times (-\beta, \beta) \to V(B^{ss}(y, r(\epsilon)), A^{su}, \beta)$$

*with respect to the measures $\lambda^{ss} \times \lambda^{su} \times dt$ on $B^{ss}(y, r(\epsilon)) \times \Xi_y A^{su} \times (-\beta, \beta)$ and the Masur Veech measure on $V(B^{ss}(y, r(\epsilon)), A^{su}, \beta)$ is contained in the interval $[1-\epsilon, (1-\epsilon)^{-1}]$.*

In the sequel if a set $V(A, B, \beta)$ with a good product structure, i.e. a product structure as in Example 3.4, fulfills the conclusion in Lemma 5.6 then we say that the Masur Veech measure on $V(A, B, \beta)$ is an $(1-\epsilon)$-*approximate product*. We will use Lemma 5.6 for subsets of $Y$ of the form $V(C^{ss}, A^{su}, \beta)$ where $C^{ss}$ is a measurable subset of $A^{ss}$ of diameter at most $r(\epsilon)$ for the Hodge distance.

The main technical tool for the control of the cocycle $\theta$ is Proposition 4.6 of [H18] which provides a quantitative control of non-uniform hyperbolicity of the Teichmüller flow.

Consider for the moment an arbitrary set $Z = V(A, B, \zeta)$ with a good product structure. Later $Z$ will always be of the form $Z = V(C^{ss}, A^{su}, \zeta)$ for some open subset $C^{ss}$ of $A^{ss}$ and some $\zeta \leq \beta$. Define the local leaf $W^s_{\mathrm{loc},Z}(y)$ through $y$ of the *stable foliation* of $Z$ to be the intersection of $Z$ with a suitably chosen neighborhood of $y$ in the leaf of the stable foliation through $y$ (which is defined as $W^s(y) = \cup_t \Phi^t W^{ss}(y)$). If $A$ is connected, this is just the connected component of $Z \cap W^s(y)$ containing $y$. If $A$ is disconnected then we assume that $V(A, B, \zeta) \subset V(A', B', \zeta') = Z'$ where $A'$ is connected, and then we define $W^s_{\mathrm{loc},Z}(y)$ to be the intersection with $Z$ of the local stable manifold of $y$ in $Z'$. The choice of $Z'$ will be clear from the context. If $z \in A$ then this local leaf is just the set $\cup_{-\zeta<t<\zeta} \Phi^t A$. The local leaf $W^u_{\mathrm{loc},Z}(y)$ of the *unstable foliation* through $y$ is defined in the same way, starting with a leaf of the strong unstable foliation.

We say that a subset $Z_0$ of $Z$ is *saturated for the local stable foliation* (or saturated for the local strong unstable foliation) if for all $y \in Z_0$, the local stable manifold $W^s_{\mathrm{loc},Z}(y)$ (or the local strong unstable manifold $W^u_{\mathrm{loc},Z}(y)$) is contained in $Z_0$.

**Lemma 5.7.** *A subset $Z_0$ of $Z = V(A, B, \zeta)$ is saturated for the local stable foliation (or the local unstable foliation) if and only if there exists a subset $E$ of $B$ (or a subset $E'$ of $A$) such that $Z_0 = V(A, E, \zeta)$ (or $Z_0 = V(E', B, \zeta)$).*

*Proof.* Let $F : A \times B \times (-\zeta, \zeta) \to V(A, B, \zeta)$ be the map which associates to a triple $(z, x, s)$ the point $F(z, x, s) = \Phi^s \Xi_z(x)$; here notations are as in Example 3.4. By construction, the set $F(A \times \{x\} \times (-\zeta, \zeta))$ equals the local stable manifold of $x$. The first part of the lemma is now immediate from the definition of the sets $V(A, B, \zeta)$.

The proof for local unstable manifolds is completely analogous and will be omitted. □

Proposition 4.6 of [H18] now states the following.

**Proposition 5.8.** *For every $\epsilon > 0$ there are open subsets $Z_1(\epsilon) \subset Z_2(\epsilon) \subset Z = V(A, B, \zeta)$, and there is a number $T(\epsilon) > 0$ such that the following properties hold true.*

(a) *For some $m > 10/\epsilon$, we have $Z_2(\epsilon) = V(A, B, \zeta/m)$.*
(b) $\lambda(Z_2(\epsilon)) \leq (1 - \epsilon)^{-1} \lambda(Z_1(\epsilon))$.
(c) *A $\Phi^t$-orbit intersects $Z_1(\epsilon)$ in arcs of length $2t_0 = 2\zeta/m$.*
(d) *Write*

$$Z_3(\epsilon) = \cup_{-t_0(m-2) \leq t \leq t_0(m-2)} \Phi^t Z_1(\epsilon) \subset V(A, B, \zeta).$$

*Let $z \in Z_1(\epsilon)$ and let $T > T(\epsilon)$ be such that $\Phi^T z \in Z_3(\epsilon)$. Then there exists an open path connected set $C(z, T) \subset Z_2(\epsilon)$ containing $z$ with the following properties.*
- $\Phi^T C(z, T) \subset Z$.
- $\lambda(C(z, T)) \in [(1 - \epsilon)^2 e^{-hT} \lambda(Z_2(\epsilon)), (1 - \epsilon)^{-2} e^{-hT} \lambda(Z_2(\epsilon))]$.
- *The characteristic curves of the periodic pseudo-orbits given by any $u \in C(z, T)$ and the time parameter $T$ determine the same parametrized periodic orbit $\gamma$ for $\Phi^t$, with basepoint contained in a distinguished component $\gamma(z, T)$ of the intersection of $\gamma$ with $Z$.*
- *The sets $C(z, T)$ are saturated for the local stable foliation of $Z_2(\epsilon)$.*
- *The sets $\Phi^T C(z, T)$ are saturated for the local strong unstable foliation of $Z$.*

(e) *If $\epsilon < \delta$ then $Z_3(\epsilon) \supset Z_3(\delta)$.*

By (a), (c) and (e) above, for each $\epsilon > 0$, the set $Z_1(\epsilon)$ is of the form

$$Z_1(\epsilon) = \cup_{-t_0 \leq t \leq t_0} \Phi^t A(\epsilon)$$

where $A(\epsilon) \subset V(A, B)$ is an open set and such that furthermore $A(\epsilon) \supset A(\delta)$ for $\epsilon < \delta$ and $\cup_\epsilon A(\epsilon) = V(A, B)$. Note however that the sets $Z_1(\epsilon)$ are not required to have a product structure. The sets $C(z, T)$ are saturated for the local stable foliation of $Z_2(\epsilon) = V(A, B, t_0)$ and hence by Lemma 5.7, they are of the form

$$C(z, T) = V(A, U(z), t_0)$$

for some path connected subset $U(z)$ of $A^{su}$. By construction, the value of $\theta(\cdot, T)$ is constant on the sets $C(z, T)$.

The main equidistribution result for the cocycle $\theta$ will be derived from a precise volume estimate for the intersection of the fixed set $Y$ as in (3) with $\Phi^{-T} Y$ for some large $T > 0$ under control of the cocycle $\theta$. The following lemma is its main technical part. Recall that the Masur Veech measure is mixing for the Teichmüller flow.

**Lemma 5.9.** *Let $S_i, T_i \subset (0,\infty)$ be sequences so that $S_i \to \infty, T_i \to \infty$ and that furthermore for each $g \in G$ there exist numbers $a(g), b(g) \geq 0$ such that*

$$\lambda\{y \in Y \mid \Phi^{S_i} y \in Y, \theta(y, S_i) = g\} \to a(g)\lambda(Y)^2 \quad (i \to \infty)$$

*and similarly*

$$\lambda\{y \in Y \mid \Phi^{T_i} y \in Y, \theta(y, T_i) = g\} \to b(g)\lambda(Y)^2 \quad (i \to \infty).$$

*Then up to passing to a subsequence, we have*

$$\lambda\{y \in Y \mid \Phi^{S_i} y \in Y, \Phi^{S_i+T_i} y \in Y, \theta(y, S_i + T_i) = g\} \to (\sum_h a(h)b(h^{-1}g))\lambda(Y)^3.$$

*Proof.* We divide the proof of the lemma into four steps.

*Step 1:*

Let $Z = V(A, B, \zeta) \subset Y$ be an arbitrary subset with a good product structure. Let $\epsilon > 0$ and let $Z_1(\epsilon), Z_2(\epsilon), Z_3(\epsilon) \subset Z$ be as in Proposition 5.8. Then $Z_2(\epsilon) = V(A, B, \zeta/m)$ for some $m > 10/\epsilon$, and $\lambda(Z_1(\epsilon)) \geq (1-\epsilon)\lambda(Z_2(\epsilon))$. As $m > 10/\epsilon$, we also have $\lambda(Z_3(\epsilon)) \geq (1-\epsilon)^2\lambda(Z)$. Let $T(\epsilon) > 0$ be as in Proposition 5.8.

As $\lambda$ is mixing for the Teichmüller flow, for large enough $T > T(\epsilon)$ we have

$$\lambda\{z \in Z_1(\epsilon) \mid \Phi^T z \in Z_3(\epsilon)\} \tag{5}$$
$$\in [(1-\epsilon)\lambda(Z_1(\epsilon))\lambda(Z_3(\epsilon)), (1-\epsilon)^{-1}\lambda(Z_1(\epsilon))\lambda(Z_3(\epsilon))]$$

and similarly

$$\lambda\{z \in Z_2(\epsilon) \mid \Phi^T z \in Z\} \in [(1-\epsilon)\lambda(Z_2(\epsilon))\lambda(Z), (1-\epsilon)^{-1}\lambda(Z_2(\epsilon))\lambda(Z)]. \tag{6}$$

For large enough $T > T(\epsilon)$ let

$$C(T, \epsilon) = \cup\{C(z, T) \mid z \in Z_1(\epsilon), \Phi^T z \in Z_3(\epsilon)\}$$

(notations as in Proposition 5.8). Since $z \in C(z, T)$ for all $z \in Z_1(\epsilon)$, by the estimates (5) and (6) we have

$$\lambda(C(T, \epsilon))/\lambda(Z_2(\epsilon))\lambda(Z) \in [(1-\epsilon)^4, (1-\epsilon)^{-4}]. \tag{7}$$

Furthermore, the sets $C(z, T)$ are saturated for the local stable foliation of $Z_2(\epsilon)$ and hence by Lemma 5.7, the same holds true for $C(T, \epsilon)$. As the value of $\theta(\cdot, T)$ is constant on each of the sets $C(z, T)$ we conclude the following.

For $g \in G$ and all large enough $T$, let

$$C(T, \epsilon, g) = \{z \in C(T, \epsilon) \mid \theta(z, T) = g\};$$

then $C(T, \epsilon, g)$ is saturated for the local stable foliation of $C(T, \epsilon)$ and hence of $Z_2(\epsilon) = V(A, B, \zeta/m)$. By Lemma 5.7, this implies that

$$C(T, \epsilon, g) = V(A, E(T, \epsilon, g), \zeta/m) \tag{8}$$

for some open subset $E(T,\epsilon,g)$ of $B$, and we have

$$\sum_{g\in G}\lambda(C(T,\epsilon,g))=\lambda(C(T,\epsilon)). \tag{9}$$

*Step 2:*

Using the notations from Step 1 above, we now apply Proposition 5.8 to the set $Z_2(\epsilon)=V(A,B,\zeta/m)$ with the same $\epsilon>0$. We find a number $n>10/\epsilon$ and a subset $\hat Z_1(\epsilon)\subset\hat Z_2(\epsilon)=V(A,B,\zeta/mn)$ so that the conclusions of Proposition 5.8 hold true for these sets and a number $\hat T(\epsilon)>0$, with

$$\hat Z_3(\epsilon)=\cup_{-(2n-2)\zeta/mn\leq t\leq(2n-2)\zeta/mn}\Phi^t\hat Z_1(\epsilon)\subset Z_2(\epsilon).$$

Let $S>\hat T(\epsilon)$ be sufficiently large and assume that $z\in\hat Z_1(\epsilon)$ is such that $\Phi^S z\in\hat Z_3(\epsilon)$. Denote by $\hat C(z,S)\subset\hat Z_2(\epsilon)$ the set constructed in Proposition 5.8. It is saturated for the local stable foliation of $\hat Z_2(\epsilon)$, and $\Phi^S\hat C(z,S)\subset Z_2(\epsilon)$. Thus by Lemma 5.7, there exists a set $U(z)\subset B$ such that

$$\hat C(z,S)=V(A,U(z),\zeta/mn).$$

Since the sets $C(T,\epsilon,g)$ are saturated for the local stable foliation of $Z_2(\epsilon)$ and the map $\Phi^S$ maps a local leaf of the stable foliation of $\hat C(z,S)$ into a local leaf of the stable foliation of $Z_2(\epsilon)$, we conclude that $\hat C(z,S)\cap\Phi^{-S}C(T,\epsilon,g)$ is saturated for the local stable foliation of $\hat Z_2(\epsilon)$ and hence this set is of the form

$$\hat C(z,S)\cap\Phi^{-S}C(T,\epsilon,g)=V(A,U(z)\cap\Phi^{-S}\Xi_{\Phi^S z}E(T,\epsilon,g),\zeta/nm).$$

Here we write $\Xi_{\Phi^S z}=\Xi_y$ if $y\in A$ and $\Phi^S z\in\cup_{-\zeta<t<\zeta}\Phi^t\Xi_y B$, and recall from the definition (8) that $E(T,\epsilon,g)\subset B$. The notation makes clear that we take the image under the map $\Phi^{-S}$ of a subset of the local strong unstable manifold of $\Phi^S z$ in $\hat Z_2(\epsilon)$.

*Step 3.*

Now let us furthermore assume that the Masur Veech measure on the set $Z=V(A,B,\zeta)$ is an $(1-\epsilon)$-approximate product. Then we have

$$\begin{aligned}\lambda(\hat C(z,S)\cap\Phi^{-S}C(T,\epsilon,g))/\lambda^{ss}(A)\lambda^{su}(U(z)\cap\Phi^{-S}\Xi_{\Phi^s z}E(T,\epsilon,g))2\zeta/nm\\ \in[1-\epsilon,(1-\epsilon)^{-1}].\end{aligned} \tag{10}$$

Since the conditional measures $\lambda^{su}$ on strong unstable manifolds transform under the Teichmüller flow by $\lambda^{su}\circ\Phi^{-S}=e^{-hS}\lambda^{su}$ and since furthermore $\Phi^S\hat C(z,S)$ is saturated for the local strong unstable foliation of $Z_2(\epsilon)$ (since the leaves of the strong unstable foliation of $Z$ which intersect $Z_2(\epsilon)$ are precisely the leaves of the local strong unstable foliation of $Z_2(\epsilon)$), we conclude that

$$\begin{aligned}\lambda(\hat C(z,S)\cap\Phi^{-S}C(T,\epsilon,g))/e^{-hS}\lambda^{ss}(A)\lambda^{su}(E(T,\epsilon,g))2\zeta/nm\\ \in[(1-\epsilon)^2,(1-\epsilon)^{-2}].\end{aligned} \tag{11}$$

The difference in the error term $(1-\epsilon)^2$ compared to the estimate (10) arises from replacing $\lambda^{su}(\Xi_{\Phi^s z}E(T,\epsilon,g))$ by $\lambda^{su}E(T,\epsilon,g)$, and these two quantities deviate by the Jacobian of the holonomy map which is contained in the interval $[1-\epsilon,(1-\epsilon)^{-1}]$ by assumption. Observe that this formula is consistent with Proposition 5.8.

From now on we use the notation $\underset{(1-\epsilon)^k}{=}$ for an equality which holds true up to multiplying one of the sides by a factor contained in $[(1-\epsilon)^k,(1-\epsilon)^{-k}]$.

The volume estimate in part (d) of Proposition 5.8 and the assumption that the measure $\lambda$ on $Z\supset Z_2(\epsilon)$ is an $(1-\epsilon)$-approximate product yield

$$\lambda(\hat{C}(z,S))\underset{(1-\epsilon)^3}{=}e^{-hS}\lambda^{ss}(A)\lambda^{su}(B)2\zeta/nm.$$

Insertion of this estimate into the estimate (11) then shows that

$$\lambda(\hat{C}(z,S)\cap\Phi^{-S}C(T,\epsilon,g))\lambda^{su}(B)\underset{(1-\epsilon)^5}{=}\lambda(\hat{C}(z,S))\lambda^{su}(E(T,\epsilon,g)). \tag{12}$$

As the measure $\lambda$ on $Z$ is a $(1-\epsilon)$-approximate product, we have

$$\lambda^{su}(E(T,\epsilon,g))\lambda^{ss}(A)2\zeta/m\underset{(1-\epsilon)}{=}\lambda(C(T,\epsilon,g)). \tag{13}$$

Now note that $\lambda^{ss}(A)2\zeta/m\underset{(1-\epsilon)}{=}\lambda(Z_2(\epsilon))/\lambda^{su}(B)$. Summing the approximate equality (13) over $g\in G$ and using the estimate (7) and the equations (8,9) yields

$$\sum_{g\in G}\lambda^{su}(E(T,\epsilon,h^{-1}g))/\lambda^{su}(B)\underset{(1-\epsilon)^6}{=}\lambda(Z). \tag{14}$$

Recall that the cocycle $\theta$ is constant on $\hat{C}(z,S)$, with constant value $\theta(z,S)$. For $g\in G$ write

$$\hat{C}(S,\epsilon,g)=\cup_z\{\hat{C}(z,S)\mid\theta(z,S)=g\}$$

and let $\hat{C}(S,\epsilon)=\cup_{g\in G}\hat{C}(S,\epsilon,g)$.

The estimate (10) is valid for all $g\in G$ and all $z\in\hat{Z}_1(\epsilon)$ such that $\Phi^Sz\in\hat{Z}_3(\epsilon)$. Moreover, for fixed $g\in G$ the set $\hat{C}(S,\epsilon,g)$ is a disjoint union of some of the sets $C(z,S)$. Thus summing the estimate (12) over all $g,h\in G$ und insertion of the estimate (14) together with the analog of the estimate (7) for the measures of the sets $\hat{C}(S,\epsilon)$ implies

$$\begin{aligned}&\sum_{h\in G}\sum_{g\in G}\lambda(\hat{C}(S,\epsilon,h)\cap\Phi^{-S}C(T,\epsilon,h^{-1}g))\\ \underset{(1-\epsilon)^5}{=}&\sum_{h\in G}\sum_{g\in G}\lambda(\hat{C}(S,\epsilon,h))\lambda^{su}(E(T,\epsilon,h^{-1}g))/\lambda^{su}(B)\\ \underset{(1-\epsilon)^{15}}{=}&\lambda(\hat{Z}_2(\epsilon))\lambda(Z_2(\epsilon))\lambda(Z).\end{aligned} \tag{15}$$

On the other hand, as the Teichmüller flow $\Phi^t$ is mixing of all orders, for sufficiently large $S,T$ we also have

$$\lambda\{z\in\hat{Z}_2(\epsilon)\mid\Phi^Sz\in Z_2(\epsilon),\Phi^{S+T}z\in Z\}\underset{(1-\epsilon)}{=}\lambda(\hat{Z}_2(\epsilon))\lambda(Z_2(\epsilon))\lambda(Z). \tag{16}$$

Comparison with the estimate (15) and using the estimate (7) for both $\hat{C}(S,\epsilon)$ and $C(T,\epsilon)$ then shows that

$$(17)\qquad \lambda\{z\in \hat{Z}_2(\epsilon)\mid \Phi^S(z)\in Z_2(\epsilon), \Phi^{S+T}z\in Z, \theta(z,S+T)=g\} \underset{(1-\epsilon)^{24}}{=} \big(\sum_{h\in G}\lambda\{z\in \hat{Z}_2(\epsilon)\mid \Phi^S z\in Z_2(\epsilon),\theta(z,S)=h\} \lambda\{z\in Z_2(\epsilon)\mid \Phi^T z\in Z, \theta(z,T)=h^{-1}g)\}\big)/\lambda(Z_2(\epsilon)).$$

*Step 4.*

In Step 3 above, for a fixed number $\epsilon>0$ and a fixed set $Z=V(A,B,\zeta)$ with a good product structure and the additional property that the Masur Veech measure on $Z$ is an $(1-\epsilon)$-approximate product, for sufficiently large $m,n$ we considered the sets $Z_2(\epsilon)=V(A,B,\zeta/m)$ and $\hat{Z}_2(\epsilon)=V(A,B,\zeta/mn)\subset Z_2(\epsilon)$ and obtained for a fixed $g\in G$ an estimate for the Masur Veech measure of the set

$$\{z\in \hat{Z}_2(\epsilon)\mid \Phi^S z\in Z_2(\epsilon), \Phi^{S+T}z\in Z, \theta(v,S+T)=g\}$$

for all sufficiently large $S,T$.

By replacing the time $S$ by $S+k\zeta/mn$ for some $k\in[-mn,mn]$, this estimate is equally valid if we replace $\hat{Z}_2(\epsilon)$ by $\Phi^{-k\zeta/mn}\hat{Z}_2(\epsilon)$ provided that $S$ is sufficiently large. Assuming now that $mn=2\ell+1$ for some integer $\ell$, we have

$$Z=\cup_{-\ell\le k\le \ell}\Phi^{2k\zeta/mn}\hat{Z}_2(\epsilon).$$

Summing the estimate (17) over all $k\in[-\ell,\ell]$ then yields that we have

$$(18)\quad \lambda\{z\in Z\mid \Phi^S(z)\in Z_2(\epsilon), \Phi^{S+T}z\in Z, \theta(z,S+T)=g\} \underset{(1-\epsilon)^{24}}{=} \sum_{h\in G}\lambda\{z\in Z\mid \Phi^S z\in Z_2(\epsilon),\theta(z,S)=h\} \lambda\{z\in Z_2(\epsilon)\mid \Phi^T z\in Z,\theta(z,T)=h^{-1}g\}/\lambda(Z_2(\epsilon)).$$

On the other hand, assuming that $m=2p+1$ for some integer $p$, the estimate is also valid if we replace $Z_2(\epsilon)$ by $\Phi^{k\zeta/m}Z_2(\epsilon)$ for $-p\le k\le p$. Then summing the estimate (18) over $k\in[-p,p]$ and using the fact that $\lambda(Z_2(\epsilon))=\lambda(Z)/2m$, we deduce that

$$(19)\qquad \lambda\{z\in Z\mid \Phi^S z\in Z, \Phi^{S+T}\in Z, \theta(z,S+T)=g\}\lambda(Z) \underset{(1-\epsilon)^{24}}{=} \sum_{h\in G}\lambda\{z\in Z\mid \Phi^S z\in Z,\theta(z,S)=h\}\lambda\{z\in Z\mid \Phi^T z\in Z,\theta(z,T)=h^{-1}g\}.$$

The above discussion depended on the choice of the number $\epsilon>0$, and it used the fact that on the subset $Z=V(A,B,\zeta)$ of $Y$, the Masur Veech measure is an $(1-\epsilon)$-approximate product. The lemma now follows from the following observation.

Fix again a number $\epsilon>0$. Recall that $Y=V(A^{ss},A^{su},\beta)$. By subdividing an open subset of $A^{ss}$ of full $\lambda^{ss}$-measure into finitely many open connected subsets of small diameter (with boundary of vanishing $\lambda^{ss}$-measure), we obtain a partition of an open subset of $Y$ of full measure into finitely many open connected sets of the form $Z_i = V(A_i, A^{su}, \beta)$ with the property that the Masur Veech measure on each of these sets is an $(1-\epsilon)$-approximate product. Let $Y = \cup_i Z_i$ be such a decomposition. Fix numbers $i, j, k$ and use the above construction for the return maps from $Z_j$ to $Z_i$ and from there to $Z_k$. For large enough $S, T$, this yields an estimate of the measure of the sets

$$\Omega(i,j,k,g) = \{z \in Z_i \mid \Phi^S z \in Z_j, \Phi^{S+T} z \in Z_k, \theta(z, S+T) = g\}$$

which is identical to the estimate (19).

As the sets $\Omega(i,j,k,g)$ are pairwise disjoint, summing their measures shows that for large enough $S, T$ we have

$$\lambda\{z \in Z \mid \Phi^S z \in Z, \Phi^{S+T} z \in Z, \theta(z, S+T) = g\}\lambda(Z) \tag{20}$$

$$\underset{(1-\epsilon)^{24}}{=} \Big(\sum_{h \in G} \lambda\{z \in Z \mid \Phi^S z \in Z, \theta(z,S) = h\}\lambda\{z \in Z \mid \Phi^T z \in Z, \theta(z,T) = h^{-1}g\}\Big)$$

But as $S, T \to \infty$ we can let $\epsilon$ tend to zero which yields the asymptotic formula stated in the lemma. ☐

Our goal is to apply Lemma 5.9 for a control of the cocycle $\theta$ in the way explained in Lemma 5.3. To this end note that by the mixing property of the Teichmüller flow, the values $a(g) \geq 0$ $(g \in G)$ obtained from a suitable chosen sequence $S_i \to \infty$ satisfy $\sum_g a(g) = 1$. For an application of the simple argument in Lemma 5.3 we need to assure that $a(g) > 0$ for all $g \in G$. That this always holds true is shown in the following lemma.

**Lemma 5.10.** *There exists a number $\sigma > 0$ with the following property. Let $T_i \subset (0, \infty)$ be a sequence so that $T_i \to \infty$ and that furthermore for each $g \in G$ there exists a number $a(g) \geq 0$ such that*

$$\lambda\{y \in Y \mid \Phi^{T_i} y \in Y, \theta(y, S_i) = g\} \to a(g)\lambda(Y)^2 \; (T_i \to \infty);$$

*then $a(g) \geq \sigma$ for all $g \in G$.*

*Proof.* We first claim that there are numbers $\kappa > 0, R_0 > 0$ with the following property. For all sufficiently large $T > 0$ and every $g \in G$ there exists a number $R = R(g, T) < R_0$ such that

$$\lambda\{y \in Y \mid \Phi^{T+R} y \in Y, \theta(y, T+R) = g\} \geq \kappa.$$

To show the claim recall that by assumption, the cocycle $\theta$ is onto $G$. Using Proposition 5.8 for $Y = V(A^{ss}, A^{su}, \beta)$, with the notations from the proposition there is a number $\epsilon > 0$, and for every $g \in G$ there exists a number $T(g) > T(\epsilon)$, and there is some $z_g \in Z_1(\epsilon)$ such that $\Phi^{T(g)} z_g \in Z_3(\epsilon)$ and $\theta(z_g, T(g)) = g$.

Now let $T_0 > T(\epsilon)$ be sufficiently large that the estimate (20) in the proof of Lemma 5.9 is valid for this $\epsilon$ and all $S, T \geq T_0$. Let $N$ be the order of the group $G$. Using the above notations, for a given number $T > T_0$ choose some $h = h(T) \in G$ so that

$$\lambda\{z \in Z_1(\epsilon) \mid \Phi^T z \in Z_3(\epsilon), \theta(z,T) = h\} \geq (1-\epsilon)\lambda(Z_1(\epsilon))\lambda(Z_3(\epsilon))/N.$$

Such an element exists by the choice of $T_0$ (which controls the mixing property of the Teichmüller flow).

Recall from Proposition 5.8 the definition of the sets $C(z,T)$ for $z \in Z_1(\epsilon)$ and $\Phi^T z \in Z_3(\epsilon)$ (here $T > T(\epsilon)$). By the reasoning used in the proof of Lemma 5.9, for large enough $T$ and putting once more $h = h(T)$ we have

$$\lambda\{y \in C(z_{gh^{-1}}, T(gh^{-1})) \mid \Phi^{T(gh^{-1})+T} y \in Y, \theta(y, T(gh^{-1}) + T) = g\} \geq \lambda(C(z_{gh^{-1}}, T(gh^{-1}))\lambda(Z_1(\epsilon))\lambda(Z_2(\epsilon))(1-\epsilon)^{24}/N. \tag{21}$$

Since the sets $C(z_g, T(g))$ are all open, and their number is finite, the right hand side of inequality (21) is bounded from below by a positive constant which is independent of $g$. This shows the claim for $R_0 = \max\{T(h) \mid h \in G\}$,

As $\lambda$ is a probability measure and as the group $G$ is finite, we can find a sequence $T_i \to \infty$ such that for every $g \in G$ the sequence

$$\chi(T_i, g) = \lambda\{z \in Y \mid \Phi^{T_i} z \in Y, \theta(z, T_i) = g\}/\lambda(Y)^2$$

converges as $i \to \infty$ to some number $\chi(g) \geq 0$. Note that by the mixing property, we have $\sum_g \chi(g) = 1$, independent of the sequence. In other words, we can view $\chi$ as a probability measure on the group $G$.

Let $\Upsilon \subset \mathcal{P}(G)$ be the closure of the set of all probability measures on $G$ obtained in this way. We claim that this set has the properties in Lemma 5.5.

Namely, let $g \in G$ be arbitrary. By the beginning of this proof, there exists a sequence $T_i \to \infty$ such that for each $i$ we have

$$\lambda\{y \in Y \mid \Phi^{T_i} y \in Y, \theta(y, T_i) = g\} \geq \kappa.$$

By passing to a subsequence we may assume that the sequence fulfills the condition in the definition of the set $\Upsilon$. This shows that for all $g \in G$ there exists some $\xi_g \in \Upsilon$ with $\xi_g(g) \geq \kappa$. The second property in Lemma 5.5 with $c = \lambda(Y)^2$ is the statement of Lemma 5.9.

From Lemma 5.5 and the definition of $\Upsilon$ we obtain the existence of a sequence $S_i \to \infty$ and a number $\kappa > 0$ such that

$$\lim_{i\to\infty} \lambda\{y \in Y \mid \Phi^{S_i} y \in Y, \theta(y, S_i) = g\} \geq \kappa$$

for all $g \in G$. Now let $R_j \to \infty$ be any sequence such that for each $g \in G$, the limit

$$\lim_{j\to\infty} \lambda\{y \in Y \mid \Phi^{R_j} y \in Y, \theta(y, R_j) = g\} = \chi(g)$$

exists. We claim that $\chi(g) \geq \kappa^2$ for all $g$. This then completes the proof of the lemma.

To this end apply Lemma 5.9 to the sequences $S_i, T_i = R_{j(i)} - S_i$ where $i \to \infty$ and where $j(i)$ is chosen in such a way that $R_{j(i)} - S_i \to \infty$. For sufficiently large $i$ the conclusion of the lemma holds true for $S_i$ and $R_{j(i)} - S_i$ up to some error of at most $\epsilon\kappa^2$. An application of Lemma 5.9 and Lemma 5.2 implies that for each $g \in G$, we have

$$\lambda\{y \in Y \mid \Phi^{S_i} y \in Y, \Phi^{R_{j(i)}-S_i} y \in Y, \theta(y, R_{j(i)}) = g\} \geq (1-\epsilon)\kappa^2.$$

As this estimate is valid for all sufficiently large $j = j(i)$ the lemma follows. □

For functions $a, b : [0, \infty) \to (0, \infty)$ define $a \sim b$ if $a(R)/b(R) \to 1$ $(R \to \infty)$. Using the earlier notations, and in particular the definition of the set $Y$ in (3) and the set $\Gamma_0$ of parametrized periodic orbits defined by $Y$, we are now ready to show

**Proposition 5.11.** *For $g \in G$ and for $R > 0$ define*

$$\mathcal{L}(R, g) = \{\gamma \in \Gamma_0(Y) \mid \ell(\gamma) \leq R, \rho(\Omega(\gamma)) = g\}.$$

*Then as $R \to \infty$,*

$$\sharp\mathcal{L}(R, g) \sim \frac{e^{hR}\lambda(Y)}{2h\beta|G|}.$$

*Proof.* By construction, each of the parametrized periodic orbits $\gamma \in \Gamma_0$ contains a distinguished subarc of length $2\beta$ with midpoint in the set $V \supset Y$. Let $\hat{\Gamma}_0 \subset \Gamma_0$ be the subset of all such orbits whose distinguished subarc is contained in $Y$.

For each $g \in G$, we construct from the periodic orbits $\gamma \in \hat{\Gamma}_0$ with $\rho(\Omega(\gamma)) = g$ a $\Phi^t$ invariant Borel measure $\lambda_g$ on $\mathcal{Q}$ which is a positive multiple of the Masur Veech measure $\lambda$.

To this end recall that any component of an intersection of an orbit of $\Phi^t$ with $Y$ is an arc of length $2\beta$. Define

$$\mathcal{C}(R - 2\beta, R, g) = \{\gamma \in \hat{\Gamma}_0 \mid R - 2\beta < \ell(\gamma) \leq R, \rho(\Omega(\gamma)) = g\}.$$

We claim that up to passing to a subsequence, for every $g \in G$ the measures

$$he^{-hR}(1 - e^{-2h\beta})^{-1} \sum_{\gamma \in \mathcal{C}(R-2\beta,R,g)} \delta_{[\gamma(-\beta),\gamma(\beta)]} \tag{22}$$

converge as $R \to \infty$ to a measure $\hat{\lambda}_g$ on $Y$ with $\hat{\lambda}_g(Y) \in [0, \lambda(Y)]$. Here $\delta_{[\gamma(-\beta),\gamma(\beta)]}$ is the restriction to $\gamma[-\beta, \beta]$ of the $\Phi^t$-invariant measure $\delta_\gamma$ supported on $\gamma$.

To show the claim it suffices to control the total mass of the measure defined in (22). This mass can be computed as follows. Let as before $\Gamma$ be the set of all (unparametrized) periodic orbits for $\Phi^t$. For each periodic orbit $\gamma \in \Gamma$, let $n(\gamma) \geq 0$ be the number of components of the intersection $\gamma \cap Y$, and let $n(\gamma, g)$ be the number of intersection components so that a parametrization of $\gamma$ with starting point in the component defines a point $\hat{\gamma} \in \Gamma_0$ with $\rho(\hat{\gamma}) = g$. Define $b(\gamma, g) = 0$ if $n(\gamma) = 0$, and define $b(\gamma, g) = n(\gamma, g)/n(\gamma)$ otherwise. Clearly $\sum_g b(\gamma, g) = 1$ for all $\gamma$ with $n(\gamma) > 0$.

For $R > 2\beta$ let $\Gamma(R - 2\beta, R)$ be the set of all (unparametrized) periodic orbits for $\Phi^t$ of length contained in the interval $(R - 2\beta, R]$. By Corollary 5.4 of [H18], the measures

$$\nu_{R,2\beta} = he^{-hR}(1 - e^{-2h\beta})^{-1} \sum_{\gamma \in \Gamma(R-2\beta,R)} \delta_\gamma$$

converge weakly to the Masur Veech measure $\lambda$, and $\lim_{R\to\infty} \nu_{R,2\beta}(\mathcal{Q}) = 1$ (which means that there is no escape of mass).

Define a $\Phi^t$-invariant Borel measure

$$\nu(R,g) = he^{-hR}(1-e^{-2h\beta})^{-1} \sum_{\gamma\in\Gamma(R-2\beta,R)} b(\gamma,g)\delta_\gamma. \tag{23}$$

By construction, we have

$$\sum_g \nu(R,g)(Y) = he^{-hR}(1-e^{-2h\beta})^{-1} \sum_{\gamma\in\Gamma(R-2\beta,R)} \delta_\gamma(Y).$$

Thus the measures $\nu(R,g)$ are precompact in the space of all $\Phi^t$-invariant Borel measures on the component $\mathcal{Q}$.

If $R_i \to \infty$ is a sequence such that for each $g \in G$ the measures $\nu(R_i,g)$ converge weakly to a measure $\lambda_g$, then $\nu(R_i,g)(\mathcal{Q}) \to \lambda_g(\mathcal{Q})$ and $\sum_g \lambda_g = \lambda$. Furthermore, the measures $\lambda_g$ are invariant under the Teichmüller flow $\Phi^t$. As the Masur Veech measure is ergodic under the action of $\Phi^t$, for each $g \in G$ there exists a number $c(g) \in [0,1]$ so that $\lambda_g = c(g)\lambda$.

A priori, the measures $\lambda_g$ depend on the choice of the sequence $R_i \to \infty$ used to construct them. By Lemma 5.10 and its proof, there is however a number $\sigma > 0$ such that $\lambda_g(\mathcal{Q}) \geq \sigma$ independent of the sequence $R_i$.

Namely, using the terminology of Proposition 5.8, each parametrized periodic orbit $\gamma \in \Gamma_0$ with $\gamma(0) \in Z_2(\epsilon)$ and period $R$ determines a component $C(\gamma(0),R)$ of $Z_2(\epsilon)\cap\Phi^{-R}Z$, and vice versa, such a component determines a parametrized periodic orbit which passes through $Y$ (see [H18] for details of this fact). The additional constraint $\rho(\gamma) = g$ for some $g \in G$ is then equivalent to stating that the value of the cocycle $\theta(\cdot,T)$ equals $g$ on such a component.

We now use an argument which is similar to the reasoning in the proof of Lemma 5.10. Namely, call a sequence $R_i \to \infty$ *admissible* if for each $g \in G$ the measures $\nu(R_i,g)$ converge weakly to a measure $\lambda_g = c(g)\lambda$.

Let $R_i$ be any admissible sequence, with limiting measures $c(g)\lambda$. Choose a subsequence $R_{i_j}$ so that for each $g \in G$ the measures $\nu(R_{i_j}/2,g)$ weakly converge as well. By Lemma 5.9, if these measures converge to measures $\hat\lambda_h$ then for each $g$ we have

$$\nu(T_{i_j},g)\{y \in Y \mid \Phi^{R_{i_j}/2} \in Y, \Phi^{R_{i_j}} z \in Y\} \to \sum_h \hat\lambda_h(Y)\hat\lambda_{h^{-1}g}(Y). \tag{24}$$

On the other hand, $\lambda_g = c(g)\lambda$ is a multiple of the Lebesgue measure and hence it is mixing of all orders. Since $\nu(T_{i_j},g) \to \lambda_g$, this implies that the limit of the left hand side of the expression (24) equals $c(g)\lambda(Y)^3$.

Now the formula (24) together with Lemma 5.2 and the definition of the limiting measures $\lambda_g$ shows that necessarily $c(g) = 1/N$ for all $N$. As $R_i$ was an arbitrary admissible sequence, we deduce that indeed, $\lambda_g(\mathcal{Q}) = 1/|G|$ for all $g \in G$, independent of the sequence $R_i$. $\square$

## 6. Stretch factors

In this section we complete the proofs of the main results of the introduction using group sieving for reductions modulo a prime of the symplectic group $Sp(2g,\mathbb{Z})$. We begin with discussing periodic orbits in a component $\mathcal{Q}$ of a stratum of abelian differentials.

Let $p \geq 3$ be an odd prime and let $F_p$ be the field with $p$ elements. Let

$$\Lambda_p : Sp(2g,\mathbb{Z}) \to Sp(2g,F_p)$$

be reduction modulo $p$. Consider a nested pair of sets $Y \subset V$ as in Section 5 and use these sets to construct the set $\Gamma_0$ of parametrized periodic orbits in $\mathcal{Q}$. We showed in Corollary 4.13 that for all but finitely many primes $p$ we have $\Lambda_p(\Psi\Omega(\Gamma_0)) = Sp(2g,F_p)$. Here as before, $\Omega(\gamma) \in \mathrm{Mod}(S)$ is the pseudo-Anosov mapping class defined by the parametrized orbit $\gamma$, and $\Psi : \mathrm{Mod}(S) \to Sp(2g,\mathbb{Z})$ is the canonical homomorphism.

The following corollary is an immediate consequence of Proposition 5.11. For its formulation, let $N(p)$ be the order of the group $Sp(2g,F_p)$.

**Corollary 6.1.** *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials and let $p \geq 3$ be an odd prime such that $\Lambda_p(\Psi \circ \Omega(\Gamma_0)) = Sp(2g,F_p)$. Let $B \in Sp(2g,F_p)$ be arbitrary and for $R > 0$ define*

$$\mathcal{B}(R,B) = \{\gamma \in \Gamma_0 \mid \ell(\gamma) \leq R, \Lambda_p(\Psi\Omega(\gamma)) = B\}.$$

*Then as $R \to \infty$,*

$$\sharp\mathcal{B}(R,B) \sim \frac{e^{hR}\lambda(Y)}{2h\beta N(p)}.$$

As in the introduction, let $\Gamma$ be the set of all periodic orbits for $\Phi^t$ in the component $\mathcal{Q}$. For a periodic orbit $\gamma$ for $\Phi^t$ denote by $A(\gamma) \in Sp(2g,\mathbb{Z})$ the image under the homomorphism $\Psi$ of some (arbitrarily chosen) pseudo-Anosov element of $\mathrm{Mod}(S)$ which preserves a flow line for the Teichmüller flow projecting onto $\gamma$. Let $[A(\gamma)]$ be the conjugacy class of $A(\gamma)$; this class not depend on any choices made.

The characteristic polynomial of a symplectic matrix $A \in Sp(2g,\mathbb{Z})$ is reciprocal of degree $2g$. The roots of such a polynomial come in pairs: If $\alpha$ is a root then so is $\alpha^{-1}$. We call the extension of $\mathbb{Q}$ defined by the characteristic polynomial of $A$ simply the *field of $A$*. It only depends on the conjugacy class of $A$. Its degree over $\mathbb{Q}$ equals $2g$ if and only if the polynomial is irreducible over $\mathbb{Q}$.

We are now ready to complete the proof of Theorem 1 from the introduction.

**Theorem 6.2.** *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials. The set of all $\gamma \in \Gamma$ such that the field of $[A(\gamma)]$ is of degree $2g$ over $\mathbb{Q}$, separable and totally real is typical.*

*Proof.* We show first that for a typical periodic orbit $\gamma \in \Gamma$ the characteristic polynomial of $[A(\gamma)]$ is irreducible.

Using the notations from Corollary 6.1, let $p_0 \geq 5$ be large enough so that $\Psi(\Omega(\Gamma_0))$ surjects onto $Sp(2g, F_p)$ for all $p \geq p_0$. Let $p \geq p_0$ and let as before $N(p)$ be the order of $Sp(2g, F_p)$. By Corollary 6.1, for every $B \in Sp(2g, F_p)$ and for all large enough $R$ the proportion of the elements $\gamma \in \Gamma_0$ of length at most $R$ which satisfy $\Lambda_p \circ \Psi \circ \Omega(\gamma) = B$ roughly equals $\frac{1}{N(p)}$. On the other hand, if we denote by $R_p(2g)$ the subset of $Sp(2g, F_p)$ of elements with reducible characteristic polynomial then

$$\frac{|R_p(2g)|}{N(p)} < 1 - \frac{1}{3g}$$

(see Theorem 6.2 of [R08] for a reference to this classical result of Borel).

We follow the proof of Theorem 6.2 of [R08]. Let $p_1, \dots, p_k$ be $k$ distinct primes bigger than $p_0$, and let $K = p_1 \cdots p_k$. Then the reduction $\Lambda_K(A)$ modulo $K$ of any element $A \in Sp(2g, \mathbb{Z})$ is defined, and we have

$$\Lambda_K(A) = \Lambda_{p_1}(A) \times \cdots \times \Lambda_{p_k}(A).$$

Namely, for distinct primes $p \neq q \geq 5$, the groups $Sp(2g, F_p)$ and $Sp(2g, F_q)$ are non-isomorphic simple groups. This implies that if $\Gamma$ is any group and if $\rho_p : \Gamma \to Sp(2g, F_p)$ and $\rho_q : \Gamma \to Sp(2g, F_q)$ are surjective homomorphisms, then the homomorphism $\rho_p \times \rho_q : \Gamma \to Sp(2g, F_p) \times Sp(2g, F_q)$ is surjective. In particular, we have

$$Sp(2g, K) = Sp(2g, F_{p_1}) \times \cdots \times Sp(2g, F_{p_k}).$$

As $\Psi\Omega(\Gamma_0)$ surjects onto $Sp(2g, F_p)$ for all $p \geq p_0$, the reduction mod $K$ defines a surjective homomorphism of the semigroup $\Psi\Omega(\Gamma_0) < Sp(2g, \mathbb{Z})$ onto the finite group $Sp(2g, K) = \Lambda_K(\Psi\Omega(\Gamma_0))$. On the other hand, if $A \in Sp(2g, \mathbb{Z})$ has a reducible characteristic polynomial, then the same holds true for $\Lambda_{p_i}(A)$ for all $i$. By the reasoning in the previous paragraph, the proportion of the number of elements in $Sp(2g, K)$ with this property is at most $(1 - \frac{1}{3g})^k$.

By Corollary 6.1, this implies that for a given number $k \geq 1$ and all large enough $R$, the proportion of all orbits $\gamma \in \Gamma_0$ of length at most $R$ with the property that the characteristic polynomial of $\Psi(\Omega(\gamma))$ is reducible is at most of the order of $(1 - \frac{1}{3g})^k$. As $k$ was arbitrarily chosen, we conclude that the degree of the field extension of $\mathbb{Q}$ defined by typical periodic orbit of $\Phi^t$ equals $2g$.

We next claim that for a typcial orbit $\gamma \in \Gamma$, the field of $A(\gamma)$ is separable and totally real. Namely, as the Lyapunov spectrum of $\mathcal{Q}$ is simple [AV07], Theorem 1 of [H18] shows that for a typical periodic orbit $\gamma$, the absolute values of the eigenvalues of $[A(\gamma)]$ are pairwise distinct. But this just means that the field of $[A(\gamma)]$ is totally real and separable. □

For a symplectic matrix $A \in Sp(2g, \mathbb{Z})$, the field of $A$ is an extension of degree at most two of its *trace field*, defined as the characteristic polynomial of $A + A^{-1}$. For a periodic orbit $\gamma \subset \mathcal{Q}$, we call the trace field of $[A(\gamma)]$ the *trace field* of $\gamma$. The trace field $\gamma$ can also be read off directly from a point $\omega \in \gamma$. Namely, let $\tilde{\omega}$

be a lift of $\omega$ to a marked abelian differential on the surface $S$. The periods of $\tilde{\omega}$ define an abelian subgroup $\Lambda = \tilde{\omega}(H_1(S,\mathbb{Z}))$ of $\mathbb{C}$ of rank two. Let $e_1, e_2 \in \Lambda$ be two points which are linearly independent over $\mathbb{R}$. Let $K$ be the smallest subfield of $\mathbb{R}$ such that every element of $\Lambda$ can be written as $ae_1 + be_2$, with $a, b \in K$; then $\Lambda \otimes_K K = K^2$ (see the appendix of [KS00] for more details).

**Definition 6.3.** The periodic orbit $\gamma$ is called *algebraically primitive* if the trace field $K$ of $\gamma$ is a totally real separable number field of degree $g$ over $\mathbb{Q}$.

The following corollary summarizes the discussion.

**Corollary 6.4.** *For every component $\mathcal{Q}$ of a stratum of abelian differentials, algebraically primitive periodic orbits for $\Phi^t$ are typical.*

We are left with the proof of Theorem 2 from the introduction. Recall that we always require that strata of quadratic differentials are not strata of squares of holomorphic one-forms. By a slight abuse of notation, for a periodic orbit $\gamma$ for the Teichmüller flow on a component $\mathcal{D}$ of the moduli space of quadratic differentials we denote by $\Omega(\gamma)$ an arbitrarily chosen pseudo-Anosov mapping class whose conjugacy class defines $\gamma$.

**Theorem 6.5.** *Let $\mathcal{D}$ be a component of a stratum of quadratic differentials with $m \geq 1$ zeros and $k \leq m$ zeros of odd order. Then for a typical periodic orbit $\gamma \subset \mathcal{D}$, the algebraic degree of the stretch factor of $\Omega(\gamma) \in \mathrm{Mod}(S)$ equals $2g-2+k$.*

*Proof.* Let $\mathcal{D}$ be a component of a stratum of quadratic differentials with $m \geq 1$ zeros and $k \leq m$ zeros of odd order. As the total orders of all zeros equals $4g-4$, the number $k$ is necessarily even. Then $\mathcal{D}$ is a complex orbifold of dimension $2g-2+m$.

For each quadratic differential $q$ on $S$ which is not the square of a holomorphic one-form, there is a two-sheeted cover $S'$ of $S$, ramified precisely at the zeros of odd orders of $q$, such that $q$ lifts to an abelian differential on $S'$. This double cover is constructed as follows.

Let $S_0$ be the surface obtained from $S$ by removing the zeros of $q$ of odd order. Then for every point $x \in S_0$, there exists a local square root of $q$ near $x$, unique up to multiplication by $-1$. Thus there exists a unique two-sheeted cover $S_0'$ of $S_0$ on which such a square root is globally defined. This cover is the cover of $S_0$ whose fibre over a point $x$ are the two choices of the square roots of $q$ at $x$. It is connected since $q$ is not the square of a holomorphic one-form. The double cover $\pi : S_0' \to S_0$ does not depend on the particular choice of $q$ in the component $\mathcal{D}$ of a stratum.

The preimages of the punctures of $S_0$ are punctures of $S_0'$. Furthermore, a loop in $S_0$ going around a puncture $p$ of $S_0$ reverses the sign of a square root of $q$ and hence the covering projection $\pi$ extends to a branched cover $S' \to S$ where $S'$ is obtained from $S_0'$ by filling in the punctures. This branched cover is ramified precisely at the punctures of $S_0$, i.e. at the zeros of $q$ of odd order. As a consequence, the cover $S' \to S$ is ramified at precisely $k$ points. The quadratic differential $q$ lifts to an abelian differential on $S'$ with $2m-k$ zeros. This shows that the component $\mathcal{D}$ lifts to an affine invariant manifold $\mathcal{C}$ in a component $\mathcal{Q}$ of a stratum in the moduli space of abelian differentials on $S'$, consisting of abelian differentials with $2m-k$ zeros.

By the Riemann Hurwitz formula, the genus $g'$ of $S'$ equals $2g-1+\frac{k}{2}$ and hence $\dim H^1(S',\mathbb{R})=4g-2+k$. The surface $S$ is the quotient of $S'$ by an involution $\iota$ which exchanges the two sheets in the cover.

The involution $\iota$ acts on the real cohomology $H^1(S',\mathbb{R})$ of $S'$. This cohomology decomposes over $\mathbb{R}$ as

$$H^1(S',\mathbb{R})=\mathcal{E}_1\oplus\mathcal{E}_2$$

where $\mathcal{E}_1$ is the eigenspace for $\iota$ with respect to the eigenvalue 1, and $\mathcal{E}_2$ is the eigenspace for $\iota$ with respect to the eigenvalue $-1$. As the action of $\iota$ on $H^1(S',\mathbb{R})$ is a symplectic transformation, this decomposition is orthogonal for the symplectic form on $H^1(S',\mathbb{R})$. The vector space $\mathcal{E}_1$ is precisely the pullback of $H^1(S,\mathbb{R})$ under the branched covering map and hence its dimension equals $2g$. Thus $\dim(\mathcal{E}_2)=2g-2+k$.

We next observe that $\mathcal{E}_2\otimes\mathbb{C}$ has a natural identification with the projection of $T\mathcal{C}_+=\mathcal{C}\times(0,\infty)$ to absolute periods. To this end note that by construction, if $q'\in\mathcal{C}_+$ then $\iota^*q'=-q'$. Hence by equivariance, the projection of $T\mathcal{C}_+$ to absolute periods is contained in $\mathcal{E}_2\otimes\mathbb{C}$.

Let $\Sigma$ be the zero set of a differential in $\mathcal{Q}_+=\mathcal{Q}\times(0,\infty)$. The set $\Sigma$ contains the $k$ ramification points of a differential in $\mathcal{C}_+$. The involution $\iota$ acts as an involution on the dual $H_1(S',\Sigma;\mathbb{Z})^*$ of the homology group of $S'$ relative to $\Sigma$. Period coordinates, for $\mathcal{Q}_+$ take values in $H_1(S',\Sigma;\mathbb{Z})^*$, and the linear equation for $\mathcal{C}_+$ is the equation $\iota^*\omega+\omega=0$. Namely, if $\omega$ is any point with this property, then $\omega^2$ is $\iota$-invariant and projects to a quadratic differential on $S$ which is not the square of a holomorphic one-form. By construction, this quadratic differential is contained in the component $\mathcal{D}$.

By naturality of period coordinates, the map which associates to an abelian differential $\omega$ with $\iota^*\omega+\omega=0$ its projection to absolute periods is a submersion into $\mathcal{E}_2\otimes\mathbb{C}$. As a consequence, the projection of $T\mathcal{C}_+$ to absolute periods equals the vector space $\mathcal{E}_2\otimes\mathbb{C}$.

Since $\iota$ descends to an element of $Sp(2g',\mathbb{Z})$ whose square is the identity, the decomposition $H^1(S',\mathbb{R})=\mathcal{E}_1\oplus\mathcal{E}_2$ is defined over $\mathbb{Z}[\frac{1}{2}]$. Thus the stabilizer of this decomposition in the group $Sp(2g',\mathbb{Z})$ projects to a lattice in the group of symplectic automorphisms of $\mathcal{E}_2$.

A periodic orbit $\gamma$ for the Teichmüller flow in $\mathcal{D}$ determines a pseudo-Anosov mapping class which preserves the zeros of odd order and hence lifts to a mapping class of the branched cover $S'$ of $S$. This mapping class projects to a Perron Frobenius automorphism of $\mathcal{E}_2$ whose Perron Frobenius eigenvalue is just the stretch factor of the pseudo-Anosov element of $\mathrm{Mod}(S)$ defining $\gamma$.

By Theorem 4.12, the affine invariant manifold $\mathcal{C}$ is locally Zariski dense (this can also be seen directly in this explicit case). Furthermore, for all but finitely many primes $p$ the local monodromy surjects onto the mod $p$ reduction of the intregral symplectic group. Using Corollary 6.1 for the Teichmüller flow on $\mathcal{D}$ and cocycles defined by the action of this flow on the mod $p$ homology on $\mathcal{C}$ in the same way as in the proof of Theorem 6.2, we conclude that the algebraic degree of the stretch factor of a pseudo-Anosov mapping class defined by a typical periodic orbit for $\Phi^t$ on $\mathcal{D}$ equals $2g-2+k$. This is what we wanted to show. □

We are left with showing the Corollary  from the introduction. To this end we proceed by induction on the genus $g$ of $S$. As there are strata of quadratic differentials with $k$ zeros of odd order for any even number $k \leq 4g-4$, the case $g=2$ follows from Theorem 6.5 applied to the numbers $k=0,2,4$. Note that the strata of quadratic differentials used in this construction are well known to be non-empty.

Assume now that the corollary is known for every genus $2 \leq h \leq g-1$. By Corollary 6.5, applied to all even numbers $k \in [0,4g-4]$, we find that for every even number $2g-2 \leq m \leq 6g-6$ there are infinitely many distinct conjugacy classes of pseudo-Anosov mapping classes with stretch factor of degree $m$ over $\mathbb{Q}$.

To cover the cases $m \leq 2g-4$ we consider first the case that $g-1=2n$ is even $(n \geq 1)$. Then a surface $S$ of genus $g$ is a double cover of a surface $S'$ of genus $n+1$. Let $\Pi : S \to S'$ be the covering projection. The pullback by $\Pi$ of a component of a stratum of abelian or quadratic differentials on $S'$ is an affine invariant manifold for the Teichmüller flow on $S$. For pseudo-Anosov mapping class $\varphi$ on $S'$ there exists some $k>0$ such that $\varphi^k$ lifts to a pseudo-Anosov mapping class on $S$ whose stretch factor is the $k$-th power of the stretch factor of $\varphi$.

By induction hypothesis, for each even number $m \leq 6(n+1)-6 = 6n = 3g-3$ there are infinitely many conjugacy classes of pseudo-Anosov mapping classes for $S'$ whose stretch factor is an algebraic integer of degree $m$. The induction step follows. In particular, we obtain the statement for $g=3$.

If $g=2n \geq 4$ is even then by the Riemann Hurwitz formula, $S$ is a double cover of a surface $S'$ of genus $n$, branched at two points. Note that as $n \geq 2$, there are no constraints for the construction of such a double branched cover. Indeed, $S$ is just the orientation cover of a quadratic differential on $S'$ with two simple zeros and all other zeros of even degree.

Let $\mathcal{D}$ be a component of a stratum of quadratic differentials on $S'$ with two simple zeros $p_1,p_2$ and all other zeros of even order. Then the points $p_1,p_2$ are the branch points of the cover. The covering map $\Pi : S \to S'$ commutes with the Teichmüller flows on $\mathcal{D}$ and on its preimage, which is an affine invariant manifold in the moduli space of quadratic differentials on $S$. The preimage of a differential $q \in \mathcal{D}$ is a differential on $S$. If $q$ is a periodic point for the Teichmüller flow on $\mathcal{D}$ then $q$ lifts to a periodic point for the Teichmüller flow on $S$ with the same stretch factor.

By induction hypothesis, for each even number $m \leq 6k-6 = 3g-6$ there are infinitely many conjugacy classes of mapping classes on $S'$ with stretch factor of algebraic degree $m$. These mapping classes lift to $S$. As $2g-2 \leq 3g-6$ for all $g \geq 4$, the induction step follows. This completes the proof of the corollary from the introduction.

**Remark 6.6.** The above results do not answer any of the more specific questions on stretch factors one might ask, and in contrast to Theorem 1, they do not imply that the extension of $\mathbb{Q}$ by a typical stretch factor is a totally real number field.

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