TOPOLOGICAL PROPERTIES OF REEB ORBITS ON BOUNDARIES OF STAR-SHAPED DOMAINS IN $\mathbb{R}^4$

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ABSTRACT. Let B^4 be the compact unit ball in $\mathbb{R}^4$ with boundary S^3. Let γ be a knot on S^3 which is transverse to the standard contact structure. We show that if there is an immersed symplectic disc $f : (D, \partial D) \to (B^4, \gamma)$ then $\mathrm{lk}(\gamma) = 2\mathrm{tan}(f) - 1$ where $\mathrm{lk}(\gamma)$ is the self-linking number of γ and $\mathrm{tan}(f)$ is the tangential self-intersection number of f. We also show that if $E \subset \mathbb{C}^2$ is compact and convex, with smooth boundary Σ, and if the principal curvatures of Σ are suitably pinched then the self-linking number of a periodic Reeb orbit on Σ of Maslov index 3 equals -1.

1. Introduction

Consider the four-dimensional euclidean space $\mathbb{R}^4$ with the standard *symplectic form* defined in standard coordinates by $\omega_0 = \sum_{i=1}^2 dx_i \wedge dy_i$. This symplectic form is the differential of the one-form

$$\lambda_0 = \frac{1}{2} \sum_{i=1}^2 (x_i dy_i - y_i dx_i).$$

For every bounded domain $\Omega \subset \mathbb{R}^4$ which is star-shaped with respect to the origin $0 \in \mathbb{R}^4$, with smooth boundary Σ, the restriction λ of λ_0 to Σ defines a smooth *contact form* on Σ. This means that $\lambda \wedge d\lambda$ is a volume form on Σ.

Let $\xi = \ker(\lambda)$ be the contact bundle. Each *transverse knot* γ on Σ, i.e. an embedded smooth closed curve on Σ which is everywhere transverse to ξ, admits a canonical orientation determined by the requirement that $\lambda(\gamma') > 0$. To such an oriented transverse knot γ we can associate its *self-linking number* $\mathrm{lk}(\gamma)$ which is defined as follows. Let $S \subset \Sigma$ be a *Seifert surface* for γ, i.e. S is a smooth embedded oriented surface in Σ whose oriented boundary equals γ. Since γ is transverse to ξ, there is a natural identification of the restriction to γ of the oriented normal bundle of S in Σ with a real line subbundle N_S of the contact bundle $\xi|\gamma$. Then N_S defines a trivialization of the oriented two-plane bundle $\xi|\gamma$. The self-linking number $\mathrm{lk}(\gamma)$ of γ is the winding number with respect to N_S of a trivialization of ξ over γ which extends to a trivialization of ξ on Σ.

Eliashberg [Eli93] showed that the self-linking number of a transverse knot in Σ is always an odd integer. If g denotes the *Seifert genus* of γ, i.e. the smallest genus of a Seifert surface for γ, then we have $\mathrm{lk}(\gamma) \leq 2g - 1$ (Theorem 4.1.1 of

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[Eli93]). Eliashberg also constructed for every $k \geq 1$ a transverse unknot ζ of self-linking number $\mathrm{lk}(\zeta) = -2k-1$ in the standard unit three-sphere $S^3 \subset \mathbb{R}^4$. This indicates that in general we can not expect additional relations between the self-linking number of a transverse knot γ on Σ and purely topological invariants of γ.

Our first goal is to relate the self-linking number of a (canonically oriented) transverse knot γ on the unit three-sphere $S^3 \subset \mathbb{R}^4$ to the symplectic topology of the closed unit ball $B^4 \subset \mathbb{R}^4$. For this let $D \subset \mathbb{C}$ be the closed unit disc with oriented boundary ∂D and let $f : (D, \partial D) \to (B^4, S^3)$ be a smooth immersion with $f^{-1}(S^3) = \partial D$. If all self-intersections of $f(D)$ are transverse then the *tangential index* $\tan(f)$ of f is the number of self-intersection points of f counted with signs and multiplicities. The disc $f : D \to B^4$ is called *symplectic* if for every $x \in D$ the restriction of the symplectic form ω_0 to $df(T_xD)$ does not vanish and defines the usual orientation of D. We show

Theorem 1. *Let γ be a transverse knot on the boundary S^3 of the compact unit ball $B^4 \subset \mathbb{R}^4$. If γ bounds an immersed symplectic disc $f : (D, \partial D) \to (B^4, \gamma)$ then* $\mathrm{lk}(\gamma) = 2\tan(f) - 1$.

Now let Σ be the boundary of an arbitrary bounded domain $\Omega \subset \mathbb{R}^4$ which is star-shaped with respect to the origin, with smooth boundary. The *Reeb vector field* of the contact form λ is the smooth vector field X on Σ defined by $\lambda(X) = 1$ and $d\lambda(X, \cdot) = 0$. Rabinowitz [Rab79] (see also [W79]) showed that the *Reeb flow* on Σ generated by the Reeb vector field X admits periodic orbits. Dynamical properties of the Reeb flow on Σ are related to properties of Ω viewed as a symplectic manifold. The proof of the following corollary is similar to the proof of Theorem 1.

Corollary. *Let γ be a periodic Reeb orbit on the boundary Σ of a star-shaped domain $\Omega \subset \mathbb{R}^4$ with compact closure C. If γ bounds an immersed symplectic disc $f : (D, \partial D) \to (C, \gamma)$ then* $\mathrm{lk}(\gamma) = 2\tan(f) - 1$.

Note that by a result of Hofer, Wysocki and Zehnder [HWZ96], there is always a periodic Reeb orbit of self-linking number -1 on Σ which is unknotted.

Even though the radial diffeomorphism $\Psi : S^3 \to \Sigma$ maps the contact bundle of S^3 to the contact bundle of Σ and hence maps a transverse knot γ on S^3 to a transverse knot $\Psi\gamma$ on Σ, the corollary is not immediate from Theorem 1. Namely, in general the radial diffeomorphism Ψ does not extend to a symplectomorphism $B^4 \to C$ and hence there is no obvious relation between symplectic immersions of discs in the unit ball and in the domain Ω.

In general, the existence of an immersed symplectic disc in C whose boundary is a given Reeb orbit γ on Σ does not seem to be known. However, we observe in Section 4 that such an immersed symplectic disc always exists if Σ is the boundary of a strictly convex domain in $\mathbb{R}^4$.

There is a second numerical invariant for a periodic Reeb orbit γ on Σ, the so-called *Maslov index* $\mu(\gamma)$. If Σ is the boundary of a compact strictly convex body $C \subset \mathbb{R}^4$ then the Maslov index of any periodic Reeb orbit on Σ is at least three

[HWZ98]. The *action* of γ is defined to be $\int_\gamma \lambda > 0$, and the orbit is *minimal* if its action is minimal among the actions of all periodic orbits of the Reeb flow. Ekeland [Eke90] showed that the Maslov index of a minimal Reeb orbit on Σ equals precisely three.

Our second result relates the Maslov index to the self-linking number for periodic Reeb orbits on the boundary of compact strictly convex bodies with geometric control.

Theorem 2. *Let C be a compact strictly convex body with smooth boundary Σ. If the principal curvatures $a \geq b \geq c$ of Σ satisfy the pointwise pinching condition $a \leq b + c$ then a periodic Reeb orbit γ on Σ of Maslov index 3 bounds an embedded symplectic disc in C. In particular, the self-linking number of γ equals -1.*

As an immediate consequence, if Σ is as in Theorem 2 then a periodic Reeb orbit γ on Σ of Maslov index 3 is a slice knot in Σ. In fact, with some additional effort it is possible to show that such an orbit is unknotted [H07].

The proofs of these results use mainly tools from differential topology and differential geometry. In Section 2 we begin to investigate topological properties of transverse knots on the three-sphere S^3. We define a self-intersection number for a (not necessarily immersed) disc in the closed unit ball B^4 with boundary γ which does not have self-intersections near the boundary and relate this self-intersection number to the self-linking number of γ. In Section 3 we study topological invariants of immersed discs in B^4 with boundary γ and show Theorem 1 and the corollary. In Section 4 we look at boundaries of strictly convex bodies in $\mathbb{R}^4$ and derive Theorem 2.

2. Self-intersection of surfaces

In this section we investigate topological invariants of smooth maps from an oriented bordered surface S with connected boundary ∂S into an arbitrary smooth oriented simply connected 4-dimensional manifold W (without boundary) whose restrictions to a neighborhood of ∂S are embeddings. For the main application, $W = \mathbb{C}^2 = \mathbb{R}^4$. We use this discussion to investigate maps from the closed unit disc $D \subset \mathbb{C}$ into the compact unit ball $B^4 \subset \mathbb{C}^2$. For maps which map the oriented boundary ∂D of D to a canonically oriented transverse knot γ on Σ we define a self-intersection number and relate this to the self-linking number of γ.

Let for the moment S be any compact oriented surface with connected boundary $\partial S = S^1$.

Definition 2.1. A smooth map $f : S \to W$, i.e. a map which is smooth up to and including the boundary, is called *boundary regular* if the singular points of f are contained in the interior of S, i.e. if there is a neighborhood A of ∂S in S such that the restriction of f to $f^{-1}(f(A))$ is an embedding.

McDuff investigated in [McD91] boundary regular *pseudo-holomorphic* discs in *almost complex* 4*-manifolds* (W, J). By definition, such a pseudo-holomorphic disc is a smooth boundary regular map f from the closed unit disc $D \subset \mathbb{C}$ into W whose differential is complex linear with respect to the complex structure on D and the almost complex structure J. She defined a topological invariant for such boundary regular pseudo-holomorphic discs which depends on a trivialization of the normal bundle over the boundary circle.

Our first goal is to find a purely topological analog of this construction. For this we say that two boundary regular maps $f, g : S \to W$ *are contained in the same boundary class* if g coincides with f near the boundary and is homotopic to f with fixed boundary. This means that there is a homotopy $h : [0, 1] \times S \to W$ connecting $h_0 = f$ to $h_1 = g$ with $h(s, z) = f(z)$ for all $s \in [0, 1]$, all $z \in \partial S$. We do not require that each of the maps $h_s : z \to h_s(z) = h(s, z)$, $s \in [0, 1]$, is boundary regular. In particular, if $\pi_2(W) = 0$ then any two boundary regular maps $f, g : S \to W$ which coincide near the boundary ∂S of S are contained in the same boundary class (recall that we require that W is simply connected).

There is also the following stronger notion of homotopy for boundary regular maps.

Definition 2.2. A homotopy $h : [0, 1] \times S \to W$ is called *boundary regular* if for each s the map h_s is boundary regular and coincides with h_0 near ∂S.

The set of boundary regular maps in the boundary class of a map $f : S \to W$ can naturally be partitioned into boundary regular homotopy classes.

A boundary regular map $f : S \to W$ is an embedding near ∂S. Since S is oriented by assumption, the normal bundle L of $f(S)$ over the embedded circle $f(\partial S)$ is an oriented real two-dimensional subbundle of $TW|f(\partial S)$.

For each trivialization ρ of this normal bundle, the self-intersection number $\mathrm{Int}(f, \rho) \in \mathbb{Z}$ is defined as follows [McD91]. Let $\overline{N}$ be a closed tubular neighborhood of $f(\partial S) = \gamma$ in W with smooth boundary ∂N such that $f(S) \cap \overline{N}$ is an embedded closed annulus A which intersects ∂N transversely. Let $E \subset W$ be an embedded submanifold with boundary which contains A and is diffeomorphic to an open disc bundle over A. One of the two connected components $(\partial E)_0$ of the boundary ∂E of E has a natural identification with the total space of the normal bundle L of $f(S)$ over γ. Remove $\overline{N} - E$ from W and glue to the boundary $(\partial E)_0$ of the resulting manifold the oriented real two-dimensional vector bundle $D \times \mathbb{C} \to D$ in such a way that $\partial D \times \{0\}$ is identified with the curve $f(\partial S) = \gamma \subset (\partial E)_0$ and that the fibres $\{x\} \times \mathbb{C}$ $(x \in \partial D)$ match up with the trivialized normal bundle $L|f(\partial S)$ of $f(S)$ over γ. Up to diffeomorphism, the resulting 4-dimensional smooth manifold W_ρ only depends on the homotopy class of the trivialization ρ and of the boundary class of f. Let S_0 be the closed oriented surface obtained by glueing a disc to the boundary of S in the usual way. The map f naturally extends to a map f_0 of S_0 into W_ρ. The *self-intersection number* $\mathrm{Int}(f, \rho)$ is then defined to be the topological self-intersection number of $f_0(S_0)$ in W_ρ. Thus $\mathrm{Int}(f, \rho)$ is the number of intersections of $f(S)$ with a surface f' which is a generic perturbation of $f(S)$ and such that $f(\partial S)$ is pushed into the direction given by ρ.

In the next lemma we determine the boundary regular homotopy classes in a fixed boundary class.

Lemma 2.3. *Let $f : S \to W$ be a smooth boundary regular map. Choose a trivialization ρ of the oriented normal bundle of $f(S)$ over $f(\partial S)$. Then the assignment which associates to a boundary regular homotopy class of maps in the boundary class of f its self-intersection number with respect to ρ is a bijection onto* $\mathrm{Int}(f,\rho) + 2\mathbb{Z}$. *Moreover, if f is an embedding then each such class can be represented by an embedding.*

Proof. Let $f : S \to W$ be a smooth boundary regular map. Write $\gamma = f(\partial S)$ and let $u : S \to W$ be a smooth boundary regular map in the boundary class of f. This means that there is a homotopy $h : [0,1] \times S \to W$ connecting $h_0 = f$ to $h_1 = u$ with fixed boundary. The maps u, f are contained in the same boundary regular homotopy class if and only if this homotopy can be chosen in such a way that there is a tubular neighborhood N of γ such that the intersection of $h_s(S)$ with N is independent of s.

Choose such an open tubular neighborhood N of γ with smooth boundary ∂N which is sufficiently small that both $f(S)$ and $u(S)$ intersect N in a smooth annulus containing γ as one of its two boundary components. We may assume that there is a compact subsurface $C \subset S$ with smooth boundary ∂C such that $S - C$ is an annulus neighborhood of ∂S and that $f(S-C) = u(S-C) = f(S) \cap N = u(S) \cap N$. Then $u|_C$ and $f|_C$ can be combined to a map into $W - N$ of the closed oriented surface $\tilde{S}$ which we obtain from C by gluing two copies of C along the boundary with an orientation reversing boundary identification. This map is homotopic in $W - N$ to a constant map if and only if u and f are contained in the same boundary regular homotopy class.

Now W is simply connected by assumption and N is homeomorphic to a 3-ball-bundle over a circle, with boundary $\partial N \sim \gamma \times S^2$. Thus by van Kampen's theorem, $W - N$ is simply connected and the second homotopy group $\pi_2(W - N)$ coincides with the second homology group $H_2(W - N, \mathbb{Z})$ via the Hurewicz isomorphism. Since two boundary regular maps in the same boundary class are homotopic with fixed boundary, we conclude that the family of boundary regular homotopy classes of maps in the boundary class of f can be identified with the kernel of the natural homomorphism $H_2(W - N, \mathbb{Z}) \to H_2(W, \mathbb{Z})$.

To compute this group, we use the long exact homology sequence of the pair $(W, W - N)$ given by

$$\cdots \to H_3(W, \mathbb{Z}) \quad \to H_3(W, W - N, \mathbb{Z}) \quad \to H_2(W - N, \mathbb{Z}) \quad \to H_2(W, \mathbb{Z}) \quad \to \dots$$

Excision shows that $H_3(W, W - N, \mathbb{Z}) = H_3(\overline{N}, \partial N, \mathbb{Z})$ where $\overline{N}$ is the closure of N. Since $\overline{N} = \gamma \times B^3 = S^1 \times B^3$ where B^3 denotes the closed unit ball in $\mathbb{R}^3$, the group $H_3(W, W - N, \mathbb{Z})$ is cyclic and generated by a ball $\{z\} \times (B^3, S^2)$ where $z \in \gamma$ is any fixed point.

Every singular homology class $v \in H_3(W, \mathbb{Z})$ can be represented by a piecewise smooth singular cycle σ whose image is nowhere dense in W. On the other hand, the curve γ is contractible in W and therefore there is a smooth isotopy of W which moves σ away from N. Thus the image of $H_3(W,\mathbb{Z})$ under the natural homomorphism $H_3(W,\mathbb{Z}) \to H_3(W,W-N,\mathbb{Z}) = H_3(\overline{N},\partial N,\mathbb{Z})$ vanishes and hence by exactness, the kernel of the natural homomorphism $H_2(W-N,\mathbb{Z}) \to H_2(W,\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and generated by a sphere $e = \{z\} \times S^2 \sim 1 \in \pi_2(\partial N) = \mathbb{Z}$ for some $z \in \gamma$. As a consequence, every boundary regular homotopy class in the boundary class of f can uniquely be represented in the form $[f] + ke$ where $[f]$ denotes the boundary regular homotopy class of f and where $k \in \mathbb{Z}$.

Next we show that if f is an embedding then each of these classes can be represented by an embedding as well. For this note that after possibly replacing N by a smaller tubular neighborhood of γ we may assume that $f^{-1}(\overline{N})$ is a closed annulus neighborhood of ∂S and that for some $z \in \gamma$, the sphere $M = \{z\} \times S^2 \subset \partial N$ intersects $f(S)$ transversely in a single point x. The orientation of the surface S defines uniquely an orientation of M such that $T_xW = T_x(f(S)) \oplus T_xM$ as oriented vector spaces. Using standard surgery near the transverse intersection point x we can attach the sphere M to the surface $f(S)$ as follows (see p.38 of [GS99]). There is a closed neighborhood V of x in $W - \gamma$ which is diffeomorphic to a closed ball and such that the intersections $f(S) \cap V$, $M \cap V$ are smooth discs which intersect transversely in the single point x. The boundaries of these discs are two disjoint oriented circles in the boundary $\partial V \sim S^3$ of V. These circles define the Hopf link in S^3 and therefore they form the oriented boundary of a smooth embedded annulus in ∂V. The surgery replaces $(f(S) \cup M) \cap V$ by such an annulus (which can be done smoothly). We obtain in this way a compact oriented bordered surface which can be represented by a boundary regular map $g : S \to W$ which coincides with $f(S)$ near the boundary. The surgery does not change relative homology classes (p.38 of [GS99]) and hence $g(S)$ is homologous to $[f] + e$ via an identification of M with a generator e of the kernel of the natural map $H_2(W-N,\mathbb{Z}) \to H_2(W,\mathbb{Z})$. In other words, the embedded surface which we just constructed represents the boundary regular homotopy class $[f] + e$ in the boundary class of f. In the same way we can also construct a surface which represents the boundary regular homotopy class $[f] - e$ by attaching to f a sphere equipped with the reverse orientation. Namely, we also can connect the boundaries of the discs $f(S) \cap V, M \cap V$ with an embedded cylinder whose oriented boundary is the union of the oriented boundary of $f(S) \cap V$ with the boundary of $M \cap V$ equipped with the reversed orientation. Repeating this procedure finitely many times with different basepoints we obtain an embedding in every boundary regular homotopy class of maps in the boundary class of f.

Let ρ be a trivialization of an oriented normal bundle of $f(S)$ along $\gamma = f(\partial S)$. We are left with showing that a boundary regular homotopy class in the boundary class of f is determined by its self-intersection number with respect to ρ. For this let again $M = \{z\} \times S^2 \subset \partial N$ be an oriented embedded sphere as above which intersects $f(S)$ transversely in a single point x. Assume that the index of intersection between $f(S)$ and M with respect to the given orientations is positive. Let $g : S \to W$ be the map constructed above with $[g] = [f] + e$. Using the above notations, it is enough to show that $\mathrm{Int}(g,\rho) = \mathrm{Int}(f,\rho) + 2$. However, this can be seen as follows.

As above, denote by W_ρ the manifold used for the definition of the self-intersection number $\mathrm{Int}(f,\rho)$. Recall that up to diffeomorphism, the manifold W_ρ only depends on ρ and the boundary class of f. In particular, we may assume that W_ρ contains the images Γ_f, Γ_g of the closed surface S_0 under the natural extensions of the maps f, g. The self-intersection numbers of the surfaces Γ_f, Γ_g in the manifold W_ρ can now be compared via

$$\begin{aligned}\mathrm{Int}(g,\rho) &= \Gamma_g \cdot \Gamma_g = (\Gamma_f + e)\cdot(\Gamma_f + e)\\ &= \Gamma_f \cdot \Gamma_f + 2\Gamma_f \cdot e + e \cdot e = \mathrm{Int}(f,\rho) + 2\end{aligned}$$

since the topological self-intersection of the sphere e in W_ρ vanishes. But this just means that the assignment which associates to a boundary regular homotopy class in the boundary class of f its self-intersection number with respect to ρ is a bijection onto $\mathrm{Int}(f,\rho) + 2\mathbb{Z}$. From this the lemma follows. □

From now on we assume that the 4-dimensional manifold W is equipped with a smooth almost complex structure J.

Definition 2.4. A smooth boundary regular map $f : S \to W$ is called *boundary holomorphic* if for each $z \in \partial S$ the tangent plane of $f(S)$ at z is a complex line in (TW, J) whose orientation coincides with the orientation induced from the orientation of S.

If $f : S \to W$ is boundary regular and boundary holomorphic then the pull-back f^*TW under f of the tangent bundle of W is a 2-dimensional complex vector bundle over S. Since f is boundary holomorphic, the restriction to ∂S of the tangent bundle TS of S is naturally a complex line-subbundle of $f^*TW|_{\partial S}$. Then the normal bundle of $f(S)$ over γ can be identified with a complex line subbundle of $(TW, J)|f(\partial S)$ as well. Every trivialization ρ of this normal bundle defines as before a smooth manifold W_ρ. This manifold admits a natural almost complex structure extending the almost complex structure on the complement of a small tubular neighborhood of $f(\partial S)$ in W. In particular, if we denote as before by f_0 the natural extension of f to the closed surface S_0 then the pull-back bundle $f_0^*TW_\rho$ is a complex two-dimensional vector bundle over S_0.

Up to homotopy, this bundle only depends on ρ and the boundary class of f. Namely, any homotopy h_s of $f = h_0$ which is the identitity near the boundary induces a homotopy of the pull-back bundles $h_s^*TW_\rho$. Now if f_0, f_1 are the extensions of h_0, h_1 to the closed surface S_0 then since the homotopy h_s is the identity near the boundary, it determines a homotopy of the complex pull-back bundle $f_0^*TW_\rho$ to the complex pull-back bundle $f_1^*TW_\rho$. Let $c(\rho)$ be the evaluation on S_0 of the first Chern class of this bundle.

Changing the trivialization ρ by a full positive (negative) twist in the group $U(1) \subset GL(1,\mathbb{C})$ changes both the self-intersection number $\mathrm{Int}(f,\rho)$ and the Chern number $c(\rho)$ by 1 (-1) (see [McD91]). In particular, there is up to homotopy a unique trivialization ρ of the complex normal bundle of $f(S)$ over $f(\partial S)$ such that $c(\rho) = 2$. We call such a trivialization a *preferred* trivialization. By the above observation, a preferred trivialization only depends on the boundary class of f but not on the boundary regular homotopy class of f. Moreover, the complex normal bundle of a boundary holomorphic boundary regular map $f : S \to W$ only depends on the oriented boundary circle $f(\partial S)$.

Definition 2.5. The *self-intersection number* $\mathrm{Int}(f)$ of a boundary holomorphic boundary regular map $f : S \to W$ is the self-intersection number $\mathrm{Int}(f, \rho)$ of f with respect to a preferred trivialization ρ of the complex normal bundle of $f(S)$ over $f(\partial S)$.

Lemma 2.3 implies

Corollary 2.6. *Let $f : S \to W$ be boundary regular and boundary holomorphic. Then a boundary regular homotopy class in the boundary class of f is uniquely determined by its self-intersection number.*

Now consider the standard unit sphere S^3 in $\mathbb{C}^2$ which bounds the standard open unit ball $B_0^4 \subset \mathbb{C}^2$. The *contact distribution* on S^3 is the unique smooth two-dimensional subbundle ξ of TS^3 which is invariant under the (integrable) complex structure J on $\mathbb{C}^2$. A smooth embedding $\gamma : S^1 \to S^3$ is *transverse* if its tangent γ' is everywhere transverse to ξ. Let N be the outer normal field of S^3. Then JN is tangent to S^3 and orthogonal to ξ. The tangent of the transverse knot γ can be written in the form

$$\gamma'(t) = a(t)JN(\gamma(t)) + B(t)$$

where $B(t) \in \xi$ for all t and where $a(t) \neq 0$. Assume that γ is oriented in such a way that $a(t)$ is positive for all t. If as in the introduction we denote by λ be the restriction to S^3 of the radial one-form λ_0 on $\mathbb{C}^2$ then this orientation of γ is determined by the requirement that the evaluation of λ on γ' is positive, and we call it *canonical*. If the transverse knot γ is canonically oriented then $J\gamma'$ points inside the ball B_0^4. Thus if $B^4 = B_0^4 \cup S^3$ denotes the closed unit ball then for every canonically oriented transverse knot γ on S^3 there is a smooth boundary regular boundary holomorphic map $f : S \to B^4$ with $f(\partial S) = \gamma$ and $f^{-1}(S^3) = \partial S$ whose restriction to a neighborhood of the boundary is an embedding: Just choose a smooth embedding of a closed annulus A into B^4 which maps one of the boundary circles ζ of A diffeomorphically onto γ and whose tangent plane at a point in ζ is J-invariant. In particular, A meets S^3 transversely along γ and hence we may assume that $A \cap S^3 = \gamma$. Extend this embedding in an arbitrary way to a smooth map of the surface S (with the annulus A as a neighborhood of ∂S) into B^4 which is always possible since B^4 is contractible.

Define a boundary regular map $f : (S, \partial S) \to (B^4, S^3)$ to be *boundary transverse* if f is transverse to S^3 along the boundary. Since B^4 is contractible, the above observation implies that every boundary regular boundary transverse map $f : (S, \partial S) \to (B^4, \gamma)$ can be homotoped within the family of such maps to a boundary regular boundary holomorphic map $f' : (S, \partial S) \to (B^4, \gamma)$. In particular, the oriented normal bundle of f over γ is naturally homotopic to the oriented normal bundle of f' over γ. The map f' is used to calculate the preferred trivialization of this normal bundle. Then the self-intersection number $\mathrm{Int}(f)$ can be defined as the self-intersection number of f with respect to the induced trivialization of the oriented normal bundle of f. This self-intersection number coincides with the self-intersection number $\mathrm{Int}(f')$ of f' and by Corollary 2.6, it only depends on f.

More generally, let Σ be the boundary of a bounded domain $\Omega \subset \mathbb{C}^2$ which contains the origin 0 in its interior and which is star-shaped with respect to 0. The contact form is the restriction λ to Σ of the radial one-form λ_0 on $\mathbb{C}^2$.

Let N be the outer normal field of $\Sigma \subset \mathbb{C}^2$. Since the one-form λ_0 can also be written in the form $(\lambda_0)_p(Y) = \frac{1}{2}\langle Jp, Y\rangle$ ($p \in \mathbb{C}^2, Y \in T_p\mathbb{C}^2$ and where $\langle , \rangle$ is the euclidean inner product), the *Reeb vector field* X on Σ is given by

$$X(p) = \varphi(p)JN(p)$$

where

$$\varphi(p) = \frac{2}{\langle p, N(p)\rangle} > 0.$$

Namely, for $p \in \Sigma$ we have

$$d\lambda_p(X, \cdot) = \varphi(p)\omega_0(JN(p), \cdot) = -\varphi(p)\langle N(p), \cdot\rangle = 0$$

on $T_p\Sigma$ and

$$\lambda_p(X) = \frac{1}{2}\langle Jp, X\rangle = \frac{1}{2}\varphi(p)\langle Jp, JN(p)\rangle = 1.$$

In particular, if γ is a Reeb orbit on Σ then $J\gamma'$ is transverse to Σ and points inside the domain Ω. As a consequence, as for transverse knots on S^3, if we denote by $C = \Omega \cup \Sigma$ the closure of Ω then for every boundary regular boundary transverse map $f : S \to C$ whose boundary $f(\partial S)$ is a periodic Reeb orbit on Σ, the self-intersection number $\mathrm{Int}(f)$ of f is well defined.

The next lemma shows that in both cases, the self-intersection number of such a map $f : (S, \partial S) \to (C, \gamma)$ only depends on γ. For a convenient formulation, let C be the closure of a bounded star-shaped domain in $\mathbb{C}^2$ and call a smoothly embedded closed curve γ in the boundary Σ of C *admissible* if either C is the unit ball, $\Sigma = S^3$ and γ is a canonically oriented transverse knot or if γ is a Reeb orbit on Σ.

Lemma 2.7. *Let $\gamma \subset \Sigma$ be an admissible curve. Then any two boundary regular boundary transverse maps $f : (S, \partial S) \to (C, \gamma), g : (S', \partial S') \to (C, \gamma)$ have the same self-intersection number.*

Proof. Let $f : S \to C, g : S' \to C$ be any two boundary regular boundary transverse maps with boundary an admissible closed curve γ. The inner normals of the surfaces $f(S), g(S')$ along $\gamma = f(\partial S) = g(\partial S')$ point strictly inside the domain C. After a small deformation through boundary regular boundary transverse maps we may assume that there is a small annular neighborhood A of the boundary of S, an annular neighborhood A' of the boundary of S' and a homeomorphism $\varphi : A \to A'$ which maps ∂S to $\partial S'$ and is such that $g(\varphi(x)) = f(x)$ for all $x \in A$. We may moreover assume that the restrictions of f, g to $f^{-1}(A), g^{-1}(A')$ are embeddings. Since f, g are boundary regular, after possibly modifying f, g once more with a small boundary regular homotopy which pushes interior intersection points of $f(S), g(S')$ with Σ into the interior Ω of C we may assume that there is a compact star-shaped set $K \subset \Omega$ such that $f(S - A) \subset K, g(S' - A) \subset K$. But then the restrictions of f, g to $S - A, S' - A'$ are maps of surfaces $S - A, S' - A'$ into K with the same boundary curve γ'. Now K is contractible and hence the maps $f|S - A, g|S' - A'$ define the same relative homology class in $H_2(K, \gamma'; \mathbb{Z})$. By Lemma 2.3 and its proof, this implies that the self-intersection numbers of f, g indeed coincide. $\square$

As a consequence, we can define:

Definition 2.8. Let $\gamma \subset \Sigma$ be an admissible curve. The *self-intersection number* $\mathrm{Int}(\gamma)$ of γ is the self-intersection number of a boundary regular boundary transverse map $f : S \to C$ with boundary $f(\partial S) = \gamma$.

The final goal of this section is to calculate the self-intersection number of an admissible curve γ on Σ. For this we begin with calculating the preferred trivialization of the normal bundle of the complex line subbundle of $T\mathbb{C}^2|\gamma$ spanned by the tangent γ' of γ. Note that this normal bundle can naturally be identified with the restriction of the contact bundle ξ to γ.

To this end let D be the closed unit disc in $\mathbb{C}$ and let $f : D \to C$ be a boundary holomorphic boundary regular *immersion* which maps ∂D diffeomorphically onto the canonically oriented admissible curve γ. The image under df of the inner normal of D along ∂D points strictly inside C. Let

$$\hat{M} : (z_1, z_2) \to (-\overline{z}_2, \overline{z}_1)$$

be a J-orthogonal $\langle \cdot, \cdot \rangle$-compatible almost complex structure on $\mathbb{C}^2$ where as usual, $z \to \overline{z}$ is complex conjugation. Then we obtain a trivialization of the complex vector bundle $(f^*T\mathbb{C}^2, J)$ over D by the sections $X_1 = df(\frac{\partial}{\partial x}), X_2 = \hat{M} df(\frac{\partial}{\partial x})$ (with a slight abuse of notation).

The trivialization of the tangent bundle $df(TD)|\gamma$ of $f(D)$ over γ defined by the tangent γ' of the admissible curve γ has rotation number one with respect to the trivialization $df(\frac{\partial}{\partial x})$. Since the tangent bundle of the two-sphere S^2 has Chern number 2 and is obtained by glueing the tangent bundles of two standard discs D_1, D_2 along the boundary using the trivializations defined by the tangent field of the boundary, the restriction of the section X_2 to γ defines the preferred trivialization of the complex normal bundle L over γ. Now $\hat{M}$ is complex anti-linear and therefore the trivialization of L over γ defined by the section $\hat{M} \circ \gamma'$ has rotation number -1 with respect to the preferred trivialization. Recall that the preferred trivialization of the complex normal bundle of γ only depends on the boundary class of an infinitesimally holomorphic immersion of a surface S into $\mathbb{C}^2$.

To the admissible curve γ we can also associate its *self-linking number* $\mathrm{lk}(\gamma)$ (see [Eli92] and the introduction). The following proposition relates these two numbers.

Proposition 2.9. *Let γ be an admissible curve on Σ. Then the self-intersection number* $\mathrm{Int}(\gamma)$ *of γ equals* $\mathrm{lk}(\gamma) + 1$.

Proof. Let N be the outer normal field of Σ and let L be the *complex subbundle* of $T\Sigma$, i.e. the 2-dimensional subbundle which is invariant under the complex structure J. If $\Sigma = S^3$ then this is just the contact bundle. The image of the outer normal N of Σ under the J-orthogonal $\langle , \rangle$-compatible almost complex structure $\hat{M}$ is a global section of the bundle L. Let $F \subset \Sigma$ be a *Seifert surface* for the admissible curve γ, i.e. F is an embedded oriented bordered surface in Σ with boundary γ. Let N_F be the oriented normal field of F in Σ with respect to the restriction of the euclidean metric $\langle , \rangle$. Since γ is transverse to L we may assume that for every $x \in \gamma$ the vector $N_F(x)$ is contained in the fibre L_x at x of the complex line bundle

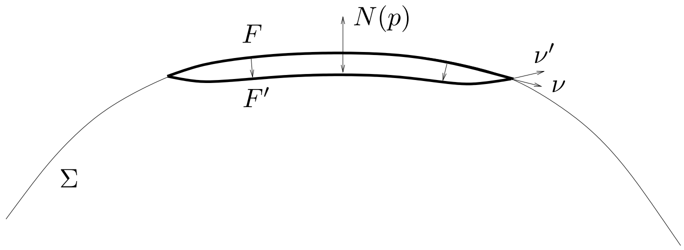

FIGURE 1. The surfaces F and F'

L. The self-linking number of γ is therefore the winding number of the section $x \to M(x) = \hat{M}N(x)$ of $L|\gamma$ with respect to the trivialization of $L|\gamma$ defined by the section $x \to N_F(x)$.

Let $F' \subset C$ be the embedded surface which we obtain by pushing F slightly in the direction $-N$ as in Figure 1. Then F' is an embedded surface in C which is boundary regular and boundary transverse. The restriction of N_F to γ extends to a global trivialization of the oriented normal bundle of the surface F and hence F' in $\mathbb{C}^2 - \gamma$. Thus the self-intersection number of F' with respect to the trivialization defined by N_F vanishes. Since by the above observation the winding number of the section M of $L|\gamma$ with respect to the preferred trivialization of $L|\gamma$ equals -1, the self-intersection number $\mathrm{Int}(\gamma)$ equals the winding number of M with respect to the trivialization of $L|\gamma$ defined by N_F plus one. This shows the proposition. □

3. TOPOLOGICAL INVARIANTS OF IMMERSED DISCS

As in Section 2, we denote by S a compact oriented surface with connected boundary $\partial S = S^1$. Let (W, J) be a smooth simply connected 4-dimensional manifold equipped with a smooth almost complex structure J. In this section we investigate topological invariants of boundary holomorphic boundary regular *immersions* $f : S \to (W, J)$. For this we use the assumptions and notations from Section 2. Recall in particular the definition of the self-intersection number $\mathrm{Int}(f)$ of f.

Let $G(2,4)$ be the Grassmannian of oriented (real) 2-planes in $\mathbb{R}^4 = \mathbb{C}^2$. This Grassmannian is just the homogeneous space $G(2,4) = SO(4)/SO(2) \times SO(2) = S^2 \times S^2$, in particular the second homotopy group of $G(2,4)$ coincides with its second homology group and is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Moreover, there are generators $\tilde{e}_1, \tilde{e}_2$ of $H_2(G(2,4); \mathbb{Z})$ such that with respect to these generators, the homological intersection form ι is the symmetric form represented by the $(2,2)$-matrix $(a_{i,j})$ with $a_{1,1} = a_{2,2} = 0$ and $a_{1,2} = a_{2,1} = \iota(\tilde{e}_1, \tilde{e}_2) = 1$.

The complex projective line $\mathbb{C}P^1 = S^2$ of *complex* oriented lines in $\mathbb{C}^2$ for the standard complex structure J is naturally embedded in $G(2,4)$. Its homotopy class is the generator of an infinite cyclic subgroup Z_1 of $\pi_2(G(2,4))$. We call this generator the *canonical generator* of Z_1 and denote it by e_1. The anti-holomorphic sphere of all complex oriented lines for the complex structure $-J$ is homotopic and hence homologous in $G(2,4)$ to the complex projective line $\mathbb{C}P^1$. Namely, the complex structures $J, -J$ define the same orientation on $\mathbb{R}^4$ and hence J can be connected to $-J$ by a continuous curve of linear complex structures on $\mathbb{R}^4$. This curve then determines a homotopy of $\mathbb{C}P^1$ onto the anti-holomorphic sphere of complex oriented lines for $-J$. Since these two spheres are disjoint, the self-intersection number of the class in $H_2(G(2,4);\mathbb{Z})$ defined by e_1 vanishes.

A second infinite cyclic subgroup Z_2 of $\pi_2(G(2,4))$ is defined as follows. Let $S^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4$ be the standard unit sphere. The map which associates to a point $y \in S^2$ the oriented tangent plane of S^2 at y, viewed as a 2-dimensional oriented linear subspace of $\mathbb{R}^4$, defines a smooth map of S^2 into $G(2,4)$. Its homotopy class e_2 generates a subgroup Z_2 of $\pi_2(G(2,4))$. We call e_2 the canonical generator of Z_2. Now the tangent bundle of $S^2 \subset \mathbb{R}^3 \subset \mathbb{C}^2$ intersects the complex projective line $\mathbb{C}P^1 \subset G(2,4)$ in precisely one point (which is the tangent space of S^2 at $(0,0,1,0)$). This intersection is transverse with positive intersection index. Therefore we have $\iota(e_1,e_2)=1$ and hence the elements e_1,e_2 generate $\pi_2(G(2,4))$.

Let W be a simply connected 4-dimensional manifold with smooth almost complex structure J. Equip the tangent bundle TW of W with a J-invariant Riemannian metric $\langle,\rangle$. Let $\mathcal{G} \to W$ be the smooth fibre bundle over W whose fibre at a point $x \in W$ consists of the Grassmannian of oriented 2-planes in T_xW. Let S be a compact oriented surface with connected boundary $\partial S \sim S^1$. Then every smooth boundary regular *immersion* $f : S \to W$ defines a smooth map Gf of S into the bundle $\mathcal{G}$ by assigning to a point $x \in S$ the oriented tangent space $df(T_xS)$ of $f(S)$ at $f(x)$. The complex pull-back bundle (f^*TW, J) over S admits a *complex* trivialization. This trivialization can be chosen to be of the form $df(X), V$ where X is a global nowhere vanishing section of the tangent bundle TS of S and V is a global section of the $\langle,\rangle$-orthogonal complement of the complex line subbundle of f^*TW which is spanned by the section $df(X)$. With respect to this complex trivialization of f^*TW, the pull-back $f^*\mathcal{G}$ of the bundle $\mathcal{G}$ can naturally be represented as a product $S \times G(2,4)$. If f is boundary holomorphic, i.e. if for every $z \in \partial S$ the tangent space $df(T_zS) \subset T_{f(z)}W$ is J-invariant and if moreover its orientation coincides with the orientation induced by J, then in the above identification of $f^*\mathcal{G}$ with $S \times G(2,4)$ the circle of tangent planes of $f(S)$ over $f(\partial S)$ is given by the curve $\partial S \times L_0$ in $\partial S \times G(2,4)$ where $L_0 = \mathbb{C} \times \{0\} \subset \mathbb{C}^2$ is a fixed complex line. Thus in this case the map Gf can be viewed as a smooth map of the surface S into the Grassmannian $G(2,4)$ which maps the boundary ∂S of S to the single complex line L_0. In other words, if we let $\tilde{S}$ be the closed oriented surface obtained from S by collapsing ∂S to a point then Gf defines a map of $\tilde{S}$ into $G(2,4)$. This map then defines a homotopy class of maps $\tilde{S} \to G(2,4)$ and a homology class $[Gf] \in H_2(G(2,4),\mathbb{Z})$.

The following definition strengthens Definition 2.2.

Definition 3.1. A smooth homotopy $h : [0,1] \times S \to W$ is *regular* if h is boundary regular and if moreover for every $s \in [0,1]$ the map h_s is a boundary holomorphic immersion.

If $f, g : S \to W$ are two boundary regular boundary holomorphic immersions which are regularly homotopic, i.e. which can be connected by a regular homotopy, then the maps Gf and Gg are homotopic. Namely, if $h : [0,1] \times S \to W$ is a regular homotopy connecting $h_0 = f$ to $h_1 = g$, then there is a complex trivialization of the complex pull-back bundle (h^*TW, J) over $[0,1] \times S$ whose restriction to $[0,1] \times \partial S$ does not depend on $s \in [0,1]$ and is determined as before by the section $dh_s(X)$ where X is a global nowhere vanishing section of TS. This trivialization then defines an identification of the bundle $h^*\mathcal{G}$ with $[0,1] \times S \times G(2,4)$. For each $s \in [0,1]$ the tangent planes of the immersion h_s define a smooth section of the bundle $h^*\mathcal{G}$ over $\{s\} \times S$ and hence a smooth map of S into $G(2,4)$. This map depends smoothly on s and maps the boundary ∂S of S to a single point. Thus by continuity, the homotopy class of the tangent map of h_s is independent of $s \in [0,1]$ and hence it is an invariant of regular homotopy.

There is another way to obtain an invariant of regular homotopy.

Definition 3.2. The *tangential index* $\tan(f)$ of a boundary regular immersion f whose only self-intersection points are transverse double points is defined to be the number of self-intersection points of f counted with signs.

If a boundary regular immersion $f : S \to W$ has self-intersection points which are not transverse double points then it can be perturbed with a regular homotopy to an immersion whose only self-intersection points are transverse double points and whose tangential index is independent of the perturbation (see e.g. [McD91]). The tangential index is invariant under regular homotopy.

If we consider more specifically boundary regular boundary holomorphic immersions of *discs* then we can derive a more precise result. For this recall from Section 2 that for every boundary regular boundary holomorphic map $f : D \to W$ there is a preferred trivialization of the normal bundle of $f(D)$ over $f(\partial D)$. On the other hand, there is a trivialization N of the oriented normal bundle of $f(D)$ over $f(\partial D)$ which extends to a global trivialization of the oriented normal bundle of $f(D)$ in TW.

Definition 3.3. The *winding number* $\mathrm{wind}(f)$ of a boundary regular boundary holomorphic immersion $f : D \to W$ is the winding number of the preferred trivialization of the normal bundle of $f(D)$ over $f(\partial D)$ with respect to a trivialization which extends to a global trivialization of the normal bundle of $f(D)$.

For the formulation of the following version of the well known *adjunction formula* for immersed boundary regular boundary holomorphic discs, denote for a boundary regular boundary holomorphic immersion $f : D \to W$ by $\mathcal{C}_2(Gf)$ the component of $[Gf]$ in the subgroup Z_2 of $H_2(G(2,4), \mathbb{Z})$, viewed as an integer.

Proposition 3.4. *For a boundary regular boundary holomorphic immersion* $f : D \to W$ *we have* $\mathrm{Int}(f) = \mathrm{wind}(f) + 2\tan(f)$*, and* $\mathrm{wind}(f) = 2\mathcal{C}_2(Gf)$*.*

Proof. Let $f : D \to W$ be a boundary regular boundary holomorphic immersion. As in Section 2, let ρ be the preferred trivialization of the complex normal bundle of $f(D)$ over $f(\partial D)$ and use this trivialization to extend f to an immersion f_0 of the two-sphere S^2 into the almost complex manifold $(W_\rho, \tilde{J})$. Then $\mathrm{Int}(f)$ is the self-intersection number of $f_0(S^2)$ in W_ρ. Since f_0 is an immersion, this self-intersection number just equals $\chi(N) + 2\tan(f_0)$ where $\chi(N)$ is the Euler number of the normal bundle of $f_0(S^2)$ in W_ρ and where $\tan(f_0) = \tan(f)$ is the tangential index defined above (see e.g. Lemma 4.2 of [McD91] or simply note that the formula is obvious if f_0 is an embedding and follows for immersions with only transverse double points by surgery at every double self-intersection point which increases the Euler class of the normal bundle by 2 if the double point has positive index and decreases it by 2 if the double point has negative index). By our definition of the winding number $\mathrm{wind}(f)$ of f, this is just the formula stated in the proposition.

To show that $\mathrm{wind}(f) = 2\mathcal{C}_2(Gf)$, note first that we have $\mathrm{wind}(f) = 0$ if $[Gf] \in Z_1$. Namely, using the above notations, recall that a preferred trivialization ρ of the normal bundle of the disc $f(D)$ over $f(\partial D) = \gamma$ is determined by the requirement that the evaluation of the first Chern class of the complex tangent bundle $(TW_\rho, \tilde{J})$ of W_ρ on the 2-sphere $f_0(S^2)$ equals two.

The tangent plane map of f can be viewed as a map $(D, \partial D) \to G(2,4)$ which maps the boundary ∂D of D to a single point and hence factors through a map $F : S^2 \to G(2,4)$. If $[Gf] \in Z_1$ then since $\pi_2(G(2,4)) = H_2(G(2,4), \mathbb{Z})$, the map F can be homotoped to a map $S^2 \to \mathbb{C}P^1$. By construction, this implies that up to homotopy, the complex vector bundle $(f_0^*TW, \tilde{J})$ decomposes as a direct sum $TS^2 \oplus N$ of two complex line bundles. The first Chern class of $(f_0^*TW, \tilde{J})$ is then the sum of the Chern classes of TS^2 and N. Therefore by our normalization, the first Chern class of the normal bundle $N = f_0^*TW/TS^2 \to S^2$ vanishes. As a consequence, the bundle $N \to S^2$ is trivial and hence the preferred trivialization of the normal bundle N over γ extends to a global trivialization of N over S^2. This shows that $\mathrm{wind}(f) = 0$ if $[Gf] \in Z_1$.

Arguing as in the proof of Lemma 2.3, if $g : D \to W$ is any boundary regular boundary holomorphic immersion with $\mathcal{C}_2(Gg) = ke_2$ for some $k \in \mathbb{Z}$ then there is a boundary regular boundary holomorphic immersion $f : D \to W$ with $f(\partial D) = g(\partial D)$, $[Gf] = [Gg] - ke_2 \in Z_1$ and such that $\mathrm{Int}(f) = \mathrm{Int}(g)$, $\tan(f) = \tan(g) + k$ and $\mathrm{wind}(f) = \mathrm{wind}(g) - 2k$. Namely, such an immersion f can be constructed as follows. Choose a point $z \in D$ such that there is a small ball $V \subset W$ about $g(z)$ which intersects $g(D)$ in an embedded disc B containing $g(z)$. Choose an embedded 2-sphere $\hat{S} \subset \partial V$ which intersects B transversely in precisely two points, one with positive and one with negative intersection index. The tangent bundle of the sphere is a generator e_2 of the subgroup Z_2 of $G(2,4)$. As in the proof of Lemma 2.3, attaching the sphere to $g(D)$ with surgery about the intersection point with positive intersection index results in a disc u which satisfies $\mathrm{Int}(u) = \mathrm{Int}(g)$, $[Gu] = [Gg] + e_2$ and $\tan(u) = \tan(g) - 1$. Similarly, attaching the sphere to $g(D)$ with surgery about the intersection point with negative intersection index results in a disc u' which satisfies $\mathrm{Int}(u') = \mathrm{Int}(g)$, $[Gu'] = [Gg] - e_2$ and $\tan(u') = \tan(g) + 1$. From this the proposition is immediate. □

A *complex point* of an immersed disc $f : D \to W$ is a point $z \in D$ such that the real two-dimensional subspace $df(T_zD)$ of TW is invariant under the almost complex structure J. The point is called *holomorphic* if the orientation of $df(T_zD)$ induced by the orientation of D coincides with the orientation induced by the almost complex structure J, and it is called *anti-holomorphic* otherwise.

Corollary 3.5. *Let $f : D \to W$ be a boundary regular boundary holomorphic immersion. If f does not have any anti-holomorphic points then* $\mathrm{Int}(f) = 2\mathrm{tan}(f)$.

Proof. Let $f : D \to W$ be a boundary regular boundary holomorphic immersion without any anti-holomorphic point. Then the tangent map Gf of f does not intersect the anti-holomorphic sphere of complex lines in $\mathbb{C}^2$ equipped with the reverse of the orientation induced by the complex structure. Since the anti-holomorphic sphere is homologous in $G(2,4)$ to the complex projection line $\mathbb{C}P^1$ and has vanishing self-intersection (see the discussion at the beginning of this section), we have $[Gf] \in Z_1$ by consideration of intersection numbers. The corollary now is immediate from Proposition 3.4. □

As in the introduction, denote by ω_0 the standard symplectic form on $\mathbb{C}^2$. An immersion $f : D \to \mathbb{C}^2$ is called symplectic if for every $z \in D$ the restriction of $f^*\omega_0$ to the tangent plane T_zD does not vanish and defines the standard orientation of T_zD. As a consequence of Corollary 3.5 we obtain Theorem 1 and the corollary from the introduction.

Corollary 3.6. *Let γ either be a transverse knot on the standard three-sphere, the boundary of the standard unit ball $C \subset \mathbb{C}^2$, or a Reeb orbit on the boundary Σ of a domain in $\mathbb{C}^2$ which is star-shaped with respect to the origin, with compact closure C. If γ bounds a boundary regular immersed symplectic disc $f : (D, \partial D) \to (C, \gamma)$ then* $\mathrm{lk}(\gamma) = 2\mathrm{tan}(f) - 1$.

Proof. By definition, a symplectic immersion does not have any anti-holomorphic points. Thus if $f : (D, \partial D) \to (C, \gamma)$ is a boundary regular boundary holomorphic immersed symplectic disc then $\mathrm{lk}(\gamma) = 2\mathrm{tan}(f) - 1$ by Proposition 2.9 and Corollary 3.5.

Now if $f : (D, \partial D) \to (C, \gamma)$ is an arbitrary boundary regular immersed symplectic disc then f can be slightly modified with a smooth homotopy to a boundary transverse symplectic disc without changing the tangential index since being symplectic is an open condition. Locally near the boundary, this disc can be represented as a graph over an embedded symplectic annulus $A \subset C$ with γ as one of its boundary components whose tangent plane is J-invariant at every point in γ. Now for a fixed nonzero vector $X \in T\mathbb{C}^2$, the set of all nonzero vectors $Y \in T\mathbb{C}^2$ such that $\omega_0(X, Y) > 0$ is convex and hence contractible and therefore locally near the boundary this graph can be deformed to a graph which coincides with the annulus A near γ and hence is a boundary regular and boundary holomorphic immersed symplectic disc with boundary γ whose tangential index coincides with the tangential index of f. □

4. Boundaries of compact convex bodies with controlled curvature

In this section we investigate periodic Reeb orbits on the boundary Σ of a compact strictly convex body $C \subset \mathbb{C}^2$. Our main goal is the proof of Theorem 2 from the introduction.

We begin with observing that Corollary 3.6 can be applied to periodic Reeb orbits on boundaries of compact convex bodies.

Lemma 4.1. *Let γ be a periodic Reeb orbit on Σ. Then there is a boundary regular symplectic immersion $f : (D, \partial D) \to (C, \gamma)$.*

Proof. Let γ be a periodic Reeb orbit on the boundary Σ of a compact strictly convex body $C \subset \mathbb{C}^2$. Choose two distinct points $a \neq b$ on γ and smooth parametrizations $\gamma_1, \gamma_2 : [0, \pi] \to \gamma$ of the two subarcs of γ connecting a to b. We assume that the orientation of γ_2 coincides with the orientation of γ and that the parametrizations γ_1, γ_2 coincide near a, b with the parametrization of γ up to translation and reflection in the real line. Define a map $f : (D, \partial D) \to (C, \gamma)$ as follows. Let $\tilde{\gamma}_1, \tilde{\gamma}_2 : [0, \pi] \to S^1$ be parametrizations by arc length of the two half-circles of the unit circle $S^1 \subset \mathbb{C}$ connecting 1 to -1, chosen in such a way that the orientation of $\tilde{\gamma}_2$ coincides with the orientation of ∂D. We require that f maps the line segment in D connecting $\tilde{\gamma}_1(t)$ to $\tilde{\gamma}_2(t)$ which is parametrized by arc length to the line segment in the convex body $C \subset \mathbb{C}^2$ connecting $\gamma_1(t)$ to $\gamma_2(t)$ and parametrized propotional to arc length on the same parameter interval. By construction, the map f is smooth, moreover it is symplectic near the points $1, -1$.

We claim that f is a symplectic immersion. For this let as before $\langle , \rangle$ be the usual euclidean inner product on $\mathbb{R}^4 = \mathbb{C}^2$. Let $t \in (0, \pi)$ and consider the straight line segment ℓ in C connecting $\gamma_1(t)$ to $\gamma_2(t)$. By strict convexity of C, the arc ℓ is contained in C and intersects Σ transversely at the endpoints. Let X, Y be the tangents of ℓ at the endpoints $\gamma_1(t), \gamma_2(t)$ and let as before N be the outer normal field of Σ. Then $\langle X, N(\gamma_1(t))\rangle < 0, \langle Y, N(\gamma_2(t))\rangle > 0$ and hence since $\gamma_1'(t) = -a_1 JN(\gamma(t)), \gamma_2'(t) = a_2 JN(\gamma_2(t))$ for some numbers $a_1 > 0, a_2 > 0$ we have $\omega_0(X, \gamma_1'(t)) > 0$ and $\omega_0(Y, \gamma_2'(t)) > 0$. Now with respect to the usual trivialization of $T\mathbb{C}^2$ we have $X = Y$. On the other hand, by the construction of the map f, for every point $s \in \ell$ the tangent space of $f(D)$ at s is spanned by $X = Y$ and a convex linear combination of $\gamma_1'(t), \gamma_2'(t)$. This shows that f is a symplectic immersion. Moreover f is clearly boundary regular whence the lemma. $\square$

We call an immersion $f : D \to C$ as in Lemma 4.1 a *linear filling* of the Reeb orbit γ. By Corollary 3.6, if $\mathrm{lk}(\gamma) = -1$ then a linear filling f of γ satisfies $\tan(f) = 0$. However, an immersed symplectic disc may have transverse self-intersection points of negative intersection index, so there is no obvious relation between the tangential index of a boundary regular immersed symplectic disc and the number of its self-intersection points. On the other hand, if γ admits an *embedded* linear filling then Corollary 3.6 implies that $\mathrm{lk}(\gamma) = -1$.

Our final goal is to relate the Maslov index of a periodic Reeb orbit γ to the geometry of the hypersurface Σ. For this consider for the moment an arbitrary bounded domain $\Omega \subset \mathbb{C}^2$ with smooth boundary Σ which is star-shaped with respect to the origin. Write $C = \Omega \cup \Sigma$. As before, denote by J the usual complex structure on $\mathbb{C}^2$ and let $\langle , \rangle$ be the *euclidean* inner product. The restriction λ of the radial one-form λ_0 on $\mathbb{C}^2$ defined by $(\lambda_0)_p(Y) = \frac{1}{2}\langle Jp, Y\rangle$ ($p \in \mathbb{C}^2, Y \in T_p\mathbb{C}^2$) defines a smooth contact structure on Σ.

Let N be the outer unit normal field of Σ. As in Section 2 write $\hat{M}(z_1, z_2) = (-\bar{z}_2, \bar{z}_1)$ and let M be the section of $T\Sigma$ defined by $M(p) = \hat{M} \circ N$. Its image is contained in the complex line subbundle L of the tangent bundle of Σ. The sections M, JM define a global trivialization of L which is symplectic with respect to the restriction of the symplectic form ω_0.

The kernel ξ of the contact form is a smooth real 2-dimensional subbundle of $T\Sigma$. Orthogonal projection P of $T\Sigma$ onto L defines a smooth bundle epimorphism whose kernel is the annihilator of the restriction of ω_0 to $T\Sigma$. Thus the morphism P preserves the restriction to $T\Sigma, L$ of the symplectic form and therefore its restriction to the subbundle ξ of $T\Sigma$ is a real symplectic bundle isomorphism. Its inverse $\pi : L \to \xi$ is a symplectic bundle morphism as well. Since by construction the sections M, JM of L form a symplectic basis of L we have

Lemma 4.2. *The smooth sections $\pi \circ M, \pi \circ JM$ of the bundle ξ define a symplectic trivialization $T : \xi \to (\mathbb{R}^2, dx \wedge dy)$.*

In other words, for each $p \in \Sigma$ the restriction T_p of T to ξ_p is an area preserving linear map $T_p : (\xi_p, \omega_0) \to (\mathbb{R}^2, dx \wedge dy)$.

Recall from Section 2 that the Reeb vector field X on Σ is given by

$$X(p) = \varphi(p) JN(p)$$

where

$$\varphi(p) = \frac{2}{\langle p, N(p) \rangle} > 0.$$

Denote by $\Psi_t : \Sigma \to \Sigma$ the Reeb-flow of (Σ, λ) and let γ be a periodic orbit for Ψ_t of period $\chi > 0$. Using the above trivialization T of the bundle ξ we obtain a curve $\Phi : [0, T] \to SL(2, \mathbb{R})$ with $\Phi(0) = \mathrm{Id}$ by defining

$$\Phi(t) := T_{\Psi_t(p)} \circ d\Psi_t(p) \circ T_p^{-1}.$$

where $p = \gamma(0)$. If the curve Φ is *non-degenerate*, which means that $\Phi(T)$ does not have one as an eigenvalue, then the Maslov index $\mu(\gamma)$ of γ is defined as the *μ-index* $\mu(\Phi)$ of the curve Φ as defined in [HWZ95] .

To estimate the μ-index of Φ define for a unit vector $X \in S^1 \subset \mathbb{R}^2$ the *rotation* of X with respect to the curve Φ as the total rotation angle $\mathrm{rot}(\Phi, X)$ (or the total winding) of the curve

$$t \to \frac{\Phi(t)X}{\|\Phi(t)X\|} \in S^1.$$

The following lemma is valid for *any* path in $SL(2, \mathbb{R})$ beginning at the identity. It uses an extension of the Maslov index to degenerate paths which is given in the proof of the lemma.

Lemma 4.3. *Let $c : [0, \chi] \to SL(2, \mathbb{R})$ be a continuous arc with $c(0) = \mathrm{Id}$. Then $\mathrm{rot}(c, X) < (\mu(c) + 1)\pi$ for every $X \in S^1$.*

Proof. We follow [RS93]. Assume that $\mathbb{R}^2$ is equipped with the standard symplectic form.

In standard euclidean coordinates let $V = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$. The *Maslov cycle* determined by V is just V, viewed as a point in the real projective line $\mathbb{R}P^1$ of all one-dimensional subspaces of $\mathbb{R}^2$. A *crossing* of a smooth curve $\Lambda : [a,b] \to \mathbb{R}P^1$ is a number $t \in [a,b]$ such that $\Lambda(t) = V$. Then locally near t, we can write $\Lambda(t) = \{x + A(s)x\}$ where $A(s) : V \to V^\perp$ is linear and vanishes for $s = t$ (and where $V^\perp$ is the euclidean orthogonal complement of V in $\mathbb{R}^2$). With respect to the standard basis of $\mathbb{R}^2 = V \oplus V^\perp$ we can view $s \to A(s)$ as a real valued function. With this interpretation, the crossing is *non-degenerate* if $A'(t) \neq 0$. The *sign* $\operatorname{sign}\Gamma(\Lambda, V, t)$ of the crossing point t then equals the sign of $A'(t)$ (p.830 of [RS93]). The *Maslov index* of the curve $\Lambda : [a,b] \to \mathbb{R}P^1$ with only non-degenerate crossings is then defined to be

$$\mu(\Lambda, V) = \frac{1}{2}\operatorname{sign}\Gamma(\Lambda, V, a) + \sum_{a<t<b} \operatorname{sign}\Gamma(\Lambda, V, t) + \frac{1}{2}\operatorname{sign}\Gamma(\Lambda, V, b)$$

(see p.831 of [RS93]).

A smooth path $\Lambda : [a,b] \to \mathbb{R}P^1$ with $\Lambda(a) = V$ and only non-degenerate crossings lifts to a smooth path $\tilde{\Lambda} : [a,b| \to S^1 \subset \mathbb{C} = \mathbb{R}^2$ beginning at $(1,0)$. Crossings of Λ are precisely those points $t \in [a,b]$ where $\tilde{\Lambda}(t) = (\pm 1, 0)$, and the sign of the crossing is the sign of the derivative of $\tilde{\Lambda}$ with respect to the usual counter-clockwise orientation of S^1. As a consequence, we have $\mu(\Lambda, V) = p$ for $p \in \mathbb{Z}$ precisely if $\operatorname{rot}(\tilde{\Lambda}) = p\pi$, and $\mu(\Lambda, V) = p + \frac{1}{2}$ precisely if $\operatorname{rot}(\tilde{\Lambda}) \in (p\pi, (p+1)\pi)$. Here we write $\operatorname{rot}(\tilde{\Lambda})$ to denote the total rotation of the path $\tilde{\Lambda}$ in S^1.

For a curve $c : [0,\chi] \to SL(2,\mathbb{R})$, the Maslov index $\mu(c, V)$ of c with respect to V is defined to be the Maslov index of the curve $\Lambda : [0,\chi] \to \mathbb{R}P^1, t \to \Lambda(t) = c(t)V$ [RS93] and hence $\operatorname{rot}(c, (1,0)) < (\mu(c,V) + \frac{1}{2})\pi$.

The above definition of a Maslov index for paths in $SL(2,\mathbb{R})$ depends on the choice of the linear subspace V (though this dependence can be removed by observing that any path in $SL(2,\mathbb{R})$ determines a path of Lagrangian subspaces in $\mathbb{R}^4$, see [RS93]). To obtain an index for paths $c : [0,\chi] \to SL(2,\mathbb{R})$ beginning at $c(0) = \mathrm{Id}$ which does not depend on such a choice we proceed as follows.

A path $c : [0,\chi] \to SL(2,\mathbb{R})$ defines a path α in the space of orientation preserving homeomorphisms of S^1 by $\alpha(s)(X) = c(s)X/\|c(s)X\|$ $(s \in [0,\chi], X \in S^1)$. For each s and each X we have $\alpha(s)(-X) = -\alpha(s)(X)$. This implies that $|\operatorname{rot}(c,X) - \operatorname{rot}(c,Y)| < \pi$ for any two points $X, Y \in S^1$. Now if there is some $p \in \mathbb{Z}$ such that $\operatorname{rot}(c,X) \in (2p\pi, 2(p+1)\pi)$ for all $X \in S^1$ then we define $\mu(c) = 2p+1$. By continuity, otherwise there is some $X \in S^1$ and some $p \in \mathbb{Z}$ with $\operatorname{rot}(c,X) = 2p\pi$. By the above discussion, the number p is unique and we define $\mu(c) = 2p$. With this definition of a Maslov index for paths in $SL(2,\mathbb{R})$ beginning at the identity, the statement of the lemma is obvious.

The fundamental group of $SL(2,\mathbb{R})$ with basepoint the identity is infinitely cyclic and generated by the loop $t \to e^{2\pi i t}$ (viewed as a loop in $U(1) \subset SL(2,\mathbb{R})$). In particular, there is a natural group isomorphism $\rho : \pi_1(SL(2,\mathbb{R})) \to \mathbb{Z}$. We claim that the Maslov index defined in the previous paragraph has the following properties.

(1) $\mu(c)$ only depends on the homotopy class of c with fixed endpoints.
(2) If $\alpha \in \pi_1(SL(2,\mathbb{R}))$ and $c : [0,1] \to SL(2,\mathbb{R})$ is any arc then

$$\mu(\alpha \cdot c) = 2\rho(\alpha) + \mu(c).$$

(3) $\mu(c^{-1}) = -\mu(c)$.
(4) For a constant invertible symmetric matrix S with $\|S\| < 2\pi$ and for $c(t) = \exp tJS$ $(t \in [0,1])$ we have

$$\mu(c) = \frac{1}{2}\text{signature}(S).$$

To see that these properties indeed hold true, note that the first statement is immediate from the definition. By definition, the Maslov index of the standard rotation $\alpha : t \to e^{2\pi i t} \in U(1)$ equals $\mu(\alpha) = 2 = 2\rho(\alpha)$ and hence the second statement follows from the definition and the first since the arc $\alpha \cdot c$ is homotopic with fixed endpoints to the concatentation of c with (a representative of) α. To see the fourth statement, observe that since S is symmetric by assumption, we have $\|S\| < 2\pi$ if and only if the absolute values of the eigenvalues of S are smaller than 2π.

Now by Theorem 3.2 of [HWZ95], the above properties uniquely determine the Maslov index of paths in $SL(2,\mathbb{R})$ beginning at the identity and ending at a matrix which does not have one as an eigenvalue as used by Hofer, Wysocki and Zehnder (Section 3 of [HWZ95], see also Chapter 2 of [S99] for alternative definitions). Together this completes the proof of the lemma. □

To calculate the total rotation angle of a vector under the curve $\Phi : [0,T] \to SL(2,\mathbb{R})$ defined above we use complex coordinates and view the vector fields N, M as $\mathbb{C}^2$-valued functions on Σ. For a curve $\gamma : [0,b] \to \Sigma$ we abbreviate $N(t) = N(\gamma(t))$ and $M(t) = M(\gamma(t))$. Define a $U(2)$-valued curve $O : [0,\chi] \to U(2)$ by the requirement that for each $t \in [0,\chi]$, $O(t)$ is given with respect to the standard basis of $\mathbb{C}^2$ by the matrix

$$O(t) := (N(t), M(t)), \quad \text{i.e. } O(t)\begin{pmatrix} a \\ b \end{pmatrix} = aN(t) + bM(t) \quad \text{for } a, b \in \mathbb{C}.$$

The image of the complex line $\{0\} \times \mathbb{C} \subset \mathbb{C}^2$ under the map $O(t)$ is just the complex line $L(\gamma(t))$. Therefore for each t, $\pi \circ O(t)$ is an $\mathbb{R}$-linear isomorphism of $\{0\} \times \mathbb{C}$ onto $\xi(\gamma(t))$.

From now on we use coordinates in $\mathbb{C}^2$. Without loss of generality we can assume that $\xi_{\gamma(0)} = L_{\gamma(0)}$ and hence we have $\pi M(0) = M(0) = M$. If we define a unitary 2×2 matrix $U(t)$ as

$$U(t) := \left(N(t), \frac{\pi^{-1} d\Psi_t M}{\|\pi^{-1} d\Psi_t M\|} \right), \tag{1}$$

then the turning angle of $O(t)^{-1}\pi^{-1}d\Psi_t M$ about zero (i.e. the rotation of M) is just the argument of $\det(U(t))$.

Define a unit vector field $\tilde{M}$ along γ by

$$\tilde{M}(t) = \frac{\pi^{-1}d\Psi_t M}{\|\pi^{-1}d\Psi_t M\|}.$$

The following lemma is the main technical tool for a calculation of the Maslov index of γ. For its formulation, recall that the *second fundamental form* of the hypersurface Σ in $\mathbb{C}^2$ is the symmetric bilinear form $\Pi : T\Sigma \times T\Sigma \to \mathbb{R}$ which is defined as follows. Let X, Y be vector fields on $\Sigma \subset \mathbb{R}^4$; then

$$\Pi(X,Y) = -\langle dY(X), N\rangle = \langle dN(X), Y\rangle.$$

The *shape operator* of Σ is the section A of the bundle $T^*\Sigma \otimes T\Sigma$ defined by $\Pi(X,Y) = \langle AX, Y\rangle$.

Lemma 4.4. *For $t_0 \in [0,T]$ we have*

$$\frac{\partial}{\partial t}\det(U(t))|_{t=t_0} = \frac{\partial}{\partial t}O(t)^{-1}\tilde{M}(t)|_{t=t_0}$$
$$= i\varphi(\gamma(t_0))(\Pi(JN(t_0), JN(t_0)) + \Pi(\tilde{M}(t_0), \tilde{M}(t_0)))\det(U(t))$$

with $\varphi(\gamma(t_0)) = \|X(\gamma(t_0))\| = \|\gamma'(t_0)\| = \frac{2}{\langle p, N(t_0)\rangle}$.

Proof. In the sequel we always view the second fundamental form Π of Σ as a bilinear form on a subspace of $\mathbb{R}^4$. Let $\pi_2 : \mathbb{C}^2 \to \{0\} \times \mathbb{C}$ be the orthogonal projection. Using the simple fact that

$$O^{-1} \circ \pi^{-1} = \pi_2 \circ O^{-1}$$

we deduce

$$\begin{aligned}
\frac{\partial}{\partial t}O(t)^{-1}\tilde{M}(t) &= \frac{\partial}{\partial t}\frac{O(t)^{-1}\pi^{-1}d\Psi_t M}{\|\pi^{-1}d\Psi_t M\|}|_{t=t_0} \\
&= \pi_2\big(\frac{\partial}{\partial t}O(t)^{-1}|_{t=t_0}\big)\frac{d\Psi_{t_0}M}{\|\pi^{-1}d\Psi_{t_0}M\|} \\
&+ \pi_2 O(t_0)^{-1}\big(\frac{\partial}{\partial t}\frac{1}{\|\pi^{-1}d\Psi_t M\|}\Big|_{t=t_0}\big)d\Psi_{t_0}M \\
&+ \pi_2 O(t_0)^{-1}\frac{1}{\|\pi^{-1}d\Psi_{t_0}M\|}\big(\frac{\partial}{\partial t}d\Psi_t M|_{t=t_0}\big).
\end{aligned} \tag{2}$$

The first term in our equation can be rewritten as

$$\pi_2\big(\frac{\partial}{\partial t}O(t)^{-1}|_{t=t_0}\big)\frac{d\Psi_{t_0}M}{\|\pi^{-1}d\Psi_{t_0}M\|}$$
$$= -\pi_2 O(t_0)^{-1}\big(\frac{\partial}{\partial t}O(t)|_{t=t_0}O(t_0)^{-1}\big)\frac{d\Psi_{t_0}M}{\|\pi^{-1}d\Psi_{t_0}M\|}$$
$$= *.$$

By definition, for every t the vectors $\{N(t), \tilde{M}(t)\}$ form a unitary basis of $\mathbb{C}^2$. Since $\{N(t), M(t)\}$ is also such a unitary basis, there is a smooth function $\psi : [0,\chi] \to \mathbb{R}$ such that $\tilde{M}(t) = e^{i\psi(t)}M(t)$ for all t. We now use the second fundamental form Π to calculate the differential of the matrix valued curve $O(t) = (N(t), M(t))$. For this recall the definition of the shape operator $A : T\Sigma \to T\Sigma$ of Σ and of the orthogonal projection $P : T\Sigma \to L$. Since $M(t) = \hat{M}N(t)$ and $\gamma'(t) = \varphi(t)JN(t)$ we have

$$\frac{\partial}{\partial t}O(t) = \varphi(t)(AJN(t), \hat{M}AJN(t)).$$

Thus with respect to the complex basis $(N(t), M(t))$ of $\mathbb{C}^2$ we have

$$\frac{\partial}{\partial t}O(t) = \varphi(t)\begin{pmatrix} i\Pi(JN(t), JN(t)) & -\overline{PA(JN(t))} \\ PA(JN(t)) & -i\Pi(JN(t), JN(t)) \end{pmatrix}$$

where we view $PA(\varphi JN(t))$ as a complex multiple of $M(t)$.

By the definition of the function ψ we can write

$$\frac{O(t_0)^{-1}d\Psi_{t_0}M}{\|\pi^{-1}d\Psi_{t_0}M\|} = \begin{pmatrix} ic(t_0) \\ e^{i\psi(t_0)} \end{pmatrix} \text{ for some } c(t_0) \in \mathbb{R}.$$

Since $\tilde{M}(t) = e^{i\psi(t)}M(t)$ and $O^{-1}(t)M(t) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), O^{-1}N(t) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ we deduce that

$$* = -\pi_2\varphi(t_0)\begin{pmatrix} -c(t_0)\Pi(JN(t_0), JN(t_0)) - e^{i\psi(t_0)}\overline{PA(JN(t_0))} \\ ic(t_0)PA(JN(t_0)) - ie^{i\psi(t_0)}\Pi(JN(t_0), JN(t_0)) \end{pmatrix}.$$

Now let exp be the exponential map of the hypersurface Σ with respect to the Riemannian metric induced by the euclidean metric. We use this exponential map to compute the third term in (2).

$$\begin{aligned} \pi_2 O(t_0)^{-1}\frac{\frac{\partial}{\partial t}d\Psi_t M|_{t=t_0}}{\|\pi^{-1}d\Psi_{t_0}M\|} &= \frac{\pi_2 O(t_0)^{-1}}{\|\pi^{-1}d\Psi_{t_0}M\|}\frac{\partial}{\partial t}\frac{\partial}{\partial s}\Psi_t(\exp_{\gamma(0)}(sM))|_{t=t_0,s=0} \\ &= \frac{\pi_2 O(t_0)^{-1}}{\|\pi^{-1}d\Psi_{t_0}M\|}\frac{\partial}{\partial s}(\varphi(\Psi_{t_0}(\exp_{x_0}(sM))JN(\Psi_{t_0}(\exp_{\gamma(0)}(sM)))) \\ = \pi_2 O^{-1}(t_0)\left(\frac{1}{\|\pi^{-1}d\Psi_{t_0}M\|}(\frac{\partial}{\partial s}\varphi(\Psi_{t_0}(sM))JN(t_0)|_{s=0}) + \varphi JA(d\Psi_{t_0}M)\right) & \\ &= \pi_2 O^{-1}(t_0)\varphi(t_0)JA(\tilde{M}(t_0) + c(t_0)JN) \\ = \varphi(t_0)e^{i\psi(t_0)}\big(i\Pi(\tilde{M}, \tilde{M}) - \Pi(\tilde{M}, J\tilde{M}) + ic\Pi(JN, \tilde{M}) - c\Pi(JN, J\tilde{M})\big). & \end{aligned}$$

The tangent vector of a curve c in $\mathbb{C}$ with constant norm always has the form $\dot{c} = irc$ for some $r \in \mathbb{R}$, so we can neglect the radial parts of the above equations. Summing up the three terms in (2) yields

$$\frac{\partial}{\partial t}\det(U(t))|_{t=t_0} = i\varphi(\Pi(JN, JN) + \Pi(\tilde{M}, \tilde{M}))\det(U(t_0)).$$

□

The following corollary is immediate from Lemma 4.4 and the definition of the rotation of a vector with respect to an arc in $SL(2, \mathbb{R})$.

Corollary 4.5. *Let γ be a closed Reeb-orbit on Σ with period T. Then*

$$\mathrm{rot}(\Phi, M(0)) = \int_0^T |\gamma'|(\Pi(JN, JN) + \Pi(\tilde{M}, \tilde{M}))dt$$

where $\tilde{M}(t) = \frac{\pi^{-1} d\Psi_t M(0)}{\|\pi^{-1} d\Psi_t M(0)\|}$.

Now we specialize again to the case that C is a compact strictly convex body in $\mathbb{C}^2$ with smooth boundary Σ which contains the origin in its interior. Recall that the *total curvature* of a smooth curve $\gamma : [0, t] \to \mathbb{C}^2$ parametrized by arc length is defined by

$$\kappa(\gamma) = \int_0^T \|\gamma''(t)\| dt.$$

The next corollary is immediate from Lemma 4.3 and lemma 4.4.

Corollary 4.6. *Let Σ be the boundary of a compact strictly convex body $C \subset \mathbb{C}^2$. If the principal curvatures $a \geq b \geq c$ of Σ satisfy the pointwise pinching condition $a \leq b + c$ then the total curvature of a periodic Reeb orbit of Maslov index 3 is smaller than 4π.*

Proof. Let Σ be the boundary of a compact strictly convex body $C \subset \mathbb{C}^2$ with principal curvatures $a \geq b \geq c$ satisfying the pinching condition $a \leq b + c$. Let $\gamma : [0, T] \to \Sigma$ be a periodic Reeb orbit of Maslov index 3. We assume that γ is parametrized by arc length on $[0, T]$. By Lemma 4.3, the rotation of the vector $M(\gamma(0))$ under the derivative of the Reeb flow is smaller than 4π.

Denote by $N(t)$ the normal field of the sphere restricted to the curve γ. Then $\gamma'(t) = JN(t)$ and therefore

$$\kappa(\gamma) = \int_0^T \|\frac{\partial}{\partial t} N(t)\| dt \leq \int_0^T a(\gamma(t)) dt \leq \int_0^T b(\gamma(t)) + c(\gamma(t)) dt < 4\pi$$

by Corollary 4.5. □

We use Corollary 4.6 to complete the proof of Theorem 2 from the introduction.

Proposition 4.7. *Let Σ be the boundary of a compact strictly convex domain $C \subset \mathbb{C}^2$. If the principal curvatures $a \geq b \geq c$ of Σ satisfy the pointwise pinching condition $a \leq b + c$ then a periodic Reeb orbit on Σ of Maslov index 3 bounds an embedded symplectic disc $f : (D, \partial D) \to (C, \gamma)$. In particular, γ has self-linking number -1.*

Proof. Define the *crookedness* of a smooth closed curve $\gamma : S^1 \to \mathbb{R}^4$ to be the minimum of the numbers $m(\gamma, v)$ where $m(\gamma, v)$ is the number of minima of the function $t \to \langle \gamma(t), v \rangle, v \in S^3$. By a result of Milnor [Mil50], the crookedness of a curve of total curvature smaller than 4π equals one. Thus by Corollary 4.6, if γ is a periodic Reeb orbit on Σ of Maslov index 3 then there is some $v \in S^3$ such that the restriction to γ of the function $\varphi : x \to \langle x, v \rangle$ assumes precisely one maximum and one minimum. We may moreover assume that these are the only critical points of the restriction of φ to γ and that they are non-degenerate (see [Mil50]).

Let a be the unique minimum of φ on γ. Assume that γ is parametrized in such a way that $\gamma(0) = a$. Let $\gamma_2 : [0, \sigma] \to \Sigma$ be the parametrized subarc of γ issuing from $\gamma_2(0) = a$ which connects a to the unique maximum b of φ on γ. Let $\gamma_1 : [0, \sigma] \to \Sigma$ be the parametrization of the second subarc of γ connecting a to b such that $\varphi(\gamma_1(t)) = \varphi(\gamma_2(t))$ for all $t \in [0, \sigma]$; this is possible by construction and by our choice of φ. Then the symplectic disc obtained from this parametrization by linear filling as in the proof of Lemma 4.1 is embedded. By Corollary 3.6, this implies that the self-linking number of γ equals -1. □

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