TOPOLOGICAL PROPERTIES OF REEB ORBITS ON BOUNDARIES OF STAR-SHAPED DOMAINS IN $\mathbb{R}^4$

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Abstract. Let C be a compact star-shaped domain in $\mathbb{R}^4$ with smooth boundary Σ. Let γ be a periodic Reeb orbit on Σ. We show that if there is an immersed symplectic disc $f : (D, \partial D) \to (C, \Sigma)$ with boundary γ then $\mathrm{lk}(\gamma) = 2\mathrm{tan}(f) - 1$ where $\mathrm{lk}(\gamma)$ is the self-linking number of γ and $\tan(f)$ the tangential self-intersection number of f. We also show that if C is convex and if the principal curvatures of Σ are suitably pinched then the self-linking number of a periodic Reeb orbit of Maslov index 3 on Σ equals -1.

1. Introduction

Consider the four-dimensional euclidean space $\mathbb{R}^4$ with the standard *symplectic form* defined in standard coordinates by $\omega_0 = \sum_{i=1}^{2} dx_i \wedge dy_i$. This symplectic form is the differential of the one-form

$$\lambda_0 = \frac{1}{2}\sum_{i=1}^{2}(x_i dy_i - y_i dx_i).$$

For every bounded domain $\Omega \subset \mathbb{R}^4$ which is star-shaped with respect to the origin $0 \in \mathbb{R}^4$, with smooth boundary Σ, the restriction λ of λ_0 to Σ defines a smooth *contact form* on Σ. This means that $\lambda \wedge d\lambda$ is a volume form on Σ.

The *Reeb vector field* of the contact structure λ is the smooth vector field X on Σ defined by $\lambda(X) = 1$ and $d\lambda(X, \cdot) = 0$. By the well-known solution of the Weinstein conjecture for boundaries of bounded star-shaped domains Ω in $\mathbb{R}^4$ [Vit87], the *Reeb flow* on Σ generated by the Reeb vector field X admits periodic orbits. Dynamical properties of the Reeb flow on Σ are related to properties of Ω viewed as a symplectic manifold.

Let $\xi = \ker(\lambda)$ be the contact bundle. To each periodic Reeb orbit γ on Σ (or, more generally, to each embedded smooth closed curve on Σ which is everywhere transverse to ξ) we can associate its *self-linking number* $\mathrm{lk}(\gamma)$ which is defined as follows. Let $S \subset \Sigma$ be a *Seifert surface* for γ, i.e. S is a smooth embedded surface in Σ whose boundary equals γ. Since γ is transverse to ξ, there is a natural identification of the restriction to γ of the oriented normal bundle of S in Σ with a real line subbundle N_S of the contact bundle $\xi|\gamma$. Then N_S defines a trivialization

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of $\xi|\gamma$. The self-linking number $\mathrm{lk}(\gamma)$ of γ is the winding number with respect to N_S of a trivialization of ξ over γ which extends to a trivialization of ξ on Σ.

Eliashberg [Eli92] showed that the self-linking number of a periodic Reeb orbit on Σ is always an odd integer. If g denotes the *Seifert genus* of γ, i.e. the smallest genus of a Seifert surface for γ, then we have $\mathrm{lk}(\gamma) \leq 2g - 1$. Eliashberg also constructed for every $k \geq 1$ a transverse unknot ζ of self-linking number $\mathrm{lk}(\zeta) = -2k - 1$ in the three-sphere equipped with the standard contact structure. This shows that in general there is no further relation between the self-linking number of a periodic Reeb orbit γ on Σ and purely topological invariants of γ. On the other hand, Hofer, Wysocki and Zehnder [HWZ96] proved that there is always a periodic Reeb orbit of self-linking number -1 on Σ which is unknotted.

Our first goal is to relate the self-linking number of a periodic Reeb orbit γ to the symplectic topology of the domain Ω. For this let $C \subset \mathbb{R}^4$ be the closure of Ω, let $D \subset \mathbb{C}$ be the closed unit disc with boundary ∂D and let $f : (D, \partial D) \to (C, \Sigma)$ be a smooth immersion with $f(\partial D) = \gamma$ and $f^{-1}(\Sigma) = \partial D$. If all self-intersections of $f(D)$ are transverse then the *tangential index* $\tan(f)$ of f is the number of self-intersection points of f counted with signs and multiplicities.

The disc $f : D \to C$ is called *symplectic* if for every $x \in D$ the restriction of the symplectic form ω_0 to $df(T_xD)$ does not vanish. A symplectic disc $f : (D, \partial D) \to (C, \gamma)$ meets the boundary Σ of C transversely along $f(\partial D) = \gamma$. We show

Theorem 1. *Let γ be a periodic Reeb orbit on the boundary Σ of a star-shaped domain $\Omega \subset \mathbb{R}^4$ with compact closure C. If γ bounds an immersed symplectic disc $f : (D, \partial D) \to (C, \gamma)$ then* $\mathrm{lk}(\gamma) = 2\tan(f) - 1$.

In Section 4 we observe that every periodic Reeb orbit on the boundary of a compact strictly convex body $C \subset \mathbb{R}^4$ bounds an immersed symplectic disc $f : (D, \partial D) \to (C, \gamma)$ to which Theorem 1 can be applied.

There is a second numircal invariant for a periodic Reeb orbit γ on Σ, the so-called *Maslov index* $\mu(\gamma)$. If Σ is the boundary of a compact strictly convex body $C \subset \mathbb{R}^4$ then the Maslov index of any periodic Reeb orbit on Σ is at least three [HWZ98]. The *action* of γ is defined to be $\int_\gamma \lambda > 0$, and the orbit is *minimal* if its action is minimal among the actions of all periodic orbits of the Reeb flow. Ekeland [Eke90] showed that the Maslov index of a minimal Reeb orbit on Σ equals precisely three.

Our second result relates the Maslov index to the self-linking number for periodic Reeb orbits on the boundary of compact strictly convex bodies with geometric control.

Theorem 2. *Let C be a compact strictly convex body with smooth boundary Σ. If the principal curvatures $a \geq b \geq c$ of Σ satisfy the pointwise pinching condition $a \leq b + c$ then a periodic Reeb orbit γ on Σ of Maslov index 3 bounds an embedded symplectic disc in C. In particular, the self-linking number of γ equals -1.*

As an immediate consequence, if Σ is as in Theorem 2 then a periodic Reeb orbit γ on Σ of Maslov index 3 is a slice knot in Σ. In fact, with some additional effort it is possible to show that such an orbit is unknotted [H07].

The proofs of these results use mainly tools from differential topology and differential geometry. In Section 2 we begin to investigate topological properties of periodic Reeb orbits γ on the boundary Σ of a bounded star-shaped domain $\Omega \subset \mathbb{C}^2$. We define a self-intersection number for a (not necessarily immersed) disc in the closure C of Ω with boundary γ and relate this self-intersection number to the self-linking number of γ. In Section 3 we study topological invariants of immersed discs in C with boundary γ and show Theorem 1. In Section 4 we look at boundaries of strictly convex bodies in $\mathbb{R}^4$ and derive Theorem 2.

2. Self-intersection of surfaces

In this section we investigate topological invariants of smooth maps from an oriented bordered surface S with connected boundary ∂S into an arbitrary smooth oriented simply connected 4-dimensional manifold W whose restrictions to a neighborhood of ∂S are embeddings. We use this discussion to investigate maps from the unit disc $D \subset \mathbb{C}$ into a compact set $C \subset \mathbb{C}^2$ which is star-shaped with respect to 0, with smooth boundary Σ. For maps which map the boundary ∂D of D to a periodic Reeb orbit γ on Σ we define a self-intersection number and relate this to the self-linking number of γ.

Let for the moment S be any compact oriented surface with connected boundary $\partial S = S^1$.

Definition 2.1. A smooth map $f : S \to W$, i.e. a map which is smooth up to and including the boundary, is called *boundary regular* if the singular points of f are contained in the interior of S, i.e. if there is a neighborhood A of ∂S in S such that the restriction of f to $f^{-1}(f(A))$ is an embedding.

McDuff investigated in [McD91] boundary regular *pseudo-holomorphic* discs in *almost complex* 4*-manifolds* (W, J). By definition, such a pseudo-holomorphic disc is a smooth boundary regular map f from the closed unit disc $D \subset \mathbb{C}$ into W whose differential is complex linear with respect to the complex structure on D and the almost complex structure J. She defined a topological invariant for such boundary regular pseudo-holomorphic discs which depends on a trivialization of the normal bundle over the boundary circle.

Our first goal is to find a purely topological analog of this construction. For this we say that two boundary regular maps $f, g : S \to W$ *are contained in the same boundary class* if g coincides with f near the boundary and is homotopic to f with fixed boundary. This means that there is a homotopy $h : [0,1] \times S \to W$ connecting $h_0 = f$ to $h_1 = g$ with $h(s,z) = f(z)$ for all $s \in [0,1]$, all $z \in \partial S$. We do not require that each of the maps $h_s : z \to h_s(z) = h(s,z)$, $s \in [0,1]$, is boundary regular. In particular, if $\pi_2(W) = 0$ then any two boundary regular maps $f, g : S \to W$ which coincide near the boundary ∂S of S are contained in the same boundary class (recall that we require that W is simply connected).

There is also the following stronger notion of homotopy for boundary regular maps.

Definition 2.2. A homotopy $h : [0,1] \times S \to W$ is called *boundary regular* if for each s the map h_s is boundary regular and coincides with h_0 near ∂S.

The set of boundary regular maps in the boundary class of a map $f : S \to W$ can naturally be partitioned into boundary regular homotopy classes.

A boundary regular map $f : S \to W$ is an embedding near ∂S. Since S is oriented by assumption, the normal bundle L of $f(S)$ over the embedded circle $f(\partial S)$ is an oriented real two-dimensional subbundle of $TW|f(\partial S)$.

For each trivialization ρ of this normal bundle, the self-intersection number $\mathrm{Int}(f,\rho) \in \mathbb{Z}$ is defined as follows [McD91]. Let $\overline{N}$ be a closed tubular neighborhood of $f(\partial S) = \gamma$ in W with smooth boundary ∂N such that $f(S) \cap \overline{N}$ is an embedded closed annulus A which intersects ∂N transversely. Let $E \subset W$ be an embedded submanifold with boundary which contains A and is diffeomorphic to an open disc bundle over A. One of the two connected componens $(\partial E)_0$ of the boundary ∂E of E has a natural identification with the total space of the normal bundle L of $f(S)$ over γ. Remove $\overline{N} - E$ from W and glue to the boundary $(\partial E)_0$ of the resulting manifold the oriented real two-dimensional vector bundle $D \times \mathbb{C} \to D$ in such a way that $\partial D \times \{0\}$ is identified with the curve $f(\partial S) = \gamma \subset (\partial E)_0$ and that the fibres $\{x\} \times \mathbb{C}$ $(x \in \partial D)$ match up with the trivialized normal bundle $L|f(\partial S)$ of $f(S)$ over γ. Up to diffeomorphism, the resulting 4-dimensional smooth manifold W_ρ only depends on the homotopy class of the trivialization ρ and of the boundary class of f. Let S_0 be the closed oriented surface obtained by glueing a disc to the boundary of S in the usual way. The map f naturally extends to a map f_0 of S_0 into W_ρ. The *self-intersection number* $\mathrm{Int}(f,\rho)$ is then defined to be the topological self-intersection number of $f_0(S_0)$ in W_ρ. Thus $\mathrm{Int}(f,\rho)$ is the number of intersections of $f(S)$ with a surface f' which is a generic perturbation of $f(S)$ and such that $f(\partial S)$ is pushed into the direction given by ρ.

In the next lemma we determine the boundary regular homotopy classes in a fixed boundary class.

Lemma 2.3. *Let $f : S \to W$ be a smooth boundary regular map. Choose a trivialization ρ of the oriented normal bundle of $f(S)$ over $f(\partial S)$. Then the assignment which associates to a boundary regular homotopy class of maps in the boundary class of f its self-intersection number with respect to ρ is a bijection onto* $\mathrm{Int}(f,\rho) + 2\mathbb{Z}$. *Moreover, if f is an embedding then each such class can be represented by an embedding.*

Proof. Let $f : S \to W$ be a smooth boundary regular map. Write $\gamma = f(\partial S)$ and let $u : S \to W$ be a smooth boundary regular map in the boundary class of f. This means that there is a homotopy $h : [0,1] \times S \to W$ connecting $h_0 = f$ to $h_1 = u$ with fixed boundary. The maps u, f are contained in the same boundary regular homotopy class if and only if this homotopy can be chosen in such a way that there is a tubular neighborhood N of γ such that the intersection of $h_s(S)$ with N is independent of s.

Choose such an open tubular neighborhood N of γ with smooth boundary ∂N which is sufficiently small that both $f(S)$ and $u(S)$ intersect N in a smooth annulus containing γ as one of its two boundary components. We may assume that there is a compact subsurface $C \subset S$ with smooth boundary ∂C such that $S - C$ is an annulus neighborhood of ∂S and that $f(S-C) = u(S-C) = f(S)\cap N = u(S)\cap N$. Then $u|_C$ and $f|_C$ can be combined to a map into $W - N$ of the closed oriented surface $\tilde{S}$ which we obtain from C by gluing two copies of C along the boundary with an orientation reversing boundary identification. This map is homotopic in $W-N$ to a constant map if and only if u and f are contained in the same boundary regular homotopy class.

Now W is simply connected by assumption and N is homeomorphic to a 3-ball-bundle over a circle, with boundary $\partial N \sim \gamma \times S^2$. Thus by van Kampen's theorem, $W - N$ is simply connected and the second homotopy group $\pi_2(W - N)$ coincides with the second homology group $H_2(W - N, \mathbb{Z})$ via the Hurewicz isomorphism. Since two boundary regular maps in the same boundary class are homotopic with fixed boundary, we conclude that the family of boundary regular homotopy classes of maps in the boundary class of f can be identified with the kernel of the natural homomorphism $H_2(W - N, \mathbb{Z}) \to H_2(W, \mathbb{Z})$.

To compute this group, we use the long exact homology sequence of the pair $(W, W - N)$ given by

$$\cdots \to H_3(W, \mathbb{Z}) \quad \to H_3(W, W - N, \mathbb{Z}) \quad \to H_2(W - N, \mathbb{Z}) \quad \to H_2(W, \mathbb{Z}) \quad \to \dots$$

Excision shows that $H_3(W, W - N, \mathbb{Z}) = H_3(\overline{N}, \partial N, \mathbb{Z})$ where $\overline{N}$ is the closure of N. Since $\overline{N} = \gamma \times B^3 = S^1 \times B^3$ where B^3 denotes the closed unit ball in $\mathbb{R}^3$, the group $H_3(W, W - N, \mathbb{Z})$ is cyclic and generated by a ball $\{z\} \times (B^3, S^2)$ where $z \in \gamma$ is any fixed point.

Every singular homology class $v \in H_3(W, \mathbb{Z})$ can be represented by a piecewise smooth singular cycle σ whose image is nowhere dense in W. On the other hand, the curve γ is contractible in W and therefore there is a smooth isotopy of W which moves σ away from N. Thus the image of $H_3(W, \mathbb{Z})$ under the natural homomorphism $H_3(W, \mathbb{Z}) \to H_3(W, W - N, \mathbb{Z}) = H_3(\overline{N}, \partial N, \mathbb{Z})$ vanishes and hence by exactness, the kernel of the natural homomorphism $H_2(W - N, \mathbb{Z}) \to H_2(W, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and generated by a sphere $e = \{z\} \times S^2 \sim 1 \in \pi_2(\partial N) = \mathbb{Z}$ for some $z \in \gamma$. As a consequence, every boundary regular homotopy class in the boundary class of f can uniquely be represented in the form $[f] + ke$ where $[f]$ denotes the boundary regular homotopy class of f and where $k \in \mathbb{Z}$.

Next we show that if f is an embedding then each of these classes can be represented by an embedding as well. For this note that after possibly replacing N by a smaller tubular neighborhood of γ we may assume that $f^{-1}(\overline{N})$ is a closed annulus neighborhood of ∂S and that for some $z \in \gamma$, the sphere $M = \{z\} \times S^2 \subset \partial N$ intersects $f(S)$ transversely in a single point x. The orientation of the surface S defines uniquely an orientation of M such that $T_xW = T_x(f(S)) \oplus T_xM$ as oriented vector spaces. Using standard surgery near the transverse intersection point x we can attach the sphere M to the surface $f(S)$ as follows (see [GS99]). There is a closed neighborhood V of x in $W-\gamma$ which is diffeomorphic to a closed ball and such that the intersections $f(S)\cap V$, $M\cap V$ are smooth discs which intersect transversely in the single point x. The boundaries of these discs are two disjoint oriented circles in the boundary $\partial V\sim S^3$ of V. These circles define the Hopf link in S^3 and therefore they form the oriented boundary of a smooth embedded annulus in ∂V. The surgery replaces $(f(S)\cup M)\cap V$ by such an annulus (which can be done smoothly). We obtain in this way a compact oriented bordered surface which can be represented by a boundary regular map $g:S\to W$ which coincides with $f(S)$ near the boundary. The surgery does not change relative homology classes [GS99] and hence $g(S)$ is homologous to $[f]+e$ via an identification of M with a generator e of the kernel of the natural map $H_2(W-N,\mathbb{Z})\to H_2(W,\mathbb{Z})$. In other words, the embedded surface which we just constructed represents the boundary regular homotopy class $[f]+e$ in the boundary class of f. In the same way we can also construct a surface which represents the boundary regular homotopy class $[f]-e$ by attaching to f a sphere equipped with the reverse orientation. Namely, we also can connect the boundaries of the discs $f(S)\cap V, M\cap V$ with an embedded cylinder whose oriented boundary is the union of the oriented boundary of $f(S)\cap V$ with the boundary of $M\cap V$ equipped with the reversed orientation. Repeating this procedure finitely many times with different basepoints we obtain an embedding in every boundary regular homotopy class of maps in the boundary class of f.

Let ρ be a trivialization of an oriented normal bundle of $f(S)$ along $\gamma=f(\partial S)$. We are left with showing that a boundary regular homotopy class in the boundary class of f is determined by its self-intersection number with respect to ρ. For this let again $M=\{z\}\times S^2\subset\partial N$ be an oriented embedded sphere as above which intersects $f(S)$ transversely in a single point x. Assume that the index of intersection between $f(S)$ and M with respect to the given orientations is positive. Let $g:S\to W$ be the map constructed above with $[g]=[f]+e$. Using the above notations, it is enough to show that $\mathrm{Int}(g,\rho)=\mathrm{Int}(f,\rho)+2$. However, this can be seen as follows.

As above, denote by W_ρ the manifold used for the definition of the self-intersection number $\mathrm{Int}(f,\rho)$. Recall that up to diffeomorphism, the manifold W_ρ only depends on ρ and the boundary class of f. In particular, we may assume that W_ρ contains the images Γ_f,Γ_g of the closed surface S_0 under the natural extensions of the maps f,g. The self-intersection numbers of the surfaces Γ_f,Γ_g in the manifold W_ρ can now be compared via

$$\begin{aligned}\mathrm{Int}(g,\rho)&=\Gamma_g\cdot\Gamma_g=(\Gamma_f+e)\cdot(\Gamma_f+e)\\&=\Gamma_f\cdot\Gamma_f+2\Gamma_f\cdot e+e\cdot e=\mathrm{Int}(f,\rho)+2\end{aligned}$$

since the topological self-intersection of the sphere e in W_ρ vanishes. But this just means that the assignment which associates to a boundary regular homotopy class in the boundary class of f its self-intersection number with respect to ρ is a bijection onto $\mathrm{Int}(f,\rho)+2\mathbb{Z}$. From this the lemma follows. □

From now on we assume that the 4-dimensional manifold W is equipped with a smooth almost complex structure J.

Definition 2.4. A smooth boundary regular map $f : S \to W$ is called *boundary holomorphic* if for each $z \in \partial S$ the tangent plane of $f(S)$ at z is a complex line in (TW, J) whose orientation coincides with the orientation induced from the orientation of S.

If $f : S \to W$ is boundary regular and boundary holomorphic then the pull-back f^*TW under f of the tangent bundle of W is a 2-dimensional complex vector bundle over S. Since f is boundary holomorphic, the restriction to ∂S of the tangent bundle TS of S is naturally a complex line-subbundle of $f^*TW|_{\partial S}$. Then the normal bundle of $f(S)$ over γ can be identified with a complex line subbundle of $(TW, J)|f(\partial S)$ as well. Every trivialization ρ of this normal bundle defines as before a smooth manifold W_ρ. This manifold admits a natural almost complex structure extending the almost complex structure on the complement of a small tubular neighborhood of $f(\partial S)$ in W. In particular, if we denote as before by f_0 the natural extension of f to the closed surface S_0 then the pull-back bundle $f_0^*TW_\rho$ is a complex two-dimensional vector bundle over S_0. Up to homotopy, this bundle only depends on ρ and the boundary class of f. Let $c(\rho)$ be the evaluation on S_0 of the first Chern class of this bundle.

Changing the trivialization ρ by a full positive (negative) twist in the group $U(1) \subset GL(1, \mathbb{C})$ changes both the self-intersection number $\mathrm{Int}(f, \rho)$ and the Chern number $c(\rho)$ by 1 (-1) (see [McD91]). In particular, there is up to homotopy a unique trivialization ρ of the complex normal bundle of $f(S)$ over $f(\partial S)$ such that $c(\rho) = 2$. We call such a trivialization a *preferred* trivialization. By the above observation, a preferred trivialization only depends on the boundary class of f but not on the boundary regular homotopy class of f.

Definition 2.5. The *self-intersection number* $\mathrm{Int}(f)$ of a boundary holomorphic boundary regular map $f : S \to W$ is the self-intersection number $\mathrm{Int}(f, \rho)$ of f with respect to a preferred trivialization ρ of the complex normal bundle of $f(S)$ over $f(\partial S)$.

Lemma 2.3 implies

Corollary 2.6. *Let $f : S \to W$ be boundary regular and boundary holomorphic. Then a boundary regular homotopy class in the boundary class of f is uniquely determined by its self-intersection number.*

In the particular case that the almost complex manifold W is just the two-dimensional complex vector space $\mathbb{C}^2$ equipped with the usual integrable complex structure J and the euclidean (real) inner product $\langle \cdot, \cdot \rangle$ we can extend the definition of the self-intersection number to a class of boundary regular maps $f : S \to \mathbb{C}^2$ which are not necessarily infinitesimally holomorphic on ∂D. Namely, call a *real* 2-dimensional subspace V of $\mathbb{C}^2$ *admissible* if $JV \cap V^\perp = \{0\}$ where $V^\perp$ denotes the orthogonal complement of V. Then for every $0 \neq X \in V$ the orthogonal projection PJX of JX into V does not vanish and spans together with X the space V. In particular, the complex structure J defines an orientation on V which we call the *canonical orientation.* Call a smooth map $f : S \to \mathbb{C}^2$ *admissible* if f is boundary regular, if for each $z \in \partial S$ the real plane $df(T_zS) \subset T\mathbb{C}^2$ is admissible and if moreover the orientation of $df(T_zS)$ induced from the orientation of S coincides with the canonical orientation. Our above construction immediately extends to admissible maps. Namely, any admissible map is naturally homotopic through admissible maps with the same boundary to a map which is infinitesimally holomorphic on the boundary.

Now consider the smooth boundary Σ of a bounded domain $\Omega \subset \mathbb{C}^2$ which contains 0 in its interior and which is star-shaped with respect to 0. The restriction λ to Σ of the *radial one-form* λ_0 on $\mathbb{C}^2$ defined as in the introduction by $(\lambda_0)_p(Y) = \frac{1}{2}\langle Jp, Y\rangle$ $(p \in \mathbb{C}^2, Y \in T_p\mathbb{C}^2)$ vanishes nowhere and defines a smooth *contact structure* on Σ. The differential $d\lambda_0$ of λ_0 is just the usual symplectic form ω_0 on $\mathbb{C}^2$.

Let N be the outer normal field of $\Sigma \subset \mathbb{C}^2$. The *Reeb vector field* X on Σ is given by

$$X(p) = \varphi(p)JN(p)$$

where

$$\varphi(p) = \frac{2}{\langle p, N(p)\rangle} > 0.$$

Namely, for $p \in \Sigma$ we have

$$d\lambda_p(X, \cdot) = \varphi(p)\omega_0(JN(p), \cdot) = -\varphi(p)\langle N(p), \cdot\rangle = 0$$

on $T_p\Sigma$ and

$$\lambda_p(X) = \frac{1}{2}\langle Jp, X\rangle = \frac{1}{2}\varphi(p)\langle Jp, JN(p)\rangle = 1.$$

In particular, an admissible map $f : S \to \mathbb{C}^2$ whose boundary $f(\partial S)$ is a periodic Reeb orbit on Σ meets Σ transversely along $f(\partial S)$ and maps a neighborhood of ∂S into the closure C of Ω.

We use this to show

Lemma 2.7. *Let γ be a periodic Reeb orbit on Σ. Then any two admissible boundary regular maps $f : (S, \partial S) \to (C, \gamma), g : (S', \partial S') \to (C, \gamma)$ have the same self-intersection number.*

Proof. Let $f : S \to C, g : S' \to C$ be any two admissible boundary regular maps with boundary a Reeb orbit γ on Σ. Then the inner normals of the surfaces $f(S), g(S')$ along $\gamma = f(\partial S) = g(\partial S')$ point strictly inside the domain C. After a small deformation through boundary regular admissible maps we may assume that there is a small annular neighborhood A of the boundary of S, an annular neighborhood A' of the boundary of S' and a homeomorphism $\varphi : A \to A'$ which maps ∂S to $\partial S'$ and is such that $g(\varphi(x)) = f(x)$ for all $x \in A$. We may moreover assume that the restrictions of f, g to $f^{-1}(A), g^{-1}(A')$ are embeddings. Since f, g are boundary regular, after possibly modifying f, g once more with a small boundary regular homotopy which pushes interior intersection points of $f(S), g(S')$ with Σ into the interior Ω of C we may assume that there is a compact star-shaped set $K \subset \Omega$ such that $f(S - A) \subset K, g(S' - A) \subset K$. But then the restrictions of f, g to $S - A, S' - A'$ are maps of surfaces $S - A, S' - A'$ into K with the same boundary curve γ'. Now K is contractible and hence the maps $f|S - A, g|S' - A'$ define the same relative homology class in $H_2(K,\gamma';\mathbb{Z})$. By Lemma 2.3 and its proof, this implies that the self-intersection numbers of f, g indeed coincide. □

As a consequence, we can define.

Definition 2.8. Let γ be a periodic Reeb orbit on the boundary Σ of the bounded star-shaped domain Ω with closure C. The *self-intersection number* $\mathrm{Int}(\gamma)$ of γ is the self-intersection number of an admissible map $f : S \to C$ with boundary $f(\partial S) = \gamma$.

The final goal of this section to calculate the self-intersection number of a periodic Reeb orbit γ on the boundary of a bounded star-shaped domain $\Omega \subset \mathbb{C}^2$. For this we begin with calculating the preferred trivialization of the normal bundle of the complex line subbundle of $T\mathbb{C}^2|\gamma$ spanned by the tangent γ' of γ.

Namely, let D be the closed unit disc in $\mathbb{C}$ and let $f : D \to C$ be an admissible boundary regular *immersion* which maps ∂D diffeomorphically onto the Reeb orbit γ. Then the image under df of the inner normal of D along ∂D points strictly inside C. Let

$$\hat{M} : (z_1, z_2) \to (-\overline{z}_2, \overline{z}_1)$$

be a J-orthogonal $\langle\cdot,\cdot\rangle$-compatible almost complex structure on $\mathbb{C}^2$ where as usual, $z \to \overline{z}$ is complex conjugation. Then we obtain a trivialization of the complex vector bundle $(f^*T\mathbb{C}^2, J)$ over D by the sections $X_1 = df(\frac{\partial}{\partial x}), X_2 = \hat{M}df(\frac{\partial}{\partial x})$ (with a slight abuse of notation). Since $\hat{M}$ is complex anti-linear and since the trivialization of the tangent bundle $df(TD)|\gamma$ of $f(D)$ over γ defined by the tangent γ' of the Reeb orbit γ has rotation number one with respect to the trivialization $df(\frac{\partial}{\partial x})$, the trivialization of the complex normal bundle L over γ given by the section $\hat{M} \circ \gamma'$ has rotation number -1 with respect to the preferred trivialization. Recall that the preferred trivialization of the complex normal bundle of γ only depends on the boundary class of an infinitesimally holomorphic immersion of a surface S into $\mathbb{C}^2$.

To the Reeb orbit γ we can also associate its *self-linking number* $\mathrm{lk}(\gamma)$ (see [Eli92] and the introduction). The following lemma relates these two numbers.

Proposition 2.9. *Let γ be a periodic Reeb orbit on Σ. Then the self-intersection number* $\mathrm{Int}(\gamma)$ *of γ equals* $\mathrm{lk}(\gamma) + 1$.

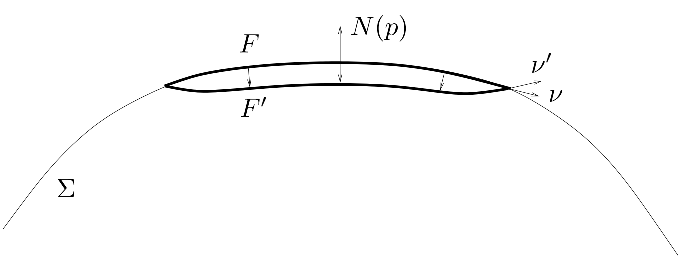

Figure 1. The surfaces F and F'

Proof. Let N be the outer normal field of Σ and let L be the *complex subbundle* of $T\Sigma$, i.e. the 2-dimensional subbundle which is invariant under the complex structure J. The image of N under the J-orthogonal $\langle,\rangle$-compatible almost complex structure $\hat{M}$ is a global section of the bundle L. Let $F \subset \Sigma$ be a *Seifert surface* for the periodic Reeb orbit γ, i.e. F is an embedded oriented bordered surface in Σ with boundary γ. Let N_F be the oriented normal field of F in Σ with respect to the restriction of the euclidean metric $\langle,\rangle$. Since γ is a Reeb orbit, for every $x \in \gamma$ the vector $N_F(x)$ is contained in the fibre L_x at x of the complex line bundle L. The orthogonal projection of the contact bundle $\xi \subset T\Sigma$ into the bundle L is a bundle isomorphism. The self-linking number of γ is therefore the winding number of the section $x \to M(x) = \hat{M}N(x)$ of $L|\gamma$ with respect to the trivialization of $L|\gamma$ defined by the section $x \to N_F(x)$.

Let $F' \subset C$ be the embedded surface which we obtain by pushing F slightly in the direction $-N$ as in Figure 1. Then F' is an embedded surface in C which is admissible in the above sense. The restriction of N_F to γ extends to a global trivialization of the oriented normal bundle of the surface F and hence F' in $\mathbb{C}^2 - \gamma$. Thus the self-intersection number of F' with respect to the trivialization defined by N_F vanishes. Since by the above observation the winding number of the section M of $L|\gamma$ with respect to the preferred trivialization of $L|\gamma$ equals -1, the self-intersection number $\mathrm{Int}(\gamma)$ equals the winding number of M with respect to the trivialization of $L|\gamma$ defined by N_F plus one. This shows the lemma. ☐

3. TOPOLOGICAL INVARIANTS OF IMMERSED DISCS

As in Section 2, we denote by S a compact oriented surface with connected boundary $\partial S = S^1$. Let (W, J) be a smooth simply connected 4-dimensional manifold equipped with a smooth almost complex structure J. In this section we investigate topological invariants of boundary holomorphic boundary regular *immersions* $f : S \to (W, J)$. For this we use the assumptions and notations from Section 2. Recall in particular the definition of the self-intersection number $\mathrm{Int}(f)$ of f.

Let $G(2, 4)$ be the Grassmannian of oriented (real) 2-planes in $\mathbb{R}^4 = \mathbb{C}^2$. This Grassmannian is just the homogeneous space $G(2, 4) = SO(4)/SO(2) \times SO(2) = S^2 \times S^2$, in particular the second homotopy group of $G(2, 4)$ coincides with its second homology group and is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

The complex projective line $\mathbb{C}P^1 = S^2$ of *complex* lines in $\mathbb{C}^2$ is naturally embedded in $G(2, 4)$. Its homotopy class is the generator of an infinite cyclic subgroup Z_1 of $\pi_2(G(2, 4))$. We call this generator the *canonical generator* of Z_1 and denote it by e_1. A second infinite cyclic subgroup Z_2 of $\pi_2(G(2, 4))$ is defined as follows. Let $S^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4$ be the standard unit sphere. The map which associates to a point $y \in S^2$ the oriented tangent plane of S^2 at y, viewed as a 2-dimensional oriented linear subspace of $\mathbb{R}^4$, defines a smooth map of S^2 into $G(2, 4)$. Its homotopy class e_2 generates a subgroup Z_2 of $\pi_2(G(2, 4))$. We call e_2 the canonical generator of Z_2.

Since $G(2,4)=S^2\times S^2$ there are generators $\tilde{e}_1,\tilde{e}_2$ of $H_2(G(2,4);\mathbb{Z})$ such that with respect to these generators, the homological intersection form ι is the symmetric form represented by the $(2,2)$-matrix $(a_{i,j})$ with $a_{1,1}=a_{2,2}=0$ and $a_{1,2}=a_{2,1}=\iota(\tilde{e}_1,\tilde{e}_2)=1$. On the other hand, the tangent bundle of $S^2\subset\mathbb{R}^3\subset\mathbb{C}^2$ intersects the complex projective line $\mathbb{C}P^1\subset G(2,4)$ in precisely one point (which is the tangent space of S^2 at $(0,0,1,0)$). This intersection is transverse with positive intersection index. Therefore we have $\iota(e_1,e_2)=1$ and hence the elements e_1,e_2 generate $\pi_2(G(2,4))$.

Let W be a simply connected 4-dimensional manifold with smooth almost complex structure J. Equip the tangent bundle TW of W with a J-invariant Riemannian metric $\langle,\rangle$. Let $\mathcal{G}\to W$ be the smooth fibre bundle over W whose fibre at a point $x\in W$ consists of the Grassmannian of oriented 2-planes in T_xW. Let S be a compact oriented surface with connected boundary $\partial S\sim S^1$. Then every smooth boundary regular *immersion* $f:S\to W$ defines a smooth map Gf of S into the bundle $\mathcal{G}$ by assigning to a point $x\in S$ the oriented tangent space $df(T_xS)$ of $f(S)$ at $f(x)$. The complex pull-back bundle (f^*TW,J) over S admits a *complex* trivialization. This trivialization can be chosen to be of the form $df(X),V$ where X is a global nowhere vanishing section of the tangent bundle TS of S and V is a global section of the $\langle,\rangle$-orthogonal complement of the complex line subbundle of f^*TW which is spanned by the section $df(X)$. With respect to this complex trivialization of f^*TW, the pull-back $f^*\mathcal{G}$ of the bundle $\mathcal{G}$ can naturally be represented as a product $S\times G(2,4)$. If f is boundary holomorphic, i.e. if for every $z\in\partial S$ the tangent space $df(T_zS)\subset T_{f(z)}W$ is J-invariant and if moreover its orientation coincides with the orientation induced by J, then in the above identification of $f^*\mathcal{G}$ with $S\times G(2,4)$ the circle of tangent planes of $f(S)$ over $f(\partial S)$ is given by the curve $\partial S\times L_0$ in $\partial S\times G(2,4)$ where $L_0=\mathbb{C}\times\{0\}\subset\mathbb{C}^2$ is a fixed complex line. Thus in this case the map Gf can be viewed as a smooth map of the surface S into the Grassmannian $G(2,4)$ which maps the boundary ∂S of S to the single complex line L_0. In other words, if we let $\tilde{S}$ be the closed oriented surface obtained from S by collapsing ∂S to a point then Gf defines a map of $\tilde{S}$ into $G(2,4)$. This map then defines a homotopy class of maps $\tilde{S}\to G(2,4)$ and a homology class $[Gf]\in H_2(G(2,4),\mathbb{Z})$.

The following definition strengthens Definition 2.2.

Definition 3.1. A smooth homotopy $h:[0,1]\times S\to W$ is *regular* if h is boundary regular and if moreover for every $s\in[0,1]$ the map h_s is a boundary holomorphic immersion.

If $f,g:S\to W$ are two boundary regular boundary holomorphic immersions which are regularly homotopic, i.e. which can be connected by a regular homotopy, then the maps Gf and Gg are homotopic. Namely, if $h:[0,1]\times S\to W$ is a regular homotopy connecting $h_0=f$ to $h_1=g$, then there is a complex trivialization of the complex pull-back bundle (h^*TW,J) over $[0,1]\times S$ whose restriction to $[0,1]\times\partial S$ does not depend on $s\in[0,1]$ and is determined as before by the section $dh_s(X)$ where X is a global nowhere vanishing section of TS. This trivialization then defines an identification of the bundle $h^*\mathcal{G}$ with $[0,1]\times S\times G(2,4)$. For each $s\in[0,1]$ the tangent planes of the immersion h_s define a smooth section of the bundle $h^*\mathcal{G}$ over

$\{s\} \times S$ and hence a smooth map of S into $G(2,4)$. This map depends smoothly on s and maps the boundary ∂S of S to a single point. Thus by continuity, the homotopy class of the tangent map of h_s is independent of $s \in [0,1]$ and hence it is an invariant of regular homotopy.

There is another way to obtain an invariant of regular homotopy.

Definition 3.2. The *tangential index* $\tan(f)$ of a boundary regular immersion f with only transverse self-intersection points is defined to be the number of self-intersection points of f counted with signs and multiplicities.

If a boundary regular immersion $f : S \to W$ has self-intersection points which are not transverse then it can be perturbed with a regular homotopy to an immersion with only transverse self-intersection points whose tangential index is independent of the perturbation (see e.g. [McD91]). The tangential index is invariant under regular homotopy.

The next lemma characterizes the regular homotopy classes in the boundary class of a fixed boundary regular boundary holomorphic immersion f.

Lemma 3.3. *Let $f : S \to W$ be a boundary holomorphic boundary regular immersion. Then the map which assigns to the regular homotopy class of a boundary holomorphic immersion g in the boundary class of f the pair $([Gg], \frac{1}{2}\mathrm{Int}(g))$ is a bijection onto $H_2(G(2,4),\mathbb{Z}) \times \mathbb{Z}$.*

Proof. Let $f : S \to W$ be a boundary regular immersion which is boundary holomorphic. The boundary regular homotopy class of f is invariant under regular homotopy and by the above observation, the same is true for the homology class $[Gf]$ of its tangent-plane map Gf. By Lemma 2.3, the assignment which associates to the boundary regular homotopy class of a map u in the boundary class of f its self-intersection number $\mathrm{Int}(u)$ is a bjection. Moreover, any two boundary holomorphic boundary regularly homotopic immersions $f, g : S \to W$ are regularly homotopic if and only if $[Gf] = [Gg]$. Namely, we can use the C^0-dense parametric h-principle for immersions in positive codimension [Gro86, Spr98]. It shows that two boundary regular immersions $g, h : S \to W$ which coincide near the boundary are regularly homotopic if and only if they are boundary regularly homotopic and if their induced tangent maps are homotopic as sections of the bundle $\mathcal{G}$.

On the other hand, for every homology class $\alpha \in H_2(G(2,4),\mathbb{Z})$ there is a boundary regular immersion $g : S \to W$ which is boundary regularly homotopic to f and such that $[Gg] = \alpha$ (since we do not need the last fact in the sequel, we omit the standard proof). This shows the lemma. ☐

If we consider more specifically boundary regular boundary holomorphic immersions of *discs* then we can derive a more precise result. For this recall from Section 2 that for every boundary holomorphic boundary regular map $f : D \to W$ there is a preferred trivialization of the normal bundle of $f(D)$ over $f(\partial D)$. On the other hand, there is a trivialization N of the oriented normal bundle of $f(D)$ over γ which extends to a global trivialization of the oriented normal bundle of $f(D)$ in TW.

Definition 3.4. The *winding number* $\mathrm{wind}(f)$ of a boundary regular boundary holomorphic immersion $f : D \to W$ is the winding number of the preferred trivialization of the normal bundle of $f(D)$ over $f(\partial D)$ with respect to a trivialization which extends to a global trivialization of the normal bundle of $f(D)$.

Note that this definition makes also sense for boundary regular immersions of discs which are admissible in the sense described in Section 2.

For the formulation of the following version of the well known *adjunction formula* for immersed boundary holomorphic (or admissible) boundary regular discs, denote for a boundary regular boundary holomorphic immersion $f : D \to W$ by $\mathcal{C}_2(Gf)$ the component of $[Gf]$ in the subgroup Z_2 of $H_2(G(2,4),\mathbb{Z})$, viewed as an integer.

Proposition 3.5. *For a boundary regular boundary holomorphic immersion* $f : D \to W$ *we have* $\mathrm{Int}(f) = \mathrm{wind}(f) + 2\mathrm{tan}(f)$*, and* $\mathrm{wind}(f) = 2\mathcal{C}_2(Gf)$*.*

Proof. Let $f : D \to W$ be a boundary holomorphic boundary regular immersion. As in Section 2, let ρ be the preferred trivialization of the complex normal bundle of $f(D)$ over $f(\partial D)$ and use this trivialization to extend f to an immersion f_0 of the two-sphere S^2 into the almost complex manifold W_ρ. Then $\mathrm{Int}(f)$ is the self-intersection number of $f_0(S^2)$ in W_ρ. Since f_0 is an immersion, this self-intersection number just equals $\chi(N) + 2\mathrm{tan}(f_0)$ where $\chi(N)$ is the Euler number of the normal bundle of $f_0(S^2)$ in $\mathbb{C}^2_\rho$ and where $\mathrm{tan}(f_0) = \mathrm{tan}(f)$ is the tangential index defined above (see e.g. [CT97]). By our definition of the winding number $\mathrm{wind}(f)$ of f, this is just the formula stated in the proposition.

To show that $\mathrm{wind}(f) = 2\mathcal{C}_2(Gf)$, note that we have $\mathrm{wind}(f) = 0$ if $[Gf] \in Z_1$. Namely, using the above notations, recall that a preferred trivialization ρ of the normal bundle of the disc $f(D)$ over $f(\partial D) = \gamma$ is determined by the requirement that the evaluation of the first Chern class of the complex tangent bundle $(TW_\rho, \tilde{J})$ of W_ρ on the 2-sphere $f_0(S^2)$ equals two.

On the other hand, if $[Gf] \in Z_1$ then the tangent plane map of f can be homotoped with fixed boundary to a map $(D, \partial D) \to (\mathbb{C}P^1, L_0)$ where $L_0 \in \mathbb{C}P^1$ is a fixed complex line. Thus up to homotopy, the complex vector bundle $(f_0^* TW, \tilde{J})$ decomposes as a direct sum $TS^2 \oplus N$ of two complex line bundles. The first Chern class of $(f_0^* TW, \tilde{J})$ is then the sum of the Chern classes of TS^2 and N. By our normalization, this means that the first Chern class of the normal bundle N over S^2 vanishes. As a consequence, the bundle $N \to S^2$ is trivial and hence the preferred trivialization of the normal bundle N over γ extends to a global trivialization of N over S^2. By definition, this means that $\mathrm{wind}(f) = 0$ if $[Gf] \in Z_1$.

Arguing as in the proof of Lemma 2.3, if we replace a boundary holomorphic boundary regular immersion $g : D \to W$ by a boundary regular immersion $u : D \to W$ with $\mathrm{Int}(u) = \mathrm{Int}(g)$ and $[Gu] = [Gg] + ke_2$ then $\mathrm{tan}(u) = \mathrm{tan}(g) - k$ and $\mathrm{wind}(u) = \mathrm{wind}(g) + 2k$. Namely, such an immersion u can be constructed as follows. Choose a point $z \in D$ such that there is a small ball $V \subset W$ about $g(z)$ which intersects $g(D)$ in an embedded disc B containing $g(z)$. Choose an embedded 2-sphere $\hat{S} \subset \partial V$ which intersects B transversely in precisely two points, one with positive and one with negative intersection index. The tangent bundle of the sphere is a generator e_2 of the subgroup Z_2 of $G(2,4)$. As in the proof of Lemma 2.3, we attach the sphere to $g(D)$ with surgery about the intersection point with positive intersection index. The resulting disc u satisfies $\mathrm{Int}(u) = \mathrm{Int}(g)$, $[Gu] = u + e_2$ and $\tan(u) = \tan(g) - 1$. From this the proposition is immediate. ☐

A *complex point* of an immersed disc $f : D \to W$ is a point $z \in D$ such that the real two-dimensional subspace $df(T_z D)$ of TW is invariant under the almost complex structure J. The point is called *holomorphic* if the orientation of $df(T_z D)$ induced by the orientation of D coincides with the orientation induced by the almost complex structure J, and it is called *anti-holomorphic* otherwise. The following corollary is immediate from the above observation.

Corollary 3.6. *Let $f : D \to W$ be a boundary holomorphic boundary regular immersion. If f does not have any anti-holomorphic points then* $\mathrm{Int}(f) = 2\tan(f)$.

Proof. Let $f : D \to W$ be a boundary holomorphic immersion without any anti-holomorphic point. Then the tangent map Gf of f does not intersect the anti-holomorphic sphere of complex lines in $\mathbb{C}^2$ equipped with the reverse of the orientation induced by the complex structure. Since the anti-holomorphic sphere is homologous in $G(2,4)$ to the complex projective line $\mathbb{C}P^1$, we have $[Gf] \in Z_1$ by consideration of intersection numbers. The corollary now is immediate from Proposition 3.5. ☐

As in the introduction, denote by ω_0 the standard symplectic form on $\mathbb{C}^2$. An immersion $f : D \to \mathbb{C}^2$ is called symplectic if for every $z \in D$ the restriction of $f^*\omega_0$ to the tangent plane $T_z D$ does not vanish and defines the standard orientation of $T_z D$. As an immediate consequence of Corollary 3.6 we obtain Theorem 1 from the introduction.

Corollary 3.7. *Let γ be a periodic Reeb orbit on the boundary Σ of a bounded star-shaped domain $\Omega \subset \mathbb{C}^2$ with closure C. If γ bounds an immersed symplectic disc $f : (D, \partial D) \to (C, \gamma)$ then* $\mathrm{lk}(\gamma) = 2\tan(f) - 1$.

Proof. By definition, a symplectic immersion does not have any anti-holomorphic points. Thus the corollary is immediate from Lemma 2.9 and Corollary 3.6. ☐

4. BOUNDARIES OF COMPACT CONVEX BODIES WITH CONTROLLED CURVATURE

In this section we investigate periodic Reeb orbits on the boundary Σ of a compact strictly convex body $C \subset \mathbb{C}^2$. Our main goal is the proof of Theorem 2 from the introduction.

The next lemms shows that Corollary 3.7 can always be applied for periodic Reeb orbits on boundaries of compact convex bodies.

Lemma 4.1. *Let γ be a periodic Reeb orbit on Σ. Then there is a boundary regular symplectic immersion $f : (D, \partial D) \to (C, \gamma)$.*

Proof. Let γ be a periodic Reeb orbit on the boundary Σ of a compact strictly convex body in $\mathbb{C}^2$. Choose two distinct points $a \neq b$ on γ and smooth parametrizations $\gamma_1, \gamma_2 : [0, \pi] \to \gamma$ of the two subarcs of γ connecting a to b. We assume that the orientation of γ_2 coincides with the orientation of γ and that the parametrizations γ_1, γ_2 coincide near a, b with the parametrization of γ up to translation and reflection in the real line. Define a map $f : (D, \partial D) \to (C, \gamma)$ as follows. Let $\tilde{\gamma}_1, \tilde{\gamma}_2 : [0, \pi] \to S^1$ be parametrizations by arc length of the two half-circles of the unit circle $S^1 \subset \mathbb{C}$ connecting 1 to -1, chosen in such a way that the orientation of $\tilde{\gamma}_2$ coincides with the orientation of ∂D. We require that f maps the line segment in D connecting $\tilde{\gamma}_1(t)$ to $\tilde{\gamma}_2(t)$ which is parametrized by arc length to the line segment in the convex body $C \subset \mathbb{C}^2$ connecting $\gamma_1(t)$ to $\gamma_2(t)$ and parametrized propotional to arc length on the same parameter interval. By construction, the map f is smooth, moreover it is symplectic near the points $1, -1$.

We claim that f is a symplectic immersion. For this let as before $\langle , \rangle$ be the usual euclidean inner product on $\mathbb{R}^4 = \mathbb{C}^2$. Let $t \in (0, \pi)$ and consider the straight line segment ℓ in C connecting $\gamma_1(t)$ to $\gamma_2(t)$. By strict convexity of C, the arc ℓ is contained in C and intersects Σ transversely at the endpoints. Let X, Y be the tangents of ℓ at the endpoints $\gamma_1(t), \gamma_2(t)$ and let as before N be the outer normal field of Σ. Then $\langle X, N(\gamma_1(t))\rangle < 0, \langle Y, N(\gamma_2(t))\rangle > 0$ and hence since $\gamma_1'(t) = -a_1 JN(\gamma(t)), \gamma_2'(t) = a_2 JN(\gamma_2(t))$ for some numbers $a_1 > 0, a_2 > 0$ we have $\omega_0(X, \gamma_1'(t)) > 0$ and $\omega_0(Y, \gamma_2'(t)) > 0$. Now with respect to the usual trivialization of $T\mathbb{C}^2$ we have $X = Y$. On the other hand, by the construction of the map f, for every point $s \in \ell$ the tangent space of $f(D)$ at s is spanned by $X = Y$ and a convex linear combination of $\gamma_1'(t), \gamma_2'(t)$. This shows that f is a symplectic immersion. Moreover f is clearly boundary regular whence the lemma. ☐

We call an immersion $f : D \to C$ as in Lemma 4.1 a *linear filling* of the Reeb orbit γ. By Corollary 3.7, if $\mathrm{lk}(\gamma) = -1$ then a linear filling f of γ satisfies $\tan(f) = 0$. However, an immersed symplectic disc may have transverse self-intersection points of negative intersection index, so there is no obvious relation between the tangential index of a boundary regular immersed symplectic disc and the number of its self-intersection points. On the other hand, if γ admits an embedded linear filling then Corollary 3.7 implies that $\mathrm{lk}(\gamma) = -1$.

Our final goal is to relate the Maslov index of a periodic Reeb orbit γ to the geometry of the hypersurface Σ. For this consider for the moment an arbitrary bounded domain $\Omega \subset \mathbb{C}^2$ with smooth boundary Σ which is star-shaped with respect to the origin. Write $C = \Omega \cup \Sigma$. As before, denote by J the usual complex structure on $\mathbb{C}^2$ and let $\langle , \rangle$ be the *euclidean* inner product. The restriction λ of the radial one-form λ_0 on $\mathbb{C}^2$ defined by $(\lambda_0)_p(Y) = \frac{1}{2}\langle Jp, Y\rangle$ $(p \in \mathbb{C}^2, Y \in T_p\mathbb{C}^2)$ defines a smooth contact structure on Σ.

Let N be the outer unit normal field of Σ. As in Section 2 write $\hat{M}(z_1, z_2) = (-\bar{z}_2, \bar{z}_1)$ and let M be the section of $T\Sigma$ defined by $M(p) = \hat{M} \circ N$. Its image is contained in the complex line subbundle L of the tangent bundle of Σ. The sections M, JM define a global trivialization of L which is symplectic with respect to the restriction of the symplectic form ω_0.

The kernel ξ of the contact form is a smooth real 2-dimensional subbundle of $T\Sigma$. Orthogonal projection P of $T\Sigma$ onto L defines a smooth bundle epimorphism whose kernel is the annihilator of the restriction of ω_0 to $T\Sigma$. Thus the morphism P preserves the restriction to $T\Sigma, L$ of the symplectic form and therefore its restriction to the subbundle ξ of $T\Sigma$ is a real symplectic bundle isomorphism whose inverse $\pi : L \to \xi$ is symplectic as well. Since by construction the sections M, JM of L form a symplectic basis of L we have.

Lemma 4.2. *The smooth sections $\pi \circ M, \pi \circ JM$ of the bundle ξ define a symplectic trivialization $T : \xi \to (\mathbb{R}^2, dx \wedge dy)$.*

In other words, for each $p \in \Sigma$ the restriction T_p of T to ξ_p is an area preserving linear map $T_p : (\xi_p, \omega_0) \to (\mathbb{R}^2, dx \wedge dy)$.

Recall from Section 2 that the Reeb vector field X on Σ is given by

$$X(p) = \varphi(p) JN(p)$$

where

$$\varphi(p) = \frac{2}{\langle p, N(p) \rangle} > 0.$$

Denote by $\Psi_t : \Sigma \to \Sigma$ the Reeb-flow of (Σ, λ) and let γ be a periodic orbit for Ψ_t of period $\chi > 0$. The Reeb orbit γ is called *non-degenerate* if the restriction of the differential $d\Psi_\chi$ of Ψ_χ to the bundle ξ does not have one as an eigenvalue. Our goal is to compute the *Maslov-index* $\mu(\gamma)$ of the Reeb orbit γ using the trivialization T of ξ along γ defined by the vector fields $\pi M, \pi JM$ as in Lemma 4.2.

Let $\mathcal{S}$ be the family of all smooth arcs $c : [0, \chi] \to SL(2, \mathbb{R})$ in the symplectic group $SL(2, \mathbb{R})$ which begin at $c(0) = \mathrm{Id}$ and such that $c(\chi)$ does not have one as an eigenvalue. Using our above trivialization T of the bundle ξ we obtain a curve $\Phi \in \mathcal{S}$ by defining

$$\Phi(t) := T_{\Psi_t(p)} \circ d\Psi_t(p) \circ T_p^{-1}.$$

where $p = \gamma(0)$. The orbit γ is non-degenerate if and only if $\Phi(\chi)$ does not have one as an eigenvalue, and in this case the Maslov index $\mu(\gamma)$ of γ is defined as the Maslov index of $\Phi \in \mathcal{S}$.

For the calculation of this index we use complex coordinates and view the vector fields N, M as $\mathbb{C}^2$-valued functions on Σ. For a curve $\gamma : [0, b] \to \Sigma$ we abbreviate $N(t) = N(\gamma(t))$ and $M(t) = M(\gamma(t))$. Define a $U(2)$-valued curve $O : [0, \chi] \to U(2)$ by the requirement that for each $t \in [0, \chi]$, $O(t)$ is given with respect to the standard basis of $\mathbb{C}^2$ by the matrix

$$O(t) := (N(t), M(t)), \quad \text{i.e. } O(t) \begin{pmatrix} a \\ b \end{pmatrix} = aN(t) + bM(t) \quad \text{for } a, b \in \mathbb{C}.$$

The image of the complex line $\{0\} \times \mathbb{C} \subset \mathbb{C}^2$ under the map $O(t)$ is just the complex line $L(\gamma(t))$. Therefore for each t, $\pi \circ O(t)$ is an $\mathbb{R}$-linear isomorphism of $\{0\} \times \mathbb{C}$ onto $\xi(\gamma(t))$, and we have

$$T_{\gamma(t)}^{-1} = \pi \circ O(t)|_{0 \times \mathbb{C}}.$$

As a consequence, we can write $\Phi(t)$ in the form

$$\Phi(t) := O(t)^{-1} \pi^{-1} d\Psi_t \pi O(0)|_{0 \times \mathbb{C}}.$$

From now on we use coordinates in $\mathbb{C}^2$. Without loss of generality we can assume that $\xi_{\gamma(0)} = L_{\gamma(0)}$ and hence $\pi M(0) = M(0) = M_0$. The Maslov index of Φ is then the variation of the argument of

$$O(t)^{-1}\pi^{-1}d\Psi_t M_0 : [0,T] \to \mathbb{C} - \{0\}.$$

If we define a unitary 2×2 matrix $U(t)$ as

$$U(t) := \left(N(t), \frac{\pi^{-1}d\Psi_t M_0}{\|\pi^{-1}d\Psi_t M_0\|} \right),$$

then the turning angle of $O(t)^{-1}\pi^{-1}d\Psi_t M_0$ about zero is just the argument of $\det(U(t))$.

Define a unit vector field $\tilde{M}$ along γ by

$$\tilde{M}(t) = \frac{\pi^{-1}d\Psi_t M_0}{\|\pi^{-1}d\Psi_t M_0\|}.$$

The following lemma is the main technical tool for a calculation of the Maslov index of γ. For its formulation, recall that the *second fundamental form* of the hypersurface Σ in $\mathbb{C}^2$ is the symmetric bilinear form $\Pi : T\Sigma \times T\Sigma \to \mathbb{R}$ which is defined as follows. Let X, Y be vector fields on $\Sigma \subset \mathbb{R}^4$; then

$$\Pi(X,Y) = -\langle dY(X), N\rangle = \langle dN(X), Y\rangle.$$

The *shape operator* of Σ is the section A of the bundle $T^*\Sigma \otimes T\Sigma$ defined by $\Pi(X,Y) = \langle AX, Y\rangle$.

Lemma 4.3. *For* $t_0 \in [0,T]$ *we have*

$$\begin{aligned} \frac{\partial}{\partial t}\det(U(t))|_{t=t_0} &= \frac{\partial}{\partial t}O(t)^{-1}\tilde{M}(t)|_{t=t_0} \\ = i\varphi(\gamma(t_0))(\Pi(JN(t_0), JN(t_0)) &+ \Pi(\tilde{M}(t_0), \tilde{M}(t_0)))\det(U(t)) \end{aligned}$$

with $\varphi(\gamma(t_0)) = \|X(\gamma(t_0))\| = \|\gamma'(t_0)\| = \frac{2}{\langle p, N(t_0)\rangle}$.

Proof. In the sequel we always view the second fundamental form Π of Σ as a bilinear form on a subspace of $\mathbb{R}^4$. Let $\pi_2 : \mathbb{C}^2 \to \{0\} \times \mathbb{C}$ be the orthogonal projection. Using the simple fact that

$$O^{-1} \circ \pi^{-1} = \pi_2 \circ O^{-1}$$

we deduce

$$\begin{aligned} \frac{\partial}{\partial t}O(t)^{-1}\tilde{M}(t) &= \frac{\partial}{\partial t}\frac{O(t)^{-1}\pi^{-1}d\Psi_t M_0}{\|\pi^{-1}d\Psi_t M_0\|}|_{t=t_0} \\ &= \pi_2\big(\frac{\partial}{\partial t}O(t)^{-1}|_{t=t_0}\big)\frac{d\Psi_{t_0}M_0}{\|\pi^{-1}d\Psi_{t_0}M_0\|} \\ &+ \pi_2 O(t_0)^{-1}\big(\frac{\partial}{\partial t}\left.\frac{1}{\|\pi^{-1}d\Psi_t M_0\|}\right|_{t=t_0}\big)d\Psi_{t_0}M_0 \\ &+ \pi_2 O(t_0)^{-1}\frac{1}{\|\pi^{-1}d\Psi_{t_0}M_0\|}\big(\frac{\partial}{\partial t}d\Psi_t M_0|_{t=t_0}\big). \end{aligned} \tag{1}$$

The first term in our equation can be rewritten as

$$\begin{aligned} &\pi_2\big(\frac{\partial}{\partial t}O(t)^{-1}|_{t=t_0}\big)\frac{d\Psi_{t_0}M_0}{\|\pi^{-1}d\Psi_{t_0}M_0\|} \\ = -\pi_2 O(t_0)^{-1}&\big(\frac{\partial}{\partial t}O(t)|_{t=t_0}\big)O(t_0)^{-1}\frac{d\Psi_{t_0}M_0}{\|\pi^{-1}d\Psi_{t_0}M_0\|} \\ &= *. \end{aligned}$$

By definition, for every t the vectors $\{N(t),\tilde{M}(t)\}$ form a unitary basis of $\mathbb{C}^2$. Since $\{N(t),M(t)\}$ is also such a unitary basis, there is a smooth function $\psi:[0,\chi]\to\mathbb{R}$ such that $\tilde{M}(t)=e^{i\psi(t)}M(t)$ for all t. We now use the second fundamental form Π to calculate the differential of the matrix valued curve $O(t)=(N(t),M(t))$. For this recall the definition of the shape operator $A:T\Sigma\to T\Sigma$ of Σ and of the orthogonal projection $P:T\Sigma\to L$. Since $M(t)=\hat{M}N(t)$ and $\gamma'(t)=\varphi(t)JN(t)$ we have

$$\frac{\partial}{\partial t}O(t)=\varphi(t)(AJN(t),\hat{M}AJN(t)).$$

Thus with respect to the complex basis $(N(t),M(t))$ of $\mathbb{C}^2$ we have

$$\frac{\partial}{\partial t}O(t)=\varphi(t)\begin{pmatrix} i\Pi(JN(t),JN(t)) & -\overline{PA(JN(t))} \\ PA(JN(t)) & -i\Pi(JN(t),JN(t)) \end{pmatrix}$$

where we view $PA(\varphi JN(t))$ as a complex multiple of $M(t)$.

By the definition of the function ψ we can write

$$\frac{O(t_0)^{-1}d\Psi_{t_0}M_0}{\|\pi^{-1}d\Psi_{t_0}M_0\|}=\begin{pmatrix} ic(t_0) \\ e^{i\psi(t_0)} \end{pmatrix} \quad \text{for some } c(t_0)\in\mathbb{R}.$$

Since $\tilde{M}(t)=e^{i\psi(t)}M(t)$ and $O^{-1}(t)M(t)=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), O^{-1}N(t)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ we deduce that

$$* = -\,\pi_2\varphi(t_0)\begin{pmatrix} -c(t_0)\Pi(JN(t_0),JN(t_0)) - e^{i\psi(t_0)}\overline{PA(JN(t_0))} \\ ic(t_0)PA(JN(t_0)) - ie^{i\psi(t_0)}\Pi(JN(t_0),JN(t_0)) \end{pmatrix}.$$

Now let exp be the exponential map of the hypersurface Σ with respect to the Riemannian metric induced by the euclidean metric. We use this exponential map to compute the third term in (1).

$$\begin{aligned} \pi_2 O(t_0)^{-1}\frac{\frac{\partial}{\partial t}d\Psi_t M_0|_{t=t_0}}{\|\pi^{-1}d\Psi_{t_0}M_0\|} &= \frac{\pi_2 O(t_0)^{-1}}{\|\pi^{-1}d\Psi_{t_0}M_0\|}\frac{\partial}{\partial t}\frac{\partial}{\partial s}\Psi_t(\exp_{\gamma(0)}(sM_0))|_{t=t_0,s=0} \\ &= \frac{\pi_2 O(t_0)^{-1}}{\|\pi^{-1}d\Psi_{t_0}M_0\|}\frac{\partial}{\partial s}(\varphi(\Psi_{t_0}(\exp_{x_0}(sM_0)))JN(\Psi_{t_0}(\exp_{\gamma(0)}(sM_0)))) \\ = \pi_2 O^{-1}(t_0)\left(\frac{1}{\|\pi^{-1}d\Psi_{t_0}M_0\|}\right.&\left.(\frac{\partial}{\partial s}\varphi(\Psi_{t_0}(sM_0))JN(t_0)|_{s=0}\right)+\varphi JA(d\Psi_{t_0}M_0)) \\ &= \pi_2 Q^{-1}(t_0)\varphi(t_0)JA(\tilde{M}(t_0)) \\ = \varphi(t_0)e^{i\psi(t_0)}\big(i\Pi(\tilde{M},\tilde{M})&-\Pi(\tilde{M},J\tilde{M})+ic\Pi(JN,\tilde{M})-c\Pi(JN,J\tilde{M})\big). \end{aligned}$$

The tangent vector of a curve c in $\mathbb{C}$ with constant norm always has the form $\dot{c} = irc$ for some $r \in \mathbb{R}$, so we can neglect the radial parts of the above equations. Summing up the three terms in (1) yields

$$\frac{\partial}{\partial t}\det(U(t))|_{t=t_0} = i\varphi(\Pi(JN, JN) + \Pi(\tilde{M}, \tilde{M}))\det(U(t_0)).$$

□

From Lemma 4.3 and the definition of the Maslov index we immediately obtain the following proposition.

Proposition 4.4. *Let γ be a closed non-degenerate Reeb-orbit on Σ with period T. Then*

$$\begin{aligned}\int_0^T |\dot{\gamma}|(\Pi(JN, JN) + \Pi(\tilde{M}, \tilde{M}))dt &\in (2k\pi, 2(k+1)\pi), \textit{ for } \mu(\gamma) = 2k+1,\\ \int_0^T |\dot{\gamma}|(\Pi(JN, JN) + \Pi(\tilde{M}, \tilde{M}))dt &= 2k\pi \qquad\qquad , \textit{ for } \mu(\gamma) = 2k,\end{aligned}$$

where $\tilde{M}(t) = \frac{\pi^{-1}d\Psi_t M_0}{\|\pi^{-1}d\Psi_t M_0\|}$.

Proof. Let γ be a non-degenerate periodic Reeb orbit on Σ of period T. Using the notations from Lemma 4.3 and its proof, the Maslov index of γ equals the Maslov index of the path $t \to \det U(t)$ $(t \in [0, T])$ and hence the proposition is immediate from Lemma 4.3. □

The Maslov index of a degenerate periodic Reeb orbit on Σ can be calculated in the same way. We restrict our attention to Reeb orbits of Maslov index 3.

Corollary 4.5. *Let γ be a periodic Reeb orbit on Σ with period T and Maslov index 3. Then*

$$\int_0^T |\dot{\gamma}|(\Pi(JN, JN) + \Pi(\tilde{M}, \tilde{M}))dt \in [2\pi, 4\pi).$$

Proof. By Proposition 4.4 we only have to show the corollary for degenerate Reeb orbits γ on Σ. Using again the notations from Lemma 4.3 and its proof, in this case the curve $t \to U(t)$ in the unitary group $U(2)$ is closed and defines a homotopy class $\alpha \in \pi_1(U(2)) = \mathbb{Z}$. With respect to the canonical generator e of the $\pi_1(U(2))$, we have $\alpha = ke$ and the Maslov index of γ equals $2k + 1$ (see [HWZ99]). As before, our claim now follows from Proposition 4.4. □

Now we specialize again to the case that C is a compact strictly convex body in $\mathbb{C}^2$ with smooth boundary Σ which contains the origin in its interior. Recall that the *total curvature* of a smooth curve $\gamma : [0, t] \to \mathbb{C}^2$ parametrized by arc length is defined by

$$\kappa(\gamma) = \int_0^T \|\gamma''(t)\|dt.$$

The next corollary is immediate from Corollary 4.5.

Corollary 4.6. *Let Σ be the boundary of a compact strictly convex domain $C \subset \mathbb{C}^2$. If the principal curvatures $a \geq b \geq c$ of Σ satisfy the pointwise pinching condition $a \leq b + c$ then the total curvature of a periodic Reeb orbit of Maslov index 3 is smaller than 4π.*

Proof. Let Σ be the boundary of a strictly convex domain $C \subset \mathbb{C}^2$ with principal curvatures $a \geq b \geq c$ satisfying the pinching condition $a \leq b+c$. Let $\gamma : [0,T] \to \Sigma$ be a periodic Reeb orbit of Maslov index 3. We assume that γ is parametrized by arc length on $[0,T]$.

Denote by $N(t)$ the normal field of the sphere restricted to the curve γ. Then $\gamma'(t) = JN(t)$ and therefore

$$\kappa(\gamma) = \int_0^T \|\frac{\partial}{\partial t} N(t)\| dt \leq \int_0^T a(\gamma(t))dt \leq \int_0^T b(\gamma(t)) + c(\gamma(t))dt < 4\pi$$

by Corollary 4.5. □

We use Corollary 4.6 to complete the proof of Theorem 2 from the introduction.

Proposition 4.7. *Let Σ be the boundary of a compact strictly convex domain $C \subset \mathbb{C}^2$. If the principal curvatures $a \geq b \geq c$ of Σ satisfy the pointwise pinching condition $a \leq b + c$ then a periodic Reeb orbit on Σ of Maslov index 3 bounds an embedded symplectic disc $f : (D, \partial D) \to (C, \gamma)$. In particular, γ has self-linking number -1.*

Proof. Define the *crookedness* of a smooth closed curve $\gamma : S^1 \to \mathbb{R}^4$ to be the minimum of the numbers $m(\gamma, v)$ where $m(\gamma, v)$ is the number of minima of the function $t \to \langle \gamma(t), v\rangle, v \in S^3$. By a result of Milnor [Mil50], the crookedness of a curve of total curvature smaller than 4π equals one. Thus by Corollary 4.6, if γ is a periodic Reeb orbit on Σ of Maslov index 3 then there is some $v \in S^3$ such that the restriction to γ of the function $\varphi : x \to \langle x, v\rangle$ assumes precisely one maximum and one minimum. We may moreover assume that these are the only critical points of the restriction of φ to γ and that they are non-degenerate (see [Mil50]).

Let a be the unique minimum of φ on γ. Assume that γ is parametrized in such a way that $\gamma(0) = a$. Let $\gamma_2 : [0, \sigma] \to \Sigma$ be the parametrized subarc of γ issuing from $\gamma_2(0) = a$ which connects a to the unique maximum b of φ on γ. Let $\gamma_1 : [0, \sigma] \to \Sigma$ be the parametrization of the second subarc of γ connecting a to b such that $\varphi(\gamma_1(t)) = \varphi(\gamma_2(t))$ for all $t \in [0, \sigma]$; this is possible by construction and by our choice of φ. Then the symplectic disc obtained from this parametrization by linear filling as in the proof of Lemma 4.1 is embedded. By Corollary 3.7, this implies that the self-linking number of γ equals -1. □

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