Chapter 1

Submanifold projections and hyperbolicity in $\mathrm{O}ut(F_n)$

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The free splitting graph of a free group F_n with $n \geq 2$ generators is a hyperbolic $\mathrm{O}ut(F_n)$-graph which has a geometric realization as a sphere graph in the connected sum of n copies of $S^1 \times S^2$. We use this realization to construct submanifold projections of the free splitting graph into the free splitting graphs of proper free factors. This is used to construct for $n \geq 3$ a new hyperbolic $\mathrm{O}ut(F_n)$-graph. If $n = 3$, then every exponentially growing element acts on this graph with positive translation length.

1.1 Introduction

The *free factor graph* $\mathcal{FF}(F_n)$ for a free group F_n of rank $n \geq 2$ is the graph whose vertices are conjugacy classes of free factors of F_n and where two such free factors A_1, A_2 are connected by an edge of length one if up to a global conjugation we have $A_1 \subset A_2$ or $A_2 \subset A_1$. The free factor graph is a locally infinite Gromov hyperbolic geodesic metric graph, and the outer automorphism group $\mathrm{O}ut(F_n)$ of F_n acts as a group of simplicial automorphisms on $\mathcal{FF}(F_n)$ [3].

There are other natural Gromov hyperbolic geodesic metric $\mathrm{O}ut(F_n)$-graphs. The best known is the so-called *free splitting graph* [7], whose first barycentric subdivision $\mathcal{FS}(F_n)$ is defined as follows. The vertices of $\mathcal{FS}(F_n)$ are graph of groups decompositions of F_n with trivial edge groups. Two such graph of groups decompositions G, G' are connected by an edge of length one if G' either is a collapse or a blow-up of G.

2020 Mathematics Subject Classification. 20M34.
Keywords. Sphere graphs, $\mathrm{O}ut(F_n)$, surgery sequences, hyperbolic graphs, uniform exponential growth, quasi-geodesics.

In view of the geometric understanding of the mapping class group of a closed surface S of genus at least 2 via its action on the curve graph of S and the curve graph of subsurfaces using subsurface projections, the graph $\mathcal{FS}(F_n)$ is significant for the geometric understanding of $\mathrm{O}ut(F_n)$. However, much less is known about $\mathcal{FS}(F_n)$ than about the free factor graph, and the action of $\mathrm{O}ut(F_n)$ is more complicated. For example, it was observed in [8] that for sufficiently large n there are free abelian subgroups of $\mathrm{O}ut(F_n)$ which act by loxodromic isometries on $\mathcal{FS}(F_n)$, with the same pair of fixed points on the Gromov boundary of $\mathcal{FS}(F_n)$.

In spite of this difficulty, it turns out that there is hyperbolicity in $\mathrm{O}ut(F_n)$ beyond the free splitting graph. This is clear for $n = 2$ since $\mathrm{O}ut(F_2) = \mathrm{G}L(2,\mathbb{Z})$ is a hyperbolic group. The following is our first result.

Theorem 1. *For $n \geq 3$ there exists a hyperbolic geodesic metric $Out(F_n)$-graph $\mathcal{PG}_n$ which admits an equivariant one-Lipschitz projection onto the free splitting graph. If $n = 3$ then every exponentially growing automorphism acts with positive translation length on $\mathcal{PG}_n$.*

For $n \geq 4$ the graph $\mathcal{PG}_n$ does not have the property that every exponentially growing automorphism acts on it with positive translation length. In fact, it seems unlikely that such a hyperbolic $\mathrm{O}ut(F_n)$-graph exists for all n. This expected distinction between $\mathrm{O}ut(F_3)$ and $\mathrm{O}ut(F_n)$ for $n \geq 4$ may correspond to the notable distinction between the automorphism groups $\mathrm{A}ut(F_3)$ and $\mathrm{A}ut(F_n)$ for $n \geq 5$: While the former group is large, that is, it virtually surjects onto a nonabelian free group (Corollary 1.3 of [5]), the group $\mathrm{A}ut(F_n)$ has property (T) for $n \geq 5$ [13].

Theorem 1 can be thought of as a strengthening in rank 3 of the following main result of [4].

Theorem 2 (Theorem 5.1 of [4]). *The group $Out(F_n)$ acts by isometries on a product $Y = Y_1 \times \cdots \times Y_k$ of $k > n$ hyperbolic spaces so that every exponentially growing automorphism has positive translation length.*

While the proof of Theorem 2 uses the free factor graph and the action of $\mathrm{O}ut(F_n)$ on Outer space as the main tool, we use a more topological viewpoint based on the so-called *sphere system graph* [9] which is defined as follows.

Let $M = S^1 \times S^2 \sharp \ldots \sharp S^1 \times S^2$ be the connected sum of n copies of $S^1 \times S^2$. Then M is a closed manifold whose fundamental group equals the free group F_n with n generators.

A *sphere* in M is an embedded sphere which is not homotopic to a point. A *sphere system* is a collection of pairwise disjoint not mutually homotopic spheres in M. The sphere system is called *simple* if it decomposes M into a union of balls.

Denote by $\mathcal{SSG}_n$ the locally finite graph whose vertices are isotopy classes of simple sphere systems in M and where two such simple sphere systems are connected by an edge of length one if they can be realized disjointly. The graph $\mathcal{SSG}_n$ is connected, and the group $\mathrm{O}ut(F_n)$ acts on the graph $\mathcal{SSG}_n$ properly and cocompactly by work of Laudenbach [14]. Thus $\mathcal{SSG}_n$ is a geometric model for $\mathrm{O}ut(F_n)$. Another geometric model is the *spine of Outer space*, where Outer space is the space of marked metric graphs of volume one, with fundamental group F_n, and its spine consists of all graphs without loops of length at most ϵ for a suitable choice of $\epsilon > 0$.

Any sphere in M defines up to conjugation a one-edge free splitting of F_n, that is, a vertex in $\mathcal{FS}(F_n)$, and two disjoint spheres S_1, S_2 define a two-edge free splitting which collapses to the free splittings defined by S_1, S_2, that is, they define an edge in $\mathcal{FS}(F_n)$. Thus the *sphere graph* $\mathcal{SG}_n$ whose set of vertices is the set of isotopy classes of spheres in M and whose edges connect spheres which can be realized disjointly is a topological model for the free splitting graph. There also is a natural coarsely well defined coarsely $\mathrm{O}ut(F_n)$-equivariant two-Lipschitz projection

$$\Theta : \mathcal{SSG}_n \to \mathcal{SG}_n$$

which associates to a simple sphere system one of its components.

As for Outer space, there are distinguished paths in $\mathcal{SSG}_n$ connecting any two simple spheres systems as follows. Let S be a sphere which intersects the simple sphere system Σ. Assume that S is in minimal position with respect to Σ; this implies that S intersects Σ in the minimal number of components, and each of these components is an embedded circle in S (see [11] for a detailed account on these facts).

An innermost such circle bounds an embedded disk D in $S - \Sigma$. Its boundary ∂D is contained in a sphere $S_0 \in \Sigma$. Replace S_0 by the spheres obtained by gluing D to each of the two components of $S_0 - D$. These spheres are disjoint from Σ. By Lemma 3.1 of [9], the sphere system Σ_1, obtained from the union of these two spheres with $\Sigma - S_0$ by removing parallel copies of the same sphere if there are any, is simple, and it has fewer intersections with S than Σ. We call Σ_1 a *sphere system obtained by surgery* of Σ along S. Note that this notion is also defined if S is a component of a sphere system Σ'.

Repetition of this construction gives rise to so-called *surgery sequences* which can be thought of as the vertices of distinguished edge paths in $\mathcal{SSG}_n$. It was shown in [11] that there exists a number $L > 1$ such that the image by the map Θ of such an edge path β in $\mathcal{SSG}_n$ is an *unparameterized* L*-quasi-geodesic* in $\mathcal{SG}_n$: If β is parameterized on $[0, m]$ then there exists an increasing homeomorphism $\rho : [a, b] \to [0, m]$ such that the path $\Theta(\beta \circ \rho)$ is an L-quasi-geodesic, that is, it satisfies

$$\begin{aligned} d_{\mathcal{SG}_n}(\Theta(\beta \circ \rho(s)), \Theta(\beta \circ \rho(t)))/L - L \leq |s - t| \\ \leq L d_{\mathcal{SG}_n}(\Theta(\beta \circ \rho(s)), \Theta(\beta \circ \rho(t))) + L \end{aligned}$$

where $d_{\mathcal{SG}_n}$ denotes the distance in the sphere graph.

While surgery sequences define unparameterized quasi-geodesics in the sphere graph, the same does not hold in general for the sphere system graph. We give an example for this in Section 1.4. In Section 1.3, we show that exponentially growing surgery sequences are uniform unparameterized quasi-geodesics in $\mathcal{SSG}_n$. For a precise formulation, we choose for each simple sphere system Σ a dual rose $R(\Sigma)$ and define a surgery sequence $(\Sigma_i)_{0 \leq i \leq m}$ to be of uniformly exponential growth if the minimal number of intersections between $R(\Sigma_i)$ and Σ_j is uniformly exponentially growing in $|i - j|$ for all i, j. The following is our second result.

Theorem 3. *A surgery sequence of uniform exponential growth is a uniform quasi-geodesic in* $\mathcal{SSG}_n$.

Hyperbolicity of the sphere graph and the fact that surgery sequences are uniform unparameterized quasi-geodesics in the sphere graph can be used to control *submanifold projections* of the sphere graph into the sphere graphs of manifolds $M(\sigma)$, obtained by cutting M open along a *non-separating* sphere σ and filling in the boundary by attaching a ball to each boundary component. These submanifold projections are defined as follows.

Let $\sigma \subset M$ be a non–separating sphere. The manifold $M(\sigma)$ is homeomorphic to the connected sum of $n - 1$ copies of $S^1 \times S^2$. Given a non-separating sphere $S \subset M$ distinct from σ, define the *projection* $p_{M(\sigma)}(S) \subset M(\sigma)$ of S into $M(\sigma)$ as follows. If $S \subset M - \sigma$ then put $p_{M(\sigma)}(S) = S \subset M(\sigma)$. This is well defined as since S is non-separating, it is essential as a sphere in $M(\sigma)$. If S intersects σ, then choose an innermost disk $D \subset S$ with boundary on σ and define $p_{M(\sigma)}(S)$ to be the sphere in $M(\sigma)$ which is the union of D with one of the two components of $\sigma - D$. We observe in Section 1.5 that this is indeed an essential sphere in $M(\sigma)$. Furthermore, it determines a point in the sphere graph of $M(\sigma)$ which coarsely does not depend on choices. This projection extends to separating spheres in the same way, with the exception of separating spheres disjoint from σ that are inessential as spheres in $M(\sigma)$.

We use this projection and its geometric properties as the main tool for the construction of the graph $\mathcal{PG}_n$ from Theorem 1.

In [4], a notion of subsurface projection of a free factor into the free splitting complex of another free factor is defined. The article [15] contains an approach similar to ours. The relation between these two constructions was worked out in [16], see in particular Proposition 5.4 of [16] for relevance to this article.

The outline of this article is as follows. In Section 1.2, which is of independent interest and not strictly needed in the sequel, we define a family of $\mathrm{O}ut(F_n)$-graphs and show that they interpolate between the free factor graph and the free splitting graph. We show that these graphs are all hyperbolic.

In Section 1.3 we introduce the notion of *exponential growth* for surgery sequences in the simple sphere system graph and show Theorem 3. Furthermore, we observe that a surgery sequence which projects to a *parameterized* quasi-geodesic in the sphere graph has exponential growth.

In Section 1.4 we give a detailed analysis of the case $n = 2$. We show that in this case, exponential growth of a surgery sequence is equivalent to stating that its projection to the sphere graph is a parameterized quasi-geodesic. This is not true any more for $n \geq 3$. Moreover, for $n \geq 3$ we construct surgery sequences which do not define quasi-geodesics in the sphere system graph.

Section 1.5 is devoted to the construction of submanifold projections. Most importantly, we show the bounded geodesic image property which is an essential tool towards the proof of Theorem 1. The proof of Theorem 1 is contained in Section 1.6.

1.2 Graphs of free factors

In this section we introduce a family of graphs which interpolate between the free factor graph and the free splitting graph. We assume that $n \geq 3$ throughout.

Definition 1.2.1. For $m \leq n-2$, the *level m free factor graph* is the graph $\mathcal{FF}_m(F_n)$ whose vertices are conjugacy classes of free factors of rank $n-1$, and where two such free factors A_1, A_2 are connected by an edge of length one if up to a global conjugation, $A_1 \cap A_2$ contains a free factor of rank m.

Clearly the graphs $\mathcal{FF}_m(F_n)$ are geodesic $\mathrm{Out}(F_n)$-graphs. Furthermore, they all have the same set of vertices, and for each $m \geq 2$ the vertex inclusion defines an embedding $\mathcal{FF}_m(F_n) \to \mathcal{FF}_{m-1}(F_n)$. In other words, $\mathcal{FF}_m(F_n)$ is obtained from $\mathcal{FF}_{m-1}(F_n)$ by deleting some edges. The next proposition justifies the terminology.

Proposition 1.2.2. *The vertex inclusion defines a* 2*-quasi-isometry*

$$\mathcal{FF}_1(F_n) \to \mathcal{FF}(F_n).$$

Proof. Since every vertex of $\mathcal{FF}(F_n)$ is of distance one to a rank $n-1$ free factor, the image of the vertex inclusion $\mathcal{FF}_1(F_n) \to \mathcal{FF}(F_n)$ is coarsely dense in $\mathcal{FF}(F_n)$. Furthermore, by construction, any edge path $(A_i)_{0\leq i\leq k} \subset \mathcal{FF}_1(F_n)$ of length k induces (non-uniquely) an edge path in $\mathcal{FF}(F_n)$ of length $2k$ with the same endpoints by replacing an edge (A_i, A_{i+1}) in $\mathcal{FF}_1(F_n)$ by an edge path (A_i, B_i, A_{i+1}) in $\mathcal{FF}(F_n)$ of length two, where B_i is a free factor contained in the intersection $A_i \cap A_{i+1}$ which exists by the definition of $\mathcal{FF}_1(F_n)$.

Thus it suffices to show the following. Let A, B be corank one free factors and let (A_i) be a geodesic in the free factor graph $\mathcal{FF}(F_n)$ connecting $A = A_0$ to $B = A_m$. Then there exists a path (A_i') in $\mathcal{FF}_1(F_n)$ connecting A to B whose length does not exceed the length of the path (A_i).

To show that this is the case, note first that if $(A_j, A_{j+1}, A_{j+2}) \subset \mathcal{FF}(F_n)$ is an edge path of length 2 and if we have $A_j \subset A_{j+1} \subset A_{j+2}$, then A_j, A_{j+2} are connected by an edge in $\mathcal{FF}(F_n)$ and hence (A_j, A_{j+1}, A_{j+2}) is not a subarc of any geodesic in $\mathcal{FF}(F_n)$. As A_0 is of corank one and hence $A_1 \subset A_0$, we thus may assume that for all i, we have $A_{2i-1} \subset A_{2i} \supset A_{2i+1}$.

Then for each i, we may replace A_{2i} by a corank 1 free factor A_{2i}' containing A_{2i}. Since $A_{2i-1} \subset (A_{2i-2} \cap A_{2i})$ for all i, the assignment $i \to A_{2i}'$ then defines an edge path in $\mathcal{FF}_1(F_n)$ of half the length and the same endpoints, which is what we wanted to show. ■

Example 1.2.3. If $n = 3$ then there is only one graph $\mathcal{FF}_1(F_3)$, and by Proposition 1.2.2, it is 2-quasi-isometric to the free factor graph.

Our next goal is to relate the graph $\mathcal{FF}_{n-2}(F_n)$ to the free splitting graph. We use a topological version of this graph which was worked out carefully in [1].

Lemma 1.2.4. *The sphere graph of* M *is a topological realization of the free splitting graph* $\mathcal{FS}(F_n)$.

Proof. (Sketch) Each sphere $S \in \mathcal{SG}_n$ determines a one-edge free splitting of F_n. Namely, if S is non-separating, then for a choice of a basepoint $x \in M - S$, the subgroup of $\pi_1(M)$ of all homotopy classes of loops which are disjoint from S is a free factor of F_n of rank $n-1$, and S defines a one-vertex one-loop free splitting (an HNN-extension) of F_n. If S is separating, then S defines a one-edge free splitting of F_n by the Seifert van Kampen theorem.

Now let S' be a sphere which is disjoint from S. Then with the same argument, $S \cup S'$ defines a two edge free splitting which collapses to both the splitting defined by S and S'. Thus the sphere graph maps 2-quasi-isometrically into $\mathcal{FS}(F_n)$, with one-dense image. We refer to [1] for a complete proof. ■

We need two technical properties of the sphere graph $\mathcal{SG}_n$.

Lemma 1.2.5. *The subgraph of $\mathcal{SG}_n$ of all non-separating spheres in M is convex embedded in $\mathcal{SG}_n$: any two non-separating spheres can be connected by a geodesic in $\mathcal{SG}_n$ consisting of non-separating spheres.*

Proof. Let A, B be non-separating spheres, connected by a geodesic $(S_j)_{0\leq j\leq m}$. For each i consider the sphere S_{2i+1}. It is disjoint from both S_{2i} and S_{2i+2}. As (S_j) is a geodesic, if S_{2i+1} is separating then S_{2i}, S_{2i+2} are contained in the same component U of $M - S_{2i+1}$ since otherwise the sphere S_{2i+1} can be deleted from the sequence. Choose a non-separating sphere S'_{2i+1} in the component $M - U$ and replace S_{2i+1} by S'_{2i+1}. The resulting path is a geodesic, and each of the spheres with odd index are non-separating, while the spheres with even index are unchanged. Proceed in the same way with the spheres S_{2i}. ■

Define a subgraph $\mathcal{NSG}_n$ of $\mathcal{SG}_n$ as follows. The vertices of $\mathcal{NSG}_n$ are non-separating spheres, and two such spheres S_1, S_2 are connected by an edge of length one if they can be realized disjointly and if moreover $M - (S_1 \cup S_2)$ is connected.

The following is the analog of a well-known result for curve graphs.

Proposition 1.2.6. *The inclusion $\mathcal{NSG}_n \to \mathcal{SG}_n$ is a 2-quasi-isometry.*

Proof. Since every separating sphere is of distance one to a non-separating sphere, the graph $\mathcal{NSG}_n$ is one-dense in $\mathcal{SG}_n$. Furthermore, by Lemma 1.2.5, two vertices of $\mathcal{NSG}_n$ can be connected by a geodesic $(S_i) \subset \mathcal{SG}_n$ consisting of non-separating spheres.

It is possible that in the path (S_i), there are two adjacent spheres, say the spheres S_i, S_{i+1}, which form a bounding pair, that is, such that $M - (S_i \cup S_{i+1})$ is disconnected. We now replace successively each such pair S_i, S_{i+1} by an edge path S_i, D_i, S_{i+1} of length two such that $M - (S_i \cup D_i)$ and $M - (D_i \cup S_{i+1})$ are both connected. To see that this is possible note that if a bounding pair exists then $n \geq 3$. Then $M - (S_i \cup S_{i+1})$ contains a component which is a non-trivial connected sum of $S^1 \times S^2$ with the interiors of two balls removed. Such a manifold contains a non-separating embedded sphere D_i. This sphere is disjoint from $S_i \cup S_{i+1}$, and $M - (S_i \cup D_i)$ and $M - (D_i \cup S_{i+1})$ are both connected.

The length of the modified path (S_i') is at most twice the length of the path (S_i) connecting the same endpoints. Furthermore, any two consecutive vertices S_i', S_{i+1}' of this path have the property that $M - (S_i' \cup S_{i+1}')$ is connected. This completes the proof of the lemma. ■

Example 1.2.7. The free group F_2 with two generators is the fundamental group of a once punctured torus T. Each oriented non-peripheral simple closed curve c on T determines the conjugacy class of a primitive element of F_2, and any conjugacy class of a primitive element arises in this way. Now primitive elements in F_2 are precisely the generators of corank one free factors of F_2. Moreover, conjugacy classes of corank one free factors of F_2 are in bijection with non-separating spheres in the manifold M. Thus the vertices of $\mathcal{NSG}_2$ correspond precisely to the simple closed curves on T.

Two such conjugacy classes are connected by an edge in $\mathcal{NSG}_2$ if they correspond to disjoint spheres in M. This is the case if and only if they define a free basis of F_2, which is the case if and only if the simple closed curves on T defining these conjugacy classes intersect up to homotopy in precisely one point. As a consequence, the graph $\mathcal{NSG}_2$ is nothing else than the familiar *Farey graph.*

The relation between the free splitting graph $\mathcal{FS}(F_n)$ and the graph $\mathcal{FF}_{n-2}(F_n)$ is now a consequence of the following observation.

Lemma 1.2.8. *There exists a one-Lipschitz simplicial map* $\mathcal{NSG}_n \to \mathcal{FF}_{n-2}(F_n)$ *which is surjective on vertices.*

Proof. If S_1, S_2 are vertices in $\mathcal{NSG}_n$ which are connected by an edge, then for a choice of a basepoint $x \in M - (S_1 \cup S_2)$, the spheres S_i define corank one free factors A_1, A_2 of $F_n = \pi_1(M, x)$ of homotopy classes of loops disjoint from S_1, S_2, and these free factors intersect in the corank 2 free factor of homotopy classes of loops disjoint from both $S_1 \cup S_2$. Thus the edge between S_1 and S_2 in $\mathcal{NSG}_n$ defines an edge in the graph $\mathcal{FF}_{n-2}(F_n)$ as claimed in the lemma. ■

As an immediate consequence of Lemma 1.2.4, Lemma 1.2.6 and Lemma 1.2.8, we obtain

Corollary 1.2.9. *There exists a coarse two-Lipschitz map*

$$\mathcal{FS}(F_n) \to \mathcal{FF}_{n-2}(F_n)$$

which is surjective on vertices.

Example 1.2.10. If $n = 3$ then Proposition 1.2.2 shows that the free factor graph is 2-quasi-isometric to the graph $\mathcal{FF}_{n-2}(F_n)$. However, it is very different from the free splitting graph. Indeed, there are elements of $\mathrm{O}ut(F_3)$ which act on the free splitting graph as loxodromic isometries, but which fix a free factor. Such an example is discussed in Example 4.2 of [8]. It can be constructed with the help of a *relative train track map*.

The example can be viewed as a family of spheres in M which are all disjoint from a fixed simple loop defining a generator of F_3, but contain tubes winding around the loop.

Recall from the introduction that a sphere system Σ can be modified to another sphere system by a *surgery move* in direction of a sphere system Σ' as follows. Let $S' \in \Sigma'$, assumed to be in minimal position with respect to Σ and not disjoint from Σ. Then each component of $S' \cap \Sigma$ is an embedded circle in S'.

An innermost such circle bounds an embedded disk D in S'. Its boundary ∂D is contained in a sphere $S \in \Sigma$. The two spheres obtained by gluing D to each of the two components of $S - \partial D$ are disjoint and disjoint from Σ. Let Σ_1 be the union of $\Sigma - S$ with these two spheres, with parallel copies of the same sphere removed. By Lemma 3.1 of [9], if the sphere system Σ was simple, then Σ_1 is simple as well, and it has fewer intersections with Σ' than Σ.

Repetition of this construction, keeping the direction Σ' fixed (and starting in a second step from Σ_1) are called *surgery sequences* or surgery paths. They terminate in a sphere system disjoint from Σ'.

There is a natural coarsely well defined projection $\tau : \mathcal{NSG}_n \to \mathcal{FF}_m(F_n)$ which factors through the composition of the map from Lemma 1.2.8 with the inclusion

$\mathcal{FF}_{n-2}(F_n) \to \mathcal{FF}_m(F_n)$. As in [11], we use the images of surgery sequences under the map τ and an argument of [12] to show

Theorem 1.2.11. *Each of the graphs $\mathcal{FF}_m(F_n)$ $(m \leq n-2)$ is hyperbolic, and the natural projections of surgery paths are uniform unparameterized quasi-geodesics in $\mathcal{FF}_m(F_n)$.*

Proof. We follow [11] (the proof of Theorem 8.3). Let S_0, S_1 be non-separating spheres and assume that $\tau(S_0)$ and $\tau(S_1)$ are connected by an edge in $\mathcal{FF}_m(F_n)$. Then we can find an embedded rose R in M with vertex p and with m petals so that the inclusion $\pi_1(R,p) \to \pi_1(M,p)$ is π_1-injective and such that both S_0 and S_1 are disjoint from R.

Namely, let $\tilde{M}$ be obtained from M by removing the interior of a small ball from M. Put a basepoint p on the boundary of $\tilde{M}$. For any non-separating sphere S in M choose a lift $\tilde{S}$ of S to $\tilde{M}$. If $\tau(S_0), \tau(S_1)$ are connected in $\mathcal{FF}_m(F_n)$ by an edge then there exists $g \in F_n$ such that $\pi_1(\tilde{M} - \tilde{S}_0, p)$ and $\pi_1(\tilde{M} - g\tilde{S}_1 g^{-1}, p)$ contain a free factor of rank m defining the conjugacy class of a free factor as in the definition of an edge in $\mathcal{FF}_m(F_n)$. Here $g\tilde{S}_1 g^{-1}$ denotes the image of $\tilde{S}_1$ under a diffeomorphism of $\tilde{M}$ realizing the conjugation by g.

It follows from Lemma 2.2 of [10] that this free factor can be represented as the fundamental group of a rose R with m petals and basepoint at p which is disjoint from both $\tilde{S}_0$ and $g\tilde{S}_1\tilde{g}^{-1}$. Projection of this rose as well as the spheres $\tilde{S}_0, \tilde{S}_1$ to M yields the statement claimed in the first paragraph of this proof.

Since neither S_0 nor S_1 intersect the rose R, any surgery path connecting S_0 to S_1 consists of spheres disjoint from R. As surgery paths are uniform unparameterized quasi-geodesics in $\mathcal{SG}_n$ [11] and hence give rise to uniform unparameterized quasi-geodesics in $\mathcal{NSG}_n$ by Proposition 1.2.6, this implies that the fibers of the projection τ are uniformly quasi-convex: Any two points in a fiber are connected by a uniform quasi-geodesic in $\mathcal{NSG}_n$ which is entirely contained in this fiber.

As a consequence, we can apply the main result of [12]. We conclude that indeed, for any $m \leq n-2$ the level m free factor graph is hyperbolic, and surgery paths in $\mathcal{SG}_n$ (that is, edge paths in $\mathcal{NSG}_n$ at distance two from surgery paths in $\mathcal{SG}_n$) project to uniform unparameterized quasi-geodesics in $\mathcal{FF}_m(F_n)$. ■

Remark 1.2.12. For $1 \leq \ell < m \leq n-2$, we believe that the inclusion $\mathcal{FF}_m(F_n) \to \mathcal{FF}_\ell(n)$ is *not* a quasi-isometric embedding. As we did not find a short verification of this fact, we leave it to other authors. More precisely, we expect that a rotationless exponentially growing element $\varphi \in \mathrm{Out}(F_n)$ acts on the graph $\mathcal{FF}_m(F_n)$ as a loxodromic isometry if φ does not preserve the conjugacy class of a free factor of rank contained in $[m, n-1]$ and if its invariant lamination fills so that φ acts as a loxodromic automorphism on the free splitting graph $\mathcal{FS}(F_n)$ by Theorem 1.1 of [8]. An example of such an element of $\mathrm{Out}(F_4)$ with $m = 2$ which fixes a point in $\mathcal{FF}_1(F_4)$ can easily be constructed as follows. Let F_3 be freely generated by a_2, a_3, a_4 and let $\psi \in \mathrm{Out}(F_3)$ be an irreducible automorphism with irreducible powers. Define an automorphism Ψ of $F_4 = \langle a_1, F_3 \rangle$ by $\Psi(a_1) = a_1$, $\Psi(a_2) = \psi(a_2) \cdot a_1$ and $\Psi(a_i) = \psi(a_i)$ for $i = 3, 4$.

1.3 Exponential growth

For any sphere system Σ and any embbeded finite graph R in $M = \sharp_n S^1 \times S^2$ let

$$\iota(\Sigma, R)$$

be the minimal number of intersection points between Σ and a homotopic realization of R, counted with multiplicity. Equivalently, this is the minimal number of intersection points between Σ and a homotopic realization of R such that every vertex of R is contained in $M - \Sigma$.

A simple sphere system Σ is *reduced* if its complement is connected. Each reduced sphere system is *dual* to a unique isotopy class of a rose $R \subset M$ which defines the conjugacy class of a basis for F_n. Here duality means that up to homotopy, each component of Σ intersects the rose R in a single point.

Recall that $\mathrm{Out}(F_n)$ can be generated by *Nielsen moves*. Such a Nielsen move either is a Nielsen twist or a permutation of two rank one free factors in a free basis (up to conjugation). A Nielsen twist replaces a marked rose R by another marked rose R'. There is a homotopy equivalence $R' \to R$ which maps a leaf of R' to a loop in R which either is a single leaf of R or passes through precisely two leaves, and the latter is the case for precisely one leaf. Thus we have

Lemma 1.3.1. *Let Σ be a simple sphere system, let R be a marked rose and assume that R' is obtained from R by a single Nielsen twist; then*

$$\iota(\Sigma, R') \in \left[\frac{1}{2}\iota(\Sigma, R), 2\iota(\Sigma, R)\right].$$

Proof. As the marked homotopy equivalence $R' \to R$ can be represented by a $2:1$ map, we have $\iota(\Sigma, R') \le 2\iota(\Sigma, R)$. On the other hand, the marked rose R' is obtained from R by a single Nielsen twist as well, which immediately shows the second part of the inequality. ∎

Let $\mathcal{R}$ be the graph whose set of vertices is the set of all marked roses and where two such roses are connected by an edge if they are related by a Nielsen move. Then $\mathcal{R}$ is an $\mathrm{O}ut(F_n)$-graph on which $\mathrm{O}ut(F_n)$ acts properly and cocompactly. In other words, $\mathcal{R}$ is a geometric realization of $\mathrm{O}ut(F_n)$.

Sphere systems define another geometric realization of $\mathrm{O}ut(F_n)$. To see this let $\mathcal{SSG}_n$ be the simple sphere system graph and let $d_{\mathcal{SSG}}$ be the distance in $\mathcal{SSG}_n$. By invariance under the action of $\mathrm{O}ut(F_n)$, cocompactness, and the fact that stabilisers of simple sphere systems are finite, the sphere system graph is equivariantly quasi-isometric to $\mathrm{O}ut(F_n)$.

Lemma 1.3.2. *There exists a number $C_0 > 0$ with the following properties. Let Σ_0, Σ_1 be reduced sphere systems and let R be a rose dual to Σ_1; then $d_{\mathcal{SSG}_n}(\Sigma_0, \Sigma_1) \geq C_0 \log_2 \iota(\Sigma_0, R)$.*

Proof. By Lemma 1.3.1, each Nielsen move decreases intersection numbers between a rose and a sphere system by at most a factor of two. Since the graph $\mathcal{SSG}_n$ is $\mathrm{O}ut(F_n)$-equivariantly quasi-isometric to $\mathcal{R}$, this shows the lemma. ∎

Given any simple sphere system Σ, we can obtain a reduced sphere system by removal of some of the spheres. Such a reduced sphere system admits a dual rose. We call a rose R obtained in this way *dual* to Σ although R may not be unique.

Let $(\Sigma_i)_{0\leq i\leq m}$ be a surgery sequence of simple sphere systems. For each i let R_i be a rose dual to Σ_i. Then R_i defines a vertex in the graph $\mathcal{R}$. The distance in $\mathcal{R}$ between R_i and R_{i+1} is bounded from above independently of i. We use this fact in the following observation.

Lemma 1.3.3. *There exists $C_1 > 0$ with the following property. Let $(\Sigma_i)_{0\leq i\leq m}$ be a surgery sequence of simple sphere systems, and let (R_i) be a sequence of dual roses; then*

$$\iota(\Sigma_m, R_1) \geq C_1 \iota(\Sigma_m, R_0).$$

Proof. By definition, the sphere system Σ_1 is obtained from Σ_0 by one surgery operation, followed by removal of at most two spheres from the resulting system. Thus the dual rose R_1 is obtained from the rose R_0 by a uniformly bounded number of Nielsen twist (which are the generators of $Out(F_n)$, permutations play no role here), say at most ℓ of such twists. The lemma now follows from Lemma 1.3.1. ∎

Lemma 1.3.3 shows that intersection numbers along a surgery path with roses dual to the sphere systems of the path decrease at most exponentially, with a fixed exponent depending only on n. We next look at such paths for which intersection numbers decrease uniformly exponentially.

Definition 1.3.4. For $a \in (0, 1)$ and $k \geq 1$ the surgery sequence $(\Sigma_i)_{0\leq i\leq m}$ has (a, k)-*exponential growth* if $\iota(\Sigma_m, R_{i+k}) \leq a\iota(\Sigma_m, R_i)$ and $\iota(\Sigma_0, R_i) \leq a\iota(\Sigma_0, R_{i+k})$ for all i.

The next proposition shows that exponential growth yields geometric control.

Proposition 1.3.5. *For all $a \in (0, 1)$, $k \geq 1$ there is a number $\ell(a, k) > 1$ with the following property. Let Σ, Λ be two simple sphere systems which are connected by a surgery sequence (Σ_i). Assume that this sequence has (a, k)-exponential growth. For each i let R_i be a rose dual to Σ_i. Then the sequence (R_i) defines an $\ell(a, k)$-quasi-geodesic in the graph $\mathcal{R}$.*

Proof. For a number $L > 1$, an *L-Lipschitz retraction* of the graph $\mathcal{R}$ onto a subset $A \subset \mathcal{R}$ is an L-Lipschitz map $\Upsilon : \mathcal{R} \to A$ such that $d(x, \Upsilon(x)) \leq L$ for all $x \in A$. If there exists an L-Lipschitz retraction $\mathcal{R} \to A$ then since $\mathcal{R}$ is a geodesic metric graph, the inclusion $A \to \mathcal{R}$ is *L-quasi-convex:* For any two points $x, y \in A$ there exists a path in the L-neighborhood of A with the same endpoints which is an L-quasi-geodesic in $\mathcal{R}$ (with additive constant larger than L).

As a consequence, it suffices to show that there is an L-Lipschitz retraction of $\mathcal{R}$ onto the sequence $(R_i)_{0\leq i\leq m}$ of roses dual to the sphere systems Σ_i for a constant $L > 1$ only depending on a, k (and, of course, the rank n).

Let $G \in \mathcal{R}$ be a marked rose. We assume that G is embedded in M. Let $\kappa = \log \frac{\iota(\Sigma,G)}{\iota(\Lambda,G)}$. We say that $P(G) = R_i$ is *roughly balanced* for G if

$$\log \frac{\iota(\Sigma, R_i)}{\iota(\Lambda, R_i)} \in [\kappa + \log C_1, \kappa - \log C_1]$$

where $C_1 \in (0, 1)$ is as in Lemma 1.3.3. If $\kappa < \log \frac{\iota(\Sigma,R_0)}{\iota(\Lambda,R_0)}$ then we put $P(G) = \Sigma$, and similarly we put $P(G) = \Lambda$ if $\kappa > \log \frac{\iota(\Sigma,R_m)}{\iota(\Lambda,R_m)}$. By Lemma 1.3.3 and the choice of the constant $C_1 > 0$, such a number i exists, and Definition 1.3.4 yields that it is coarsely unique: If R_j is another such point then $|j - i| \leq 2k \log C_1/\log a$.

Now let us assume that G' is obtained from G by a single Nielsen twist. Then Lemma 1.3.1 shows that

$$\left|\log \frac{\iota(\Sigma, G')}{\iota(\Lambda, G')} - \log \frac{\iota(\Sigma, G)}{\iota(\Lambda, G)}\right| \leq 2 \log 2 \leq 2.$$

Thus as a consequence of (a, k)-exponential growth, we obtain that the intrinsic distance between $P(G)$ and $P(G')$ is at most L where $L = L(a, k) > 0$ is a universal constant. In other words, the map P is coarsely L-Lipschitz.

If G is dual to one of the sphere systems Σ_i then it follows from the construction that $P(G)$ is contained in a uniformly bounded neighborhood of Σ_i. As a consequence, P is indeed a Lipschitz retraction. The lemma follows. ■

While Lemma 1.3.1 shows that intersection numbers change at most exponentially with a fixed rate along a one-Lipschitz path in the graph $\mathcal{R}$, the next observation yields that the distance in $\mathcal{SG}_n$ yields a lower bound on intersection numbers.

Lemma 1.3.6. *Let $S \subset M$ be a sphere and let $R \subset M$ be an embedded rose with n petals and vertex p such that the inclusion $R \to M$ defines an isomorphism $\pi_1(R, p) \to \pi_1(M, p)$. Let $S' \subset M$ be a sphere which intersects R in precisely one point; then*

$$d_{\mathcal{SG}_n}(S, S') \leq 2 \log_2 \iota(S, R) + 3.$$

Proof. If S' is disjoint from S then

$$d_{\mathcal{SG}_n}(S, S') = 1$$

and there is nothing to show. Thus assume that S', S intersect and that R intersects S in $k \geq 1$ points.

There are at least two innermost components of $S' - S$. Up to homotopy of R, we may assume that the intersection point between R and S' is contained in one of these components, say the component D'. Let D be an innermost component of $S' - S$ different from D'; its boundary ∂D decomposes S into two disks D_1, D_2. Assume by renaming that the disk D_1 has fewer intersections with R than D_2. Then R intersects D_1 in at most $k/2$ points.

Surger S at D so that the surgered sphere S_1 is the union $D_1 \cup D$. Then $\iota(S_1, R) \leq k/2$. Note that $d_{\mathcal{SG}}(S, S_1) \leq 1$ since S, S_1 are disjoint. The lemma now follows by induction on the length of a surgery sequence connecting S to a sphere disjoint from S'. ■

Recall the coarsely well defined map Θ which associates to a simple sphere system one of its components. For a number $B > 1$, define two reduced sphere systems Σ_0, Σ_1 to be in *B-tight position* if $Bd_{\mathcal{SG}_n}(\Theta(\Sigma_0), \Theta(\Sigma_1)) \geq d_{\mathcal{SSG}_n}(\Sigma_0, \Sigma_1)$.

Corollary 1.3.7. *For every $B > 1$ there is a number $a = a(B) > 0$ with the following property. Let Σ_0, Σ_1 be two reduced sphere systems which are in B-tight position. Let R_1 be a rose dual to S_1; then*

$$d_{\mathcal{SSG}_n}(\Sigma_0, \Sigma_1) \in [\log_2 \iota(\Sigma_0, R_1)/a, a \log_2 \iota(\Sigma_0, R_1)].$$

Proof. Since Σ_0, Σ_1 are in B-tight position, we have

$$d_{\mathcal{SSG}_n}(\Sigma_0, \Sigma_1) \leq Bd_{\mathcal{SG}_n}(\Theta(\Sigma_0), \Theta(\Sigma_1)).$$

Thus the corollary follows from Lemma 1.3.6 and Lemma 1.3.2. ■

1.4 Growth and quasigeodesics

The goal of this section is to have a closer look at surgery sequences in relation to growth in the case of rank 2 and rank 3. Throughout we only consider particular surgery sequences called *full surgery sequences*, defined by the property that we always use all spheres (and remove multiple copies). That is, we replace a sphere by both spheres obtained from surgery at a fixed innermost disk.

Recall the map $\Theta : \mathcal{SSG}_n \to \mathcal{SG}_n$ which associates to a simple sphere system one of its components. In the statement of the following proposition, exponential growth means strong (a, k)-exponential growth for some $a > 0, k > 0$. The constants depend on each other, but we do not make this dependence quantitative.

Proposition 1.4.1. *For the free group of rank $n = 2$ and a full surgery sequence $\Sigma_i \subset \mathcal{SSG}_2$ of simple sphere systems the following are equivalent.*

(1) *Σ_i is of exponential growth.*

(2) *The image sequence $\Theta(\Sigma_i)$ is a parameterized quasi-geodesic in $\mathcal{SG}_2$.*

Proof. Since (2) implies (1) by Lemma 1.3.2 and Lemma 1.3.6 and hence is valid for any rank, it suffices to show that (1) implies (2) if the rank equals two.

We observed in Example 1.2.7 that up to uniform quasi-isometry, the graph $\mathcal{SG}_2$ can be identified with the Farey graph, where this identification is via viewing the free group F_2 as the fundamental group of a once punctured torus T and viewing the Farey graph as the curve graph of T.

Furthermore, we have $\mathrm{O}ut(F_2) = \mathrm{G}L(2,\mathbb{Z})$, which is a hyperbolic group with respect to some (and hence any) finite symmetric generating set. Thus any uniform (that is, with fixed constants) quasi-geodesic γ in $\mathrm{O}ut(F_2)$ is *stable*: Any other uniform quasi-geodesic with the same endpoints is contained in a uniformly bounded neighborhood of γ. Since the surgery sequence Σ_i is of (a,k)-exponential growth by assumption, Proposition 1.3.5 shows that it determines a uniform quasi-geodesic in $\mathrm{G}L(2,\mathbb{Z})$ and hence it is at uniformly bounded distance from a geodesic.

To understand the relation between the geometry of $\mathrm{O}ut(F_2)$ and the geometry of the Farey graph we first pass to the quotient $\mathrm{P}SL(2,\mathbb{Z})$ of the index two subgroup $\mathrm{S}L(2,\mathbb{Z})$ of $\mathrm{G}L(2,\mathbb{Z})$ with fiber of order 2. It acts as a group of isometries on the hyperbolic plane $\mathbb{H}^2$. The quotient of $\mathbb{H}^2$ by this action is a finite volume orbifold with one cusp. There exists a $\mathrm{P}SL(2,\mathbb{Z})$-invariant collection $\mathcal{H}$ of open *horoballs* with pairwise disjoint closure which are centered at the rational numbers and ∞ in $\partial\mathbb{H}^2 = \mathbb{R}\cup\infty$ (here we use the upper half-plane model for the hyperbolic plane). This system of horoballs is precisely invariant under the action of the group $\mathrm{P}SL(2,\mathbb{Z})$: if $H \in \mathcal{H}$ is such a horoball, and if $g \in \mathrm{P}SL(2,\mathbb{Z})$ is such that $gH \cap H \neq \emptyset$, then $gH = H$. Furthermore, the action of $\mathrm{P}SL(2,\mathbb{Z})$ on $\mathcal{H}$ is transitive. Up to adjusting the system $\mathcal{H}$, the complement $X = \mathbb{H}^2 - \mathcal{H}$ is a path connected two-dimensional space on which $\mathrm{P}SL(2,\mathbb{Z})$ acts properly and cocompactly.

Let $\mathrm{S}tab(H) \subset \mathrm{P}SL(2,\mathbb{Z})$ be the stabilizer of a component $H \in \mathcal{H}$. Then $\mathrm{S}tab(H)$ is virtually infinite cyclic. The hyperbolic group $\mathrm{P}SL(2,\mathbb{Z})$ is hyperbolic relative to its system of pairwise conjugate parabolic subgroups $\mathrm{S}tab(H)$ $(H \in \mathcal{H})$. Up to uniform quasi-isometry, the Farey graph is then obtained by adding for each $H \in \mathcal{H}$ a point to the Cayley graph of $\mathrm{P}SL(2,\mathbb{Z})$ and connect this point to each element in $\mathrm{S}tab(H)$. Equivalently, up to quasi-isometry, the Farey graph can be thought of as the space obtained by equivariantly coning off each component $H \in \mathcal{H}$ by adding a point and connecting this point to each point in H by an edge of length one.

As a consequence, a (uniform) quasi-geodesic $\gamma \subset \mathrm{P}SL(2,\mathbb{Z})$ projects to a uniform quasi-geodesic in the Farey graph if and only if the length of any subsegment which is contained in a uniform neighborhood of $\mathrm{S}tab(H)$ for some $H \in \mathcal{H}$ is uniformly bounded. Equivalently, the geodesic segment in $\mathbb{H}^2$ connecting a point $x \in X$ to a point $y \in \gamma(x)$ remains in a uniformly bounded neighborhood of X. We have to show that this is a consequence of exponential growth.

View $\mathbb{H}^2$ as the Teichmüller space of marked punctured tori equipped with a finite volume hyperbolic metric. Then $X \subset \mathbb{H}^2$ parameterizes such marked tori whose *systole*, that is, the length of a shortest closed geodesic, is bounded from below by a universal positive constant $\epsilon > 0$.

Choose a basepoint $x \in X$ and a rose $R \subset x$ such that the inclusion $R \hookrightarrow x$ is an isomorphism on π_1 and that the length of R with respect to the metric x is uniformly bounded. Since the systole of $x \in X$ is at least ϵ, such a rose exists, and it is essentially unique: If a_1, a_2 is a free basis of F_2 defined by the petals of the rose, then any other such free basis of F_2 can be obtained from a_1, a_2 by a uniformly bounded number of Nielsen moves.

Let $\gamma : [0, u] \to PSL(2, \mathbb{Z})$ be a uniform quasi-geodesic through $\gamma(0) = Id$. Put $\psi = \gamma(u)$ and consider the Teichmüller geodesic segment $\eta : [0, \tau] \to \mathbb{H}^2$ connecting x to $\psi(x)$, which is just the hyperbolic geodesic parameterized by arc length (for a suitable uniform normalization). The length τ of the hyperbolic geodesic η can be determined as follows.

For $t \in [0, \tau]$ let $q(t)$ be the area one singular euclidean metric on T defined by the area one quadratic differential which is the cotangent vector of η at $\eta(t)$. Any conjugacy class in F_2 can be represented by a closed curve α which is a $q(t)$-geodesic for each t. The tangent of this geodesic decomposes into a *horizontal component* α^h and an orthogonal vertical component α^v. The $q(t)$-length of α then equals $\int_\alpha (e^{2t}|\alpha^h|^2 + e^{-2t}|\alpha^v|^2)^{1/2}$ for each t, where $|\alpha^h|^2$ and $|\alpha^v|^2$ are the square norms of α^h, α^v with respect to the metric $q(0)$. In particular, the length of α grows at most exponentially with t.

For points $x \in X$, the singular euclidean metric defined by an area one quadratic differential is uniformly bi-Lipschitz equivalent to the hyperbolic metric in the complement of the cusp. Furthermore, with respect to a basis of F_2 consisting of two simple closed geodesics of uniformly bounded x-length, the length of any simple closed curve α on x is uniformly proportional to the word length with respect to that basis. This follows from the fact that the geodesic representative of a *simple* closed curve is contained in a subset of x of uniformly bounded diameter. This subset is uniformly quasi-isometric to an embedded rose $R \subset x$ of uniformly bounded x-length so that the inclusion $R \hookrightarrow x$ is an isomorphism on π_1. Thus (a, k)-exponential growth implies that the length of any sufficiently long subsegment of the Teichmüller geodesic η whose endpoints are contained in X and hence uniformly close to a point $\psi(x)$ on the $PSL(2, \mathbb{Z})$-orbit of x is bounded from below by a uniform multiple of the word norm of ψ in $PSL(2, \mathbb{Z})$.

Now let $\zeta : [0, p] \to \mathbb{H}^2$ be a geodesic arc of length $p > 0$ connecting two points on the boundary of a horoball $H \in \mathcal{H}$. Since close-by points in X define hyperbolic tori which are marked uniformly bi-Lipschitz, we may assume that the endpoints of ζ are contained in the same $PSL(2, \mathbb{Z})$-orbit. This means that there exists an element $\sigma \in Stab(H)$ with $\sigma(\zeta(0)) = \zeta(p)$. Let $\ell > 0$ be the word norm of σ in the infinite cyclic group $Stab(H)$. Note that this word norm is uniformly proportional to the word norm in $PSL(2, \mathbb{Z})$. Then the length p of ζ is bounded from above by $b \log \ell + b$ where $b > 0$ is a universal constant. As a consequence, for large enough ℓ the condition of (a, k)-exponential growth is violated. In other words, (a, k)-exponential growth implies property (2) stated in the proposition, which is what we wanted to show. ■

We next give an example which shows that for $n \geq 3$, a surgery sequence which violates the exponential growth condition in Proposition 1.3.5 does *not* necessarily define a uniform quasi-geodesic in $Out(F_n)$. For simplicity of exposition, we only carry out the case $n = 3$. It will be clear from the discussion that the construction is valid for any $n \geq 3$.

Example 1.4.2. Consider the free group F_3 with a free basis $\mathcal{A}_0 = \{a_1, a_2, a_3\}$. Let $R \subset M$ be a marked rose whose petals define these generators and let Σ_0 be the corresponding dual simple sphere system in $M = \sharp_3 S^1 \times S^2$. Denote by $S_1 \in \Sigma_0$ the sphere which intersects a_1.

Choose a hyperbolic element $\alpha \in GL(2, \mathbb{Z}) = Out(F_2)$ where we view $F_2 \subset F_3$ is the free factor generated by a_2, a_3 and extend it to an element of $Out(F_3)$ which preserves a_1 (up to a global conjugation). Denote the thus defined element of $Out(F_3)$ again by α. It preserves the conjugacy class of the one-edge free splitting $F_3 = \langle a_1 \rangle * F_2$. The element α acts on the sphere system graph, preserving the sphere S_1. By Proposition 1.4.1, we know that the intersection $\iota(\alpha^k(R), \Sigma_0)$ is uniformly exponentially growing in k: there exists a number $c > 0$ such that $\iota(\alpha^k(R), \Sigma_0) \geq e^{ck}$.

Let m be the word length of α with respect to the standard generating set of $Out(F_2) = GL(2, \mathbb{Z})$ consisting of Nielsen twists for the generators a_2, a_3 of F_2. For each k consider the free basis $\mathcal{A}_k = \{a_1 \cdot \alpha^k(a_2), a_2, a_3\}$ of F_3. There exists a path in $Out(F_3)$ of length $2km + 1$ which transforms the basis $\mathcal{A}_0$ to $\mathcal{A}_k$. This path consists in first applying α^k, which contributes km to the length of the path. The image of $\mathcal{A}_0$ by this automorphism is the basis $a_1, \alpha^k(a_2), \alpha^k(a_3)$ (up to a global conjugation). Perform a Nielsen twist to replace a_1 by $a_1 \cdot \alpha^k(a_2)$ and apply α^{-k}, extended to F_3 by fixing the free splitting $F_3 = \langle a_1 \cdot \alpha^k(a_2) \rangle * F_2$. The thus defined path in $Out(F_3)$ has length $2km + 1$, and its endpoint ψ_k maps $\mathcal{A}_0$ to $\mathcal{A}_k$.

Consider the reduced edge path in the rose R defining the element $\alpha^k(a_2)$, which is given as a word w in the generators a_2, a_3 and their inverses. The length ℓ of this edge path is at least $e^{ck} \leq \iota(\psi_k(R), \Sigma_0)$ for a constant $c > 0$ not depending on k. We claim that the length of a surgery path connecting a sphere system dual to $\mathcal{A}_k$ to a sphere system disjoint from Σ_0 equals $\ell - 2$. To this end we describe explicitly the sphere system $\mathcal{A}_k$. It contains the sphere $S_1 \in \Sigma_0$ as a component.

Represent the loop $a_1 \cdot w$ in R by an embedded closed curve ρ in M which is disjoint from R except at its endpoints and which has precisely one essential intersection with S_1 and one essential intersection with the sphere $S_i \in \Sigma_0$ dual to a_i for any transition of ρ through the edge a_i in either direction $(i = 2, 3)$. Choose a parameterization of ρ so that $\rho(1) \in S_1$, that $\rho(j) \in S_2 \cup S_3$ and that $\rho(j-1, j)$ is disjoint from Σ_0 for all $j \geq 1$.

Assume for simplicity that the word w begins with the letter a_2. Replace $\rho[1,2]$ by a small closed tubular neighborhood meeting S_1, S_2 in a small compact embedded disk D_1, D_2 each. The union of $S_1 \cup S_2$ with the boundary of this tubular neighborhood, with the interiors of the disks D_1, D_2 removed, is a sphere S_2' which intersects the petal a_2 in precisely one point (which is the intersection point between a_2 and S_2, which is contained in $S_2 \setminus D_2$) and is disjoint from a_3, $\rho[0,3)$ and Σ_0. Thus $\Sigma_1 = \{S_1, S_2', S_3\}$ is a reduced sphere system dual to the rose with petals $a_1 \cdot a_2, a_2, a_3$. From the point of view of homotopy classes, S_2' is obtained from S_2 by adding the class of S_1 (for a suitable orientation).

Repeat this construction. If the second intersection point $\rho(3)$ of ρ with $S_2 \cup S_3$ is an intersection point with S_2, then the sphere S_2' is modified to a sphere S_2'' which consists of the sphere S_1 connected to S_2' by a tube which passes through the tube of S_2' and ends in a small disk neighborhood of ρ_3 in $S_2 \cap S_2'$. Otherwise the sphere S_3 is modified to a sphere S_3' in a similar way. In ℓ such steps we construct in this way a path $\Sigma_0, \Sigma_1, \dots, \Sigma_\ell$ in the sphere graph, where Σ_ℓ is dual to the free basis $\mathcal{A}_k$. The intersection $\Sigma_\ell \cap S_2$ consists of finitely many collections of nested circles, one for each intersection point of ρ with S_2, except perhaps the last intersection point. The number of circles in the collection corresponding to the intersection point $\rho(i)$ of ρ with S_2 consists of $\ell - i$ circles. A full surgery of Σ_ℓ at any innermost intersection circle in $S_2 \cap \Sigma_\ell$ gives rise to the disjoint union $\Sigma_{\ell-1} \cup S_1$. Proceeding inductively, this shows that the length of a surgery path connecting the sphere system Σ_ℓ dual to $\mathcal{A}_k$ to a sphere system disjoint from $\mathcal{A}_0$ has length $\ell - 2$. As a consequence, the surgery paths connecting $\mathcal{A}_k$ to $\mathcal{A}_0$ do not define a family of uniform quasi-geodesics in $\mathrm{O}ut(F_3)$.

1.5 Submanifold projection

Let σ_0 be a nonseparating sphere in $M = M_n = \#_n S^1 \times S^2$. The metric completion $\hat{N}$ of $M_n - \sigma_0$ with respect to some path metric on M_n is a compact manifold with two boundary components, corresponding to the two sides of σ_0. The manifold N obtained by gluing a 3-ball to each boundary component of $\hat{N}$ is homeomorphic to M_{n-1}. Our goal is to analyze intersections of spheres with $\hat{N}$ and use this to define a *submanifold projection* of the sphere graph of M into the sphere graph of N.

We begin with a topological observation.

Lemma 1.5.1. *Let S be any sphere in normal position with respect to σ_0 which is not disjoint from σ_0. Let $D \subset S$ be any innermost disk of $S - \sigma_0$, and let $D_0 \subset \sigma_0$ be an embedded disk in σ_0 with the same boundary circle: $\partial D = \partial D_0$. Then the sphere $S' = D \cup D_0$ is essential in N.*

Proof. Assume by contradiction that S' is inessential in N. Denote the boundary component of $\hat{N}$ which intersects the disk D by $\partial^+\hat{N}$ and the other by $\partial^-\hat{N}$. Equip σ_0 with the orientation of the oriented boundary component $\partial^+\hat{N}$ of $\hat{N}$ (for a choice of an orientation of N). Since σ_0 is non-separating by assumption, this choice of orientation determines a choice of a generator of $H_2(M, \mathbb{Z})$, given by the oriented inclusion $\sigma_0 \to M$, again denoted by σ_0. Furthermore, this choice of orientation restricts to an orientation of D_0 and hence defines an orientation of S'.

Since S' is an inessential embedded sphere in N, it bounds a ball in N. Because σ_0 and S are in minimal position, the sphere S' does not bound a ball in the manifold $\hat{N}$. Similarly, the sphere S' does not bound a ball in the manifold $\hat{N}_+$ obtained from $\hat{N}$ by gluing a ball to $\partial^+\hat{N}$. Namely, otherwise D would be homotopic in $\hat{N}$ into $\partial^+\hat{N}$, violating as before normal position. As a consequence, S' bounds a region in $\hat{N}$ whose second boundary component is $\partial^-\hat{N}$. Thus S' is homologous to $\pm\sigma_0$ in M. Inspecting orientations, we obtain that S' defines the homology class σ_0 in M.

Let $\hat{S}$ be the sphere in N obtained by gluing $\sigma_0 - D_0$ to D and equipped with the orientation inherited from the boundary orientation of $\partial^+\hat{N}$. For this choice of orientation, σ_0 is the oriented connected sum of S' and $\hat{S}$. Thus as homolopy classes in M, we have $\sigma_0 = S' + \hat{S} = \sigma_0 + \hat{S}$ and hence $\hat{S}$ is homologically trivial in M. In other words, the embedded sphere $\hat{S}$ in M is separating. Furthermore, it is not homotopically trivial in M, again by minimal position.

Now the second homotopy group $\pi_2(M)$ of M is a free $\pi_1(M)$-module which is the direct sum of two submodules $V_1 \oplus V_2$, where V_1 is spanned by nonseparating embedded spheres and V_2 is spanned by separating embedded spheres. In other words, V_2 is the kernel of the map $\pi_2(M) \longrightarrow H_2(M,\mathbb{Z})$ as $\pi_1(M)$-modules, where the action of $\pi_1(M)$ on $H_2(M,\mathbb{Z})$ is the trivial action. By the above, the spheres σ_0 and S' are contained in the submodule V_1, and the sphere $\hat{S}$ is contained in V_2. As $\sigma_0 - S' = \hat{S}$ (connected sum and hence sum in $\pi_2(M)$), and all elements are non-zero, this is impossible. ■

Call a sphere $S \subset M - \sigma_0$ (where we tacitly identify $M - \sigma_0$ with $\hat{N}$) *non-peripheral* if its image in the manifold N is non-trivial. The set of all non-peripheral spheres defines a subgraph $\mathcal{NP}(\sigma_0)$ of the sphere graph of M consisting of spheres disjoint from σ_0 up to homotopy. Let also $\mathcal{P}(\sigma_0)$ be the set of all spheres in M which are disjoint from σ_0 and *peripheral*, that is, whose images in N are homotopically trivial. Note that any such sphere (with σ_0 excluded) is separating since the natural map $M - \sigma_0 \longrightarrow N$ induces an isomorphism of fundamental groups and any nonseparating sphere in $M - \sigma_0$ intersects some loop in $M - \sigma_0$ in a single point.

Lemma 1.5.1 allows to define a *submanifold projection*

$$p_{\sigma_0} : \mathcal{SG}_n - \mathcal{P}(\sigma_0) \to \mathcal{NP}(\sigma_0)$$

(more precisely, the target of the projection is the family of all non-empty finite subsets of $\mathcal{NP}(\sigma_0)$) in the following way. For a sphere S in M distinct from σ_0 and not peripheral we put $p_{\sigma_0}(S) = S$ if S is disjoint from σ_0, and if S intersects σ_0, then we let $p_{\sigma_0}(S)$ be the union of all spheres which are obtained by surgery at an innermost disk of $S - \sigma_0$. By Lemma 1.5.1, each such surgery yields a non-peripheral sphere in $M - \sigma_0$. The projection $p_{\sigma_0}(\Sigma)$ of a sphere system Σ with more than one component is defined to be the union $\cup_{S\in\Sigma} p_{\sigma_0}(S)$.

There may be spheres in the set $p_{\sigma_0}(\Sigma)$ which intersect, but as a subset of $\mathcal{NP}(\sigma_0)$, the diameter of $p_{\sigma_0}(\Sigma)$ is uniformly bounded. Namely, all innermost disks of $\Sigma - \sigma_0$ are disjoint. Hence if $S_1, S_2 \in p_{\sigma_0}(\Sigma)$ are any two spheres constructed from innermost disks D_1, D_2, then there exist disjoint spheres $S_1', S_2' \in p_{\sigma_0}(\Sigma)$ such that S_i' is disjoint from S_i $(i = 1, 2)$. Just choose S_i' to be the spheres constructed from the innermost disks D_i of $\Sigma - \sigma_0$ and from two disjoint disks in σ_0 bounded by the disjoint boundary circles of D_1, D_2.

Let $\mathcal{SG}_N = \mathcal{SG}_{n-1}$ be the sphere graph of the manifold N obtained by cutting M open along σ_0 and capping off the boundary. There exists a natural simplicial projection

$$\Upsilon_{\sigma_0} : \mathcal{NP}(\sigma_0) \to \mathcal{SG}_N.$$

Consider the composition

$$p_N = \Upsilon_{\sigma_0} \circ p_{\sigma_0} : \mathcal{SG}_n - \mathcal{P}(\sigma_0) \to \mathcal{SG}_N.$$

Our next goal is to establish a control of the images of suitably chosen surgery paths under the projections p_N. This will follow from a stability property of normal position along such surgery sequences. Note that the disk D_2 in the formulation of the lemma below need not be innermost, which corresponds to the second possibility listed.

Lemma 1.5.2. *Let Σ_1 be a sphere system, and let Σ_2 be a sphere which is in normal position with respect to Σ_1. Let $D_i \subset \Sigma_i, i = 1, 2$ be two embedded disks such that $\partial D_1 = \partial D_2$, and such that the interiors of the D_i are disjoint. Let $S = D_1 \cup D_2$. Then either*

(1) *up to homotopy, S is disjoint from Σ_1, or*
(2) *the normal position of S with respect to Σ_1 has an innermost disk component which is (with boundary gliding on Σ_1) isotopic to an innermost disk component of Σ_2.*

Proof. Let $\widetilde{M}$ be the universal cover of M. We let $\widetilde{\Sigma}_1$ be the full preimage of Σ_1, and let $\overline{\Sigma}_2$ be a connected lift of Σ_2. This contains a unique lift $\overline{D}_2$ of D_2. We denote by $\overline{D}_1$ the unique lift of D_1 which intersects $\overline{D}_2$. Then the sphere

$$\overline{S} = \overline{D}_1 \cup \overline{D}_2$$

is a connected lift of S. We modify S and this lift by pushing $\overline{D}_1$ slightly off $\widetilde{\Sigma}_1$ in order to make every intersection of $\overline{S}$ with $\widetilde{\Sigma}_1$ transverse. Note that every such intersection circle is then contained in $\overline{D}_2$. In particular, every innermost disk component of $\overline{S}$ with respect to $\widetilde{\Sigma}_1$ is either contained in $\overline{\Sigma}_2$, or contains $\overline{D}_1$ (and there is at most one of the latter type).

If there is no innermost disk containing $\overline{D}_1$, or if the innermost disk containing $\overline{D}_1$ is not homotopic (relative to its boundary) into $\widetilde{\Sigma}_1$, then S is in normal position with respect to Σ_1. Namely, any other pathology is excluded by normal position of Σ_1 and Σ_2. In that case we find an innermost disk component of $\overline{S}$ which is contained in $\overline{D}_2$ and satisfies property ii).

If there is an innermost disk component $D \supset \overline{D}_1$ which is homotopic relative to its boundary into $\widetilde{\Sigma}_1$, then there is a ball B whose boundary is the union of D with a disk contained in $\widetilde{\Sigma}_1$. In this case, we can homotope S by pushing it through this ball. As a result we obtain a sphere S' which is again of the form $D_1' \cup D_2'$ with disks contained in Σ_1, Σ_2, and whose lift intersects $\widetilde{\Sigma}_1$ in one less circle. Iterating this argument, we either terminate in a sphere which is disjoint from $\widetilde{\Sigma}_1$ and therefore has property i), or we obtain property ii) as above. ■

Lemma 1.5.2 allows to define *nested* surgery sequences Σ_i as follows. Let Σ_0, Σ be sphere systems, and let S_0, S be components of Σ_0, Σ which intersect. Choose an innermost disk $D \subset S \in \Sigma$ with boundary on $S_0 \in \Sigma_0$ and let $D_0 \subset S_0$ be a disk with boundary $\partial D_0 = \partial D$. Perform surgery of S_0 by replacing S_0 by $S_1 = D_0 \cup D$.

Assume that S_1 is not disjoint from S. Choose an innermost disk D' of $S - S_1$, with boundary on S_1. By Lemma 1.5.2, up to homotopy, the boundary of D' is contained in D_0 and hence bounds a unique disk $D_1 \subset D_0$. Perform surgery by replacing S_1 by $D_1 \cup D'$ and iterate this construction.

Lemma 1.5.3. *Let (S_i) be a nested surgery sequence of a sphere S_0 towards a sphere S. Then each sphere S_i in the sequence is a union of a disk $D_i \subset S_0$ and a disk $D_i' \subset S$, with $D_{i+1} \subset D_i$.*

Proof. We proceed by induction on the length m of the sequence. The statement is clear in the case $m = 1$, so assume that the statement holds true for $m - 1$. Let (S_i) be a nested surgery sequence of length m. By induction hypothesis, S_{m-1} is a union of a disk $D_{m-1} \subset S_0$ and a disk $D'_{m-1} \subset S$. By Lemma 1.5.2, an innermost disk $D \subset S$ of $S - S_{m-1}$ has its boundary in D_{m-1} and hence bounds a disk $D_m \subset D_{m-1}$. Moreover up to homotopy, either D contains the disk D'_{m-1} and hence the sphere S_m obtained from S_{m-1} by nested surgery with innermost component D is a union of $D \supset D'_{m-1}$ and $D_m \subset D_{m-1}$, or it is disjoint from D'_{m-1} and once again, the statement of the lemma is true for S_m. ■

Let as before $d_{\mathcal{SG}}$ be the distance in the sphere graph of M.

Lemma 1.5.4. *Let (S_i) be a nested surgery sequence connecting a non-separating sphere σ_0 to a different sphere S. Let S_k be any point of this surgery sequence which satisfies $d_{\mathcal{SG}_n}(S_k, \sigma_0) \geq 2$. Let $\hat{N}$ be obtained from M by removal of σ_0, and let N be obtained from $\hat{N}$ by capping off the boundary spheres. Then*

$$p_N(S_k) \cap p_N(S) \neq \emptyset.$$

Consequently, the projections $p_N(S_k), p_N(S)$ are 2-close in the sphere graph of N.

Proof. Assume without loss of generality that we have chosen representatives of σ_0, S which are in normal position. We will denote these representatives by the same symbol again.

Let S_i be a sphere on the nested surgery sequence. By Lemma 1.5.3, S_i is a union of two disks $S_i = D_i^- \cup D_i^+$ with $D_i^- \subset \sigma_0$ and $D_i^+ \subset S$.

By Lemma 1.5.2, either S_i is disjoint from σ_0, or its normal position has an innermost disk component which is also an innermost disk component of S. In the latter case, the projections $p_N(S_i), p_N(S)$ intersect as stated in the lemma.

The final statement of the lemma follows from Lemma 1.5.1. ■

Now we can show the *bounded geodesic projection theorem* using an argument of Webb from [17].

Theorem 1.5.5. *There is a number* $q > 0$ *with the following property. Let* σ_0 *be a nonseparating sphere, and let* N *the capped off complement of* σ_0 *as before, with projection* $p_N : \mathcal{SG}_n - \mathcal{P}(\sigma_0) \to \mathcal{SG}_N$.

Let $(S_i)_{0\le i\le m}$ *be any geodesic in* $\mathcal{SG}_n$ *which is disjoint from* $\mathcal{P}(\sigma_0) \cup \{\sigma_0\}$. *Then*

$$d(p_N(S_0), p_N(S_m)) < q.$$

Proof. Theorem 1.2 of [11] shows that there exists a number $K > 0$ such that surgery sequences are unparameterized K-quasigeodesics in the sphere graph. Since the sphere graph is Gromov hyperbolic, there is a constant $D > 0$ such that a triangle with K-quasigeodesic sides is D-thin.

For ease of exposition, we distinguish between two cases. First assume that the geodesic (S_i) never enters the $(2D + 2)$-neighborhood of σ_0. Consider nested surgery sequences P and Q joining σ_0 to spheres disjoint from S_0 and S_m, respectively.

By the thin triangle property, there is a sphere S_k which is of distance at most D to both P and Q. Furthermore, every point S_i for $i < k$ is of distance at most D to P, and every point S_i for $i > k$ is of distance at most D to Q. By Lemma 1.5.4, the projections to N of any point on P of distance at least 2 from σ_0 intersect and hence are coarsely the same. Since $d_{\mathcal{SG}_n}(S_i, \sigma_0) \ge 2D + 2$ for all i, for $i \le k$ the sphere S_i is of distance at most D from a point S_i' on P of distance at least $D + 2$ from σ_0. Thus a geodesic connecting S_i' to S_i does not enter the 1-neighborhood of σ_0. Hence the projection p_N

is defined on such a geodesic, and since p_N-is 2-Lipschitz, the projection of $S_i, i \leq k$ is coarsely equal to the projection of S_k, and similarly for $S_i, i \geq k$. This shows that the diameter of the projection is bounded from above by a universal constant as claimed.

If (S_i) does enter the $(2D + 2)$-ball around σ_0, then the argument needs to be modified in the following way. Let $(S_i)_{j \leq i \leq u}$ be the minimal connected segment in (S_i) which contains all intersection points of (S_i) with the $(2D + 2)$ -ball around σ_0. The diameter of this segment is at most $4(D + 1)$, and since (S_i) is a geodesic, the same is true for its length. By our assumption that (S_i) is disjoint from $\mathcal{P}(\sigma_0) \cup \{\sigma_0\}$, the projection $p_N(S_i)$ is defined for all i, and the assignment $i \mapsto p_N(S_i)$ is 2–Lipschitz. Hence, we have

$$\mathrm{diam}(\{p_N(S_i), \quad j \leq i \leq u\}) \leq 8(D + 1).$$

On the complement of $(S_i)_{j \leq i \leq u}$ the argument used in the first case applies. Together this completes the proof. ■

The condition in the theorem simplifies for nonseparating spheres. To exploit this, recall from Lemma 1.2.5 that any two non-separating spheres in $\mathcal{SG}_n$ can be connected by a geodesic consisting of non-separating spheres. We use this in the following

Corollary 1.5.6. *Suppose that σ_1, σ_2 are two non-separating spheres, and that (S_i) is a geodesic in the sphere graph connecting σ_1 to σ_2 and consisting of non-separating spheres.*

If N is the capped off complement of a non-separating sphere σ_0 (as in Theorem 1.5.5), and

$$d(p_N(\sigma_1), p_N(\sigma_2)) \geq q,$$

then $\sigma_0 = S_i$ for some i.

Proof. As remarked above, any sphere in $\mathcal{P}(\sigma_0)$ distinct from σ_0 is separating. Hence, if $\sigma_0 \neq S_i$ for all i, then the geodesic (S_i) consisting of non-separating spheres satisfies the assumption in Theorem 1.5.5. This yields the desired contradiction. ■

A useful more general version of this corollary is the following

Corollary 1.5.7. *For every $L > 0$ there exists a number $q(L) > 0$ with the following property. Let $(S_i)_{0 \leq i \leq m}$ be an L-quasi-geodesic edge path in the graph of non-separating spheres. If N is the capped off complement of a non-separating sphere σ_0 and if $d(p_N(S_0), p_N(S_m)) \geq q(L)$ then $\sigma_0 = S_i$ for some i.*

Proof. We know that a uniform quasi-geodesic in $\mathcal{SG}_n$ avoiding $\mathcal{P}(\sigma_0)$ has uniformly small diameter projection into $\mathcal{SG}_N$. Thus if the diameter of the projection is large, it has to pass through $\mathcal{P}(\sigma_0)$. As any point in $\mathcal{P}(\sigma_0)$ is separating, if the path consists of non-separating spheres then it has to pass through σ_0. ■

1.6 Actions of $\mathrm{O}ut(F_n)$ on products of hyperbolic spaces

This final section is devoted to the proofs of the results stated in the introduction. We follow the strategy developed in [2] as used in [4]. The starting point is the following result of [2].

Theorem 1.6.1. *Let $\mathcal{Y}$ be a collection of δ-hyperbolic spaces, and for every pair $A, B \in \mathcal{Y}$ of distinct elements suppose that we are given a uniformly bounded subset $\pi_A(B) \subset A$, called the projection of B to A. Denoting by $d_A(B, C)$ the diameter of $\pi_A(B) \cup \pi_A(C)$, assume that the following holds: there is a constant $K > 0$ such that*

(1) *if $A, B, C \in \mathcal{Y}$ are distinct, then at most one of the three numbers*

$$d_A(B,C), \quad d_B(A,C), \quad d_C(A,B)$$

is greater than K and

(2) *for any distinct A, B the set*

$$\{C \in \mathcal{Y} - \{A,B\} \mid d_C(A,B) > K\}$$

is finite.

Then there is a hyperbolic space Y and an isometric embedding of each $A \in \mathcal{Y}$ onto a convex set in Y so that the images are pairwise disjoint and the nearest point projection of any B to any $A \neq B$ is within uniformly bounded distance of $\pi_A(B)$. Moreover, the construction is equivariant with respect to any group acting on $\mathcal{Y}$ by isometries.

For a non-separating sphere $S \subset M$ let $\mathcal{Y}(S)$ be the following graph. The set of vertices of $\mathcal{Y}(S)$ is the set $\mathcal{NP}(S)$ of non-peripheral spheres in $M - S$. Two such spheres are connected by an edge of length one if their projections into the sphere graph $\mathcal{SG}_S$ of the manifold obtained from $M - S$ by capping off the boundary are of distance at most one. With this definition, the graph $\mathcal{Y}(S)$ is a geodesic metric graph which is 2-quasi-isometric to the graph $\mathcal{SG}_S$ and hence it is δ-hyperbolic for a constant

$\delta > 0$ not depending on S. The group $\mathrm{O}ut(F_n)$ acts on the collection $\mathcal{Y} = \{\mathcal{Y}(S) \mid S\}$ by isometries.

For S let $p_S : \mathcal{SG}_n - \mathcal{P}(S) \to \mathcal{Y}(S)$ be the submanifold projection defined in Section 1.5. Note that in contrast to the construction in Section 1.5, the target of the map p_S equals the set $\mathcal{NP}(S)$ equipped with a metric inherited from $\mathcal{SG}_S$. Thus it makes sense to project the image into the complement of other spheres which may intersect S. If A is a non-separating sphere and if S is contained in $M - A$, then p_S is not defined on all of $\mathcal{NP}(A) = \mathcal{Y}(A)$, but the only exceptions are points in $\mathcal{P}(S)$. Extend the definition of p_S to $\mathcal{Y}(A)$ by putting

$$p_S(\mathcal{P}(S) \cap \mathcal{NP}(A)) = p_S(A).$$

Note that this should be viewed as an extension of p_S to all of $\mathcal{Y}(A)$. This extension depends on A, but the collection of these extension is equivariant with respect to the action of $\mathrm{O}ut(F_n)$.

Proposition 1.6.2. *The collection* $(\mathcal{Y}(S), p_S)$ *satisfies the conditions in Theorem* 1.6.1.

Proof. Let B be a non-separating sphere different from S. We begin with showing that the diameter of the set $p_S(\mathcal{Y}(B)) \subset \mathcal{Y}(S)$ is uniformly bounded, independent of B and S.

To this end we distinguish two cases. In the first case we have $d_{\mathcal{SG}_n}(B, S) \geq 2$. Then B intersects S, furthermore for every $C \in \mathcal{NP}(B) - \mathcal{P}(S)$, the projections $p_S(B), p_S(C)$ contain components which are disjoint and hence whose distance in $\mathcal{Y}(S)$ equal one. By the definition of p_S, this implies that $p_S(\mathcal{NP}(B))$ is contained in a uniformly bounded neighborhood of $p_S(B)$.

If $d_{\mathcal{SG}}(B, S) = 1$ then $B \in \mathcal{NP}(S)$ since B is non-separating. In particular, we have $p_S(B) = B$. If $C \in \mathcal{NP}(B) \setminus \mathcal{P}(S)$, then by definition, C is disjoint from B and hence the projection $p_S(C)$ is disjoint from B as well. Once again, by the definition of the projection p_S, we conclude that $p_S(\mathcal{NP}(B))$ is contained in a uniformly bounded neighborhood of B. This completes the proof that the diameters of the sets $p_S(\mathcal{Y}(B))$ $(B \neq S)$ are bounded from above by a constant not depending on B, S.

We next verify property (1) in Theorem 1.6.1. Thus let A, B, C be pairwise distinct non-separating spheres and suppose that $d_A(B, C) > 2q$ where $q > 0$ is as in Theorem 1.5.5. Choose a geodesic γ connecting B to C consisting of non-separating spheres. By Corollary 1.5.6, the geodesic γ has to pass through A. Let $i \geq 1$ be such that $\gamma(i) = A$.

Then Theorem 1.5.5 shows that $d_B(C, \gamma(i+1)) \leq q$. As A and $\gamma(i+1)$ are disjoint, the distance between $p_B(A), p_B(\gamma(i+1))$ is uniformly bounded and hence the same holds true for $d_B(A, C)$.

As the roles of B, C can be exchanged, this shows that condition (1) in Theorem 1.6.1 is fulfilled.

Property (2) follows immediately from Corollary 1.5.6: the only spheres C so that the projection $d_C(A, B)$ is large appear along a (fixed) geodesic consisting only of nonseparating spheres, which has finite length. ■

As a fairly immediate consequence, we obtain a more precise version of the main result of [4] in rank 3.

Corollary 1.6.3. *The group* $Out(F_3)$ *admits an isometric action on a product* $Y = Y_1 \times Y_2$ *of two hyperbolic metric spaces so that every exponentially growing automorphism has positive translation length.*

Proof. By Theorem 1.6.1 and Proposition 1.6.2, the group $\mathrm{O}ut(F_3)$ admits an isometric action on $Y = Y_1 \times Y_2$ where Y_1 is the free splitting complex or, equivalently, the sphere graph of $M = M_3$, and where Y_2 is a hyperbolic space containing for each non-separating sphere S the graph $\mathcal{Y}(S) = \mathcal{SG}_2$ as a convex isometrically embedded subspace.

A non-separating sphere S in the manifold M corresponds precisely to the conjugacy class of a corank one free factor, consisting of homotopy classes of loops based at a point $p \in M - S$ which do not intersect S. As a consequence, any element of $\mathrm{O}ut(F_3)$ which preserves such a corank one free factor, defined by the sphere S, and acts as an exponentially growing automorphism on it acts with positive translation length on the graph $\mathcal{Y}(S)$, which is uniformly quasi-isometric to the Farey graph. Then such an element acts with positive translation length on Y_2 and hence on Y. We refer to Section 1.4 for a detailed discussion.

On the other hand, by [8], if φ is an exponentially growing automorphism of F_3 then there exists a number $j \geq 1$ such that either φ^j acts with positive translation length on the sphere graph Y_1 of M, or φ^j preserves a corank one free factor A and acts with positive translation length on the free splitting complex of A. Note that the conclusion on the corank stems from the fact that a corank 2 free factor of F_3 is infinite cyclic and hence does not admit any exponentially growing automorphisms. Together this yields the proof of the corollary. ■

The above construction can be interpreted in the following way. Let $n \geq 3$ and let $\mathcal{PG}_n$ be the graph whose vertices are ordered pairs (S_1, S_2) of disjoint non-separating spheres. Two such pairs (S_1, S_2) and (S_1', S_2') are connected by an edge of length one if either $S_1 = S_1'$ and the second spheres S_2, S_2' are connected by an edge in the graph $\mathcal{Y}(S_1)$, or if $(S_1', S_2') = (S_2, S_1)$. Note that in contrast to similar constructions for graphs of curves or graphs of disks (see for example [6]), the spheres (S_1, S_2) and (S_1', S_2') may be connected by an edge although their union can not be realized disjointly. The group $\mathrm{O}ut(F_n)$ acts on the graph $\mathcal{PG}_n$ as a group of simplicial automorphisms.

Since two spheres in the first factor of the points in $\mathcal{PG}_n$ only are exchanged if they are disjoint, the first factor projection $\Pi_1 : \mathcal{PG}_n \to \mathcal{SG}_n$ is an $\mathrm{O}ut(F_n)$-equivariant one-Lipschitz projection onto the 1-dense convex subgraph of non-separating spheres. Recall that $\mathcal{PG}_n$ is only defined for $n \geq 3$.

Theorem 1.6.4. *The graph $\mathcal{PG}_n$ of non-separating pairs is a hyperbolic $Out(F_n)$-graph.*

Proof. Given what we achieved so far, the proof is fairly standard. For each non-separating sphere $S \subset M$ consider the subgraph

$$\Pi_1^{-1}(S) = \{(S, S') \mid S'\} = H(S) \subset \mathcal{PG}_n$$

of pairs with one component equal to S. This graph is 2 -quasi-isometric to $\mathcal{Y}(S)$ and hence it is δ-hyperbolic for a number $\delta > 0$ not depending on S.

For $S \neq S'$, the intersection $H(S) \cap H(S')$ can be viewed as a graph of non-separating spheres which are disjoint from both S, S'. Thus the diameter of this intersection in both $H(S), H(S')$ is uniformly bounded.

Let $\mathcal{EG}$ be the electrification of $\mathcal{PG}_n$ with respect to the family $\mathcal{H}$ of subgraphs $H(S)$. This electrification is the graph obtained from $\mathcal{PG}_n$ by adding a vertex v_S for each of the graphs $H(S)$ and connecting v_S to each vertex in $H(S)$ by an edge. By construction, this electrification is two-quasi-isometric to the graph of non-separating spheres and hence it is hyperbolic. In particular, any L-quasi-geodesic in $\mathcal{EG}$ defines a $2L$-quasi-geodesic in $\mathcal{SG}_n$.

The *bounded penetration property* in this context states that for every $L > 1$ there exists a number $p(L) > 0$ with the following property [6]. Call an L-quasi-geodesic edge path in $\mathcal{EG}$ *efficient* if for every non-separating sphere S we have $\gamma(k) = v_S$ for at most one k. Let $\gamma \subset \mathcal{EG}$ be an efficient L-quasi-geodesic and let S and k be such that $\gamma(k) = v_S$. If the distance in $H(S)$ between $\gamma(k-1)$ and $\gamma(k+1)$ is at least $p(L)$ then every efficient L-quasi-geodesic γ' in $\mathcal{EG}$ with the same endpoints as γ passes through v_S. Moreover, if $\gamma'(k') = v_S$ then the distance in $H(S)$ between $\gamma(k-1), \gamma'(k'-1)$ and $\gamma(k+1), \gamma'(k'+1)$ is at most $p(L)$.

By Corollary 1.5.7 and the fact that $\mathcal{EG}$ is 2-quasi-isometric to the graph of non-separating spheres, the bounded penetration property holds true for the subspaces $H(S)$. Thus it follows from Theorem 1 of [6] that $\mathcal{PG}_n$ is hyperbolic. ∎

The graph $\mathcal{PG}_n$ also has the following description. Its vertices are conjugacy classes of pairs $A_1 > A_2$ of free factors, where A_1 is of corank 1 and A_2 is of corank 2. There are two types of edges. The first type preserves A_1 and exchanges A_2 by a corank two free factor contained in A_1, equivalently a corank one free factor of A_1, that is connected to A_2 by an edge in the free splitting graph of A_1. The second type preserves A_2 and replaces A_1 by a corank one free factor containing A_2 which is connected to A_1 by an edge in the free splitting graph of F_n.

Note that the group $\mathrm{O}ut(F_n)$ naturally acts on $\mathcal{PG}_n$ as a group of simplicial isometries. Using this graph we can complete the proof of Theorem 1.

Theorem 1.6.5. *The group Out(F_3) admits an isometric action on a hyperbolic metric graph such that every exponentially growing automorphism has positive translation length.*

Proof. The proof is immediate from the proof of Corollary 1.6.3 via noting that by Theorem 1 of [6] and the construction of the graph $\mathcal{PG}_n$, for each non-separating sphere S the subgraph $H(S)$ is uniformly quasi-convex and isometric to the graph $\mathcal{Y}(S)$. Thus any exponentially growing automorphism of F_3 acts with positive translation length on $\mathcal{PG}_3$. ∎

Acknowledgments. The authors are grateful to Sam Taylor for pointing out the reference [16], and to Mladen Bestvina for useful information. We also thank the referee for helpful comments.

Funding. This work was partially supported by the Hausdorff Center Bonn.

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