APPLICATIONS OF TEICHMÜLLER THEORY TO HYPERBOLIC 3-MANIFOLDS

URSULA HAMENSTÄDT

Lecture 1: Teichmüller geodesics and the curve complex

1. Introduction. In these lectures, S always denotes a *closed* oriented surface of genus $g \geq 2$.

We begin with collecting some differential-geometric facts about such surfaces.

Fact 1 S admits a hyperbolic metric, i.e. a Riemannian metric of constant curvature -1.

Fact 2 There is a constant $\chi_0 = \chi_0(S)$ such that for every hyperbolic metric g on S there is a *simple closed* g-geodesic of length at most χ_0.

Definition 1. The *mapping class group* $\mathcal{M}(S)$ of S is the group of isotopy classes of orientation preserving diffeomorphisms of S.

A basic example of a closed 3-manifold:

The *mapping torus* M of $\varphi \in \mathcal{M}(S)$ is defined by

$$M = S \times [0,1]/\sim \text{ where } (x,1) \sim (\varphi(x),0).$$

This mapping torus has the following properties.

(1) M is a $K(\pi,1)$-space.
(2) There is an exact sequence
$$0 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 0$$
i.e. $\pi_1(M)$ is a $\mathbb{Z}$-extension of $\pi_1(S)$.

In particular: $\pi_1(M)$ admits $\pi_1(S)$ as a normal subgroup, and the natural $\mathbb{Z}$-cover of M is diffeomorphic to $S \times \mathbb{R}$.

Basic question: How are
- the properties of φ
- the "geometry" of M (for a suitable choice of a metric)
- the topology of M
related?

Date: June 1, 2007.

Basic idea: Try to use an "easy to understand" combinatorial model for the geometry of M and relate this model to the topology of M.

Definition 2. An L-quasi-isometric embedding of a metric space (X,d) into a metric space (Y,d) is a map $F : X \to Y$ such that

$$d(x,y)/L - L \leq d(Fx,Fy) \leq Ld(x,y) + L$$

for all $x, y \in X$. F is an L-quasi-isometry if moreover for every $y \in Y$ there is some $x \in X$ with $d(F(x),y) \leq L$.

There is a fundamental geometric relation between the structure of a finitely presented group and the structure of a space on which this group can act cocompactly. To explain this fact we recall some basic properties of a finitely generated group.

Namely, let Γ be a finitely generated group with finite *symmetric* generating set $\mathcal{G}$. The *word norm* $|g|$ of $g \in \Gamma$ is the minimum of a word in the generators representing g. Then

$$d(g,h) = |g^{-1}h|$$

defines a Γ-invariant metric on Γ, i.e. we have $d(ug,uh) = d(g,h) \forall u, g, h$.

Any two such metrics are quasi-isometric: If $|\,|'$ is another word norm then there is a constant $c > 0$ such that $|g|/c \leq |g|' \leq c|g| \forall g$.

Proposition 1. *(Svarc-Milnor): Let M be a closed 3-manifold with universal covering $\mathbb{R}^3$ and fundamental group $\pi_1(M)$ of M. If g is any Riemannian metric on M then there is a $\pi_1(M)$-equivariant quasi-isometry $F : \tilde{M} \to \Gamma$.*

Proof. The fundamental group $\pi_1(M)$ of M acts on $\tilde{M} = \mathbb{R}^3$ freely, properly discontinuously and cocompactly.

Fix a point $x \in \tilde{M}$ and define $F : \pi_1(M) \to \tilde{M}$ by $F(g) = gx$. Then F is equivariant with respect to the action of $\pi_1(M)$ on Γ and on $\tilde{M}$. We claim that F is a quasi-isometry with respect to the word norm on $\pi_1(M)$ defined by a finite symmetric set $\mathcal{G}$ of generators of $\pi_1(M)$ and the distance defined by any Riemannian metric on M.

For this observe that there is a constant $L > 0$ such that $d(gx,x) \leq L \forall g \in \mathcal{G}$. If $h = g_1 \dots g_k$ is any word in $\mathcal{G}$ of length k then

$$d(hx,x) \leq d(hx, g_2 \dots g_k x) + d(g_2 \dots g_k x, x) \leq kL$$

by an inductive use of the triangle inequality which shows that F is coarsely Lipschitz.

The fact that distances in $\tilde{M}$ on the orbit $\pi_1(M)x$ are controlled by distances on $\pi_1(M)$ is a little bit more difficult and will not be explained here. $\square$

As a consequence of our argument, the lifts to $\tilde{M}$ of any two metrics on M are quasi-isometric.

Basic example: Let M be the mapping torus of $\varphi \in \mathcal{M}(S)$. Choose a hyperbolic metric on S and a metric on M so that $S \to S \times \{0\}/\sim$ is an isometric embedding. Let $\hat{M}$ be the $\mathbb{Z}$-cover of M.

Let c be a simple closed geodesic on S of length at most χ_0. Then for each $i \in \mathbb{Z}$, the minimal length of a closed curve in $\hat{M}$ representing $\varphi^i(c)$ is *uniformly bounded, independent of* i.

Question: Can we recover from the "collection of shorts curves" in $\hat{M}$ the mapping class φ and hence the topology of M?

2. The complex of curves.

Definition 3. The *complex of curves* is the simplicial complex whose *vertex set* $\mathcal{C}(S)$ is the set of all nontrivial free homotopy classes of simple closed curves on S. A collection $c_1, \ldots, c_k \subset \mathcal{C}(S)$ spans a simplex if and only if $c_1, \ldots, c_k$ can be realized disjointly. The *curve graph* $\mathcal{CG}(S)$ is the one-skeleton of the curve complex.

Facts:

(1) The complex of curves is connected and of dimension $3g-4$.
(2) A simplex of maximal dimension is a *pants decomposition* of S: After cutting S open along the curves of the simplex we obtain $2g-2$ *pairs of pants.*

The curve graph is a naturally a *locally infinite metric graph.*

The following picture shows a pants decomposition for a surface of genus 2.

Fact: If $c, d \in \mathcal{C}(S)$ and if $d(c,d) \geq 3$ then c, d *fill up* S, i.e. c, d decompose S into topological discs.

Definition 4. The *intersection number* $i(c,d)$ between two simple closed curves c, d is the minimal number of intersections between two curves freely homotopic to c, d.

Proposition 2. *There is a number* $\kappa = \kappa(S)$ *such that* $d(c,d) \leq \kappa \log i(c,d) + \kappa$ *for all* $c, d \in \mathcal{C}(S)$.

Proof. Let c, d be simple closed curves on S with minimal intersection number in their free homotopy classes.

Then d intersects $S - c$ in simple arcs with both endpoints on c. Thus there are at most m homotopy classes rel c of such arcs.

Let δ be an arc whose class contains the maximal number of components and let

$$b = \text{ a component of } \partial(N(c \cup \delta)).$$

Then $i(c,b) = 0 \Rightarrow d(c,b) = 1$ and $i(b,d) \leq (m-1)i(c,d)/m$. □

Fix a hyperbolic metric on S.

Definition 5. A *geodesic lamination* on S is a *closed* subset of S foliated into simple geodesics.

Recall: The *Hausdorff-distance* $d_H(A,B)$ of compact subsets A, B of S is defined by

$$d_H(A,B) = \inf\{\epsilon > 0 \mid A \subset U_\epsilon(B), B \subset U_\epsilon(A)\}.$$

The space of compact subsets of S equipped with the Hausdorff distance is compact.

Facts:

(1) The space $\mathcal{L}(S)$ of geodesic laminations with the Hausdorff topology is compact.
(2) For every geodesic lamination λ, $S - \lambda$ is a hyperbolic surface with geodesic boundary and area $4\pi(g-1)$.

Definition 6. A geodesic lamination λ is *maximal* if $S - \lambda$ is a union of ideal triangles. It is *complete* if in addition λ can be approximated in the Hausdorff topology by simple closed geodesics. It is *minimal* if every half-leaf is dense.

Proposition 3. *The diameter of $\mathcal{CG}(S)$ is infinite.*

Proof. (Luo) By contradiction: *Assume* that $\mathrm{diam}(\mathcal{CG}(S)) = D < \infty$.

Let μ be a complete minimal geodesic lamination. After passing to a subsequence we may assume that $c_i \to \mu$ $(i \to \infty)$ in the Hausdorff topology.

Since $d(c_0, c_i) \leq D$ we may assume that $d(c_0, c_i) = N \forall i$.

Choose some simple closed curves b_i $(i > 0)$ such that $d(c_0, b_i) = N-1, d(b_i, c_i) = 1 \forall i$. After passing to a subsequence we may assume that $b_i \to \mu'$ in the Hausdorff topology. But $\mu = \mu'$ *because* $i(b_i, c_i) = 0$ means: $b_i \cup c_i$ is as geodesic lamination and hence it converges up to passing to a subsequence to a geodesic lamination.

Repeat with (b_i). After N steps we conclude that $c_0 \to \mu$, a contradiction. □

3. Pseudo-Anosov mapping classes. The mapping class group acts on $\mathcal{CG}(S)$ as a group of simplicial isometries.

Definition 7. A mapping class $\varphi \in \mathcal{M}(S)$ is *pseudo-Anosov* if the orbit of $< \varphi >< \mathcal{M}(S)$ on $\mathcal{CG}(S)$ is *unbounded*.
A mapping class φ is *periodic* if $| < \varphi > | < \infty$. A mapping class which is neither periodic nor pseudo-Anosov is *reducible*.

Our basic example:

(1) $\varphi \in \mathcal{M}(S)$ periodic $\Leftrightarrow$ the mapping torus $M = \text{Maptorus}(\varphi)$ of φ has a *finite* covering diffeomorphic to $S \times S^1$
$\Rightarrow$ for *every* smooth metric on M and every $L > 0$ the number of elements in $\pi_1(\hat{M}) = \pi_1(S)$ which can be realized by a curve of length at most L in M is *finite*.
Also: By the *Nielsen realization problem*, φ can be realized as an biholomorphic map of a hyperbolic surface
$\Rightarrow S \to S/ < \varphi >$ is a branched covering $\Rightarrow M$ is foliated into smooth circles $\Leftrightarrow M$ is a *Seifert fibered space*.
(2) φ is pseudo-Anosov $\Rightarrow$ for $M = \text{Maptorus}(\varphi)$ and every smooth metric on M there is some $L > 0$ such that the number of elements of $\pi_1(\hat{M})$ which can be realized by a curve of length at most L is *infinite*.

Afternoon discussion:

Construction of a minimal and complete geodesic lamination on a closed surface S.

Lecture 2: Singular euclidean metrics

As before, S denotes a *closed* oriented surface of genus $g \geq 2$.
Recall: Two simple closed curves $\alpha, \beta \in \mathcal{C}(S)$ with $d(\alpha, \beta) \geq 3$ *fill up* S.

Define a *singular euclidean metric* using the simple closed curves α, β and numbers $a, b > 0$ with $abi(\alpha, \beta) = 1$ as follows. Choose representatives of α, β with the minimal number of intersection points. Place each intersection point in the interior of an euclidean rectangle with sides a, b such that the sides of length b are parallel to α and the sides of length a are parallel to β. These rectangles can be arranged in such a way that the fill any two of them meet at most along a side or a vertex and that they cover S. The area of the resulting singular euclidean metric on S equals one.

The *width* of an *annulus* in a surface with singular euclidean metric is the distance between its boundary curves.

Proposition 4. *There is $w > 0$: Every area one singular euclidean metric on S has an annulus of width $\geq w$.*

Proof. (Bowditch) Use *isoperimetric inequality*:
In the euclidean plane,

$$\text{length}(\gamma) \geq 2\sqrt{\pi \text{area}(\text{disc enclosed by } \gamma)}.$$

In a piecewise euclidean metric with one p-pronged singularity:
if γ is a Jordan curve enclosing the singularity then divide the enclosed disc along boundaries of sectors
$\Rightarrow$ use euclidean isoperimetric inequality on sectors to deduce:

There exists a distance decreasing function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\text{area}(D) \leq f(\text{length}(\partial D))$$

for every disc D in the singular euclidean surface.

A *spine* of S is a *embedded graph* G s.th. $S - G =$ union of discs.

Consequence: The length of a spine is uniformly bounded from below.

Let c be a shortest simple closed geodesic on S (of length η_0).

Fact: If η_0 is large then c is the core curve of a cylinder of width $\geq \chi$ where $\chi > 0$ is a universal constant.

Namely: Let

$$\epsilon_0 = \sup\{\epsilon \mid N_\epsilon(c) = \text{annulus}\}.$$

Then there is a geodesic arc of length $2\epsilon_0$ with both endpoints on c
$\Rightarrow$ if $\epsilon_0 < \eta_0/4$ then c is not shortest.

Second case:

Assume η_0 is small.
Choose a universal number $\eta_1 > 0$ (depending on S).
Choose a maximal collection of simple arcs, of length $\leq 2\eta_1$, with endpoints on c
$\Rightarrow$ the *number* of these arcs is bounded by topology
$\Rightarrow$ their *total length* uniformly bounded
$\Rightarrow$ the collection can not be a spine of $S - N_{\eta_1}(c)$
$\Rightarrow$ there exists a nontrivial component of $S - N_{\eta_1}(c)$
$\Rightarrow$ there exists a nontrivial simple closed curve d in $S - N_{\eta_1}(c)$.

Choose a distance decreasing function $f : S \to [0, \infty)$ with $f(c) = 0, f(d) = \eta_1$.
Then for no subinterval $[a, b]$ of $[0, \eta_0]$ the set $f^{-1}[a, b]$ is a disc
$\Rightarrow$ we can find an annulus with large width in preimages of disjoint intervals. $\square$

Conclusion: $\forall \alpha, \beta$, $\forall a, b > 0$ with $abi(\alpha, \beta) = 1$ there is $\delta \in \mathcal{C}(S)$ with

$$i(\rho, \delta) \leq \text{const}\big(ai(\rho, \alpha) + bi(\rho, \beta)\big)\forall \rho.$$

Also: The *length* of δ with respect to the singular euclidean metric q defined by c, d is uniformly bounded.

Corollary 5. (1) *The diameter in $\mathcal{CG}(S)$ of the set of short q-curves is uniformly bounded.*
(2) *The distance in $\mathcal{CG}(S)$ of a short q-curve and a short curve for the hyperbolic metric defined by q is uniformly bounded.*

The above now allows us to *find* a path $c_0, \dots, c_k \subset \mathcal{C}(S)$ with

(1) $c_0 = \alpha, c_k = \beta$.
(2) $d(c_i, c_{i+1}) \leq k$.
(3) $i(\alpha, c_i)i(c_i, \beta) \leq ki(\alpha, \beta)$.

Construction: For $t \in \mathbb{R}$ consider the euclidean metric q_t determined by the pair $(e^t c, e^{-t} d/i(c, d))$. Then:

(1) t very large $\Rightarrow$ c is q_t-short.
(2) t very small $\Rightarrow$ d is q_t-short.

Define $\eta_{c,d}(t) =$ a short curve for q_t.
Then: This definition is unique up to a uniformly bounded error.
These curve vary "coarsely continuously" with t.

Corollary 6. *There is $k > 0$ and for $\alpha, \beta, \gamma \in \mathcal{C}(S)$ there is $\delta \in \mathcal{C}(S)$:*

$$\delta \in U_k(\eta_{\alpha,\beta}(\mathbb{R})) \cap U_k(\eta_{\beta,\gamma}(\mathbb{R})) \cap U_k(\eta_{\gamma,\alpha}(\mathbb{R})).$$

Proof. Let a, b, c s.th.

$$abi(\alpha, \beta) = bci(\beta, \gamma) = cai(\gamma, \alpha) = 1.$$

Let $\delta \in \mathcal{C}(S)$,

$$i(\delta, \rho) \leq \text{const} \max\{ai(\alpha, \rho), bi(\beta, \rho)\}\forall \rho.$$

Then $i(\delta, \gamma) \leq \text{const}/c$ shows:

$$\max\{ci(\gamma, \delta), ai(\alpha, \delta)\} \leq \text{const}$$

$\Rightarrow$ δ is short for the metric defined by $a\alpha, c\gamma$ and similarly for the metric defined by $b\beta, c\gamma$. □

Very important fact: If we replace c by a multi-curve containing c as a component then we find *nearby short curves*.

Homework: Prove this.

This implies:

Proposition 7. *There is a number $C > 0$ and for any two points $c, d \in \mathcal{C}(S)$ there is a path $\eta_{c,d} : [0, 1] \to \mathcal{CG}(S)$ with the following properties.*

(1) *If $d(c, d) = 1$ then* $\operatorname{diam}(\eta[0, 1]) \leq C$.
(2) *If $c, d \in \mathcal{C}(S)$ and if $s < t \in [0, 1]$ then for the Hausdorff-distance*

$$d_H(\eta_{c,d}[s, t], \eta_{\eta_{c,d}(s),\eta_{c,d}(t)}[0, 1]) \leq C.$$

(3) *For a, b, c, $\eta_{[a,b]}[0, 1] \subset N_C(\eta_{[b,c]}[0, 1]) \cup \eta_{[c,a]}[0, 1])$.*

0.1. **Hyperbolicity of the curve graph.**

Definition 8. A geodesic metric space X is δ-*hyperbolic* if $\forall$ triangles with geodesic sides a, b, c we have $c \subset N_\delta(a \cup b)$.

Basic property: In a hyperbolic geodesic metric space, an L-quasi-geodesic is contained in a uniformly bounded neighborhood of a geodesic.

Theorem 1. *(Masur-Minsky) The curve graph is hyperbolic.*

Lecture 3: Teichmüller geodesics

Definition 9. A *measured geodesic lamination* on S is a geodesic lamination together with a *transverse translation invariant measure.* A *projective measured geodesic lamination* is the projectivization of the space of measured geodesic laminations.

Example: A *weighted simple multi-curve* is a geodesic multicurve whose components carry a positive weight.

Theorem 2. *(Thurston)*

(1) *The space $\mathcal{PML}$ of projective measured geodesic laminations equipped with the weak*-topology is homeomorphic to S^{6g-7}.*
(2) *The intersection form i extends continuously to a symmetric function on the space $\mathcal{ML}$ of measured geodesic laminations.*
(3) *If $\eta, \zeta \in \mathcal{ML}$* fill up *$S$, i.e. if $i(c, \eta) + i(c, \zeta) > 0$ for all simple closed curves then (η, ζ) defines a* complex structure *$R(\eta, \zeta)$ on S and a* singular euclidean metric.
(4) *If moreover $i(\eta, \zeta) = 1$ then $t \to R(e^t\eta, e^{-t}\zeta)$ is a* Teichmüller geodesic.

(5) *(Teichmüller) Any two points in the* Teichmüller space $\mathcal{T}(S)$ *can be connected by a unique Teichmüller geodesic segment depending smoothly on the point.*
(6) *Teichmüller space can naturally and* $\mathcal{M}(S)$*-equivariantly be compactified by adding* $\mathcal{PML}$.

Homework: Recall that a mapping class is *periodic* if the subgroup of $\mathcal{M}(S)$ it generates is finite, and it is *pseudo-Anosov* if the subgroup it generates acts on $\mathcal{C}(S)$ with unbounded orbits.

Show: A reducible mapping class preserves a multi-curve.

Lecture 4: Cobounded Teichmüller geodesics and the curve graph

As before, S denotes a *closed* oriented surface of genus $g \geq 2$ with set $\mathcal{C}(S)$ of free homotopy classes of simple closed curves defining the vertex set of the curve graph $\mathcal{CG}(S)$.

A pair (λ, μ) of measured geodesic laminations with

$$i(\lambda, \mu) = 1, i(\lambda, c) + i(\mu, c) > 0 \forall c \in \mathcal{C}(S)$$

defines an *area one singular euclidean metric* and a *Teichmüller geodesic*

$$t \to \text{ the complex structure defined by } (e^t\lambda, e^{-t}\mu)$$

in Teichmüller space $\mathcal{T}(S)$.

Definition 10. A sequence of simple closed geodesics $c_i \subset \mathcal{C}(S)$ *converges in the coarse Hausdorff topology* to a *minimal geodesic lamination* λ *which fills up* S if every accumulation point of (c_i) in the Hausdorff topology contains λ as a subset.

Theorem 3. *(Klarreich) If* $\gamma : [0, \infty) \to \mathcal{C}(S)$ *is a quasi-geodesic then* $(\gamma(i))$ *converges in the coarse Hausdorff topology to a minimal geodesic lamination which fills up* S.

The ϵ*-thick part* $\mathcal{T}(S)_\epsilon$ of Teichmüller space $\mathcal{T}(S)$ is the subset of all metrics whose *systole* (length of shortest closed geodesic) is at least ϵ.

Recall: There is a map $\Phi : \mathcal{T}(S) \to \mathcal{CG}(S)$ defined as follows. $\forall x, \Phi(x)$ is a simple closed x-geodesic of length at most χ_0 ($\chi_0 > 0$ is a Bers constant).

The map Φ is
- *coarsely Lipschitz continuous*: Exists L s.th.

$$d(\Phi x, \Phi y) \leq L d(x, y) + L \forall x, y$$

(here d is the Teichmüller distance on $\mathcal{T}(S)$.)
- *coarsely $\mathcal{M}(S)$-equivariant*: Exists C s.th.

$$d(g\Phi x, \Phi g x) \leq C \,\forall x \in \mathcal{T}(S),\, \forall g \in \mathcal{M}(S).$$

Definition 11. For $\epsilon > 0$, a *quasi-convex curve* in $\mathcal{T}(S)_\epsilon$ is a closed subset of $\mathcal{T}(S)$ whose *Hausdorff distance* to the image of a geodesic arc $\zeta : J \to \mathcal{T}(S)_\epsilon$ is at most $1/\epsilon$.

Theorem 4. *For every $\nu > 1$ there is a constant $\epsilon = \epsilon(\nu) > 0$ with the following properties. Let $J \subset \mathbb{R}$ be a closed connected set of diameter at least $1/\epsilon$.*
Let $\gamma : J \to \mathcal{T}(S)$ be a ν-quasi-geodesic.
If $\Phi \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{GC}(S)$ then $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}(S)_\epsilon$.

Lemma 8. *For every $\nu > 1$ there is a number $\epsilon_0 = \epsilon_0(\nu) > 0$ s.th:*
Let $\gamma : [0, n] \to \mathcal{T}(S)$ be a ν-quasi-geodesic whose projection $\Phi\gamma$ to $\mathcal{GC}(S)$ is a ν-quasi-geodesic. If $n \geq 1/\epsilon_0$ then $\gamma[0, n] \subset \mathcal{T}(S)_{\epsilon_0}$.

Proof. Wolpert:

$$d(h, h') \geq |\log \ell_h(\alpha) - \log \ell_{h'}(\alpha)| \forall \alpha \in \mathcal{C}(S), \forall h, h' \in \mathcal{T}(S).$$

□

Proof of the theorem:

Step 1: For $\nu > 1$, define a *ν-Lipschitz curve* in $\mathcal{T}(S)$ to be a ν-Lipschitz map $\gamma : J \to \mathcal{T}(S)$.

It is enough to show the statement of the theorem for ν-Lipschitz ν-quasi-geodesics s.th. $\Phi \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{GC}(S)$.

Also: By the above Lemma, assume that $\gamma(J) \subset \mathcal{T}(S)_{\epsilon_0}$.

Step 2:

Since $\mathcal{C}(S)$ is hyperbolic and $\Phi \circ \gamma$ is a ν-quasi-geodesic there is a geodesic arc ζ in $\mathcal{C}(S)$ with

$$d_{\text{Hausdorff}}(\zeta, \Phi \circ \gamma) \leq C.$$

If J is one-sided infinite, say if $[0, \infty) \subset J$ then $\Phi(\gamma(t))$ converges in the coarse Hausdorff topology to a minimal geodesic lamination χ which fills up S.

Denote by $\mathcal{ML}$ the space of measured geodesic laminations. The projective measured lamination $[\alpha]$ defined by $\alpha \in \mathcal{C}(S)$ is *realized* at some $t \in J$ if $\ell_{\gamma(t)}(\alpha) \leq \chi$.

The projectivization $[\lambda]$ of a measured geodesic lamination λ is realized at an infinite "endpoint" of J if the support of λ equals the coarse Hausdorff limit of the quasi-geodesic $\Phi\gamma(J)$.

Call a projective measured lamination which is realized at a (finite or infinite) endpoint of J an *endpoint lamination*.

Transport the distance on $\mathcal{GC}(S)$ to a *continuous* distance function on $\mathcal{T}(S)$:

Choose a number $R > 2\chi$ and a smooth function $\sigma : [0, \infty) \to [0, 1]$ with $\sigma[0, \chi] \equiv 1$ and $\sigma[R, \infty) \equiv 0$.

For every $h \in \mathcal{T}(S)$ define a finite Borel measure μ_h on $\mathcal{C}(S)$ by

$$\mu_h = \sum_\beta \sigma(\ell_h(\beta))\delta_\beta$$

where δ_β denotes the Dirac mass at β.

Properties: There is $C > 0$ such that the following holds true.

(1) $1/C \le \mu_h(\mathcal{C}(S)) \le C$.
(2) $\mathrm{diam}(\mathrm{supp}(\mu_\mathrm{h})) \le \mathrm{C}$.
(3) μ_h depends continuously on h.

Define a new "distance" function ρ on $\mathcal{T}(S)$ by

$$\rho(h, h') = \int_{\mathcal{C}(S)\times\mathcal{C}(S)} d(\cdot, \cdot)d\mu_h \times d\mu'_h/\mu_h(\mathcal{C}(S))\mu_{h'}(\mathcal{C}(S)).$$

Properties:

(1) ρ is positive and continuous on $\mathcal{T}(S)\times\mathcal{T}(S)$ and invariant under the action of $\mathcal{M}(S)$.
(2) There is $a > 0$ s.th.

$$\rho(h, h')/a - a \le d(\Phi(h), \Phi(h')) \le a\rho(h, h') + a.$$

As a consequence, for every $\nu > 1$ there is a number $p = p(\nu) > 1$ with the following properties. If $\gamma : J \to \mathcal{T}(S)$ is such that $\Phi\gamma$ is a ν-quasi-geodesic then γ is a p-quasi-geodesic with respect to ρ:

$$\rho(\gamma(s), \gamma(t))/p - p \le |s - t| \le p\rho(\gamma(s), \gamma(t)) + p\forall s, t \in J.$$

For $h \in \mathcal{T}(S), \mu \in \mathcal{ML}$ the product of the transverse measure for μ together with the length element of h defines a measure on $\mathrm{supp}(\mu)$.
Its total mass is called the *h-length* $\ell_h(\mu)$ of μ.

For $p > 1$ define Γ_p to be the set of all triples $(\gamma : J \to \mathcal{T}(S), \lambda_+, \lambda_-)$ s.th.

(1) $0 \in J$ and $\mathrm{diam}(J) \ge 1/\epsilon_0$.
(2) $\gamma : J \to \mathcal{T}(S)$ is a p-Lipschitz curve which is p-quasi-geodesic w.r.to ρ.
(3) $\lambda_+, \lambda_- \in \mathcal{ML}$ are of $\gamma(0)$-length 1, and the projectivizations $[\lambda_+], [\lambda_-]$ are realized at the ends.

Equip Γ_p with the product topology of the weak*-topology on $\mathcal{ML}$ and the compact-open topology for the arc $\gamma \subset \mathcal{T}(S)$.
This topology is metrizable.

Claim: The action of $\mathcal{M}(S)$ on Γ_p is cocompact.

Proof of the claim: By equivariance and cocompactness of the action of $\mathcal{M}(S)$ on $\mathcal{T}(S)_\epsilon$:
Enough to show the claim for the subset of Γ_p consisting of triples with the additional property that $\gamma(0) \in A$, $A \subset \mathcal{T}(S)$ compact.

For this: Use the Arzela-Ascoli theorem.

Each point $(\gamma, \lambda_+, \lambda_-) \in \Gamma_p$ determines the geodesic $\eta([\lambda_+], [\lambda_-])$ in $\mathcal{T}(S)$.

For $(\gamma, \lambda_+, \lambda_-)$ define $\sigma(\gamma, \lambda_+, \lambda_-)$ to be the point on the geodesic $\eta([\lambda_+], [\lambda_-])$ which corresponds to the quadratic differential defined by the measured geodesic laminations λ_+, λ_-.

The map taking $(\gamma, \lambda_+, \lambda_-)$ to $(\gamma(0), \sigma(\gamma, \lambda_+, \lambda_-)) \in \mathcal{T}(S) \times \mathcal{T}(S)$ is continuous and equivariant with respect to the natural action of $\mathcal{M}(S)$ on Γ_p and on $\mathcal{T}(S) \times \mathcal{T}(S)$
$\Rightarrow$ the action of $\mathcal{M}(S)$ on the image of our map is cocompact
$\Rightarrow d(\gamma(0), \sigma(\gamma, \lambda_+, \lambda_-)) \leq p > 0$.

Let $(\gamma, \lambda_+, \lambda_-) \in \Gamma_p$. For each $s \in J$ define

$$a_-(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_-)}, \quad a_+(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_+)}$$

where: $\ell_{\gamma(s)}(\lambda_\pm)$ is the $\gamma(s)$-length of $\lambda_\pm$.

For $s \in \mathbb{R}$ define $\gamma'(t) = \gamma(t+s)$
$\Rightarrow$ the triple $(\gamma'(0), a_+(s)\lambda_+, a_-(s)\lambda_-)$ lies in the $\mathcal{M}(S)$-cocompact set Γ_p
$\Rightarrow d(\gamma(s), \eta([\lambda_+], [\lambda_-])) \leq p$
$\Rightarrow \gamma$ is contained in the p-neighborhood of $\eta([\lambda_+], [\lambda_-])$. □

Lecture 5: Hyperbolic 3-manifolds diffeomorphic to $S \times \mathbb{R}$

As before, S denotes a *closed* oriented surface of genus $g \geq 2$ with set $\mathcal{C}(S)$ of free homotopy classes of simple closed curves defining the vertex set of the curve graph $\mathcal{CG}(S)$.

Let $\rho : \Gamma \to PSL(2, \mathbb{C})$ be a discrete faithful representation s.th.
i) $M = \mathbf{H}^3/\Gamma$ is diffeomorphic to $S \times \mathbb{R}$
ii) There is $\epsilon > 0$: $\forall x \in M, \text{inj}(x) \geq \epsilon$.

Recall: The *convex core* **Core**(M) of M is the quotient under Γ of the *convex hull of the limit set* Λ of Γ.

3 Cases:

(1) **Core**(M) is compact
$\Leftrightarrow$ Γ is *convex cocompact*
$\Leftrightarrow$ Γ is *quasifuchsian*
$\Leftrightarrow$ the limit set Λ is an embedded circle in S^2
$\Leftrightarrow$ if $\Omega = S^2 - \Lambda$ then Ω/Γ consists of a pair of Riemann surfaces diffeomorphic to S which determine M up to isometry.
(2) ∂**Core**(M) is a connected surface diffeomorphic to S
$\Leftrightarrow$ M has precisely one geometrically finite end
$\Leftrightarrow$ $\Omega = S^2 - \Lambda$ is a disc, Ω/Γ is a Riemann surface diffeomorphic to S.
(3) $\Lambda = S^2$, i.e M is *completely degenerate*.

Definition 12. An end E is *simply degenerate* if it is contained in Core(M).

Fact: Every free homotopy class in S can be represented by a unique geodesic in M.

Theorem 5. *(Bonahon): For every simply degenerate end E there is a sequence (c_i) of simple closed curves whose geodesic representatives in M exit the end.*

Idea: Try to control the curves c_i.

Definition 13. A *pleated surface* in M is a map $f : (S, x) \to M$ for some hyperbolic surface structure $x \in \mathcal{T}(S)$ which is a homotopy equivalence and s.th.

(1) f is a path-isometry.
(2) There is a geodesic lamination λ on S such that $f(\lambda)$ is geodesic in M and $f|S - \lambda$ is totally geodesic.

A lamination λ which arises as above from a pleated surface S is called *realized.*

Proposition 9. (1) *Every simple multicurve on c is realized.*
(2) *If the injectivity radius of M is bounded from below then the diameter in M of a pleated surface is uniformly bounded.*

Important question: For which $x \in \mathcal{T}(S)$ there is a pleated surface $f : (S, x) \to M$?

Proposition 10. *(Minsky): $\forall a > 0$ there exists a number $b = b(a, S, \epsilon) > 0$ with the following property. If M is a hyperbolic 3-manifold which is homeomorphic to $S \times \mathbb{R}$ and with $\text{inj}(M) \geq \epsilon$, if $g : (S, \sigma) \to M, h : (S, \rho) \to M$ are pleated surfaces with $d_M(g(S), h(S)) \leq a$ then $d_{\mathcal{T}(S)}(\sigma, \rho) \leq b$.*

Proof. Choose a filling short curve systems X_g for g. Since $\text{inj}(M) \geq \epsilon$, the distance in M between α and the geodesic representative of α in M is bounded from above by a universal constant. Thus it is enough to show: If $g : (S, \sigma) \to M$ is any pleated surface and if $d(g(S), \alpha) \leq a$ for some closed geodesic α in M of length $\leq \kappa$ then length(α on $g(S)$) $\leq b$.

To show that this is indeed the case assume otherwise.

Then there is a sequence (M_i) of 3-manifolds homeomorphic to $S \times \mathbb{R}$ with $\text{inj}(M_i) \geq \epsilon$, a sequence of geodesics α_i on M_i of length $\leq \kappa$ and a sequence of pleated surfaces $g : (S_i, \sigma_i) \to M_i$ with $d_{M_i}(g_i(S), \alpha_i) \leq a, \ell_{\sigma_i}(\alpha_i) \to \infty$.

Thurston: After passing to a subsequence we may assume that the manifolds M_i converge *geometrically* to a hyperbolic 3-manifold M containing a closed geodesic α of uniformly bounded length. After passing to another subsequence we may assume that the sequence of pleated surfaces $g_i(S_i, \sigma_i) \to M_i$ as above has a convergent subsequence to a pleated surface $g : (S, \sigma) \to M$.

We then get lengthbound for α_i on (S, σ_i) from the length of α near the limiting pleated surface g.

Now let $(c_i) \to M$ be a sequence of simple closed curves on S whose geodesic representatives in M exit the end E.

Choose i large and choose a geodesic $\alpha_0 = c_0, \alpha_1, \ldots, \alpha_k = c_i$ a in $\mathcal{C}(S)$ connecting c_0 to c_i.

For each j, the curves α_j, α_{j+1} are contained in a pleated surface (S, σ_j) with image set $S_j \subset M$.

Since $S_j \cap S_{j+1} \neq \emptyset$ the distances $d_{\mathcal{T}(S)}(\sigma_j, \sigma_{j+1})$ in Teichmüller space are bounded from above by b. Thus the assignment $j \to \sigma_j$ is a *quasi-convex curve in* $\mathcal{T}(S)$ and hence there is a Teichmüller geodesic arc γ_i in $\mathcal{T}(S)_\epsilon$ containing (σ_j) in a uniformly bounded neighborhood. □

As a consequence, $d_M(c_0, c_i) \leq \text{const} d_{\mathcal{C}(S)}(c_0, c_i)$
$\Rightarrow d_{\mathcal{C}(S)}(c_0, c_i) \to \infty \ (i \to \infty)$.

By passing to a limit we can assume that the geodesics γ_i converge to Teichmüller geodesic ray $\gamma : [0, \infty)$ "following pleated surfaces".

Masur: The horizontal measured geodesic lamination of γ is uniquely ergodic; its support is called the *ending lamination* of E.

By construction: Every point in E is contained in uniformly bounded neighborhood of the pleated surfaces defining γ.

Conclusion: The ending lamination is unique.

Definition 14. The *end-invariants of M* consist of:
1) If M is convex cocompact: The Riemann surface Ω/Γ.
2) If M has one degenerated end E: The Riemann surface Ω/Γ and the ending lamination of E.
3) If M is totally degenerate: Both ending laminations.

Theorem 6. *(Minsky 1994) Let M_1, M_2 be two hyperbolic 3-manifolds which are diffeomorphic to $S \times \mathbb{R}$, with $\mathrm{inj}(M_i) \geq \epsilon$. If the end-invariants of M_1, M_2 coincide then M_1 and M_2 are isometric.*

References

[B02] B. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, J. Reine Angew. Math. 598 (2006), 105–129.

[BH99] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer Grundlehren 319, Springer, Berlin 1999.

[Bu92] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser, Boston 1992.

[FLP91] A. Fathi, F. Laudenbach, V. Poénaru, *Travaux de Thurston sur les surfaces,* Astérisque 1991.

[H06] U. Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves*, in "Spaces of Kleinian groups", (Y. Minski, M. Sakuma and C. Series, eds.) London Math. Soc. Lecture Notes 329, Cambride University Presse, Cambridge 2006, 187–207.

[H07] U. Hamenstädt, *Geometry of the complex of curves and of Teichmüller space*, in "Handbook of Teichmüller theory, (A. Papadopolous, ed.), European Math. Soc. 2007, 447–467.

[K99] E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space,* preprint 1999.

[Ma82] H. Masur, *Interval exchange transformations and measured foliations*, Ann. Math. 115 (1982), 169-200.

[MM99] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103–149.

[M93] Y. Minsky, *Teichmüller geodesics and ends of hyperbolic 3-manifolds*, Topology 32 (1993), 625–647.

[M94] Y. Minsky, *On rigidity, limit sets, and end invariants of hyperbolic 3-manifolds*, J. Amer. Math. Soc. .7 (1994), 539–588.