

# DISTANCE IN THE CURVE GRAPH

URSULA HAMENSTÄDT

ABSTRACT. We estimate the distance in the curve graph of a surface  $S$  of finite type up to a fixed multiplicative constant using Teichmüller geodesics.

## 1. INTRODUCTION

The *curve graph* of an oriented surface  $S$  of genus  $g \geq 0$  with  $m \geq 0$  punctures and  $3g - 3 + m \geq 2$  is the metric graph  $(\mathcal{CG}, d_{\mathcal{CG}})$  whose vertices are isotopy classes of essential (i.e. non-contractible and not puncture parallel) simple closed curves on  $S$ . Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a hyperbolic geodesic metric space of infinite diameter [MM99]. It turned out to be an important tool for understanding the geometry of the *mapping class group* of  $S$  [MM00].

The *Teichmüller space*  $\mathcal{T}(S)$  of  $S$  is the space of all complete finite volume marked hyperbolic metrics on  $S$ . Let  $P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$  be the bundle of marked area one *quadratic differentials*. The bundle  $\mathcal{Q}^1(S)$  can naturally be identified with the unit cotangent bundle of  $\mathcal{T}(S)$  for the *Teichmüller metric*. In particular, every  $q \in \mathcal{Q}^1(S)$  determines the unit cotangent line of a *Teichmüller geodesic*.

An area one quadratic differential  $q$  defines a *singular euclidean metric* on  $S$  of area one. Call a simple closed curve  $\alpha$   $\delta$ -wide for  $q$  if  $\alpha$  is the core curve of an embedded annulus of width at least  $\delta$  with respect to the metric  $q$ . Here the width of an annulus is the minimal distance between the two boundary circles. For small enough  $\delta$ , a  $\delta$ -wide curve exists for all  $q \in \mathcal{Q}^1(S)$  (Lemma 5.1 of [MM99]).

A map  $\Upsilon : \mathcal{Q}^1(S) \rightarrow \mathcal{CG}$  which associates to a marked area one quadratic differential  $q$  a  $\delta$ -wide curve for  $q$  relates the Teichmüller metric on  $\mathcal{T}(S)$  to the metric on  $\mathcal{CG}$ . Namely, there is a number  $L > 1$  such that for every unit cotangent line  $(q_t) \subset \mathcal{Q}^1(S)$  of a Teichmüller geodesic, the assignment  $t \rightarrow \Upsilon(q_t)$  is a coarsely Lipschitz *unparametrized  $L$ -quasi-geodesic* (this is an immediate consequence of Theorem 2.6 of [MM99]). This implies that there is a number  $\kappa > 0$  not depending on the geodesic such that

$$(1) \quad \begin{aligned} d_{\mathcal{CG}}(\Upsilon(q_s), \Upsilon(q_t)) &\leq \kappa|t - s| + \kappa \text{ and} \\ d_{\mathcal{CG}}(\Upsilon(q_s), \Upsilon(q_t)) + d_{\mathcal{CG}}(\Upsilon(q_t), \Upsilon(q_u)) &\leq d_{\mathcal{CG}}(\Upsilon(q_s), \Upsilon(q_u)) + \kappa \end{aligned}$$

for  $s \leq t \leq u$  (see Lemma 2.5 of [H10] for a detailed discussion).

In general these unparametrized quasi-geodesics are not quasi-geodesics with their proper parametrizations. More precisely, for  $\epsilon > 0$  let  $\mathcal{T}(S)_\epsilon \subset \mathcal{T}(S)$  be the set of all complete finite volume marked hyperbolic metrics on  $S$  whose *systole* (i.e. the shortest length of a simple closed geodesic) is at least  $\epsilon$ . Then for a given number

---

*Date:* May 17, 2012.

Partially supported by the Hausdorff Center Bonn  
AMS subject classification: 30F60.

$L' > L$ , there exists a number  $\epsilon = \epsilon(L') > 0$  such that the unparametrized quasi-geodesic in  $\mathcal{CG}$  defined by the image under  $\Upsilon$  of the cotangent line of a Teichmüller geodesic  $\gamma$  is a *parametrized*  $L'$ -quasi-geodesic only if  $\gamma$  entirely remains in  $\mathcal{T}(S)_\epsilon$  [MM99, H10]. In particular,  $d_{\mathcal{CG}}(\Upsilon(q_s), \Upsilon(q_t))$  may be uniformly bounded even though  $|s - t|$  is arbitrarily large.

Nevertheless, the estimate (1) allows to construct for any two points  $\xi, \zeta \in \mathcal{CG}$  a parametrized uniform quasi-geodesic connecting  $\xi$  to  $\zeta$  as follows.

Note first that it is very easy to decide whether the distance in the curve graph between  $\xi$  and  $\zeta$  is at most two. Namely, in this case there is an essential simple closed curve  $\alpha$  which can be realized disjointly from  $\xi, \zeta$ .

If the distance between  $\xi, \zeta$  is at least three then  $\xi, \zeta$  define the unit cotangent line  $(q_t)$  of a Teichmüller geodesic whose *horizontal* and *vertical* measured foliations are one-cylinder foliations with core curves homotopic to  $\xi, \zeta$ , respectively. For any sequence  $t_0 < \dots < t_n$  so that  $d_{\mathcal{CG}}(\Upsilon(q_{t_i}), \Upsilon(q_{t_{i+1}})) \geq 2\kappa$  for all  $i$  and  $d_{\mathcal{CG}}(\Upsilon(q_s), \Upsilon(q_t)) < 2\kappa$  for all  $s, t \in (t_i, t_{i+1})$ , a successive application of the second part of the estimate (1) implies that

$$d_{\mathcal{CG}}(\Upsilon(q_{t_0}), \Upsilon(q_{t_n})) \geq n\kappa.$$

Combined with the first part of (1), the map  $s \in [i, i+1) \rightarrow \Upsilon(q_t)$  is a uniform quasi-geodesic in  $\mathcal{CG}$ .

Unfortunately, up to date no bounds for the number  $\kappa > 0$  are known. Moreover, the curve graph is locally infinite, and although there are algorithms which compute the distance between two curves in the curve graph explicitly (see [Sh04] for an example of such an algorithm), these algorithms depend on the calculation of so-called *tight geodesics*. Unlike the problem of deciding whether or not the distance in the curve graph is at least three, determining whether or not the distance in  $\mathcal{CG}$  between two given curves is bigger than a given number  $\beta \geq 4$  does not seem to be easier than calculating their distance explicitly.

The goal of this note is to show that we can use Teichmüller geodesics to calculate the distance between two points in  $\mathcal{CG}$  up to a fixed multiplicative constant by only detecting pairs of curves of distance at least three. We show

**Theorem.** *There is a number  $\theta > 0$  with the following property. Let  $(q_t) \subset \mathcal{Q}^1(S)$  be the unit cotangent line of a Teichmüller geodesic. Assume that there are numbers  $0 = t_0 < \dots < t_n = T$  such that for all  $i < n$  we have*

- (1)  $t_{i+1} \geq t_i + 1$
- (2)  $d_{\mathcal{CG}}(\Upsilon(q_s), \Upsilon(q_t)) \geq 3$  for  $s \leq t_i, t \geq t_{i+1}$ .

*Then  $d_{\mathcal{CG}}(\Upsilon(q_0), \Upsilon(q_T)) \geq n/\theta - \theta$ .*

Let again  $(q_t)$  be the unit cotangent line of a Teichmüller geodesic. For  $T > 0$ , construct recursively a sequence  $0 = t_0 < \dots < t_u = T$  as follows. If  $t_i$  has already been determined, let  $t_{i+1}$  be the smallest number contained in the interval  $[t_i + 1, T]$  so that  $d_{\mathcal{CG}}(\Upsilon(q_s), \Upsilon(q_t)) \geq 3$  for all  $s \leq t_i, t \geq t_{i+1}$ . If no such number exists put  $u = i$ . By the first part of the estimate (1), for each  $s \leq t_\ell$  there is at least one  $t \leq t_{\ell-1}$  so that  $d_{\mathcal{CG}}(\Upsilon(q_s), \Upsilon(q_t)) \leq 3 + 2\kappa$ . A successive application of the triangle inequality shows that  $d_{\mathcal{CG}}(\Upsilon(q_T), \Upsilon(q_0)) \leq u(3 + 2\kappa) + 3$ . On the other hand, we have  $d_{\mathcal{CG}}(\Upsilon(q_T), \Upsilon(q_0)) \geq u\theta - 1/\theta$  by the Theorem.

The proof of this result relies on the work of Minsky [M92] and Rafi [R10]. The basic idea is that  $\delta$ -wide curves for a quadratic differential  $q$  detect "active" subsurfaces in the sense of [R10] for the Teichmüller geodesic defined by  $q$ . However,

such active subsurfaces may intersect in a complicated way, and making this simple idea rigorous and quantitative is the main contribution of this work. Unfortunately the proof of the theorem does not give an explicit bound for the number  $\theta$ .

## 2. TEICHMÜLLER GEODESICS AND THE CURVE GRAPH

Let  $S$  be an oriented surface of genus  $g \geq 0$  with  $m \geq 0$  punctures and of complexity  $\xi(S) = 3g - 3 + m \geq 1$ . In the sequel we always mean by a simple closed curve on  $S$  an essential simple closed curve. Moreover, we identify curves which are isotopic unless specifically stated otherwise.

We denote by  $\mathcal{CG}$  the curve graph of  $S$  and by  $d_{\mathcal{CG}}$  the distance in  $\mathcal{CG}$ . If  $\xi(S) = 1$  then  $S$  is a once-punctured torus or a four-punctured sphere and  $\mathcal{CG}$  is the Farey graph (i.e. two simple closed curves are connected by an edge if they intersect in the smallest possible number of points).

Let  $P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$  be the bundle of marked area one quadratic differentials over the Teichmüller space  $\mathcal{T}(S)$  of  $S$ . Such a quadratic differential on a surface  $x \in \mathcal{T}(S)$  is a meromorphic section of  $T'(x) \otimes T'(x)$  with at most simple poles at the punctures and no other poles. It defines a singular euclidean metric on  $S$  of finite area, and we require that the area of this metric equals one (see [S84] for details). Any two simple closed curves  $\xi, \zeta$  of distance at least 3 in  $\mathcal{CG}$  define a one-parameter family  $(q_t)$  of area one quadratic differentials which form the unit cotangent line of a Teichmüller geodesic. These differentials are determined by the requirement that the *vertical* and the *horizontal* measured foliation of  $q_t$  is a one-cylinder foliation with cylinder curves homotopic to  $\xi, \zeta$ , respectively (see Proposition 3.10 of [H12] for one of the many places in the literature where such quadratic differentials are constructed explicitly).

For  $x \in \mathcal{T}(S)$  and an essential simple closed curve  $\alpha$  on  $S$  let  $\text{Ext}_x(\alpha)$  be the *extremal length* of  $\alpha$  with respect to the conformal structure defined by  $x$ . If  $q$  is a quadratic differential on  $x$  then we also write  $\text{Ext}_q(\alpha)$  instead of  $\text{Ext}_x(\alpha)$ . We refer to [G87] for more and references.

In the sequel we always mean by a quadratic differential a marked area one quadratic differential. The  $q$ -length  $\ell_q(\alpha)$  of an essential simple closed curve  $\alpha$  on  $S$  is the infimum of the lengths for the singular euclidean metric defined by  $q$  of all curves freely homotopic to  $\alpha$ . With this terminology, we have

$$\text{Ext}_q(\alpha) \geq \ell_q(\alpha)^2$$

for every simple closed curve  $\alpha$ .

There is a number  $\delta > 0$  such that each quadratic differential  $q$  admits a  $\delta$ -wide curve [MM99] (see also [B06] for a more explicit and detailed discussion), i.e. a simple closed curve  $\alpha$  which is the core curve of an embedded annulus of width at least  $\delta$ . Then the  $q$ -length of  $\alpha$  is at most  $1/\delta$ . Define a map

$$\Upsilon : \mathcal{Q}^1(S) \rightarrow \mathcal{CG}$$

by associating to an area one quadratic differential  $q$  a  $\delta$ -wide simple closed curve for  $q$ .

There is a number  $\kappa > 0$  such that for each cotangent line  $(q_t)$  of a Teichmüller geodesic, the estimate (1) from the introduction holds true. In particular, if  $t > s, k \geq 0$  are such that  $d_{\mathcal{CG}}(\Upsilon(q_s), \Upsilon(q_t)) \geq k + 2\kappa$  then

$$d(\Upsilon(q_u), \Upsilon(q_v)) \geq k$$

for all  $u \leq s < t \leq v$ .

For a quadratic differential  $q \in \mathcal{Q}^1(S)$  let

$$\nu(q) = \inf\{\text{Ext}_q(\alpha) \mid \alpha\}$$

where  $\alpha$  ranges through the essential simple closed curves. Note that the function  $\nu : \mathcal{Q}^1(S) \rightarrow (0, \infty)$  is continuous. Moreover, for every  $\epsilon > 0$ , the set  $\{\nu \geq \epsilon\}$  is closed and invariant under the action of the *mapping class group*  $\text{Mod}(S)$  of  $S$ , and the quotient  $\{\nu \geq \epsilon\}/\text{Mod}(S)$  is compact.

Denote by  $\lambda$  the Lebesgue measure on the real line. The following observation can be thought of as a quantitative account on progress in the curve graph along Teichmüller geodesic segments which spend a definitive proportion of time in the thick part of Teichmüller space.

**Proposition 2.1.** *For every  $\epsilon > 0, k > 0$  and every  $b \in (0, 1)$  there is a number  $R_0 = R_0(\epsilon, k, b) > 0$  with the following property. Let  $(q_t)$  be the unit cotangent line of a Teichmüller geodesic. If  $\lambda\{t \in [0, R] \mid \nu(q_t) \geq \epsilon\} \geq bR$  for some  $R \geq R_0$  then  $d_{CG}(\Upsilon(q_0), \Upsilon(q_R)) \geq k$ .*

*Proof.* Let  $\epsilon > 0, k > 0, b \in (0, 1)$ . We show first that there is a number

$$\tau = \tau(\epsilon, k, b) > 1$$

with the following property. For every Teichmüller geodesic  $(q_t)$  there is a number  $\sigma \in [1, \tau]$  so that either  $d_{CG}(\Upsilon(q_0), \Upsilon(q_\sigma)) \geq k$  or

$$\lambda\{s \in [0, \sigma] \mid \nu(q_s) < \epsilon\} \geq \sigma(1 - b).$$

For this we argue by contradiction and we assume that the claim does not hold. Then there is a sequence  $(q_t^i)$  of unit cotangent lines of Teichmüller geodesics so that for each  $i > 0$ , the smallest subinterval of  $[0, \infty)$  containing  $[0, 1]$  with the property that the above alternative holds true for  $(q_t^i)$  is of length at least  $i$ .

Since for each  $i \geq 2$  the above alternative does not hold for  $(q_t^i)$  and  $\sigma = 1$ , there is at least one  $s \in [0, 1]$  so that  $\nu(q_s^i) \geq \epsilon$ . This implies that the quadratic differentials  $q_0^i$  project to a compact subset of the moduli space of quadratic differentials. Thus by invariance under the mapping class group, after passing to a subsequence we may assume that the quadratic differentials  $q_0^i$  converge as  $i \rightarrow \infty$  in  $\mathcal{Q}^1(S)$  to a quadratic differential  $q$ .

Let  $\chi > 0$  be sufficiently large that for every quadratic differential  $z$ , the distance in the curve graph between any two curves on  $S$  of  $z$ -length at most  $2/\delta$  is smaller than  $\chi$  (see Lemma 2.1 of [H10] for a detailed proof of the existence of such a number). Let  $(q_t)$  be the unit cotangent line of the Teichmüller geodesic with initial velocity  $q_0 = q$ .

If  $(q_t)$  is *recurrent* (i.e. its projection to the moduli space of quadratic differentials returns to a fixed compact set for arbitrarily large times) then there is some  $T > 0$  so that

$$d_{CG}(\Upsilon(q_0), \Upsilon(q_T)) \geq k + 2\chi$$

(see Proposition 2.4 of [H10] for details). However, by continuity, for all sufficiently large  $i$  the  $q$ -length (or  $q_T$ -length) of a curve of  $q_0^i$ -length at most  $1/\delta$  (or of a curve of  $q_T^i$ -length at most  $1/\delta$ ) does not exceed  $2/\delta$ . Thus by the choice of  $\chi$ ,

$$d_{CG}(\Upsilon(q_0^i), \Upsilon(q_T^i)) \geq k$$

for all sufficiently large  $i$  which violates the assumption on the sequence  $(q_t^i)$ .

On the other hand, if  $(q_t)$  is not recurrent then there is some  $t_0 > 0$  so that  $\nu(q_t) \leq \epsilon/2$  for all  $t \geq t_0$ . Then the proportion of time the arc  $q_t$  ( $t \in [0, t_0/b]$ ) spends in the region  $\{\nu \leq \epsilon/2\}$  is at least  $1 - b$ . By continuity, for all sufficiently large  $i$  the proportion of time the arc  $q_t^i$  ( $t \in [0, t_0/b]$ ) spends in the region  $\{\nu < \epsilon\}$  is at least  $1 - b$  as well. Once again, this is a contradiction.

Let  $\kappa > 0$  be as in (1). Let  $b \in (0, 1), k > 0, \epsilon > 0$ , let  $\tau = \tau(\epsilon, b/2, k + 2\kappa)$  and let  $R > 2\tau/b$ . Let  $(q_t)$  be the unit cotangent line of any Teichmüller geodesic. There are successive numbers  $0 = t_0 < t_1 < \dots < t_u \leq R$  so that  $R - t_u \leq \tau$  and that for all  $i < u$ ,  $1 \leq t_{i+1} - t_i \leq \tau$  and either

$$(2) \quad d_{\mathcal{CG}}(\Upsilon(q_{t_i}), \Upsilon(q_{t_{i+1}})) \geq k + 2\kappa$$

or  $\lambda\{t \in [t_i, t_{i+1}] \mid \nu(q_t) < \epsilon\} \geq (t_{i+1} - t_i)(1 - b/2)$ .

By the choice of  $\kappa$ , if there is some  $i < u$  such that the inequality (2) holds true then  $d_{\mathcal{CG}}(\Upsilon(q_0), \Upsilon(q_R)) \geq k$ . Otherwise since  $R - t_u \leq \tau < bR/2$ , we have  $\lambda\{t \in [0, R] \mid \nu(t) < \epsilon\} > R(1 - b)$ . The proposition follows.  $\square$

For small  $\epsilon > 0$ , the (hyperbolic) length of a simple closed curve on a hyperbolic surface  $x \in \mathcal{T}(S)$  is roughly proportional to its extremal length. Thus by the collar lemma, there is a number  $\epsilon_0 < \delta^2/2$  such that for  $x \in \mathcal{T}(S)$ , any two simple closed curves on  $(S, x)$  of extremal length at most  $\epsilon_0$  can be realized disjointly.

For  $\epsilon \leq \epsilon_0$  let

$$\mathcal{A}_\epsilon = \mathcal{A}_\epsilon(x)$$

be the set of all simple closed curves on  $(S, x)$  of extremal length less than  $\epsilon$ . Then  $S - \mathcal{A}_\epsilon$  is a union  $\mathcal{Y}$  of connected components (this is meant to be a topological decomposition, i.e. we cut  $S$  open along disjoint representatives of the curves of small extremal length). A component  $Y \in \mathcal{Y}$  is called an  $\epsilon$ -thick component for  $x$ , and  $(\mathcal{A}_\epsilon, \mathcal{Y})$  is the  $\epsilon$ -thin-thick decomposition of  $(S, x)$ . Note that for  $\chi < \epsilon$ , a  $\chi$ -thick component is a union of  $\epsilon$ -thick components.

If  $q$  is a quadratic differential with underlying conformal structure  $x$  then we also call  $(\mathcal{A}_\epsilon, \mathcal{Y})$  the  $\epsilon$ -thin-thick decomposition for  $q$ , and we call a component  $Y$  of  $\mathcal{Y}$  an  $\epsilon$ -thick component for  $q$ . More generally, we call a non-peripheral incompressible open connected subsurface  $Y$  of  $S$   $\epsilon$ -semi-thick for  $q$  if the extremal length of each boundary circle of  $Y$  is at most  $\epsilon$ . Then the result of cutting  $Y$  along all the curves of extremal length (as curves in  $S$ ) at most  $\epsilon$  is a union of  $\epsilon$ -thick components.

Let again  $q$  be a quadratic differential on  $S$ . For  $\epsilon \leq \epsilon_0$  and an  $\epsilon$ -semi-thick subsurface  $Y \subset S$  for  $q$  let  $\mathbf{Y}$  be the representative of  $Y$  with  $q$ -geodesic boundary which is disjoint from the interiors of the (possibly degenerate) flat cylinders foliated by simple closed  $q$ -geodesics homotopic to the boundary components of  $Y$ . We call  $\mathbf{Y}$  the *geometric representative* of  $Y$ . Following Rafi [R07], if  $Y$  is not a pair of pants then we define  $\text{size}_q(Y)$  to be the shortest  $q$ -length of an essential simple closed curve in  $\mathbf{Y}$ . If  $Y$  is a pair of pants then we define  $\text{size}_q(Y)$  to be the diameter of  $Y$ .

An *expanding annulus* for a simple closed curve  $\alpha \subset (S, q)$  is an embedded annulus  $A \subset S$  homotopic to  $\alpha$  with the following property. One boundary component of  $A$  is a  $q$ -geodesic  $\psi$ , and the second boundary component is a curve which is equidistant to  $\psi$ . Moreover,  $A$  does not intersect the interior of a flat cylinder foliated by closed  $q$ -geodesics homotopic to  $\alpha$  (see [M92] for details).

By Theorem 3.1 of [R10], up to making  $\epsilon_0$  smaller we may assume that for every quadratic differential  $q$  and every  $\alpha \in \mathcal{A}_{\epsilon_0} = \mathcal{A}_{\epsilon_0}(q)$  (this means that we calculate extremal length for the conformal structure underlying  $q$ ) the following holds true.

Assume that  $Y, Z$  are the  $\epsilon_0$ -thick components of  $(S, q)$  which contain  $\alpha$  in their boundary. Note that  $Y, Z$  are not necessarily distinct. Let  $E, G$  be the *maximal expanding annuli* in the geometric representatives  $\mathbf{Y}, \mathbf{Z}$  of  $Y, Z$  homotopic to  $\alpha$ . This means that  $E, G$  are expanding annuli contained in  $\mathbf{Y}, \mathbf{Z}$  which are as big as possible, i.e. which are not proper subsets of another expanding annulus. If  $Y = Z$  then these annuli are required to lie on the two different sides of  $\alpha$  with respect to an orientation of  $\alpha$  and the orientation of  $S$ . Then

$$(3) \quad \frac{1}{\text{Ext}_q(\alpha)} \asymp \log \frac{\text{size}_q(Y)}{\ell_q(\alpha)} + \frac{\text{size}_q(F_q(\alpha))}{\ell_q(\alpha)} + \log \frac{\text{size}_q(Z)}{\ell_q(\alpha)} \\ \asymp \text{Mod}_q(E) + \text{Mod}_q(F_q(\alpha)) + \text{Mod}_q(G)$$

where  $F_q(\alpha)$  is the (possibly degenerate) flat cylinder foliated by simple closed geodesics homotopic to  $\alpha$  and where  $\text{Mod}_q$  is the modulus with respect to the conformal structure defined by  $q$ . Also,  $\asymp$  means equality up to a uniform multiplicative constant.

Let  $M_0 > 0$  be sufficiently large so that Theorem 5.3 and Corollary 5.4 of [R10] hold true for this  $M_0$ . For  $M \geq M_0$  and for  $\epsilon \leq \epsilon_0$ , a boundary curve  $\alpha$  of an  $\epsilon$ -semi-thick subsurface  $Y$  of  $S$  is called  *$M$ -large* if  $\text{Mod}(E) \geq M$  where as before,  $E$  is a maximal expanding annulus homotopic to  $\alpha$  in the geometric representative  $\mathbf{Y}$  of  $Y$ . An  $\epsilon$ -semi-thick subsurface  $Y$  is  *$M$ -large* if  $Y$  is not a pair of pants and if each of its boundary circles is  $M$ -large. We also require that  $Y$  is non-trivial, i.e. that  $Y \neq S$ .

Since  $\epsilon_0 < \delta^2/2$  by assumption and since the  $q$ -length of a simple closed curve  $\alpha$  is at most  $\sqrt{\text{Ext}_q(\alpha)}$ , a  $\delta$ -wide simple closed curve  $\alpha$  can not have an essential intersection with a boundary curve of an  $\epsilon_0$ -thick component of  $S$ . As a consequence, a geodesic representative of  $\alpha$  either is contained in the geometric representative  $\mathbf{Y}$  of an  $\epsilon_0$ -thick component  $Y$  of  $S$ , or it is homotopic to a boundary circle of such a component.

In the case that  $\alpha$  is contained in the geometric representative  $\mathbf{Y}$  of an  $\epsilon_0$ -thick component  $Y$  of  $S$  and is not homotopic to a boundary component of  $Y$ , the embedded annulus of width  $\delta$  which is homotopic to  $\alpha$  can intersect a boundary component of  $Y$  only in a set of small diameter. As a consequence, up to adjusting the number  $\epsilon_0$ , the  $q$ -diameter of  $Y$  is bounded from below by a universal constant.

If  $\text{Ext}_q(\alpha) \leq \epsilon_0$  then there are two (not mutually exclusive) possibilities. In the first case,  $\alpha$  is the core curve of a flat cylinder of small circumference whose width is uniformly bounded from below. Such a cylinder has a large modulus. The second possibility is that there is an expanding annulus homotopic to  $\alpha$  whose width is uniformly bounded from below. Then this annulus is contained in the geometric representative  $\mathbf{Y}$  of an  $\epsilon_0$ -thick component  $Y$  of  $S$ . The  $q$ -diameter of  $\mathbf{Y}$  is bounded from below by a universal constant.

This observation is used to show

**Lemma 2.2.** *For every  $M > 0, \chi > 0$  there is a number  $\epsilon_1(\chi, M) < \epsilon_0$  with the following property. Let  $q \in \mathcal{Q}^1(S)$  and assume that there is a simple closed curve  $\beta$  with  $\text{Ext}_q(\beta) < \epsilon_1(\chi, M)$ . Let  $\alpha$  be a  $\delta$ -wide curve for  $q$ ; then up to isotopy, either*

$\alpha$  is contained in an  $M$ -large subsurface of  $S - \beta$  or  $\alpha$  is contained in a flat cylinder of modulus at least  $\chi$ .

*Proof.* For  $\chi > 0$  let  $\nu \leq \epsilon_0$  be sufficiently small that the modulus of a flat cylinder of circumference  $\sqrt{\nu}$  and width  $\delta/3$  is at least  $\chi$ .

Let  $\alpha$  be a  $\delta$ -wide curve for the quadratic differential  $q$ . By the choice of  $\epsilon_0$ , there is a  $q$ -geodesic representative of  $\alpha$  which is contained in a  $\nu$ -thick component  $Y$  for  $q$ .

If  $\alpha$  is homotopic to a boundary circle of  $Y$  and if moreover  $\alpha$  is the core curve of a flat cylinder of width at least  $\delta/3$  then  $\alpha$  is the core curve of a flat cylinder of modulus at least  $\chi$  and we are done.

For the remainder of this proof suppose that no such cylinder exists. The discussion preceding this lemma shows that in this case we may assume without loss of generality that the  $q$ -diameter of  $Y$  is bounded from below by a universal constant  $\sigma_0 = \sigma_0(\chi) > 0$ .

By the main result of [R07], since there are no essential simple closed curves in  $Y$  of extremal length smaller than  $\nu$ ,  $\text{size}_q(Y)$  is comparable to the  $q$ -diameter of  $Y$  (with a comparison factor depending on  $\nu$ ). Now the  $q$ -diameter of  $Y$  is at least  $\sigma_0$  and therefore  $\text{size}_q(Y)$  is bounded from below by a number  $\sigma_1 \leq \sigma_0$  only depending on  $\nu$  and hence only depending on  $\chi$ .

Let  $M > 0$  be arbitrary. If each boundary circle of  $Y$  is  $M$ -large then we are done. Otherwise let  $\beta$  be a boundary component of  $Y$  which is not  $M$ -large. Since  $\text{size}_q(Y) \geq \sigma_1$ , the estimate (3) shows that  $\ell_q(\beta) \geq \sigma_2$  for a number  $\sigma_2 > 0$  only depending on  $\sigma_1$ . In particular, we have  $\text{Ext}_q(\beta) \geq \sigma_2^2$ .

If  $\beta$  is non-separating and defines two distinct free homotopy classes in  $Y$  then  $Y \cup \beta = Y_1$  is a subsurface of  $S$  containing  $\alpha$ . Since the length of  $\beta$  is bounded from below by a universal constant, the size of  $Y \cup Y_1$  is bounded from below by a universal constant  $\sigma_3 > 0$  as well.

Otherwise there is a  $\chi$ -thick component  $Y'$  of  $S$  which contains  $\beta$  in its boundary and which is distinct from  $Y$ . Using again the fact that the length of  $\beta$  is uniformly bounded from below, the diameter of  $Y'$  and hence the size of  $Y'$  is uniformly bounded from below. Let  $Y_1 = Y \cup Y'$  and note that  $Y_1$  contains  $\alpha$ , and its size is bounded from below by a universal constant  $\sigma_4 > 0$ . In particular, any very short boundary curve of  $Y_1$  is contained in an expanding cylinder of modulus at least  $M$ .

Repeat this reasoning with  $Y_1$ . In at most  $3g - 4 + m$  steps we either find an  $M$ -large subsurface  $Y$  of  $S$  containing  $\alpha$  whose size is bounded from below by a universal constant, or we conclude that the shortest  $q$ -length of an essential simple closed curve on  $S$  is bounded from below by a universal constant. Together this shows the lemma.  $\square$

Let  $X \subset S$  be a non-peripheral, incompressible, open connected subsurface which is distinct from  $S$ , a three-holed sphere and an annulus. The *arc and curve complex*  $\mathcal{C}'(X)$  of  $X$  is defined to be the complex whose vertices are isotopy classes of arcs with endpoints on  $\partial X$  or essential simple closed curves in  $X$ . Two such vertices are connected by an edge of length one if they can be realized disjointly. There is a *subsurface projection*  $\pi_X$  of  $\mathcal{CG}$  into the space of subsets of  $\mathcal{C}'(X)$  which associates to a simple closed curve the homotopy classes of its intersection components with  $X$  (see [MM00]). For every simple closed curve  $c$ , the diameter of  $\pi_X(c)$  in  $\mathcal{C}'(X)$  is at most one. Moreover, if  $c$  can be realized disjointly from  $X$  then  $\pi_X(c) = \emptyset$ .

There also is an arc complex  $\mathcal{C}'(A)$  for an essential annulus  $A \subset S$ , and there is a subsurface projection  $\pi_A$  of  $\mathcal{CG}$  into the space of subsets of  $\mathcal{C}'(A)$ . We refer to [MM00] for details of this construction. In the sequel we call a subsurface  $X$  of  $S$  *proper* if  $X$  is non-peripheral, incompressible, open and connected and different from a three-holed sphere or  $S$ .

As before, let  $\delta > 0$  be such that for every  $q \in \mathcal{Q}^1(S)$  there is a  $\delta$ -wide curve for  $q$ . The next lemma is a version of Proposition 2.1 for Teichmüller geodesic arcs which are allowed to be entirely contained in the thin part of Teichmüller space.

**Lemma 2.3.** *For every  $\chi > 0, k > 0$  there is a number  $T = T(\chi, k) > 0$  with the following property. Let  $q_t$  be the cotangent line of a Teichmüller geodesic defined by two simple closed curves  $\xi, \zeta$ . For  $t \in \mathbb{R}$  let  $\alpha(t) \in \mathcal{CG}$  be a  $\delta$ -wide curve for  $q_t$ . Then one of the following (not mutually exclusive) possibilities is satisfied.*

- (1)  $d_{\mathcal{CG}}(\Upsilon(q_0), \Upsilon(q_T)) \geq k$ .
- (2) There is a number  $t \in [0, T]$  and a proper subsurface  $Y$  of  $S$  containing  $\alpha(t)$  such that  $\text{diam}(\pi_Y(\xi \cup \zeta)) \geq k$ .
- (3) There is a number  $t \in [0, T]$  such that  $\alpha(t)$  is the core curve of a flat cylinder of modulus at least  $\chi$ .

*Proof.* Define a *shortest marking*  $\mu$  for a quadratic differential  $q \in \mathcal{Q}^1(S)$  to consist of a pants decomposition with pants curves of the shortest extremal length for the conformal structure underlying  $q$  constructed using the greedy algorithm. There is a system of spanning arcs, one for each pants curve  $\alpha$  of  $\mu$ . Such an arc is a geodesic arc in the annular cover of  $S$  with fundamental group generated by  $\alpha$  which intersects a  $q$ -geodesic representing  $\alpha$  perpendicularly. We refer to Section 5 of [R10] for details- the purpose of using such shortest markings here is for ease of reference to the results of [R10]. A shortest marking defines a subset of  $\mathcal{CG}$ .

In the remainder of this proof we denote the diameter of a subset  $X$  of  $\mathcal{CG}$  by  $\text{diam}_{\mathcal{CG}}(X)$ . Moreover, whenever we use constants for a given surface  $S$  of complexity  $n$ , we assume that they are also valid in the same context for surfaces of complexity smaller than  $n$ .

For  $q \in \mathcal{Q}^1(S)$  and  $\rho > 0$  call a curve  $\alpha$   $\rho$ -*slim* if  $\alpha$  is *not* the core curve of a flat cylinder for  $q$  of modulus at least  $\rho$ . Let  $\rho > 0, k > 0$ , let  $b \in (0, 1)$  and let  $\lambda$  be the Lebesgue measure on the real line. Let  $(q_t)$  be the cotangent line of a Teichmüller geodesic on  $S$ . For each  $t$  let  $\mu_t$  be a shortest marking for  $q_t$ . Let  $n = \xi(S) \geq 1$  be the complexity of  $S$ . Let  $\delta > 0$  be as before. We show by induction on  $n$  the following

**Claim:** There is a number  $T_n = T_n(\delta, \rho, k, b) > 0$  and a number  $f(b, n) \in (0, b]$  with the following property. Assume that there is a set  $A \subset [0, T_n]$  with  $\lambda(A) \geq bT_n$  such that for every  $t \in A$  there is a  $\delta$ -wide  $\rho$ -slim curve  $\alpha(t)$  for  $q_t$ . Then either

$$\text{diam}_{\mathcal{CG}}(\mu_0 \cup \mu_{T_n}) \geq k$$

or there is a set  $A' \subset A$  with  $\lambda(A') \geq f(b, n)T_n$  so that for all  $t \in A'$ , the curve  $\alpha(t)$  is contained in a proper subsurface  $Y$  of  $S$  with  $\text{diam}(\pi_Y(\mu_0 \cup \mu_{T_n})) \geq k$ .

The claim easily yields the lemma. Namely, by Theorem 5.3 of [R10], for every  $k > 0$  there is a number  $\ell = \ell(k) > k$  with the following property. Let  $(q_s)$  be the cotangent line of a Teichmüller geodesic defined by simple closed curves  $\xi, \zeta$ . Let  $s < t$ , let  $\mu_s, \mu_t$  be shortest markings for  $q_s, q_t$  and assume that there is a subsurface  $Y$  of  $S$  such that  $\text{diam}(\pi_Y(\mu_s \cup \mu_t)) \geq \ell$ ; then

$$\text{diam}(\pi_Y(\xi \cup \zeta)) \geq k.$$



Let  $T = T(\chi, k) = T_n(\delta, \chi, \ell, \frac{1}{2})$  and let  $q_t$  be the cotangent line of a Teichmüller geodesic defined by two simple closed curves  $\xi, \zeta$ . For  $t \in [0, T]$  let  $\alpha(t)$  be a  $\delta$ -wide curve for  $q_t$ . If there is some  $t \in [0, T]$  such that  $\alpha(t)$  is *not*  $\chi$ -slim then the third property in the statement of the lemma is satisfied and we are done. Otherwise each of the curves  $\alpha(t)$  ( $t \in [0, T]$ ) is  $\chi$ -slim and hence we can apply the above claim to obtain the statement of the lemma.

For the inductive proof of the claim, note that if  $\xi(S) = 1$  then  $S$  either is a once punctured torus or a four times punctured sphere. Consider first the case of a one-punctured torus. Quadratic differentials on such a torus are just squares of abelian differentials on the torus with the puncture filled in. As a consequence, a  $\delta$ -wide  $\rho$ -slim simple closed curve in  $S$  is the core curve of a flat cylinder (i.e. a cylinder foliated by simple closed geodesics) whose modulus is at most  $\rho$ . Such flat structures on tori project to a compact subset of the moduli space of tori.

As a consequence, there is a number  $\epsilon > 0$  (depending on  $\delta, \rho$ ) with the following property. If there is a  $\delta$ -wide  $\rho$ -slim curve for a quadratic differential  $q$  on  $S$  then  $\nu(q) > \epsilon$ . The above claim now follows from Proposition 2.1 with  $f(b, 1) = b$  (here only the first alternative in the claim occurs). The case of the four punctured sphere is completely analogous and will be omitted.

Now assume that the claim holds true for all surfaces of complexity at most  $n - 1 \geq 1$ . Let  $S$  be a surface of complexity  $n$ . Theorem 4.2 and Theorem 5.3 of [R10] and their proofs (see also [M92]) show the following.

Let  $(q_t)$  be the cotangent line of a Teichmüller geodesic. Assume that there is an interval  $[a, b] \subset \mathbb{R}$  and a proper  $\epsilon_0$ -semi-thick subsurface  $Y$  of  $S$  which is  $M_0$ -large for every  $t \in [a, b]$ . Here as before,  $M_0 > 0$  is a constant only depending on the complexity of  $S$ , chosen in such a way that the results of [R10] hold true for this number. Let  $\mathbf{Y}_t$  be the geometric representative of  $Y$  for the quadratic differential  $q_t$  and define  $q_{t,Y}$  to be the quadratic differential obtained by capping off the boundary components with (perhaps degenerate) discs or punctured discs (see the proof of Theorem 4.2 in [R10] for details). Then  $(q_{t,Y})$  ( $t \in [a, b]$ ) is the cotangent line of a Teichmüller geodesic on  $Y$  (with the boundary circles of  $Y$  closed by discs or replaced by punctures). Moreover, there is a number  $\delta' < \delta$  and a number  $\rho' < \rho$  such that every essential simple closed curve in  $\mathbf{Y}_t$  which is  $\delta$ -wide and  $\rho$ -slim for  $q_t$  is  $\delta'$ -wide and  $\rho'$ -slim for  $q_{t,Y}$ .

Let  $T_{n-1} = T_{n-1}(\delta', \rho', k', b/4n)$  be the number found for surfaces of complexity at most  $n - 1$  and for the numbers  $\delta' < \delta, \rho' < \rho$  and a number  $k' > k$  which will be determined below. Let  $M_1 > M_0$  be sufficiently large that the following holds true. Let  $s \in \mathbb{R}$  and let  $Y$  be an  $\epsilon_0$ -semi-thick subsurface for  $q_s$  with geometric representative  $\mathbf{Y}_s$ . Let  $\beta$  be a boundary circle of  $Y$  and let  $E \subset \mathbf{Y}_s$  be a maximal expanding annulus for  $q_s$  homotopic to  $\beta$ . If  $\text{Mod}_{q_s}(E) \geq M_1$  then for  $|t - s| \leq T_{n-1}$  the modulus of the maximal expanding annulus for  $q_t$  homotopic to  $\beta$  which is contained in the geometric representative  $\mathbf{Y}_t$  of  $Y$  for  $q_t$  and lies on the same side of  $\beta$  as  $E$  is not smaller than  $M_0$ .

For this number  $M_1$  let  $\epsilon_1(\rho/2, M_1) > 0$  be as in Lemma 2.2. Let  $\epsilon_2 < \epsilon_1(\rho/2, M_1)$  be sufficiently small that whenever  $\beta$  is a curve with  $\text{Ext}_{q_s}(\beta) \leq \epsilon_2$  then  $\text{Ext}_{q_t}(\beta) \leq \epsilon_1(\rho/2, M_1)$  for all  $t$  with  $|s - t| \leq T_{n-1}$ .

Let  $\ell > 0$  be sufficiently large that for every  $q \in \mathcal{Q}^1(S)$  the diameter in  $\mathcal{CG}$  of the union of the set of curves from a shortest marking for  $q$  with the set of all  $\delta$ -wide curves does not exceed  $\ell$  (such a number exists since the extremal length and

hence the hyperbolic length of a  $\delta$ -wide curve is uniformly bounded from above, see [H10]). Let  $T_n = R(\epsilon_2, k + 2\ell, b/8)$  be as in Proposition 2.1. We may assume that

$$bT_n/16 \geq bT_n/8n \geq T_{n-1}.$$

Let  $(q_t)$  be the cotangent line of a Teichmüller geodesic on  $S$ . Proposition 2.1 shows that either  $d_{\mathcal{CG}}(\Upsilon(q_0), \Upsilon(q_{T_n})) \geq k + 2\ell$  or there is a subset  $B$  of  $[0, T_n]$  with  $\lambda(B) \geq (1-b/8)T_n$  and for every  $s \in B$  there is a curve of extremal length at most  $\epsilon_2$  for  $q_s$ . In the first case we conclude from the choice of  $\ell$  that  $\text{diam}_{\mathcal{CG}}(\mu_0 \cup \mu_{q_{T_n}}) \geq k$  and we are done. Thus assume that the second possibility holds true.

Let us now suppose that there exists a set  $A \subset [0, T_n]$  with  $\lambda(A) \geq bT_n$  such that for every  $t \in A$  there is a  $\delta$ -wide  $\rho$ -slim curve  $\alpha(t)$  for  $q_t$ . Let  $\chi_B, \chi_A$  be the characteristic function of  $B, A$ , respectively. Since  $bT_n/16 \geq T_{n-1}$  and  $\lambda(B) \geq (1-b/8)T_n$ , for every  $s \in [0, T_{n-1}]$  we have

$$\lambda\{u \in [T_{n-1}, T_n - T_{n-1}] \mid u - s \in B\} \geq (1-b/4)T_n.$$

Now  $\lambda(A) \cap [T_{n-1}, T_n - T_{n-1}] \geq 3bT_n/4$  and hence Fubini's theorem shows that

$$\begin{aligned} & \int_0^{T_n - T_{n-1}} \left( \frac{1}{T_{n-1}} \int_s^{s+T_{n-1}} \chi_B(s)\chi_A(u) du \right) ds \\ & \geq \frac{1}{T_{n-1}} \int_0^{T_{n-1}} \int_{T_{n-1}}^{T_n - T_{n-1}} \chi_B(u-s)\chi_A(u) du ds \geq bT_n/2. \end{aligned}$$

As a consequence, the Lebesgue measure of the set

$$C = \{s \in B \cap [0, T_n - T_{n-1}] \mid \int_s^{s+T_{n-1}} \chi_A(u) du \geq bT_{n-1}/4\}$$

is at least  $bT_n/4$ .

If  $s \in C$  then  $s \in B \cap [0, T_n - T_{n-1}]$  and hence there is a simple closed curve  $\beta_s$  with  $\text{Ext}_{q_s}(\beta_s) \leq \epsilon_2$ . Moreover, there is a set

$$D \subset [s, s + T_{n-1}]$$

of Lebesgue measure at least  $bT_{n-1}/4$  so that for every  $t \in D$  there is a  $\delta$ -wide  $\rho$ -long curve  $\alpha(t)$  for  $q_t$ .

By Lemma 2.2 and the choice of  $\epsilon_2$ , for every  $t \in D$  the curve  $\alpha(t)$  is contained in an  $M_1$ -large subsurface  $Y \subset S - \beta_s$  for  $q_t$ .

A subsurface  $Y$  of  $S$  which is  $M_1$ -large for some  $t \in [s, s + T_{n-1}]$  is  $M_0$ -large for every  $t' \in [s, s + T_{n-1}]$ . Now  $M_0$ -large subsurfaces for  $q_t$  are pairwise disjoint, and their number is at most  $n$ . As a consequence, there is a subsurface  $Y \subset S - \beta_s$  which is  $M_0$ -large for every  $t \in [s, s + T_{n-1}]$ , and there is a subset  $D_s \subset D$  of Lebesgue measure at least  $T_{n-1}b/4n$  so that  $\alpha(t) \in Y$  for every  $t \in D_s$ .

Let  $\psi(s), \psi(s + T_{n-1}) \subset Y$  be shortest markings for  $q_{s,Y}, q_{s+T_{n-1},Y}$ . By the choice of  $T_{n-1}$ , we can apply the induction hypothesis to the cotangent line  $(q_{t,Y})$  ( $t \in [s, s + T_{n-1}]$ ) of the induced Teichmüller geodesic on  $Y$ .

Let  $H_s \subset [s, s + T_{n-1}]$  be the set of all  $t \in D_s$  such that the curve  $\alpha(t)$  is contained in a proper subsurface  $Z_t$  of  $Y$  with the additional property that

$$\text{diam}(\pi_{Z_t}(\psi(s) \cup \psi(s + T_{n-1}))) \geq k'.$$

By induction assumption, either the diameter of  $\psi(s) \cup \psi(s + T_{n-1})$  in the curve graph of  $Y$  is at least  $k'$  or

$$\lambda(H_s) \geq f(b/4n, n-1)T_{n-1}.$$

Theorem 5.3 of [R10] shows that there is a number  $p > 0$  with the following property. If  $\text{diam}(\pi_Y(\psi(s) \cup \psi(s + T_{n-1}))) \geq k'$  then  $\text{diam}(\pi_Y(\mu_0 \cup \mu_{T_n})) \geq k' - p$ . Moreover, for  $t \in H_s$  the diameter of the subsurface projection of  $\mu_0 \cup \mu_{T_n}$  into  $Z_t$  is at least  $k' - p$ .

Define  $k' = k + p$  and let  $H'_s = D_s$  if  $\text{diam}(\pi_Y(\psi(s) \cup \psi(s + T_{n-1}))) \geq k'$  (which implies that  $\text{diam}(\pi_Y(\mu_0 \cup \mu_{T_n})) \geq k$ ), and define  $H'_s = H_s$  otherwise. Then

$$\lambda(H'_s) \geq f(b/4n, n-1)T_{n-1}$$

for every  $s \in C$ , moreover for all  $t \in H'_s$  the curve  $\alpha(t)$  is contained in a subsurface  $Y_s$  of  $S$  with  $\text{diam}(\pi_{Y_s}(\mu_0 \cup \mu_{T_n})) \geq k$ .

For  $s \in C$  let  $\chi_s$  be the characteristic function of  $D_s$ , viewed as a subset of  $\mathbb{R}$ , and let  $\chi(t) = \max\{\chi_s(t) \mid s\}$ . Since  $\lambda(C) \geq bT_n/4$  we have

$$\int_0^{T_n} \chi(s) ds \geq f(b/4n, n-1)bT_n/4$$

and hence the claim holds true with  $f(b, n) = bf(b/4n, n-1)/4$ .  $\square$

Now we are ready for the proof of the Theorem from the introduction. To this end choose a number  $K_0 > 0$  such that the following holds true. Let  $\xi, \zeta$  be simple closed curves. If there is a proper subsurface  $Y$  of  $S$  such that  $\text{diam}(\pi_Y(\xi \cup \zeta)) \geq K_0$  then a geodesic  $\gamma$  in  $\mathcal{CG}$  connecting  $\xi$  to  $\zeta$  passes through the complement of  $Y$ . The existence of such a number  $K_0 > 0$  follows from Theorem 3.1 of [MM00].

Let  $(q_t)$  be the cotangent line of the Teichmüller geodesic connecting  $\xi$  to  $\zeta$ . By the results of [MM00], there is a number  $\chi > 0$  with the following property. If  $\alpha \neq \xi, \zeta$  is a simple closed curve so that  $\alpha$  is the core curve of a flat cylinder for  $q_t$  of modulus at least  $\chi$  then the diameter of the subsurface projection of  $(\xi, \zeta)$  into an annulus with core curve  $\alpha$  is at least  $K_0$ .

Let  $\kappa > 0$  be as in (1). For these numbers  $\chi, \kappa, K_0$  let  $R = T(\chi, K_0 + 4\kappa)$  be as in Lemma 2.3.

Let  $t_0 < \dots < t_n = T$  be as in the statement of the theorem. Let  $u$  be an integer bigger than  $R$ . By Lemma 2.3 and the above discussion, for all  $\ell$  we either have

- i)  $d_{\mathcal{CG}}(\Upsilon(q_{t_\ell}), \Upsilon(q_{t_{\ell+u}})) \geq K_0 + 4\kappa$  and hence  $d_{\mathcal{CG}}(\Upsilon(q_{t_\ell}), \Upsilon(q_{t_w})) \geq 2\kappa$  for all  $w \geq \ell + u \geq \ell \geq v$ , or
- ii) there is a subsurface  $Y$  of  $S$  (perhaps an annulus) so that the diameter of the subsurface projection of  $\xi, \zeta$  into  $Y$  is at least  $K_0$  and that  $\Upsilon(q_s) \subset Y$  for some  $s \in [\ell, \ell + u]$ .

It now suffices to show that  $d_{\mathcal{CG}}(\Upsilon(q_\ell), \Upsilon(q_{t_{\ell+u(8\kappa+8)}})) \geq 2\kappa$  for all  $\ell$ . By the choice of  $\kappa$ , this holds true if there is some  $j \leq u(8\kappa + 6)$  so that the alternative i) above is satisfied for  $q_{t_j}$ .

Otherwise for every  $i \leq 8\kappa + 6$  there is some  $s \in [ui, u(i+1)]$  such that  $\Upsilon(q_s)$  is contained in a subsurface  $Y$  with the property that the diameter of the subsurface projection of  $\xi, \zeta$  into  $Y$  is at least  $K_0$ . By the choice of  $K_0$  (see Theorem 3.1 of [MM00]), there is a curve  $c_s$  on  $\gamma$  disjoint from  $\Upsilon(q_s)$ . In particular, we have  $d_{\mathcal{CG}}(c_s, \Upsilon(q_s)) \leq 1$ .

Now if  $v \in [uj, u(j+1)]$  for some  $j$  with  $|i-j| \geq 2$  then  $d_{\mathcal{CG}}(c_s, c_v) \geq 1$  by assumption. In particular,  $c_s, c_v$  are distinct. As a consequence, there are at least  $4\kappa + 3$  distinct points on the geodesic  $\gamma$  arising in this way. Then there are at least two of these points, say the points  $c_s, c_v$ , whose distance is at least  $4\kappa + 2$ . This shows  $d_{\mathcal{CG}}(\Upsilon(q_{t_s}), \Upsilon(q_{t_v})) \geq 4\kappa$  which implies the required estimate.

## REFERENCES

- [B06] B. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, J. Reine Angew. Math. 598 (2006), 105–129.
- [G87] F. Gardiner, *Teichmüller theory and quadratic differentials*, Pure and Appl. Math., Wiley-Interscience, New York 1987.
- [H06] U. Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves*, in “Spaces of Kleinian groups” (Y. Minsky, M. Sakuma, C. Series, eds.), London Math. Soc. Lec. Notes 329 (2006), 187–207.
- [H10] U. Hamenstädt, *Stability of quasi-geodesics in Teichmüller space*, Geom. Dedicata 146 (2010), 101–116.
- [H12] U. Hamenstädt, *Teichmüller theory*, Lecture Notes of the Park City Summer School 2011.
- [MM99] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103–149.
- [MM00] H. Masur, Y. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000), 902–974.
- [M92] Y. Minsky, *Harmonic maps, length, and energy in Teichmüller space*, J. Diff. Geom. 35 (1992), 151–217.
- [R07] K. Rafi, *Thick-thin decomposition for quadratic differentials*, Math. Res. Lett. 14 (2007), 333–341.
- [R10] K. Rafi, *Hyperbolicity in the Teichmüller space*, arXiv:1011.6004
- [Sh04] K. Shackleton, *Tightness and computing distances in the curve complex*, arXiv:math/0412078.
- [S84] K. Strebel, *Quadratic differentials*, Ergebnisse der Mathematik 5, Springer, Berlin 1984.

MATH. INSTITUT DER UNIVERSITÄT BONN  
ENDENICHER ALLEE 60  
53115 BONN  
GERMANY

e-mail: ursula@math.uni-bonn.de