SYMBOLIC DYNAMICS FOR THE TEICHMÜLLER FLOW

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Abstract. Let $\mathcal{Q}$ be a component of a stratum of abelian or quadratic differentials on an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$. We construct a subshift of finite type (Ω, σ) and a Borel suspension of (Ω, σ) which admits a finite-to-one semi-conjugacy into the Teichmüller flow Φ^t on $\mathcal{Q}$. This is used to show that the Φ^t-invariant Lebesgue measure λ on $\mathcal{Q}$ is the unique measure of maximal entropy.

1. Introduction

A surface S *of finite type* is a closed oriented surface of genus $g \geq 0$ with $m \geq 0$ marked points, so-called *punctures*. We assume that $3g-3+m \geq 2$, that is, S is not a sphere with at most 4 punctures or a torus with at most 1 puncture. We then call the surface S *nonexceptional*. The Euler characteristic of S is negative.

The *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all complete finite area hyperbolic metrics on the complement of the punctures in S under the action of the group of diffeomorphisms of S which are isotopic to the identity. The sphere bundle

$$\tilde{\mathcal{Q}}(S) \to \mathcal{T}(S)$$

of all *holomorphic quadratic differentials* of area one can naturally be identified with the unit cotangent bundle for the *Teichmüller metric*. If the surface S has punctures, that is, if $m > 0$, then we define a holomorphic quadratic differential on S to be a meromorphic quadratic differential on the closed Riemann surface obtained from S by filling in the punctures, with a simple pole at each of the punctures and no other poles.

The *mapping class group* $\mathrm{Mod}(S)$ of all isotopy classes of orientation preserving diffeomorphisms of S naturally acts on $\tilde{\mathcal{Q}}(S)$. The quotient

$$\mathcal{Q}(S) = \tilde{\mathcal{Q}}(S)/\mathrm{Mod}(S)$$

is the *moduli space of area one quadratic differentials*. It can be partitioned into so-called *strata*. Namely, let $1 \leq m_1 \leq \cdots \leq m_\ell$ ($\ell \geq 1$) be a sequence of positive integers with

$$\sum_i m_i = 4g-4+m.$$

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The stratum $\mathcal{Q}(m_1,\dots,m_\ell;-m)$ defined by the ℓ-tuple $(m_1,\dots,m_\ell)$ is the moduli space of pairs (C,φ) where C is a closed Riemann surface of genus g and where φ is an area one meromorphic quadratic differential on C with ℓ zeros of order m_i and m simple poles and which is not the square of a holomorphic one-form.

A stratum $\mathcal{Q}(m_1,\dots,m_\ell;-m)$ is a real hypersurface in a complex orbifold of complex dimension

$$h = 2g + \ell + m - 2.$$

Strata need not be connected, however they have at most two connected components [L08]. The closure in $\mathcal{Q}(S)$ of a component of a stratum $\mathcal{Q}(m_1,\dots,m_\ell;-m)$ is a union of components of strata $\mathcal{Q}(n_1,\dots,n_s;-m')$ where $s\leq \ell, m'\leq m$. Note here that it is natural to allow a simple pole to merge with a zero in the closure, thus decreasing the number m of marked points.

If the surface S is closed, that is, if $m=0$, then we can also consider the bundle

$$\tilde{\mathcal{H}}(S)\to\mathcal{T}(S)$$

of area one *abelian differentials*. It descends to the moduli space $\mathcal{H}(S)$ of holomorphic one-forms defining a singular euclidean metric of area one. Again this moduli space decomposes into a union of strata $\mathcal{H}(k_1,\dots,k_s)$ corresponding to the orders of the zeros of the differentials. Strata are in general not connected, but there are at most three connected components [KZ03]. The stratum $\mathcal{H}(k_1,\dots,k_s)$ is a real hypersurface in a complex orbifold of dimension

$$h = 2g + s - 1.$$

The *Teichmüller flow* Φ^t acts on $\mathcal{Q}(S)$ (or $\mathcal{H}(S)$) preserving the strata. If $\mathcal{Q}$ is a component of a stratum of abelian differentials then *Rauzy induction* for interval exchange transformations can be used to construct a *symbolic coding* for the Teichmüller flow on $\mathcal{Q}$ ([V82], see also [AGY06] for a discussion and references). Rauzy induction has been extended to strata of quadratic differentials by Boissy and Lanneau [BL09]. It was used to identify connected components of strata of quadratic differentials.

Our main goal is to construct a new coding for the Teichmüller flow on any component $\mathcal{Q}$ of a stratum. This coding is based on the perspective on components of strata of abelian or quadratic differentials developed in [H24], and it is well suited to study the space $\mathcal{M}_{\rm inv}(\mathcal{Q})$ of all Φ^t-invariant Borel probability measures on $\mathcal{Q}$, equipped with the weak*-topology.

For the formulation of our main result, recall that a *biinfinite subshift of finite type* (Ω,σ) is defined by a finite alphabet $\mathcal{A}=\{1,\dots,p\}$ and a (p,p)-matrix (a_{ij}) with entries in $\{0,1\}$ such that

$$\Omega=\{(x_i)\in\mathcal{A}^{\mathbb{Z}}\mid a_{x_ix_{i+1}}=1 \text{ for all } i\}.$$

The *shift* σ acts on Ω by $\sigma(x_i)=(x_{i+1})$. The set Ω carries a natural σ-invariant topology, and for this topology, Ω is compact.

The shift is called *topologically transitive* if the σ-action has a dense orbit. A sequence $(x_i)\subset\Omega$ is called *normal* if every finite string $(y_i)_{1\leq i\leq k}$ with $a_{y_iy_{i+1}}=1$

for all $0 \leq i \leq k-1$ occurs infinitely often in forward and backward direction as a substring of (x_i).

In the statement of the following result, spaces of probability measures are equipped with the weak*-topology.

Theorem 1. *Let $\mathcal{Q}$ be a component of a stratum of quadratic or abelian differentials. Then there exists*

- *a topologically transitive subshift of finite type (Ω, σ),*
- *a σ- invariant dense Borel set $\mathcal{U} \subset \Omega$ containing all normal sequences,*
- *a suspension (X, Θ^t) of σ over $\mathcal{U}$, given by a positive bounded continuous roof function on $\mathcal{U}$*

and a finite-to-one semi-conjugacy $\Xi : (X, \Theta^t) \to (\mathcal{Q}, \Phi^t)$ which maps the space of Θ^t-invariant Borel probability measures on X continuously onto $\mathcal{M}_{\rm inv}(\mathcal{Q})$.

As is the case for Rauzy induction or, more precisely, the zippered rectangle flow considered in [V86], the coding constructed in Theorem 1 can be thought of as a finite cover of the Teichmüller flow on $\mathcal{Q}$. In particular, it maps the collection of all periodic orbits for σ contained in $\mathcal{U}$ onto the collection of all periodic orbits in $\mathcal{Q}$. However, a periodic orbit in $\mathcal{Q}$ may have more than one preimage in the suspension flow (X, Θ^t), and the restriction of Ξ to any such preimage may be a nontrivial finite covering of the periodic orbit.

Our construction is valid for strata of abelian differentials, but it is different from Rauzy induction. A dictionary between these two codings has yet to be established.

A specific example of a Φ^t-invariant Borel probability measure on a component $\mathcal{Q}$ of a stratum in the Lebesgue measure class was constructed by Masur and Veech [M82, V86]. This measure λ is ergodic [M82, V86] and of full support, and its *entropy* h_λ coincides with the complex dimension $2g+\ell+m-2$ (or $2g+s-1$) of the complex orbifold defining the stratum (note that we use a normalization for the Teichmüller flow which is different from the one used by Masur and Veech). In particular, the entropy of the Lebesgue measure on the open connected stratum $\mathcal{Q}(1,\dots,1;-m)$ equals $6g-6+2m$.

Denote by h_ν the entropy of a measure $\nu \in \mathcal{M}_{\rm inv}(\mathcal{Q})$. Define

$$h_{\rm top}(\mathcal{Q}) = \sup\{h_\nu \mid \nu \in \mathcal{M}_{\rm inv}(\mathcal{Q})\}.$$

A *measure of maximal entropy* for the component $\mathcal{Q}$ is a measure $\mu \in \mathcal{M}_{\rm inv}(\mathcal{Q})$ such that $h_\mu = h_{\rm top}(\mathcal{Q})$. Since $\mathcal{Q}$ is non-compact, a priori such a measure need not exist. However, using Rauzy induction and the work of Buzzi and Sarig [BS03], Bufetov and Gurevich [BG11] showed that for components of strata of abelian differentials, the Φ^t-invariant probability measure in the Lebesgue measure class is the unique measure of maximal entropy for the component. We use Theorem 1 and [BS03] to extend this result to all components of strata of quadratic or abelian differentials, with a different proof.

Theorem 2. *For every component $\mathcal{Q}$ of a stratum in $\mathcal{Q}(S)$ or $\mathcal{H}(S)$, the Φ^t-invariant Borel probability measure in the Lebesgue measure class is the unique measure of maximal entropy.*

In view of the groundbreaking work of Eskin and Mirzakhani [EM18] and of Eskin, Mirzakhani and Mohammadi [EMM15], we expect that the analog of Theorem 2 also holds for arbitrary affine invariant manifolds in $\mathcal{Q}(S)$. However our methods do not apply in this generality. Instead they are very well suited to study the dynamics of the Teichmüller flow near the *principal boundary* of a stratum as initiated in [H24]. This analysis is will be made precise in a sequel to this article.

Organization of the article and outline of the proofs. The organization of the article is as follows. In Section 2 we collect some results from [H24] relating *train tracks* to components of strata of abelian or quadratic differentials. This is used in Section 3 to construct for every connected component $\mathcal{Q}$ of a stratum an associated topologically transitive subshift of finite type (Ω, σ).

In Section 4 we define a *roof function* ρ on a σ-invariant dense Borel subset $\mathcal{U}$ of Ω containing all normal sequences and use this roof function to define a suspension flow (X, Θ^t) over $\mathcal{U}$. It fairly immediately follows from the construction that there is a (partial) semi-conjugacy $\Xi : (X, \Theta^t) \to (\mathcal{Q}, \Phi^t)$, that is, the map Ξ is continuous and commutes with the action of Θ^t on X and Φ^t on $\mathcal{Q}$, however it is not surjective. We establish that this semi-conjugacy is finite-to-one, which is the most elaborate part of the proof of Theorem 1. This is used to show that every ergodic Φ^t-invariant Borel probability measure on $\mathcal{Q}$ is the push-forward under Ξ of a finite invariant measure on (X, Θ^t) and hence on (Ω, σ), which completes the proof of Theorem 1.

Theorem 1 is not sufficient for the proof of Theorem 2. Namely, the semi-conjugacy Ξ is only finite-to-one but not bounded-to-one, and it is defined on a suspension over a Borel subset of Ω which is not closed, reflecting the fact that the component $\mathcal{Q}$ is not compact, and the Teichmüller flow Φ^t is not hyperbolic. Instead, in Section 5 we start with a point $q \in \mathcal{Q}$ which is contained in both the α- and ω limit set of its own orbit under Φ^t. By the Poincarë recurrence theorem, for any invariant Borel probability measure on $\mathcal{Q}$, the set of such points has full measure. We then use the point q and the subshift (Ω, σ) to construct a new Markov shift, now over a countably infinite alphabet. We also construct a continuous roof function which is bounded from below by a universal positive constant, but is unbounded. We then show that the corresponding suspension flow admits a bounded-to-one semi-conjugacy onto the restriction of the Teichmüller flow Φ^t to the invariant set of all points whose orbit under Φ^t contain q in its α- and ω limit set.

It follows from the work of Buzzi and Sarig [BS03] that the suspension flow over the countable alphabet admits at most one measure of maximal entropy provided that some technical conditions are fullfilled. We then verify that these technical assumptions are indeed satisfied for the flow constructed earlier, which leads to the proof of Theorem 2.

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2. Strata and train tracks

In this section we summarize some results from [H24] which will be used throughout the paper.

2.1. Geodesic laminations.

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ marked points and where $3g-3+m \geq 2$. A *geodesic lamination* for a complete hyperbolic structure of finite volume on S (which means in the sequel that the metric is defined on the complement of the marked points in S) is a *compact* subset of S (with the marked points removed) which is foliated into simple geodesics. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ.

A geodesic lamination λ on S is said to *fill up* S if its complementary regions are all topological discs or once punctured monogons.

Definition 2.1 (Definition 2.1 of [H24])**.** A geodesic lamination λ is called *large* if λ fills up S and if moreover λ can be approximated in the *Hausdorff topology* by simple closed geodesics.

Since every minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed geodesics (Theorem 4.2.14 of [CEG87]), a minimal geodesic lamination which fills up S is large. However, there are large geodesic laminations with finitely many leaves.

The *topological type* of a large geodesic lamination ν is a tuple

$$(m_1,\dots,m_\ell;-m) \text{ where } 1\leq m_1\leq\dots\leq m_\ell,\ \sum_i m_i = 4g-4+m$$

such that the complementary regions of ν which are topological discs are m_i+2-gons and the complementary regions which are once punctured discs are once punctured monogons. Let

$$\mathcal{LL}(m_1,\dots,m_\ell;-m)$$

be the space of all large geodesic laminations of type $(m_1,\dots,m_\ell;-m)$ equipped with the restriction of the Hausdorff topology for compact subsets of S.

A *measured geodesic lamination* is a geodesic lamination λ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in S with endpoints in the complementary regions of λ which intersects λ nontrivially and transversely. The geodesic lamination λ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. The space $\mathcal{ML}$ of all measured geodesic laminations on S

equipped with the weak*-topology is homeomorphic to $S^{6g-7+2m} \times (0,\infty)$. Its projectivization is the space $\mathcal{PML}$ of all *projective measured geodesic laminations.*

The measured geodesic lamination $\mu \in \mathcal{ML}$ is said to *fill up* S if its support fills up S. This support is then necessarily connected and hence minimal, and for some tuple $(m_1,\dots,m_\ell;-m)$, it defines a point in the set $\mathcal{LL}(m_1,\dots,m_\ell;-m)$. The projectivization of a measured geodesic lamination which fills up S is also said to fill up S. We call $\mu \in \mathcal{ML}$ *strongly uniquely ergodic* if the support of μ fills up S and admits a unique transverse measure up to scale.

There is a continuous symmetric pairing

$$\iota : \mathcal{ML} \times \mathcal{ML} \to [0,\infty),$$

the so-called *intersection form*, which extends the geometric intersection number between simple closed curves. We refer to Section 8.2.10 of [Mar16] for more information.

2.2. **Train tracks.** A *train track* on S is defined to be an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*), the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. A simple closed curve component of τ is required to contain a unique bivalent switch, and all other switches must be at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with $0,1$ or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. We always identify train tracks which are isotopic. Throughout we use the book [PH92] as the main reference for train tracks.

A train track is called *generic* if all switches are at most trivalent. For each switch v of a generic train track τ which is not contained in a simple closed curve component, there is a unique half-branch b of τ which is incident on v and which is *large* at v. This means that every germ of an immersed arc of class C^1 on τ which passes through v also passes through the interior of b. A half-branch which is not large is called *small.* A branch b of τ is called *large* (or *small*) if each of its two half-branches is large (or small). A branch which is neither large nor small is called *mixed.*

Remark 2.2. As in [H09a], all train tracks are assumed to be generic. Unfortunately this leads to a small inconsistency of our terminology with the terminology found in the literature.

A generic train track τ is *orientable* if there is a consistent orientation of the branches of τ such that at any switch s of τ, the orientation of the large half-branch incident on s extends to the orientation of the two small half-branches incident on s. If C is a complementary polygon of an oriented train track, then the number of sides of C is even. In particular, a train track which contains a once punctured monogon component, that is, a once punctured disc with one cusp at the boundary, is not orientable (see p.31 of [PH92] for a more detailed discussion).

A train track or a geodesic lamination η is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is homotopic to the identity and maps η into τ in such a way that the restriction of the differential of F to the tangent space of η vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of F to η a *carrying map* for η. Write $\eta \prec \tau$ if the train track η is carried by the train track τ. Then every geodesic lamination ν which is carried by η is also carried by τ.

A train track *fills up* S if its complementary components are topological discs or once punctured monogons. Note that such a train track τ is connected. Let $\ell \geq 1$ be the number of those complementary components of τ which are topological discs. Each of these discs is an $m_i + 2$-gon for some $m_i \geq 1$ $(i = 1, \dots, \ell)$. The *topological type* of τ is defined to be the ordered tuple $(m_1, \dots, m_\ell; -m)$ where $1 \leq m_1 \leq \cdots \leq m_\ell$; then $\sum_i m_i = 4g-4+m$. If τ is orientable then $m = 0$ and m_i is even for all i. A train track of topological type $(m_1, \dots, m_\ell; -m)$ is called *fully recurrent* if it carries a minimal large geodesic lamination of type $(m_1, \dots, m_\ell; -m)$ (Definition 2.6 of [H24]).

A *transverse measure* on a generic train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: for every trivalent switch s of τ, the sum of the weights of the two small half-branches incident on s equals the weight of the large half-branch. The space

$$\mathcal{V}(\tau)$$

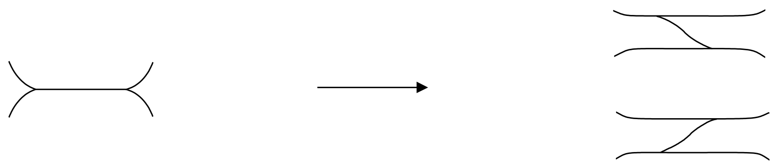
of all measured geodesic laminations whose supports are carried by τ can naturally be identified with the space of all transverse measures on τ. Thus $\mathcal{V}(\tau)$ has the structure of a cone in a finite dimensional real vector space. The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. A fully recurrent train track is recurrent [PH92, H24].

There are two simple ways to modify a fully recurrent train track τ to another fully recurrent train track. Namely, if b is a mixed branch of τ then we can *shift* τ along b to a new train track τ'. This new train track carries τ and hence it is fully recurrent since it carries every geodesic lamination which is carried by τ, see p.118 of [PH92] and also [H09a].

Similarly, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure A below. A (right or left) split τ' of a train track τ is carried by τ. If τ is of topological type $(m_1, \dots, m_\ell; -m)$, if $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -m)$ is carried by τ and if e is a large branch of τ, then there is a unique choice of a right or left split of τ at e such that the split track η carries ν. In particular, η is fully recurrent. We refer to p.48-49 of [H24] for a more detailed discussion. Note that unlike in the standard reference [PH92], we do *not* allow to modify a train track with a *collision*, which is defined to be a split followed by the removal of the diagonal branch of the split and which strictly reduces the number of branches of the train track.

To each train track τ which fills up S one can associate a *dual bigon track* τ^* (Section 3.4 of [PH92]). There is a bijection between the complementary components of τ and those complementary components of τ^* which are not *bigons*, i.e.

Figure A



discs with two cusps at the boundary. This bijection maps a component C of τ which is an n-gon for some $n \geq 3$ to an n-gon component of τ^* contained in C, and it maps a once punctured monogon C to a once punctured monogon contained in C. If τ is orientable then the orientation of S and an orientation of τ induce an orientation on τ^*, that is, τ^* is orientable.

There is a notion of carrying for bigon tracks which is analogous to the notion of carrying for train tracks.

Definition 2.3 (Definition 2.8 of [H24])**.** A train track τ of topological type $(m_1, \dots, m_\ell; -m)$ is called *large* if both τ, τ^* carry a minimal large geodesic lamination of the same topological type as τ.

For a large train track τ let $\mathcal{V}^*(\tau) \subset \mathcal{ML}$ be the set of all measured geodesic laminations whose support is carried by τ^*. Then $\mathcal{V}^*(\tau)$ is naturally homeomorphic to a convex cone in a real vector space. The dimension of this cone coincides with the dimension of $\mathcal{V}(\tau)$.

Denote by

$$\mathcal{LT}(m_1, \dots, m_\ell; -m)$$

the set of all isotopy classes of large train tracks on S of type $(m_1, \dots, m_\ell; -m)$.

2.3. **Strata.** For a closed oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures let $\tilde{\mathcal{Q}}(S)$ be the bundle of marked area one holomorphic quadratic differentials with a simple pole at each puncture over the Teichmüller space $\mathcal{T}(S)$ of marked complex structures on S. For a complete hyperbolic metric on S of finite area, an area one quadratic differential $q \in \tilde{\mathcal{Q}}(S)$ is determined by a pair (λ^+, λ^-) of measured geodesic laminations which *bind* S, that is, we have

$$\iota(\lambda^+, \mu) + \iota(\lambda^-, \mu) > 0$$

for every measured geodesic lamination μ, moreover $\iota(\lambda^+, \lambda^-) = 1$ as the area of q equals one. The *vertical* measured geodesic lamination λ^+ for q corresponds to the equivalence class of the vertical measured foliation of q. The *horizontal* measured geodesic lamination λ^- for q corresponds to the equivalence class of the horizontal measured foliation of q.

Recall from the introduction the definition of the *stratum* $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m)$ of $\tilde{\mathcal{Q}}(S)$ where $(m_1, \dots, m_\ell)$ of positive integers $1 \leq m_1 \leq \dots \leq m_\ell$ with $\sum_i m_i = 4g-4+m$. Also consider as before the bundle $\tilde{\mathcal{H}}(S)$ of marked area one holomorphic one-forms over Teichmüller space $\mathcal{T}(S)$ of S with its strata $\tilde{\mathcal{H}}(k_1,\dots,k_\ell)$ of marked area one holomorphic one-forms on S with ℓ zeros of order k_i $(i=1,\dots,\ell)$.

For a large train track $\tau\in\mathcal{LT}(m_1,\dots,m_\ell;-m)$ let

$$\mathcal{Q}(\tau)\subset\tilde{\mathcal{Q}}(S) \tag{1}$$

be the set of all marked area one quadratic differentials whose vertical measured geodesic lamination is carried by τ and whose horizontal measured geodesic lamination is carried by the dual bigon track τ^* of τ. By definition of a large train track, we have $\mathcal{Q}(\tau)\neq\emptyset$.

The next proposition relates $\mathcal{Q}(\tau)$ to components of strata.

Proposition 2.4 (Proposition 3.2 and Proposition 3.3 of [H24])**.** (1) *For any large non-orientable train track $\tau\in\mathcal{LT}(m_1,\dots,m_\ell;-m)$ there is a component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-m)$ such that $\mathcal{Q}(\tau)$ is the closure in $\tilde{\mathcal{Q}}(S)$ of an open path connected subset of $\tilde{\mathcal{Q}}$.*

(2) *For every large orientable train track $\tau\in\mathcal{LT}(m_1,\dots,m_\ell;0)$ there is a component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{H}}(m_1/2,\dots,m_\ell/2)$ such that $\mathcal{Q}(\tau)$ is the closure in $\tilde{\mathcal{H}}(S)$ of an open path connected subset of $\tilde{\mathcal{Q}}$.*

(3) *For every component $\tilde{\mathcal{Q}}$ of a stratum of $\tilde{\mathcal{Q}}(S)$ (or of a stratum of $\tilde{\mathcal{H}}(S)$) and for every $q\in\tilde{\mathcal{Q}}$ there is a large train track τ such that $q\in\mathcal{Q}(\tau)$ and that $Q(\tau)$ is the closure of the open dense path connected subset $\tilde{\mathcal{Q}}\cap\mathcal{Q}(\tau)$.*

3. A symbolic system

In this section we construct a subshift of finite type which is used in the following sections to construct a symbolic coding for the Teichmüller flow on a component of a stratum in the moduli space of quadratic or abelian differentials. We continue to use the assumptions and notations from Section 2. The section is divided into two subsections.

3.1. **A shift space constructed from train tracks.** Let $\mathcal{Q}$ be a connected component of a stratum $\mathcal{Q}(m_1,\dots,m_\ell;-m)$ of $\mathcal{Q}(S)$ (or of a stratum $\mathcal{H}(m_1/2,\dots,m_\ell/2)$ of $\mathcal{H}(S)$). Let $\tilde{\mathcal{Q}}$ be the preimage of $\mathcal{Q}$ in $\tilde{\mathcal{Q}}(S)$ (or in $\tilde{\mathcal{H}}(S)$). Let

$$\mathcal{LT}(\mathcal{Q})\subset\mathcal{LT}(m_1,\dots,m_\ell;-m) \tag{2}$$

be the set of all marked large train tracks τ of the same topological type as $\mathcal{Q}$ such that the set $\mathcal{Q}(\tau)$ defined in (1) is the closure of the open dense subset $\mathcal{Q}(\tau)\cap\tilde{\mathcal{Q}}$. Proposition 2.4 shows that this is well defined and that furthermore

$$\tilde{\mathcal{Q}}=\cup_{\tau\in\mathcal{LT}(\mathcal{Q})}(\mathcal{Q}(\tau)\cap\tilde{\mathcal{Q}}).$$

The set $\mathcal{LT}(\mathcal{Q})$ is invariant under the action of the mapping class group.

Fix a complete finite volume metric on S. For ease of notation, define

$$\mathcal{LL}(\mathcal{Q})\subset\mathcal{LL}(m_1,\dots,m_\ell;-m) \tag{3}$$

to be the closure (in the restriction of the Hausdorff topology defined by the hyperbolic metric) of the set of all minimal large geodesic laminations which can be represented as the support of the vertical measured geodesic lamination of some quadratic differential $q \in \tilde{\mathcal{Q}}$. Then $\mathcal{LL}(\mathcal{Q})$ is invariant under the action of $\mathrm{Mod}(S)$, and for every $\tau \in \mathcal{LT}(\mathcal{Q})$ it contains the set of all large geodesic laminations of topological type $(m_1, \dots, m_\ell; -m)$ carried by τ. We refer to Section 3 of [H24] for a more detailed discussion.

Our goal is to use the train tracks from the collection $\mathcal{LT}(\mathcal{Q})$ for a symbolic coding of the Teichmüller flow on $\mathcal{Q}$. However, the mapping class group $\mathrm{Mod}(S)$ does not act freely on $\mathcal{LT}(\mathcal{Q})$. To overcome this difficulty we extend the definition of a large train track as follows.

Definition 3.1. A *numbered marked large train track* is a marked large train track τ together with a numbering of the branches of τ.

The set

$$\mathcal{NT}(\mathcal{Q}) \tag{4}$$

of all isotopy classes of numbered marked large train tracks on S whose underlying unnumbered large train track is contained in $\mathcal{LT}(\mathcal{Q})$ is invariant under the natural action of the mapping class group.

A mapping class which preserves a marked large train track τ as well as each of its branches is the identity. Namely, such a mapping class can be represented by a homeomorphism of S whose restriction to τ is the identity. Since all complementary components of τ are discs or once punctured discs, such a homeomorphism is homotopic to the identity. Thus the action of the mapping class group on $\mathcal{NT}(\mathcal{Q})$ is free.

Define a *(numbered) combinatorial type* to be an orbit of a (numbered) marked large train track in $\mathcal{LT}(\mathcal{Q})$ (or in $\mathcal{NT}(\mathcal{Q})$) under the action of the mapping class group. Thus the set of numbered combinatorial types is the quotient of $\mathcal{NT}(\mathcal{Q})$ by the action of the mapping class group. Let

$$\mathcal{E}_0(\mathcal{Q})$$

be the set of all numbered combinatorial types which are $\mathrm{Mod}(S)$-orbits of elements of $\mathcal{NT}(\mathcal{Q})$.

Note that if the large train track τ' can be obtained from a large train track τ by a single split, then a numbering of the branches of τ naturally induces a numbering of the branches of τ' and therefore such a numbering defines a *numbered split.*

Definition 3.2. A *full split* of a (numbered) large train track τ is a (numbered) large train track τ' which can be obtained from τ by splitting τ at each large branch precisely once.

A *full (numbered) splitting sequence* is a sequence (τ_i) of (numbered) large train tracks such that for each i, the (numbered) large train track τ_{i+1} can be obtained from τ_i by a full (numbered) split.

Definition 3.3. A numbered combinatorial type $x \in \mathcal{E}_0(\mathcal{Q})$ is *splittable* to a numbered combinatorial type x' if there is a numbered large train track τ contained in x which can be connected to a numbered large train track τ' contained in x' by a *full* numbered splitting sequence.

In general it is unclear whether a given numbered combinatorial type is splittable to another type. This issue is addressed in Lemma 3.4 below which is a main technical ingredient towards the construction of a subshift of finite type with the properties stated in Theorem 1.

Given a numbered marked train track τ and a subset $\mathcal{W}$ of $\mathcal{E}_0(\mathcal{Q})$, we write $[\tau] \in \mathcal{W}$ if the $\mathrm{Mod}(S)$-orbit of τ is contained in $\mathcal{W}$. The first statement in the following lemma is crucial for topological transitivity of the subshift of finite type which defines our coding of the Teichmüller flow.

Lemma 3.4. *For every connected component $\mathcal{Q}$ of a stratum of $\mathcal{Q}(S)$ (or of a stratum of $\mathcal{H}(S)$) there is a set $\mathcal{E}(\mathcal{Q}) \subset \mathcal{E}_0(\mathcal{Q})$ of numbered combinatorial types with the following properties.*

1. *For all $x, x' \in \mathcal{E}(\mathcal{Q})$, x is splittable to x'.*
2. *If τ is contained in $\mathcal{E}(\mathcal{Q})$ and if (τ_i) is any full numbered splitting sequence issuing from $\tau_0 = \tau$ then τ_i is contained in $\mathcal{E}(\mathcal{Q})$ for all $i \geq 0$.*

Proof. For $[\beta] \in \mathcal{E}_0(\mathcal{Q})$ let $\mathcal{A}([\beta]) \subset \mathcal{E}_0(\mathcal{Q})$ be the set of all combinatorial types of numbered large train tracks ξ with the following additional property. There is a representative $\beta \in \mathcal{NT}(\mathcal{Q})$ of $[\beta]$ which can be connected to ξ by a (possibly trivial) full numbered splitting sequence. Since the concatenation of two full splitting sequences is a full splitting sequence, if $[\xi] \in \mathcal{A}([\beta])$ then $\mathcal{A}([\xi]) \subset \mathcal{A}([\beta])$.

Since $\mathcal{E}_0(\mathcal{Q})$ is a finite set, there exists some $[\sigma] \in \mathcal{E}_0(\mathcal{Q})$ such that the cardinality of $\mathcal{A}([\sigma])$ is minimal among the cardinalities of the sets $\mathcal{A}([\beta])$ where $[\beta]$ ranges through $\mathcal{E}_0(\mathcal{Q})$. Let $[\xi] \in \mathcal{A}([\sigma])$. Since $\mathcal{A}([\xi]) \subset \mathcal{A}([\sigma])$ and the cardinality of $\mathcal{A}([\sigma|)$ is minimal, we know that $\mathcal{A}([\xi]) = \mathcal{A}([\sigma])$ and, in particular, $[\sigma] \in \mathcal{A}([\xi])$. As $[\xi] \in \mathcal{A}([\sigma])$ was arbitrary, we conclude that for all $\xi, \xi' \in \mathcal{NT}(\mathcal{Q})$ which are contained in $\mathcal{A}([\sigma])$, there is a full numbered splitting sequence connecting ξ to a train track $g\xi'$ for some $g \in \mathrm{Mod}(S)$.

Define

$$\mathcal{E}(\mathcal{Q}) = \mathcal{A}([\sigma]) \subset \mathcal{E}_0(\mathcal{Q}). \tag{5}$$

Let $\theta \in \mathcal{NT}(\mathcal{Q})$ with $[\theta] \in \mathcal{E}(\mathcal{Q})$. By the above discussion, the train track θ can be connected to a numbered train track σ' in the $\mathrm{Mod}(S)$-orbit of σ by a full numbered splitting sequence. Thus the first property in the lemma holds true for $\mathcal{E}(\mathcal{Q})$, and the second is true by the definition of $\mathcal{E}(\mathcal{Q})$. This completes the proof of the lemma. $\square$

Let $k > 0$ be the cardinality of the set $\mathcal{E}(\mathcal{Q}) \subset \mathcal{E}_0(\mathcal{Q})$ as in Lemma 3.4. Number the k elements of $\mathcal{E}(\mathcal{Q})$ in an arbitrary way. Identify each element of $\mathcal{E}(\mathcal{Q})$ with its number. Define $a_{ij} = 1$ if the numbered combinatorial type i can be split with a single full numbered split to the numbered combinatorial type j and define $a_{ij} = 0$ otherwise. The matrix $A = (a_{ij})$ defines a *subshift of finite type.* Its phase space is the set of biinfinite sequences

$$\Omega = \{(x_i) \in \prod_{i=-\infty}^{\infty} \{1, \dots, k\} \mid a_{x_i x_{i+1}} = 1 \text{ for all } i\}.$$

Every biinfinite full numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(\mathcal{Q})$ contained in $\mathcal{E}(\mathcal{Q})$ defines a point in Ω. Vice versa, since the action of $\mathrm{Mod}(S)$ on the set of numbered large train tracks is free, a point $(x_i) \in \Omega$ determines a $\mathrm{Mod}(S)$-orbit of biinfinite full numbered splitting sequences. We say that such a full numbered splitting sequence *realizes* (x_i).

The shift map $\sigma : \Omega \to \Omega, \sigma(x_i) = (x_{i+1})$ acts on Ω. For $n > 0$ write $A^n = (a_{ij}^{(n)})$; the shift σ is *topologically transitive* if for all i, j there is some $n > 0$ such that $a_{ij}^{(n)} > 0$. Define a finite sequence $(x_i)_{0 \leq i \leq n}$ of points $x_i \in \{1, \ldots, k\}$ to be *admissible* if $a_{x_i x_{i+1}} = 1$ for all $i \leq n-1$. Then $a_{ij}^{(n)}$ equals the number of all admissible sequences of length n connecting i to j [Mn87]. The following observation is immediate from the definitions.

Lemma 3.5. *The shift (Ω, σ) is topologically transitive.*

Proof. Let $i, j \in \{1, \ldots, k\}$ be arbitrary. By Lemma 3.4, there is a nontrivial finite full numbered splitting sequence $\{\tau_i\}_{0 \leq i \leq n} \subset \mathcal{NT}(\mathcal{Q})$ connecting a train track τ_0 of numbered combinatorial type i to a train track τ_n of numbered combinatorial type j. This splitting sequence then defines an admissible sequence $(x_i)_{0 \leq i \leq n}$ connecting i to j. $\square$

Remark 3.6. Without loss of generality, we can in fact assume that the shift (Ω, σ) is *topologically mixing*, that is, there exists a number $n > 0$ so that the matrix A^n is positive. Namely, by the discussion on p.55 of [HK95], otherwise there are numbers $\ell, n > 0$ such that $\ell n = k$ and that the following holds true. The elements of $\mathcal{E}(\mathcal{Q})$ are divided into n disjoint sets $C_1, \ldots, C_n$ of ℓ elements each so that for $x_i \in C_j$ we have $a_{x_i x_{i+1}} = 1$ only if $x_{i+1} \in C_{j+1}$ (indices are taken modulo n). Moreover, the restriction of σ^n to C_1 is topologically mixing. However, in this case we can repeat the argument in the proof of Lemma 3.4 with a single full numbered split replaced by a full numbered splitting sequence of length n. This amounts to replacing (Ω, σ) by the topologically mixing subshift (C_1, σ^n) which has all properties stated above.

3.2. **Relation to the Teichmüller flow.** In this subsection we connect the subshift of finite type (Ω, σ) constructed in Subsection 3.1 to the Teichmüller flow on $\mathcal{Q}$. To this end we shall make use of the following simple consequence of a celebrated result of Masur (Theorem 1.1 of [M92]) about the structure of differentials $q \in \mathcal{Q}$ which are recurrent under the Teichmüller flow.

Lemma 3.7. *Let $q \in \mathcal{Q}$ be a point whose forward orbit $\Phi^t q$ $(t \geq 0)$ under the Teichmüller flow returns to a compact set $K \subset \mathcal{Q}$ for arbitrarily large times. Then the vertical measured geodesic lamination of a lift of q to $\tilde{\mathcal{Q}}$ is uniquely ergodic, with support in $\mathcal{LL}(\mathcal{Q})$.*

Proof. Unique ergodicity of the vertical measured geodesic lamination of a forward recurrent differential $q \in \mathcal{Q}$ is Theorem 1.1 of [M92].

To see that the support of the measured geodesic lamination of a lift $\tilde{q} \in \tilde{\mathcal{Q}}$ of q is contained in $\mathcal{LL}(\mathcal{Q})$ assume otherwise. Then q admits at least one vertical saddle connection. This saddle connection has finite length.

By continuity, for every compact set $K \subset \mathcal{Q}$ there exists a number $\kappa(K) > 0$ which bounds from below the minimal length of any saddle connection for a differential $u \in K$. Now the length of a vertical saddle connection is exponentially decreasing under the Teichmüller flow and therefore the orbit of q does not return to K for arbitrarily large times. This completes the proof. $\square$

The union $\cup_{[\tau]\in\mathcal{E}(\mathcal{Q})}\mathcal{Q}(\tau)\cap\tilde{\mathcal{Q}}$ is clearly invariant under the Teichmüller flow and under the action of the mapping class group, and it contains a non-empty open invariant subset of $\tilde{\mathcal{Q}}$. However, this does not imply that it projects to a subset of $\mathcal{Q}$ which contains the support of every Φ^t-invariant Borel probability measure. The next lemma provides the relevant additional information we need,

Lemma 3.8. *For every $q \in \mathcal{Q}$ without vertical saddle connections and every lift $\tilde{q}$ of q to $\tilde{\mathcal{Q}}$ there is some $\tau \in \mathcal{NT}(\mathcal{Q})$ such that $[\tau] \in \mathcal{E}(\mathcal{Q})$ and $\tilde{q} \in \mathcal{Q}(\tau)$.*

Proof. Let $q \in \mathcal{Q}$ and assume that q does not have vertical saddle connections. Let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of q. By Proposition 2.4 and using the notation (2), there is a train track $\eta \in \mathcal{LT}(\mathcal{Q})$ so that $\tilde{q} \in \mathcal{Q}(\eta)$. Recall that $\mathcal{Q}(\eta)\cap\tilde{\mathcal{Q}}$ contains an open subset of $\tilde{\mathcal{Q}}$ which projects to an open subset of $\mathcal{Q}$. Since the set of all points in $\mathcal{Q}$ whose Φ^t-orbits are dense in $\mathcal{Q}$ has full Lebesgue measure and hence is dense, there is a point $z \in \mathcal{Q}$ whose Φ^t-orbit is dense in $\mathcal{Q}$ in forward and backward direction and which admits a lift $\tilde{z} \in \mathcal{Q}(\eta)$.

By the definition of $\mathcal{Q}(\eta)$, the support of the horizontal measured geodesic lamination $\tilde{z}^h$ of $\tilde{z}$ is carried by the dual η^* of η. Furthermore, by Lemma 3.7, applied to the time reversal $t \to \Phi^{-t}$ of the Teichmüller flow, the support of $\tilde{z}^h$ fills S. Thus since the support of the vertical measured geodesic lamination $\tilde{q}^v$ of $\tilde{q}$ is carried by η, the measured geodesic laminations $\tilde{q}^v, \tilde{z}^h$ bind S. Hence there is a quadratic differential $\tilde{u} \in \mathcal{Q}(\eta)$ whose horizontal measured geodesic lamination equals $\tilde{z}^h$ and whose vertical measured geodesic lamination equals $c\tilde{q}^v$ for a number $c > 0$. Proposition 2.4 implies that $\tilde{u} \in \tilde{\mathcal{Q}}$.

Since the orbit of z under the time reversal of the Teichmüller flow is dense in $\mathcal{Q}$, and since $\tilde{u}, \tilde{z}$ have the same horizontal measured laminations, Theorem 2 of [M80] shows that we have $d(\Phi^{-t}\tilde{u}, \Phi^{-t}\tilde{z}) \to 0$ $(t \to \infty)$ for any distance function d on $\tilde{\mathcal{Q}}$ which is induced by a complete $\mathrm{Mod}(S)$-invariant Riemannian metric on $\tilde{\mathcal{Q}}$.

On the other hand, let $\sigma \in \mathcal{NT}(\mathcal{Q})$ be contained in $\mathcal{E}(\mathcal{Q})$. Then $\mathcal{Q}(\sigma)$ contains an open subset of $\tilde{\mathcal{Q}}$. Using once more the fact that the backward orbit of z is dense in $\mathcal{Q}$, there is some $g \in \mathrm{Mod}(S)$ and some $T > 0$ so that $g\Phi^{-T}\tilde{u} \in \mathcal{Q}(\sigma)$ and hence $g\tilde{u} \in \mathcal{Q}(\sigma)$ by invariance. In particular, the vertical measured geodesic lamination $g(c\tilde{q}^v)$ of $g\tilde{u}$ is carried by σ. Equivalently, $\tilde{q}^v$ is carried by $g^{-1}\sigma$.

Now $\tilde{q}$ does not have any vertical saddle connection by assumption and hence the complementary components of the vertical measured geodesic lamination $\tilde{q}^v$ of $\tilde{q}$ are in bijection with the zeros of $\tilde{q}$. Then $\tilde{q}^v$ is minimal, with support in the set $\mathcal{LL}(\mathcal{Q})$ defined in (3). Together with Lemma 5.1 of [H09a], we obtain that there is a unique full numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(\mathcal{Q})$ beginning at $\tau_0 = g^{-1}\sigma$

which consists of train tracks carrying $\tilde{q}^v$. By the second statement in Lemma 3.4, each of the train tracks τ_i is contained in $\mathcal{E}(\mathcal{Q})$.

We claim that for sufficiently large i we have $\tilde{q} \in \mathcal{Q}(\tau_i)$. This then completes the proof of the lemma. To this end note that $\tilde{q}^v$ is carried by both η, τ_0, and the topological type of its support coincides with the topological type of η, τ_0. Thus by Corollary 2.4.3 of [PH92], there exists a train track ν which can be obtained from both η, τ_0 by a splitting and collision sequence (where a collision is a split followed by the removal of the diagonal) and which carries $\tilde{q}^v$. Since the topological type of $\tilde{q}^v$ coincides with the topological type of η and $\tilde{q}^v$ is carried by ν, the topological type of ν coincides with the topological type of η, τ_0 and hence there is no collision in the transformation of τ_0 to ν.

Now by uniqueness of splitting sequences as established in Lemma 5.1 of [H09a] and the fact that the (numbered) splitting sequence (τ_i) is full, there exists some $j > 0$ such that ν is splittable to τ_j and hence η is splittable to τ_j. As $\tilde{q} \in \mathcal{Q}(\eta)$, the horizontal measured geodesic lamination $\tilde{q}^h$ of $\tilde{q}$ is carried by η^* and hence by τ_j^*. But this means that $\tilde{q} \in \mathcal{Q}(\tau_j)$. As $[\tau_j] \in \mathcal{E}(\mathcal{Q})$, this completes the proof of the lemma. □

Remark 3.9. We do not know whether there are components $\mathcal{Q}$ with $\mathcal{E}(\mathcal{Q}) = \mathcal{E}_0(\mathcal{Q})$. It seems likely that such components do not exist as we expect that for $\eta \in \mathcal{E}(\mathcal{Q})$, not all permutations of the numberings of the branches of η are contained in $\mathcal{E}(\mathcal{Q})$.

4. Symbolic dynamics for the Teichmüller flow

In this section we relate the subshift of finite type (Ω, σ) constructed in Section 3 to the Teichmüller flow Φ^t on the component $\mathcal{Q}$ of a stratum. The section is divided into four subsections.

In Section 4.1, we define a σ-invariant Borel subset $\mathcal{U}_+$ of the biinfinite shift space (Ω, σ) called *uniquely ergodic sequences*, and we define a continuous bounded roof function ρ on the set $\mathcal{U}_+$ only depending on the future. The set $\mathcal{U}_+$ will be a superset of the Borel set $\mathcal{U}$ which appears in Theorem 1, but a priori, the set may be empty.

In Section 4.2, we establish that normal sequences in (Ω, σ) are contained in $\mathcal{U}_+$. We then choose $\mathcal{U}$ to be the intersection of $\mathcal{U}_+$ with its mirror image defined by time reversal of the shift. All normal sequences are contained in $\mathcal{U}$, in particular, $\mathcal{U}$ is not empty, and the suspension of $\mathcal{U}$ with respect to the roof function ρ is defined.

In Section 4.3 we construct a map $\mathcal{U} \to \mathcal{Q}$ and establish that is defines a finite-to one semi-conjugacy Ξ of the suspension flow over $\mathcal{U}$ with roof function ρ onto a Φ^t-invariant Borel subset of $\mathcal{Q}$. Finally in Section 4.4 we study the action of Ξ on invariant measures for the suspension flow and complete the proof of Theorem 1. Throughout, we continue to use the assumptions and notations from Section 2 and Section 3.

4.1. **A roof function for the shift space.** Let as before $\mathcal{Q}$ be a component of a stratum. Let $\mathcal{NT}(\mathcal{Q})$ be as in (4) and let $p > 0$ be the number of branches of a train track in $\mathcal{NT}(\mathcal{Q})$. For $\tau \in \mathcal{NT}(\mathcal{Q})$ let as before $\mathcal{V}(\tau)$ be the space of all measured geodesic laminations carried by τ, equipped with the topology as the subcone of $\mathbb{R}^p$ of nonnegative functions on the branches of τ which satisfy the switch condition. For each $\mu \in \mathcal{V}(\tau)$ we denote by

$$\mu(\tau) = \sum_{b \subset \tau} \mu(b) \tag{6}$$

the total mass of μ, that is, the sum of the weights of μ over all branches of τ. Define

$$\mathcal{V}^{\mathcal{P}}(\tau) \subset \mathcal{V}(\tau)$$

to be the subspace of transverse (probability) measures of total mass 1.

Denote by $P\mathcal{V}(\tau)$ the space of all *projective* measured geodesic laminations which are carried by τ. Note that $P\mathcal{V}(\tau)$ is a *compact* subset of the compact space $\mathcal{PML}$ of all projective measured geodesic laminations on S.

Let $(\tau_i)_{0 \leq i} \subset \mathcal{NT}(\mathcal{Q})$ be any full numbered splitting sequence. Then we have $\emptyset \neq P\mathcal{V}(\tau_{i+1}) \subset P\mathcal{V}(\tau_i)$ and hence $\bigcap_i P\mathcal{V}(\tau_i)$ is a non-empty compact subset of $\mathcal{PML}$. If $\bigcap_i P\mathcal{V}(\tau_i)$ consists of a unique point, and the support of this projective measured lamination is of the same topological type as τ_0, then we call (τ_i) *uniquely ergodic.*

Definition 4.1. The sequence $(x_i) \in \Omega$ is called *uniquely ergodic* if some (and hence every) full numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(\mathcal{Q})$ which realizes (x_i) is uniquely ergodic.

If $(\tau_i) \subset \mathcal{NT}(\mathcal{Q})$ realizes a uniquely ergodic sequence $(x_i) \in \Omega$ then for every i, a transverse measure on τ_i defined by a point $\zeta \in \bigcap_i \mathcal{V}(\tau_i)$ is positive on every branch of τ_i. Moreover, by Lemma 5.1 of [H09a], the sequence (τ_i) is uniquely determined by τ_0 and ζ.

Let

$$\mathcal{U}_+ \subset \Omega \tag{7}$$

be the set of all uniquely ergodic sequences. We define a function $\rho : \mathcal{U}_+ \to \mathbb{R}$ as follows. For $(x_i) \in \mathcal{U}_+$ choose a full numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(\mathcal{Q})$ which realizes (x_i). Let $\mu \in \bigcap_{i \geq 0} \mathcal{V}(\tau_i)$ be carried by each of the train tracks τ_i and note that μ is uniquely determined by this requirement up to scaling. Define

$$\rho(x_i) = -\log(\mu(\tau_1)/\mu(\tau_0)). \tag{8}$$

By equivariance under the action of the mapping class group, the number $\rho(x_i) \in (0, \infty)$ only depends on the sequence $(x_i) \in \mathcal{U}_+$. In other words, ρ is a function defined on $\mathcal{U}_+$. We have

Lemma 4.2. *The function ρ maps $\mathcal{U}_+$ to $(0, p \log 2]$.*

Proof. Let $(x_i) \in \mathcal{U}_+$ and choose a full numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(\mathcal{Q})$ which realizes (x_i). Let $\mu \in \mathcal{V}^{\mathcal{P}}(\tau_0) \cap \bigcap_{i\geq 0} \mathcal{V}(\tau_i)$. Thus μ, viewed as a measured geodesic lamination, is carried by each of the train tracks τ_i, and it defines a transverse measure μ on τ_0 of total weight one.

Let e be a large branch of τ_0 and let τ' be the large train track which is obtained from τ_0 by a single split at e and which is splittable to τ_1 (compare [H09a] for details). Let e' be the branch in τ' which is the *diagonal* of the split of τ_0 at e. This means that e' is the small branch in τ' which is the image of e under the natural bijection Λ of the branches of τ_0 onto the branches of τ'. Let μ' be the transverse measure on τ' defined by the measured geodesic lamination μ. There are two branches b, d in τ incident on the two endpoints of e such that

$$\mu(e) = \mu'(e') + \mu'(\Lambda(b)) + \mu'(\Lambda(d)). \tag{9}$$

Moreover, we have $\mu(a) = \mu'(\Lambda(a))$ for every branch $a \neq e$ of τ_0 and hence $\mu(\tau_0) = 1 = \mu'(\tau') + \mu'(\Lambda(b)) + \mu'(\Lambda(d)) \leq 2\mu'(\tau')$ and

$$\mu'(\tau') \in [1/2, 1].$$

Since τ_1 can be obtained from τ_0 by at most p splits, this immediately implies that ρ is nonnegative and bounded from above by $p \log 2$.

On the other hand, by the definition of $\mathcal{U}_+$, the measure μ' is positive on every branch of τ'. Then the equation (9) shows that $\rho > 0$ on $\mathcal{U}_+$. □

In the following lemma, we view $\mathcal{U}_+$ as a subspace of the shift space (Ω, σ) with its standard topology, generated by the clopen cylinder sets $C_\ell(x_i) = \{(y_i) \mid y_i = x_i \text{ for } -\ell \leq i \leq \ell\}$.

Lemma 4.3. *The function $\rho : \mathcal{U}_+ \to \mathbb{R}$ is continuous and only depends on the future.*

Proof. By construction, if $x_i = y_i$ for all $i \geq 0$, then we have $\rho(x_i) = \rho(y_i)$, that is, ρ only depends on the future.

To show continuity of ρ let $(x_i) \in \mathcal{U}_+$. By the definition of the topology on the shift space (Ω, σ) with basis the cylinder sets, it suffices to show that for every $\epsilon > 0$ there is some $j \geq 0$ (depending on (x_i)) such that

$$|\rho(y_i) - \rho(x_i)| \leq \epsilon$$

whenever $(y_i) \in \mathcal{U}_+$ is such that $x_i = y_i$ for $0 \leq i \leq j$.

For this let (τ_i) be a full numbered splitting sequence which realizes (x_i). Then (τ_i) determines a measured geodesic lamination

$$\mu = \mathcal{V}^{\mathcal{P}}(\tau_1) \cap \bigcap_{i\geq 0} \mathcal{V}(\tau_i).$$

By definition, $\rho(x_i) = \log \mu(\tau_0)$ where μ is viewed as a weight function on τ_0 via a carrying map $\tau_1 \to \tau_0$. Note that here we normalize μ at τ_1 for ease of exposition.

Let as before $p > 0$ be the number of branches of a train track in $\mathcal{NT}(\mathcal{Q})$. The set $\mathcal{V}^{\mathcal{P}}(\tau_1)$ of all transverse measures on τ_1 of total mass one can be identified with a compact convex subset of $\mathbb{R}^p$. The natural projection

$$\pi : \mathcal{V}^{\mathcal{P}}(\tau_1) \to \mathcal{PML}$$

is a homeomorphism onto its image with respect to the weak*-topology on $\mathcal{PML}$ [PH92]. Since by equation (9) in the proof of Lemma 4.2, a carrying map $\tau_1 \to \tau_0$ induces a linear and hence continuous map $\mathcal{V}(\tau_1) \to \mathcal{V}(\tau_0)$, there is an open neighborhood $V \subset \mathcal{PML}$ of $\pi(\mu)$ with the following property. Every $\nu \in \mathcal{V}^{\mathcal{P}}(\tau_1)$ with $\pi(\nu) \in V$ defines a transverse measure on τ_0 whose total weight is contained in the interval $(e^{\rho(x_i)-\epsilon}, e^{\rho(x_i)+\epsilon})$.

Now for every $j > 0$, the set $P\mathcal{V}(\tau_j)$ of all projective measured geodesic laminations which are carried by τ_j is a compact subset of $\mathcal{PML}$ containing $\pi(\mu)$, and we have $P\mathcal{V}(\tau_j) \subset P\mathcal{V}(\tau_i)$ for $j \geq i$ and $\bigcap_j P\mathcal{V}(\tau_j) = \pi(\mu)$. As a consequence, there is some $j_0 > 0$ such that $P\mathcal{V}(\tau_{j_0}) \subset V$. By the definition of ρ, this implies that the value of ρ on the intersection with $\mathcal{U}_+$ of the cylinder $\{(y_i) \mid y_j = x_j \text{ for } 0 \leq j \leq j_0\}$ is contained in the interval $(\rho(x_i) - \epsilon, \rho(x_i) + \epsilon)$. This shows the lemma. □

4.2. **Normal sequences are uniquely ergodic.** Our next goal is to obtain a better understanding of the set $\mathcal{U}_+ \subset \Omega$ of uniquely ergodic sequences. It follows from the definitions that $\mathcal{U}_+$ is a Borel subset of Ω.

Call a biinfinite sequence $(x_j) \in \Omega$ *normal* if every finite admissible sequence occurs in (x_j) infinitely often in forward and backward direction. The following is the main result of this subsection.

Proposition 4.4. *A normal sequence $(x_i) \in \Omega$ is uniquely ergodic.*

The proof of Proposition 4.4 relies on some technical concepts and results which will also be important tools in Section 5.

We first define a norm-like quantity which measures the difference between two projective measured geodesic laminations $[\mu], [\nu]$ which are carried by a train track τ and define positive weight functions on τ. Namely, choose representatives $\mu, \nu \in \mathcal{V}(\tau)$ of $[\mu], [\nu]$ and put

$$(10) \qquad \begin{aligned} (\mu \mid \nu)^0_\tau &= \max\{\mu(b)/\nu(b), \nu(b)/\mu(b) \mid b \text{ is a branch of } \tau\} \text{ and} \\ ([\mu] \mid [\nu])_\tau &= \min\{(\mu \mid a\nu)^0_\tau \mid a > 0\}. \end{aligned}$$

Note that $([\mu] \mid [\nu])_\tau$ indeed only depends on the projective classes of μ, ν. Moreover, we have $([\mu] \mid [\nu])_\tau = ([\nu] \mid [\mu])_\tau \geq 1$ for all $[\mu], [\nu]$, with equality if and only if $[\mu] = [\nu]$.

The following simple observation compares the behavior of these norms under splitting operations.

Lemma 4.5. *Let $(\tau_i)_{0\leq i\leq n}\subset\mathcal{NT}(\mathcal{Q})$ be any finite full splitting sequence, let $a_0>0$ and suppose that $\mu,\nu\in\mathcal{V}(\tau_n)$ fulfill $\mu(b)\leq a_0\nu(b)$ for every branch b of τ_n. Then the transverse measures μ_0,ν_0 on τ_0 defined by μ,ν via a carrying map $\tau_n\to\tau_0$ satisfy $\mu_0(e)\leq a_0\nu_0(e)$ for every branch e of τ_0. In particular, we have*

$$([\mu]\mid[\nu])_{\tau_n}\geq([\mu]\mid[\nu])_{\tau_0}.$$

Proof. The lemma follows immediately from the fact that the natural map $\mathcal{V}(\tau_n)\to\mathcal{V}(\tau_0)$ induced by a carrying map $\tau_n\to\tau_0$ is the restriction of a linear map, explicitly given in (9), from the finite dimensional vector space of weight functions on the branches of τ_n to the vector space of weight functions on the branches of τ_0 which preserves positivity. □

Using the assumptions and notations from Sections 2-3 and the above notions, we can now formulate the main technical tool towards the proof of Proposition 4.4.

Lemma 4.6. *Let $\tau_0\in\mathcal{NT}(\mathcal{Q})$ and let $\zeta\in\mathcal{V}(\tau)$ be a uniquely ergodic geodesic lamination with support in $\mathcal{LL}(\mathcal{Q})$. Let $(\tau_i)\subset\mathcal{NT}(\mathcal{Q})$ be the full splitting sequence starting at τ_0 with $\bigcap_i\mathcal{V}(\tau_i)=(0,\infty)\zeta$. Then there is some $n>0$ with the following properties.*

1. *There exists a number $\beta>0$ such that $\mu(b)/\mu(b')\geq\beta$ for every $\mu\in\mathcal{V}(\tau_0)$ which is carried by τ_n and all branches b,b' of τ_0.*
2. *There is a number $\delta>0$ with the following property. Let $\mu,\nu\in\mathcal{V}(\tau_n)$ be positive transverse measures; then*
$$([\mu]\mid[\nu])_{\tau_0}^{-1}\geq([\mu]\mid[\nu])_{\tau_n}^{-1}(1-\delta)+\delta.$$

Proof. Let $\tau_0\in\mathcal{NT}(\mathcal{Q})$ and let $\zeta\in\mathcal{V}(\tau_0)$ be a uniquely ergodic measured geodesic lamination whose support is contained in $\mathcal{LL}(\mathcal{Q})$. Then the ζ-weight of every branch of τ_0 is positive.

Let (τ_i) be the full (numbered) splitting sequence such that for every i, the train track τ_i carries ζ. We first claim that there exists a number $\ell>0$ so that the image of any branch z of τ_ℓ under a carrying map $F:\tau_\ell\to\tau_0$ which maps switches to switches and branches to edge paths on τ_0 is all of τ_0.

To show the claim we use the fact that every half-leaf of ζ is dense in ζ and the unzipping procedure for train tracks introduced on p.134 of [PH92]. This construction can be described as follows. Choose an arbitrary switch v of τ_0. The two small branches a_1,a_2 incident on v are contained in the boundary of some complementary region C of τ_0. There is a corresponding complementary region C_ζ of ζ, and there are two oriented forward asymptotic boundary half-leaves ℓ_1,ℓ_2 of C_ζ which correspond to the two sides of the complementary component C of τ_0 determined by the two small half-branches a_1,a_2 incident on v.

The two boundary half-leaves ℓ_1,ℓ_2 are mapped by a carrying map $\zeta\to\tau_0$ to a train path (an edge path of class C^1) on τ_0, given by a chain of oriented branches $b_1,b_2,\cdots$ where b_1 is the branch incident on v and large at v. Cut τ_0 successively open along the branches b_i as follows. Replace first b_1 by a graph consisting of two edges (two copies of b_1) which are attached at a common vertex, which is the switch of τ_0 on which the forward endpoint of the oriented edge b_1 is incident. Connect the two univalent vertices of this graph, which correspond to the switch v, to the small branches a_i in such a way that the resulting graph is a train track τ' as described on pages 135–137 of [PH92]. The train track τ' is carried by τ_0 and it carries ζ, and for $i = 1, 2$ it contains a branch a_1' which is mapped by a carrying map $\tau' \to \tau_0$ to the union of a_i with b_1. The train track τ' may be not generic, but one can nevertheless repeat this construction with τ' and the branch b_2 etc.

Since the two half-leaves ℓ_1, ℓ_2 are mapped by a carrying map $\zeta \to \tau_0$ onto τ_0, there is a number $\ell > 0$ so that after cutting τ_0 open in this way along the branches $b_1, \dots, b_\ell$, the resulting train track ξ is carried by τ_0 and has the property that its two branches that correspond to a_1, a_2 are mapped by a carrying map $\xi \to \tau_0$ onto τ_0. The train track ξ may not be generic but can be modified with a sequence of shifts to a generic train track. Repeat this procedure successively with all branches of ξ which are not mapped onto τ_0. In finitely many steps, one obtains a train track η which is carried by τ_0 and so that a carrying map $\eta \to \tau_0$ maps every branch of η onto τ_0.

If η' is obtained from η by a split, then η' is carried by τ_0 and every branch of η' is mapped onto τ_0 by a carrying map. By Corollary 2.4.3 of [PH92], the train tracks τ_0, η can be split to the same train track σ which carries ζ. Uniqueness of splitting sequences of train tracks which carry ζ up to ordering of the splits, established in Lemma 5.1 of [H09a], yields that σ is splittable to τ_ℓ for some $\ell > 0$. The train track τ_ℓ has the properties stated in the claim.

By rescaling, assume now that $\zeta \in \mathcal{V}^{\mathcal{P}}(\tau_\ell)$. For each $i > \ell$ let

$$A_i = \mathcal{V}^{\mathcal{P}}(\tau_\ell) \cap \mathcal{V}(\tau_i)$$

be the set of all normalized transverse measures on τ_ℓ defined by measured geodesic laminations which are carried by τ_i. Then $A_{i+1} \subset A_i$ and moreover $\bigcap_i A_i = \{\zeta\}$. Since the ζ-weight of every branch of τ_ℓ is positive, there is a number $\kappa > 0$ so that for sufficiently large i, say for all $i \geq n$, and for every $\nu \in A_n$ we have

$$\min\{\nu(b)/\nu(b') \mid b, b' \text{ are branches of } \tau_\ell\} \geq \kappa. \tag{11}$$

Together with Lemma 4.5, this yields the first part of the lemma.

Let again $F : \tau_\ell \to \tau_0$ be the carrying map which maps switches to switches. To show the second part of the lemma let $n \geq \ell$ and assume that $\mu, \nu \in \mathcal{V}(\tau_n)$ are positive and let

$$([\mu] \mid [\nu])_{\tau_n} = \max\{\mu(b)/\nu(b), \nu(b)/\mu(b) \mid b \text{ is a branch of } \tau_n\} = a_0 > 1.$$

By Lemma 4.5, we have $([\mu] \mid [\nu])_{\tau_\ell} = a_1 \leq a_0$. By the definition of $([\mu] \mid [\nu])_{\tau_\ell}$, up to exchanging μ and ν, there exists a branch e of τ_ℓ with $\nu(e)/\mu(e) = a_1$.

For a branch z of τ_ℓ and a branch u of τ_0 let $\beta(z)(u) \geq 1$ be the number of times the trainpath $F(z)$ passes through u. Let

$$m_1 = \max\{\beta(z)(u) \mid z, u\} \text{ and } m_0 = \min\{\beta(z)(u) \mid z, u\} \geq 1.$$

Denote as before by $p > 2$ the number of branches of τ_0. Then for every branch z of τ_ℓ and u of τ_0 we have

$$\beta(z)(u) \geq \frac{m_0}{m_1 p} \cdot \sum_s \beta(s)(u). \tag{12}$$

By linearity, for any branch u of τ_0 we have

$$\begin{aligned}\nu(u) &= \sum_{s\neq e} \nu(s)\beta(s)(u) + \nu(e)\beta(e)(u) \geq a_1^{-1} \sum_{s\neq e} \mu(s)\beta(s)(u) + a_1\mu(e)\beta(e)(u)\\ &= a_1^{-1}\mu(u) + (a_1 - a_1^{-1})\mu(e)\beta(e)(u).\end{aligned}$$

Since by the estimate (11) it holds $\mu(e) \geq \kappa \max\{\mu(z) \mid z\}$, together with the estimate (12) we conclude that

$$\mu(e)\beta(e)(u) \geq \frac{m_0\kappa}{m_1 p}\mu(u).$$

Now for $\delta = m_0\kappa/m_1p$ we obtain

$$\nu(u) \geq a_1^{-1}\mu(u) + \delta(a_1 - a_1^{-1})\mu(u) = (a_1^{-1} + \delta(a_1 - a_1^{-1}))\mu(u).$$

Since $1 \leq a_1 \leq a_0$, the branch u of τ_0 was arbitrary and we can exchange the roles of μ, ν, this yields the desired estimate. □

Definition 4.7. Call a finite full splitting sequence $(\tau_i)_{0\leq i\leq n}$ *weakly tight* if the train tracks $\tau_n \prec \tau_0$ have properties (1) and (2) stated in Lemma 4.6.

It follows from the definitions and Lemma 4.5 that if $(\tau_i)_{0\leq i\leq n}$ is weakly tight, then the same holds true for the sequence $(\tau_i)_{0\leq i\leq n+1}$ where τ_{n+1} is obtained from τ_n by a full split.

To take full advantage of this concept we isolate a simple lemma which will be used several times in the sequel.

Lemma 4.8. *Let $\beta \in (0, \frac{1}{2})$, let $\delta \in (0, \beta)$ and put $\kappa = 1 - \delta \in (\frac{1}{2}, 1)$. Define recursively a sequence (a_n) by $a_0 = \beta$ and $a_n = (1-\delta)a_{n-1} + \delta$; then $a_n \geq 1 - \kappa^n$ for all $n \geq 0$.*

Proof. We proceed by induction on n. The case $n = 0$ follows from the requirement that $\delta \leq \beta$ and hence $\beta \geq 1 - \kappa$.

By induction, assume that the required inequality holds true for $n-1 \geq 0$. Then

$$a_n = (1-\delta)a_{n-1} + \delta \geq \kappa(1 - \kappa^{n-1}) + (1-\kappa) = 1 - \kappa^n$$

completing the induction step. □

Proof of Proposition 4.4. Choose a weakly tight sequence $(y_i)_{0\leq i\leq n} \subset \mathcal{E}(\mathcal{Q})$ and let $\beta > 0$ and $\delta > 0$ be as in Definition 4.7. Assume without loss of generality that $\delta < \beta^2$. Let $(x_i) \in \Omega$ be a normal sequence; then there are numbers $i_0 < i_1 < \cdots$ so that the following holds true.

(1) $i_{j+1} > i_j + n$ for all $j \geq 0$.
(2) For each j and each $\ell \leq n$, we have $x_{i_j+\ell} = y_\ell$.

Choose a number $j > 0$ and let $\mu, \nu \in \mathcal{V}^{\mathcal{P}}(x_{i_j+n})$. By the first condition in the definition of a weakly tight sequence, the measures on τ_{i_j} defined by μ, ν are positive, and we have $([\mu],[\nu])^{-1}_{\tau_{i_j}} \geq \beta^2$.

By the second property of a weakly tight sequence, for all $\ell < j$ we have

$$([\mu],[\nu])^{-1}_{\tau_{i_\ell}} \geq ([\mu],[\nu])^{-1}_{\tau_{i_{\ell+1}}}(1-\delta)+\delta. \tag{13}$$

Lemma 4.8 now yields that if μ, ν are carried by $\tau_{i_\ell+n}$ then $([\mu] \mid [\mu])^{-1}_{\tau_0} \geq 1-\kappa^\ell$. But this implies that $([\mu] \mid [\nu])_{\tau_0} = 1$ for $[\mu],[\nu] \in \bigcap_{i\geq 0} P\mathcal{V}(\tau_i)$ and hence $[\mu]=[\nu]$. As a consequence, $\bigcap_{i\geq 0} P\mathcal{V}(\tau_i)$ consists of a unique point $[\mu]$. The transverse measure defined by $[\mu]$ on any of the train tracks τ_i is positive on every branch.

To complete the proof we have to verify that the support of $[\mu]$ is of the same topological type as τ_0. This follows from another application of Corollary 2.4.3 of [PH92]. Namely, otherwise there exists a splitting sequence (η_j) starting at $\eta_0 = \tau_0$ (here the transition from η_j to η_{j+1} is a split at a single large branch) which consists of train tracks carrying $[\mu]$ and such that this splitting sequence contains a collision, that is, a split followed by the diagonal of the split. By uniqueness of splitting sequences starting from τ_0 which consist of train tracks carrying $[\mu]$ as established in Lemma 5.1 of [H09a], this violates the fact that the measure $[\mu]$ is positive on every branch of any of the train tracks τ_i. Together this shows that indeed, we have $(x_i) \in \mathcal{U}_+$ as claimed. □

As in Section 2, for $\tau \in \mathcal{NT}(\mathcal{Q})$ let $\mathcal{V}^*(\tau)$ be the set of all measured geodesic laminations carried by τ^* and denote by $P\mathcal{V}^*(\tau)$ the projectivization of $\mathcal{V}^*(\tau)$. If $\tau' \in \mathcal{NT}(\mathcal{Q})$ is obtained from $\tau \in \mathcal{NT}(\mathcal{Q})$ by a single split at a large branch e and if C is the matrix which describes the transformation $\mathcal{V}(\tau') \to \mathcal{V}(\tau)$ then the dual transformation $\mathcal{V}^*(\tau) \to \mathcal{V}^*(\tau')$ is given by the transposed matrix C^t (Section 3.4 of [PH92]). Thus Proposition 4.4 also applies to time reversal and shows the following corollary.

Corollary 4.9. *Let $(x_i) \in \Omega$ be normal and let $(\tau_i) \subset \mathcal{NT}(\mathcal{Q})$ be a full numbered splitting sequence which realizes (x_i); then $\bigcap_{i<0} P\mathcal{V}^*(\tau_i)$ consists of a single uniquely ergodic projective measured geodesic lamination whose support is contained in $\mathcal{LL}(\mathcal{Q})$.*

We call the sequence $(x_i) \in \Omega$ *doubly uniquely ergodic* if (x_i) is uniquely ergodic as defined above and if moreover for one (and hence every) full numbered splitting sequence $(\tau_i) \in \mathcal{NT}(\mathcal{Q})$ which realizes (x_i), the intersection $\bigcap_{i<0} \mathcal{PV}^*(\tau_i)$ consists of a unique projective measured geodesic lamination whose support is contained in $\mathcal{LL}(\mathcal{Q})$. By Lemma 4.4 and Lemma 4.9, every normal sequence is doubly uniquely ergodic and hence the Borel set

$$\mathcal{U} \subset \Omega \tag{14}$$

of all doubly uniquely ergodic sequences $(x_i) \in \Omega$ is dense.

4.3. **Mapping doubly uniquely ergodic sequences into $\mathcal{Q}$.** Let $(x_i) \in \mathcal{U}$ and let (τ_i) be a full numbered splitting sequence which realizes (x_i). Then (τ_i) determines a pair (μ, ν) of measured geodesic laminations by the requirement that $\mu \in \mathcal{V}^{\mathcal{P}}(\tau_0) \cap \bigcap_{i\geq 0} \mathcal{V}(\tau_i)$, that $\nu \in \bigcap_{i\leq 0} \mathcal{V}^*(\tau_i)$ and that $\iota(\mu, \nu) = 1$. Since the supports of both μ, ν are contained in $\mathcal{LL}(\mathcal{Q})$, by equivariance under the action of the mapping class group, this implies that every sequence $(x_i) \in \mathcal{U}$ determines a quadratic differential

$$\Xi(x_i) \in \mathcal{Q} \subset \mathcal{Q}(m_1, \dots, m_\ell; -m). \tag{15}$$

More specifically, this quadratic differential is contained in the subset

$$\mathcal{UQ} \subset \mathcal{Q} \tag{16}$$

of all area one quadratic differentials in $\mathcal{Q}$ whose vertical and horizontal measured geodesic laminations are uniquely ergodic, with support in $\mathcal{LL}(\mathcal{Q})$. Note that $\mathcal{UQ}$ is a Φ^t-invariant Borel subset of $\mathcal{Q}$. Its preimage $\mathcal{U}\tilde{\mathcal{Q}}$ in $\tilde{\mathcal{Q}}(S)$ (or in $\tilde{\mathcal{H}}(S)$) is a Φ^t-invariant Borel subset of $\tilde{\mathcal{Q}}$.

Recall that the roof function ρ is defined on the dense shift invariant Borel set $\mathcal{U} \subset \Omega$, and by Lemma 4.3, it is continuous, positive and bounded from above by $p \log 2$. The *suspension* for the shift σ on the invariant subspace $\mathcal{U}$ with roof function ρ is the space

$$X = \{(x_i) \times [0, \rho(x_i)] \mid (x_i) \in \mathcal{U}\}/\sim$$

where the equivalence relation $\sim$ identifies the point $((x_i), \rho(x_i))$ with the point $(\sigma(x_i), 0)$. Note that $\sim$ is a closed equivalence relation on $\mathcal{U}$ since the function ρ is continuous. There is a natural flow Θ^t on X defined by $\Theta^t(x, s) = (\sigma^j x, \tilde{s})$ (for $t \geq 0$) where $j \geq 0$ is such that $0 \leq \tilde{s} = t + s - \sum_{i=0}^{j-1} \rho(\sigma^i x) < \rho(\sigma^j x)$.

A *semi-conjugacy* of (X, Θ^t) into a flow space (Y, Φ^s) is a surjective continuous map $\Psi : X \to Y$ such that $\Phi^t \Psi(x) = \Psi(\Theta^t x)$ for all $x \in X$ and all $t \in \mathbb{R}$. We call a semi-conjugacy Ψ *finite-to-one* if the number of preimages of any point is finite.

By construction, there is a natural extension of the map Ξ defined in equation (15) to the suspension flow (X, Θ^t), again denoted by Ξ, which commutes with the flows Θ^t and Φ^t. We call such a map a *partial semi-conjugacy.* Its image is contained in the Φ^t-invariant Borel subset $\mathcal{UQ}$.

The goal of this subsection is to establish the following result.

Proposition 4.10. *The map $\Xi : (X, \Theta^t) \to (\mathcal{UQ}, \Phi^t)$ is a finite-to-one semi-conjugacy.*

In particular, the image of the map Ξ equals the Φ^t-invariant subset $\mathcal{UQ} \subset \mathcal{Q}$.

The proof of this proposition uses a technical result which constructs the preimage of a point $q \in \mathcal{UQ}$ under the map Ξ and shows that it is finite. Before we can establish this technical tool we have to invoke some results from [H09b].

The *curve graph* $\mathcal{C}(S)$ of S is the metric graph whose vertices are the essential simple closed curves on S and where two such vertices are connected by an edge of length one if and only if they can be realized disjointly. The mapping class group acts on the curve graph as a group of simplicial isometries. The curve graph is a hyperbolic geometric metric graph [MM99] which can be used for navigation in train tracks and in Teichmüller space in the following way.

Any area one quadratic differential $\tilde{q} \in \tilde{\mathcal{Q}}$ defines a singular flat metric on S of area one. The injectivity radius of this metric is bounded from above by a universal constant. Thus there exists a number $\chi_0 > 0$ (depending on S) so that any such metric admits a simple closed geodesic of $\tilde{q}$-length (that is, length with respect to the flat metric defined by $\tilde{q}$) at most χ_0.

Fix a number $\chi > \chi_0$ and define a map $\Upsilon_{\tilde{\mathcal{Q}}} : \tilde{\mathcal{Q}} \to \mathcal{C}(S)$ by associating to a differential $\tilde{q}$ a simple closed curve of $\tilde{q}$-length at most χ. The following lemma is well known and can explicitly be extracted from [R14]. It implies that the map $\Upsilon_{\tilde{\mathcal{Q}}}$ coarsely does not depend on choices.

Lemma 4.11. *For every $\chi > \chi_0$ there exists a number $R(\chi) > 0$ so that for any $\tilde{q} \in \tilde{\mathcal{Q}}$, the distance in $\mathcal{C}(S)$ between any two simple closed curves of $\tilde{q}$-length at most χ does not exceed $R(\chi)$.*

Proof. As the distance in the curve graph between two simple closed curves a, b on S is bounded from above by the geometric intersection number $\iota(a, b) + 1$ between a, b [MM99], it suffices to show the following. For any $\chi > 0$ there exists a number $E(\chi) > 0$, and for any singular flat metric on S defined by an area one quadratic differential $\tilde{q}$, there exists a simple closed curve c on S so that $\iota(a, c) \leq E(\chi)$ for every simple closed curve a of $\tilde{q}$-length at most χ.

Theorem 3.1 of [R14] shows that for any $\tilde{q}$ there exists a subsurface Y of S bounded by simple closed curves of uniformly bounded *extremal length* so that the $\tilde{q}$-length of any essential arc in Y with endpoints on ∂Y or any simple closed curve in Y is bounded from below by a constant not depending on $\tilde{q}$. Thus the number of intersections with ∂Y of any simple closed curve of $\tilde{q}$-length at most χ is bounded from above by a constant only depending on χ. This shows the lemma. ☐

Remark 4.12. The proof of Lemma 4.11 shows more specifically that for any area one quadratic differential $\tilde{q}$, the distance in the curve graph of any simple closed curve on S of flat length at most χ to a curve on S of small extremal length for the conformal structure on S defined by $\tilde{q}$ is uniformly bounded. Additionally, curves of small extremal length are of small length for the hyperbolic metric defining the same conformal structure (see [R14] for a discussion).

Let $\mathcal{T}(S)$ be the Teichmüller space of all complete finite volume hyperbolic metrics on S, equipped with the *Teichmüller metric* $d_{\mathcal{T}}$. The mapping class group $\mathrm{Mod}(S)$ acts properly discontinuously and isometrically on $(\mathcal{T}(S), d_{\mathcal{T}})$. By the construction on p.251 of [H09b], there exists a coarsely well defined map

$$\Lambda : \mathcal{NT}(\mathcal{Q}) \to \mathcal{T}(S) \tag{17}$$

which is coarsely equivariant under the action of the mapping class group $\mathrm{Mod}(S)$. This means that the map Λ depends on choices, but there exists a number $q > 0$ so that if Λ' is any other choice, then for all $\tau \in \mathcal{NT}(\mathcal{Q})$ and all $g \in \mathrm{Mod}(S)$, it holds $d_{\mathcal{T}}(\Lambda(\tau), \Lambda'(\tau)) \leq q$ and $d_{\mathcal{T}}(\Lambda(g\tau), g(\Lambda(\tau)) \leq q$.

A *vertex cycle* for a large train track $\tau \in \mathcal{NT}(\mathcal{Q})$ is a simple closed curve carried by τ whose counting measure defines an extreme point for the space of all transverse measures on τ. The distance in $\mathcal{C}(S)$ between any two vertex cycles of a train track on S is bounded from above by a universal constant.

Recall the canonical projection $P : \tilde{\mathcal{Q}}(S) \to \mathcal{T}(S)$. The following statement is Lemma 4.3 of [H09b] which will be used as an essential tool for the proof of Proposition 4.10. For its formulation and later use, for $\tau \in \mathcal{NT}(\mathcal{Q})$ define

$$\mathcal{Q}^{\mathcal{P}}(\tau) \tag{18}$$

to be the set of all differentials $\tilde{q} \in \mathcal{Q}(\tau)$ with the property that the vertical measured geodesic lamination of $\tilde{q}$ is contained in $\mathcal{V}^{\mathcal{P}}(\tau)$, that is, it deposits the total mass one on τ.

Lemma 4.13 (Lemma 4.3 of [H09b])**.** *There is a number $\ell > 0$, and for every $\epsilon > 0$ there is a number $m(\epsilon) > 0$ with the following property. Let $\sigma, \tau \in \mathcal{NT}(\mathcal{Q})$ and assume that σ is carried by τ and that the distance in $\mathcal{C}(S)$ between a vertex cycle of σ and a vertex cycle of τ is at least ℓ. Let $\tilde{q} \in Q^{\mathcal{P}}(\tau)$ be such that the vertical measured geodesic lamination $\tilde{q}^v$ of $\tilde{q}$ is carried by σ. If the total weight of the transverse measure on σ defined by $\tilde{q}$ is not smaller than ϵ, then $d_{\mathcal{T}}(\Lambda(\tau), P\tilde{q}) \leq m(\epsilon)$.*

We are now ready to prove the main technical result of this subsection. Recall from Lemma 4.2 that the roof function ρ is bounded from above by $p \log 2$.

Lemma 4.14. *For $\tilde{q} \in \mathcal{U}\tilde{\mathcal{Q}}$ there is a neighborhood V of $\tilde{q}$ in $\tilde{\mathcal{Q}}$, and there are finitely many train tracks $\tau_1, \dots, \tau_n \in \mathcal{NT}(\mathcal{Q})$ (where $n \geq 1$ depends on $\tilde{q}$) with the following property. If $\eta \in \mathcal{NT}(\mathcal{Q})$ is such that $\Phi^t \tilde{z} \in \mathcal{Q}^{\mathcal{P}}(\eta)$ for some $\tilde{z} \in V$ and some $t \in [0, p\log 2]$ then $\eta \in \{\tau_1, \dots, \tau_n\}$.*

Proof. By coarse equivariance under the action of the mapping class group and proper discontinuity of this action on $\mathcal{T}(S)$, for every $R > 0$ the number of all elements $\eta \in \mathcal{NT}(\mathcal{Q})$ so that $d_{\mathcal{T}}(\Lambda(\eta), P\tilde{q}) \leq R$ is finite where $\Lambda : \mathcal{NT}(\mathcal{Q}) \to \mathcal{T}(S)$ is as in equation (17). Thus for the proof of the lemma it suffices to show that for $\tilde{q} \in \mathcal{U}\tilde{\mathcal{Q}}$ there is a neighborhood V of $\tilde{q}$ in $\tilde{\mathcal{Q}}$ and a number $R > 0$ with the following property. If $\tilde{z} \in V$, if $t \in [0, p\log 2]$ and $\eta \in \mathcal{NT}(\mathcal{Q})$ are such that $\Phi^t \tilde{z} \in \mathcal{Q}^{\mathcal{P}}(\eta)$ then $d_{\mathcal{T}}(\Lambda(\eta), P\tilde{q}) \leq R$.

Let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a differential with the property that the vertical measured geodesic lamination $\tilde{q}^v$ of $\tilde{q}$ is uniquely ergodic, with support in $\mathcal{LL}(\mathcal{Q})$.

By [MM99] and Remark 4.12, there exists a number $d > 0$ not depending on γ so that the map $t \to \Upsilon_{\tilde{\mathcal{Q}}}(\Phi^t \tilde{q})$ is an *unparameterized d-quasi-geodesic* in $\mathcal{C}(S)$ (see Theorem 2.3 of [H10b] for an explicit statement). This means that there exists a (perhaps finite) interval $(a, b) \subset \mathbb{R}$ and an increasing homeomorphism $\psi : (a, b) \to \mathbb{R}$ so that the map $t \to \Upsilon_{\tilde{\mathcal{Q}}}(\Phi^t \tilde{q})$ is a d-quasi-geodesic in $\mathcal{C}(S)$. Here we subsume additive and multiplicative control constants in the single number d.

As the measured lamination $\tilde{q}^v$ is uniquely ergodic and fills S, it follows from Theorem 1.1 of [H06] that the unparameterized quasi-geodesic $\Upsilon_{\tilde{\mathcal{Q}}}(\Phi^t \tilde{q})$ $(t \in [0, \infty))$

is of infinite length (and the support of $\tilde{q}^v$ is its boundary point in the boundary of the curve graph, see also the argument on p.124 of [MM99]). Since uniform quasi-geodesics in a hyperbolic space do not backtrack (see Lemma 2.4 of [H10b] for a precise account in the situation at hand), since the Teichmüller flow is continuous, and since nearby differentials define uniformly bi-Lipschitz equivalent flat metrics, there is a neighborhood V of $\tilde{q}$ in $\tilde{\mathcal{Q}}$, and there is a number $T > 0$ such that

$$d(\Upsilon_{\tilde{\mathcal{Q}}}(\Phi^t \tilde{z}), \Upsilon_{\tilde{\mathcal{Q}}}(P\Phi^s \tilde{z})) \geq \ell + 2b + 2p \log 2$$

for all $t \geq T$, all $s \in [0, p\log 2]$ and for all $\tilde{z} \in V$ where $\ell > 0$ is as in Lemma 4.13, $b > 0$ is a constant which will be determined below and where as before, $p \log 2$ is an upper bound for the roof function ρ.

To determine the constant b consider for the moment an arbitrary train track $\eta \in \mathcal{NT}(\mathcal{Q})$ and a differential $\zeta \in \mathcal{Q}^{\mathcal{P}}(\eta)$. Let ζ^v be the vertical measured lamination of ζ. Vertex cycles of η are precisely the extreme points for $\mathcal{V}(\eta)$ and are furthermore represented by an embedded trainpath which passes through every branch at most twice (Lemma 2.2 of [H06]). In particular, their number is bounded from above by a universal constant. Moreover, there exists a number $\kappa > 0$ not depending η or η, and there is a vertex cycle $\chi \subset \eta$ so that $\zeta^v(b) > \kappa$ for every branch $b \in \chi$.

Now the ζ-length of a closed curve α on S is bounded from above by $2(\iota(\zeta^h, \alpha) + \iota(\zeta^v, \alpha))$ where as before, ι stands for intersection number. To estimate these quantities note that it follows from the continuity of the intersection form, the fact that transverse measures on η supported on weighted geodesic multicurves are dense and Corollary 2.3 of [H06] that $\iota(\zeta^v, \chi) \leq 2$. Moreover, by p.197 of [PH92], the horizontal measured foliation ζ^h defines a *tangential measure* on η (up to some equivalence relation) so that $1 = \iota(\zeta^v, \zeta^h) = \sum_b \zeta^v(b)\zeta^v(b)$. As $\zeta^v(b) \geq \kappa$ for all $b \in \chi$, it follows that $\iota(\zeta^v, \chi) \leq \frac{1}{\kappa}$. Together we conclude that the ζ-length of χ is bounded from above by a universal constant $\chi > 0$. Thus by Lemma 4.11, the distance in the curve graph between the vertex cycle χ and $\Upsilon_{\tilde{\mathcal{Q}}}(\tilde{q})$ is bounded from above by a unviersal constant. We let $b > 0$ be such an upper bound.

Let $\tilde{z} \in V$, let $s \in [0, p\log 2]$ and let $\eta \in \mathcal{NT}(\mathcal{Q})$ be such that $\Phi^s \tilde{z} \in \mathcal{Q}^{\mathcal{P}}(\eta)$. We claim that $d_{\mathcal{T}}(\tilde{z}, \Lambda(\eta)) \leq m(\epsilon)$ where $\epsilon > e^{-T - p\log 2}$ and $m(\epsilon) > 0$ is as in Lemma 4.13. This claim completes the proof of the lemma.

To show the claim modify η with a full splitting sequence to a train track $\sigma \in \mathcal{NT}(\mathcal{Q})$ so that σ carries the vertical measured geodesic lamination $\tilde{z}^v$ of $\tilde{z}$ and such that, more precisely, we have $\Phi^t \tilde{z} \in \mathcal{Q}^{\mathcal{P}}(\sigma)$ for some $t \in [T, T + p\log 2]$. Such a train track σ exists by Lemma 4.2. By the choice of the constants $b > 0, T > 0$, the distance in $\mathcal{C}(S)$ between a vertex cycle of η and a vertex cycle of σ is at least ℓ. Thus σ, η satisfy the hypothesis in Lemma 4.13 with a number $\epsilon \geq e^{-T - p\log 2}$. This implies that $d_{\mathcal{T}}(P\tilde{z}, \Lambda(\eta)) \leq m(\epsilon)$ which completes the proof of the lemma. □

Proof of Proposition 4.10. For every $\tau \in \mathcal{NT}(\mathcal{Q})$ and every $\tilde{q} \in \mathcal{Q}(\tau)$ whose vertical measured geodesic lamination $\tilde{q}^v$ has support $\nu \in \mathcal{LL}(\mathcal{Q})$, there is a *unique* full numbered splitting sequence $(\tau_i)_{i>0}$ issuing from $\tau_0 = \tau$ which consists of train tracks carrying $\tilde{q}^v$ (see the paragraph before Section 3 of [H09a] for uniqueness). As by Lemma 4.14, there are only finitely many $\eta \in \mathcal{NT}(\mathcal{Q})$ which can arise as the starting point for such a splitting sequence and controlled weight, it follows that for every $x \in X$ the cardinality of $\Xi^{-1}(\Xi(x))$ is finite. By construction, the map Ξ commutes with the suspension flow Θ^t defined by the roof function ρ and the Teichmüller flow on $\mathcal{Q}$ and hence it is a semi-conjugacy of X onto a Φ^t-invariant subset of $\mathcal{UQ}$ provided it it is continuous.

Continuity of the map Ξ follows from the arguments used in the proof of Lemma 4.3. Namely, as the roof function ρ is continuous, the flow Θ^t is continuous. Since Ξ conjugates the flow Θ^t to the flow Φ^t and both flows are continuous, if now suffices to show the following. Let $(x_i) \in \mathcal{U}$ and let $s \in (0, \rho(x_i))$; then for every neighborhood $V \subset \tilde{\mathcal{Q}}$ of $\tilde{q} = \Xi((x_i), s)$, there exists a number $\epsilon > 0$ and a cylinder set $C_i = \{(y_i) \mid y_i = x_i \text{ for } -m \leq i \leq m\}$ so that $\Xi(C \cap (\mathcal{DU}, (t-\epsilon, t+\epsilon)) \subset V$.

To this end recall from the proof of Lemma 4.3 that $\bigcap_i \Xi(C_i) = \tilde{q}$. Since the cylinder sets C_i are open and closed, for a compact neighborhood K of $\tilde{q}$ contained in the interior of V there exists some i so that $\Xi(C_i, s) \subset K$. Choose $\epsilon > 0$ so that $\cup_{-\epsilon<t<\epsilon}\Phi^t K \subset V$ and note that by equivariance, we have $\Xi(C_i, (s-\epsilon, s+\epsilon)) \subset V$. Since a cylinder set in Ω is open and closed, this completes the proof of continuity.

We are left with showing that $\Xi(X)$ is all of $\mathcal{UQ}$. For this let $q \in \mathcal{UQ}$ and let $\tilde{q}$ be a lift of q to $\tilde{\mathcal{Q}}$. By Lemma 3.8, there is some $\tau \in \mathcal{NT}(\mathcal{Q})$ which is contained in $\mathcal{E}(\mathcal{Q})$ and such that $\tilde{q} \in \mathcal{Q}(\tau)$. If $\tilde{q}^v, \tilde{q}^h$ are the vertical and horizontal measured geodesic laminations of $\tilde{q}$, respectively, then there is a biinfinite full numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(\mathcal{Q})$ issuing from τ such that the intersection $\bigcap_{i>0} P\mathcal{V}(\tau_i)$ consists of a unique point which is just the class of $\tilde{q}^v$, and that the intersection $\bigcap_{i<0} P\mathcal{V}^*(\tau)$ consists of a unique point which is the class of $\tilde{q}^h$. Since the suspension of the orbit of a point in $\mathcal{U}$ under the shift is mapped to a biinfinite flow line of the Teichmüller flow, this implies that $q \in \Xi(X)$. □

Remark 4.15. The Teichmüller flow is not hyperbolic, and the map Ξ is not bounded-to-one.

4.4. Surjectivity of the semi-conjugacy on probability measures.

Since the roof function ρ on $\mathcal{U} \subset \Omega$ is continuous, uniformly bounded and positive, every σ-invariant Borel probability measure ν on Ω which gives full mass to $\mathcal{U}$ induces an invariant measure $\tilde{\nu}$ for the suspension flow (X, Θ^t) of total mass $\int \rho d\nu < \infty$. The measure $\tilde{\nu}$ is defined by $d\tilde{\nu} = d\nu \times dt$ where dt is the Lebesgue measure on the flow lines of the suspension flow.

Since by Lemma 4.14, every $q \in \mathcal{UQ}$ has a neighborhood V in $\mathcal{Q}$ so that the cardinality of the preimage under Ξ of any point $q \in V$ in bounded from above by a constant not depending on q, we can push forward the measure $\tilde{\nu}$ with the semi-conjugacy Ξ and obtain a nontrivial finite Φ^t-invariant Borel measure on $\mathcal{Q}$ which we may normalize to have total mass one. Thus if $\mathcal{M}_\sigma(\mathcal{U})$ denotes the space of all σ-invariant Borel probability measures on Ω which give full measure to $\mathcal{U}$ then Ξ induces a map

$$\Xi_* : \mathcal{M}_\sigma(\mathcal{U}) \to \mathcal{M}_{\mathrm{inv}}(\mathcal{Q})$$

where $\mathcal{M}_{\mathrm{inv}}(\mathcal{Q})$ is the space of Φ^t-invariant Borel probability measures on $\mathcal{Q}$. We equip both spaces with the weak*-topology. We have

Lemma 4.16. *The map Ξ_* is continuous.*

Proof. Since Ω is a compact metrizable space, the space of all Borel probability measures on Ω equipped with the weak*-topology is compact and metrizable. Thus we only have to show that whenever $\mu_i \to \mu$ in $\mathcal{M}_\sigma(\mathcal{U})$ then $\Xi_*(\mu_i) \to \Xi_*(\mu)$.

Now by Lemma 4.2, the function ρ is continuous on $\mathcal{U}$, bounded and positive and hence if $\mu_i \to \mu$ in $\mathcal{M}_\sigma(\mathcal{U})$ then $\int \rho d\mu_i \to \int \rho d\mu > 0$. In particular, we have $\tilde{\mu}_i(X) \to \tilde{\mu}(X)$ where $\tilde{\mu}_i, \tilde{\mu}$ are the finite Borel measures on the suspension space (X, Θ^t) defined by the measures μ_i, μ. Therefore it holds $\Xi_*(\mu_i) \to \Xi_*(\mu)$ if and only if for every continuous function f on $\mathcal{Q}$ with compact support we have $\int f \circ \Xi \, d\tilde{\mu}_i \to \int f \circ \Xi \, d\tilde{\mu}$. However, since Ξ is continuous this is immediate. □

The next result completes the proof of Theorem 1 from the introduction. For its formulation, denote by $\mathcal{M}_\Theta(X)$ the space of Θ-invariant Borel probability measures on X. Denote again by $\Xi_* : \mathcal{M}_\Theta(X) \to \mathcal{M}_{\rm inv}(\mathcal{Q})$ the natural map.

Lemma 4.17. *The map $\Xi_* : \mathcal{M}_\Theta(X) \to \mathcal{M}_{\rm inv}(\mathcal{Q})$ is surjective.*

Proof. It suffices to show that every ergodic Φ^t-invariant Borel probability measure on $\mathcal{Q}$ is contained in the image of Ξ_*.

Thus let ν be an ergodic Φ^t-invariant Borel probability measure on $\mathcal{Q}$. By the Birkhoff ergodic theorem, there is a density point $q \in \mathcal{Q}$ for ν such that the Borel probability measures

$$\nu_T = \frac{1}{T}\int_0^T \delta_{\Phi^t q} dt$$

converge weakly to ν as $T \to \infty$ where δ_x denotes the Dirac mass at x. By Lemma 3.7 and the Poincaré recurrence theorem, we have $q \in \mathcal{UQ}$. Hence by Corollary 4.10, up to possibly replacing q by $\Phi^t q$ for some $t \in \mathbb{R}$ there is some $(x_i) \in \mathcal{U}$ with $\Xi(x_i) = q$.

By Lemma 4.14, there is a neighborhood V of q in $\mathcal{Q}$ such that the preimage of $V \cap \mathcal{UQ}$ under the map Ξ is a *finite* union of Borel sets $W_j \subset X$ $(j = 1, \dots, n)$ with the property that the restriction of Ξ to each of the sets W_j is injective. Namely, note first that as $q \in \mathcal{UQ}$, a preimage $\tilde{q} \in \tilde{\mathcal{Q}}$ of q can not be a fixed point for a non-trivial element of Mod(S). Thus there exists a neighborhood $\tilde{V}$ of $\tilde{q}$ which is mapped homeomorphically into $\mathcal{Q}$. By making $\tilde{V}$ smaller if necessary, we may assume that $\tilde{V}$ has the properties stated in Lemma 4.14. Then the projection V of $\tilde{V}$ to $\mathcal{Q}$ has the finiteness property we are looking for.

As $\nu_T|V \to \nu|V$ weakly as $T \to \infty$ and as the map Ξ is equivariant with respect to the suspension flow and the Teichmüller flow, we conclude that the restriction to $\cup_{j=1}^n W_j$ of the Borel probability measures

$$\tilde{\nu}_T = \frac{1}{T}\int_0^T \delta_{\Theta^t(x_i)} dt$$

converge weakly to a measure on $\cup_i W_i$ which projects to the measure ν on V.

Since q was an arbitrary density point for ν, we deduce that the measures $\tilde{\nu}_T$ converge weakly to a Θ^t-invariant Borel probability measure on X whose image under the map Ξ_* equals ν. To be more precise, choose a countable partition $\mathcal{V} =$

$\cup_j V_j$ of a measurable subset of $\mathcal{UQ}$ of full ν-mass so that each V_j has the properties stated in the second paragraph of this proof. These sets define a countable collection of sets W_j^ℓ so that for each j, the collection $\{W_j^\ell \mid \ell\}$ is finite and its union equals the preimage of V_j under Ξ. By the discussion in the previous paragraph, the restriction of the probability measures $\tilde\nu_T$ to $\cup_{j,\ell} W_j^\ell$ converges weakly to a measure $\tilde\nu$ which projects to ν. But then $\tilde\nu$ is a probability measure and hence a weak limit of the measures $\tilde\nu_T$. The measure $\tilde\nu$ is invariant under the flow Θ^t and is mapped by Ξ to ν. This completes the proof that Ξ_* is surjective. □

Consider again the shift space (Ω,σ). Let $f:\Omega\to\mathbb{R}$ be a Hölder continuous function. Then f defines an *equilibrium state* μ_f which is an ergodic mixing σ-invariant Borel probability measure on Ω. It is defined to be the unique invariant probability measure which maximizes the quantity $h_\mu + \int f d\mu$ where h_μ denotes the entropy of the invariant measure μ. The measure μ_f gives full mass to normal sequences and hence it gives full support to the domain of the map Ξ. As a consequence, the following holds true.

Theorem 4.18. *Any Gibbs equilibrium state on Ω induces via the map Ξ_* a Φ^t-invariant mixing Borel probability measure on $\mathcal{Q}$.*

5. The measure of maximal entropy

In this section we use the subshift of finite type constructed in Sections 2-4 to show that for every component $\mathcal{Q}$ of a stratum, the Φ^t-invariant probability measure λ in the Lebesgue measure class is the unique measure of maximal entropy. For strata of abelian differentials, this was earlier shown by Bufetov and Gurevich [BG11].

The strategy is as follows. Let $q\in\mathcal{Q}$ be any birecurrent point which is contained in its own α-and ω-limit set for the flow Φ^t. By the Poincaré recurrence theorem, for every Φ^t-invariant Borel probability measure μ the set of such points is of full μ-mass. Starting from the symbolic system constructed in Section 3, we construct a topological Markov shift on a countable set $\mathcal{S}$ of symbols, given by a transition matrix $A=(a_{ij})_{\mathcal{S}\times\mathcal{S}}$. The phase space of this shift is the space

$$\Sigma = \{(y_i)\in\mathcal{S}^{\mathbb{Z}} \mid a_{y_i y_{i+1}} = 1 \text{ for all } i\}.$$

We find a positive roof function $\varphi:\Sigma\to(0,\infty)$ of bounded variation and only depending on the future such that the suspension of the shift $T:\Sigma\to\Sigma$ with roof function φ admits a *bounded-to-one* semi-conjugacy into $(\mathcal{Q},\Phi^t)$. Its image $\mathcal{D}$ is Φ^t-invariant and contains all points $z\in\mathcal{Q}$ which contain the fixed quadratic differential q in their α-and ω-limit set. Since the Lebesgue measure λ on $\mathcal{Q}$ has full support and is ergodic under the Teichmüller flow, λ- almost every orbit for Φ^t is dense in $\mathcal{Q}$. Thus the set $\mathcal{D}$ is of full Lebesgue measure.

We use this coding and a result of Sarig [S99] to show that the supremum of the entropies of all Φ^t-invariant Borel probability measures on $\mathcal{Q}$ which give full mass to $\mathcal{D}$ is the supremum of the entropies of all such measures which are supported in some compact invariant subset of $\mathcal{Q}$. That the supremum of the entropies of *all* invariant probability measures on $\mathcal{Q}$ supported in a compact subset of $\mathcal{Q}$ equals the entropy h of the Lebesgue measure λ was established in [H10a]. Thus λ is a measure of maximal entropy for the Teichmüller flow on $\mathcal{Q}$. We then apply a result of Buzzi and Sarig [BS03] to conclude that there exists at most one such measure. Together this shows Theorem 2 from the introduction. The implementation of this strategy is carried out in three subsections.

5.1. **A shift with countably many symbols.** Let $q \in \mathcal{Q}$ be any point which is contained in both the α- and ω-limit set of its orbit under the flow Φ^t. The goal of this subsection is to construct a shift (Σ, T) with countably many symbols, a suspension over Σ and a partial semi-conjugacy of this suspension into $(\mathcal{Q}, \Phi^t)$ so that the image of this map is the subset of $\mathcal{UQ}$ of differentials which contain q in their α- and ω limit set.

Let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of q. By Lemma 3.7, the vertical and horizontal measured geodesic lamination of $\tilde{q}$ is strongly uniquely ergodic, with support in $\mathcal{LL}(\mathcal{Q})$. Let $p > 0$ be as in Lemma 4.2. By Lemma 3.4 and Lemma 4.14, there is a number $\ell \geq 1$, and there are ℓ large numbered train tracks $\tau_1, \dots, \tau_\ell \in \mathcal{E}(\mathcal{Q})$ such that $\Phi^t \tilde{q} \in \mathcal{Q}^{\mathcal{P}}(\eta)$ for some $t \in [0, p \log 2]$ and some $\eta \in \mathcal{NT}(\mathcal{Q})$ which is contained in $\mathcal{E}(\mathcal{Q})$ if and only if $\eta \in \{\tau_1, \dots, \tau_\ell\}$ (see (18) for notations).

By Lemma 4.6, there is a number $n > 0$ such that the following holds true. Let $i \leq \ell$ and let $(\sigma_j^i)_{0 \leq j \leq n}$ be a full numbered splitting sequence of length n issuing from $\sigma_0^i = \tau_i$ with the property that σ_n^i carries the support ζ of the vertical measured geodesic lamination of $\tilde{q}$. Then the sequence $(\sigma_j^i)_{0 \leq j \leq n}$ is weakly tight as defined in Definition 4.7.

Recall the definition of the transition matrix for the subshift of finite type (Ω, σ) which defines admissible sequences. Define $\mathcal{S}$ to be the set of all finite admissible sequences $(x_i)_{0 \leq i \leq s}$ with the following additional properties.

(1) $s \geq 2n$ and the sequences $(x_j)_{0 \leq j \leq n}$ and $(x_j)_{s-n \leq j \leq s}$ are realized by one of the full splitting sequences $(\sigma_j^i)_{0 \leq j \leq n}$ $(i \leq n)$.
(2) There is no number $t \in [n, s-n)$ such that the sequence $(x_j)_{t \leq j \leq t+n}$ is realized by one of the full splitting sequences $(\sigma_j^i)_{0 \leq j \leq n}$.

Note that $\mathcal{S}$ is a countable set.

Define a transition matrix $A = (a_{ij})_{\mathcal{S} \times \mathcal{S}}$ by requiring that $a_{ij} = 1$ if and only if the sequence $(x_\ell)_{0 \leq \ell \leq s}$ representing the symbol i and the sequence $(y_t)_{0 \leq t \leq u}$ representing the symbol j satisfy $y_t = x_{s-n+t}$ for every $t \in \{0, \dots, n\}$. By construction and the properties of the set $\mathcal{E}(\mathcal{Q})$ established in Lemma 3.4,

(19) there are $i_1, \dots, i_N \in \mathcal{S}$ such that for every $u \in \mathcal{S}$, $\exists j, v$ with $a_{i_j u} a_{u i_v} = 1$.

In other words, the transition matrix has the *big images and preimages (BIP) property* as defined in [S03].

Let Σ be the set of all biinfinite sequences $(y_i) \subset \mathcal{S}^{\mathbb{Z}}$ with $a_{y_i y_{i+1}} = 1$ for all i, equipped with the (biinfinite) shift $T : \Sigma \to \Sigma$. There is a natural continuous injective map

$$G : \Sigma \to \Omega$$

whose image contains the set of all normal sequences. Here Ω is as in Lemma 3.5. This map is defined by associating to a sequence $(y_i) \in \Sigma$ the sequence $(x_j) \in \Omega$ obtained by viewing a symbol y_i as a finite admissible word and noticing that the words represented by y_i and y_{i+1} overlap in the sense that the subword of y_i consisting of the last n letters coincides with the subword of y_{i+1} consisting of the first n letters. Thus we can combine these two words to an admissible word obtained by erasing the last n letters of y_i and concatenating the resulting word with the word y_{i+1}.

Lemma 5.1. *It holds*

$$G(\Sigma) \subset \mathcal{U}.$$

Proof. Let $(x_i) \in \Omega$ be a sequence in the image of a sequence in Σ under the map G. Then there are numbers $i_1 < i_2 < \cdots$ so that for each j, there exists some $s_j \in \mathcal{S}$ so that $s_j = (x_\ell)_{i_j - n \leq \ell \leq i_{j+1}}$.

Let $(\tau_i) \subset \mathcal{NT}(\mathcal{Q})$ be a full splitting sequence which realizes (x_i). Let $[\mu], [\nu] \in P\mathcal{V}(\tau_0) \cap \bigcap_j P\mathcal{V}(\tau_j)$. We have to show that $[\mu] = [\nu]$ and that the support of $[\mu]$ is contained in $\mathcal{LL}(\mathcal{Q})$. By the first requirement in the definition of weak tightness, applied to the finite admissible sequence $(x_\ell)_{i_j \leq \ell \leq i_{j+1}}$, the measures $[\mu], [\nu]$ are positive on each τ_n, and there exists a number $\beta > 0$ not depending on j so that

$$([\mu] \mid [\nu])^{-1}_{\tau_{i_j}} \geq \beta. \tag{20}$$

The second property in the definition of a weakly tight sequence shows that there exists a constant $\delta > 0$ not depending on j so that

$$([\mu] \mid [\nu])^{-1}_{\tau_{i_j}} \geq ([\mu] \mid [\nu])^{-1}_{\tau_{i_{j+1}}}(1-\delta) + \delta \tag{21}$$

for all j. An application of Lemma 4.5 and Lemma 4.8 and letting j tend to infinity yields $([\mu] \mid [\nu])_{\tau_0} = 1$ and hence $[\mu] = [\nu]$. The fact that the support of $[\mu]$ is contained in $\mathcal{LL}(\mathcal{Q})$ is a consequence of positivity as explained at the end of the proof of Proposition 4.4.

The same argument is also valid by reversing the time, equivalently for $\cap \mathcal{V}^*(\tau_i)$. This shows the lemma. ☐

By the BIP-property (19) and the discussion at the end of Section 3, we may assume that the topological Markov chain (Σ, T) is topologically mixing.

5.2. **A roof function of bounded variation.** Recall that the roof function ρ is defined on $\mathcal{U} \subset \Omega$. By Lemma 5.1, we then can define a roof function φ on Σ by associating to an infinite sequence $(y_i) \in \Sigma$ with $y_0 = (x_i)_{0 \leq i \leq s}$ the value

$$\varphi(y_i) = \sum_{i=0}^{s-n-1} \rho(\sigma^i(G(y_i))).$$

Note that by the definition of the roof function ρ, the value of φ on (y_i) is just the logarithm of the total mass that the transverse measure $\mu \in \mathcal{V}^{\mathcal{P}}(x_{s-n}) \cap \bigcap_{\ell \geq 0} \mathcal{V}(x_\ell)$ deposits on x_0. By the first requirement in the definition of a weakly tight sequence, if $p > 0$ is as before the number of branches of a train track in $\mathcal{NT}(\mathcal{Q})$, then this mass is at least p. In other words, the function φ is bounded from below by a positive constant $\log p > 0$, is unbounded and only depends on the future.

For $m \geq 1$ define the m-th variation of φ by

$$\mathrm{var}_m(\varphi) = \sup\{\varphi(y) - \varphi(z) \mid y_i = z_i \text{ for } i = 0, \dots, m-1\}.$$

The following is the main technical tool towards the proof of Theorem 2.

Lemma 5.2. *There are numbers $\theta \in (0,1)$ and $L > 0$ such that $\mathrm{var}_m(\varphi) \leq L\theta^m$ for all $m \geq 1$. In particular,*

$$\sum_{m \geq 1} \mathrm{var}_m(\varphi) < \infty.$$

Proof. Let $m \geq 1$ and let $(y_i), (z_i) \in \Sigma$ be such that $y_i = z_i$ for $i = 0, \dots, m$. By definition, there is a finite full numbered splitting sequence $(\tau_i)_{0 \leq i \leq u} \subset \mathcal{E}(\mathcal{Q})$, there are numbers $n \leq \ell_1 - n < \ell_1 \leq \cdots < \ell_m = u$, and there are two uniquely ergodic measured geodesic laminations $\mu, \nu \in \mathcal{V}^{\mathcal{P}}(\tau_{\ell_1 - n})$ with support in $\mathcal{LL}(\mathcal{Q})$ such that the following holds.

(1) The measured geodesic lamination μ, ν projects to the vertical measured geodesic lamination of $\Xi G(y_i), \Xi G(z_i)$, respectively (where Ξ is as in Section 4).
(2) μ, ν are carried by τ_u.
(3) The sequence $(\tau_i)_{0 \leq i \leq n}$ and each of the sequences $(\tau_i)_{\ell_j - n \leq i \leq \ell_j}$ $(j \leq m)$ is weakly tight.
(4) $\varphi(y_i) = \sum_{j=0}^{\ell_1 - n - 1} \rho(\sigma^j G(y_i))$ and $\varphi(z_i) = \sum_{j=0}^{\ell_1 - n - 1} \rho(\sigma^j G(z_i))$.

Let $\beta \in (0, 1/2)$ and $\delta \in (0, \beta^2)$ be the control constants for the weakly tightness property of the finite admissible sequences which were used for the construction of the alphabet $\mathcal{S}$. We claim that for $\kappa = 1 - \delta \in (1/2, 1)$ we have

$$1 - \kappa^m \geq ([\mu] \mid [\nu])^{-1}_{\tau_{\ell_1 - n}}. \tag{22}$$

For this recall from the definition of a weakly tight sequence that for all j we have $\mu(b)/\mu(b') \geq \beta$ for all branches b, b' of $\tau_{\ell_j - n}$, and similarly for ν.

Let b_0 be any branch of $\tau_{\ell_j - n}$ and let $\hat{\nu}$ be the multiple of ν so that $\hat{\nu}(b_0) = \mu(b_0)$. Then for all branches b of $\tau_{\ell_j - n}$ we have

$$\mu(b)/\hat{\nu}(b) \geq \beta^2 \mu(b_0)/\hat{\nu}(b_0) = \beta^2$$

and similarly $\hat{\nu}(b)/\mu(b) \geq \beta^2$. This implies that indeed, $([\mu] \mid [\nu])^{-1}_{\tau_{\ell_j - n}} \geq \beta^2 \geq 1 - (1 - \delta)$ for all j. The claim now follows from Lemma 4.8, Lemma 4.5 and Lemma 4.6.

Let for the moment $\tau \in \mathcal{NT}(\mathcal{Q})$ be arbitrary and let $\zeta, \xi \in \mathcal{V}^{\mathcal{P}}(\tau)$ be positive. Assume that $([\zeta] \mid [\xi])_\tau = a \geq 1$. Let $\hat{\xi} = c\xi \in \mathcal{V}(\tau)$ be such that $\max\{\zeta(b)/\hat{\xi}(b), \hat{\xi}(b)/\zeta(b) \mid b\} = a$. Then we have

$$a^{-1} \leq \hat{\xi}(\tau) \leq a$$

and hence

$$\max\{\zeta(b)/\xi(b),\xi(b)/\zeta(b)\mid b\}\leq a^2.$$

An application of this simple estimate to normalized positive transverse measures $\mu,\nu\in\mathcal{V}^{\mathcal{P}}(\tau_{\ell_1-n})$ on τ_{ℓ_1-n} with $([\mu]\mid[\nu])_{\tau_{\ell_1-n}}=a\geq 1$ together with Lemma 4.5 yields $\mu(\tau_0)/\nu(\tau_0)\in[a^2,a^{-2}]$ and therefore if $\mu_0,\nu_0\in\mathcal{V}^{\mathcal{P}}(\tau_0)$ are the normalizations of μ,ν, then

$$|\varphi(\mu_0)-\varphi(\nu_0)|=|\log\mu(\tau_0)-\log\nu(\tau_0)|\leq -2\log a.$$

As a consequence, the estimate (22) implies that

$$|\varphi(y_i)-\varphi(z_i)|\leq -2\log(1-\kappa^m)\text{ if } y_i=z_i \text{ for } 0\leq i\leq m.$$

As $\frac{-\log(1-t)}{t}\to 1$ $(t\to 0)$, from this the lemma follows. ☐

By Lemma 5.1, the function φ is defined on the entire space Σ. This is equivalent to stating that the image of Σ under the map G is contained in the set $\mathcal{U}$ on which the roof function ρ is defined. In particular, we can construct the suspension (Y,Ψ^t) over Σ with roof function φ.

Lemma 5.3. *There is a bounded-to-one partial semiconjugacy $\Upsilon:(Y,\Psi^t)\to(\mathcal{Q},\Phi^t)$. Its image contains the set of all points z whose α- and ω limit set contains q.*

Proof. By construction of the roof function φ and Lemma 5.1, there is an obvious partial semi-conjugacy $(Y,\Psi^t)\to(X,\Theta^t)$ whose image is contained in the domain of the semi-conjugacy $\Xi:(X,\Theta^t)\to(\mathcal{UQ},\Phi^t)$ defined in Section 3. The composition

$$\Upsilon:(Y,\Psi^t)\to(\mathcal{Q},\Phi^t)$$

of these maps is a partial semi-conjugacy. By construction, its image is the subset $\mathcal{D}$ of $\mathcal{UQ}$ consisting of all points whose orbit contains q in its α- and ω limit set.

We are left with showing that Υ is bounded-to-one. Now note that by the construction of the alphabet $\mathcal{S}$, if $(y_i)\in\Sigma$ then the image of (y_i) under the map $\Xi\circ G:\Sigma\to\mathcal{Q}$ is contained in a fixed small neighborhood V of the differential q used in the construction with the additional property that the cardinality of the preimage of V in Ω is bounded from above by a fixed constant $k>0$. But this yields that the map Υ is a most k-to-one. ☐

5.3. **Entropy computation.** As before, let h be the entropy of the Φ^t-invariant Lebesgue measure on $\mathcal{Q}$. Let $\mathcal{M}_T(\Sigma)$ be the space of all T-invariant Borel probability measures on Σ. For $\mu\in\mathcal{M}_T(\Sigma)$ let

$$\mathrm{pr}_\mu(-h\varphi)=h_\mu-h\int\varphi d\mu$$

where h_μ is the entropy of μ. By [S99], under the assumptions at hand, the *Gurevich pressure* of the function $-h\varphi$ is given by

(23) $$\mathrm{pr}_G(-h\varphi)=\sup\{\mathrm{pr}_\mu(-h\varphi)\mid\mu\in\mathcal{M}_T(\Sigma),\mathrm{pr}_\mu(-h\varphi)\text{ is well-defined}\}.$$

The following observation relies on the results of Sarig [S99] and on [H10a].

Lemma 5.4. $\mathrm{pr}_G(-h\varphi) \leq 0$.

Proof. By Theorem 3 of [H10a], the entropy h of the invariant Lebesgue measure λ on $\mathcal{Q}$ equals the supremum of the topological entropies of all compact invariant subsets of $\mathcal{Q}$.

On the other hand, by Theorem 2 of [S99], $\mathrm{pr}_G(-h\varphi)$ equals the supremum of the quantity defined in (23) but restricted to invariant measures μ supported in compact subsets of Σ.

By Abramov's formula, if $A \subset \Sigma$ is any compact invariant set and if μ is a T-invariant Borel probability measure supported in A, then the entropy of the induced invariant measure for the suspension flow (Y, Ψ^t) equals

$$h_\mu / \int \varphi d\mu.$$

As a consequence, the Gurevich pressure of $-h\varphi$ is nonpositive if the entropy of every Ψ^t-invariant Borel probability measure on Y which is supported in a compact set does not exceed h.

The partial semi-conjugacy $\Upsilon : (Y, \Psi^t) \to (\mathcal{Q}, \Phi^t)$ is bounded-to-one, and it maps the suspension of a T-invariant compact set $A \subset \Sigma$ to a compact Φ^t-invariant subset of $\mathcal{Q}$. This implies that the entropy of a Ψ^t-invariant Borel probability measure on Y supported in a compact set is bounded from above by the supremum of the topological entropies of the restriction of the Teichmüller flow to compact invariant subsets of $\mathcal{Q}$. This quantity equals h by the first paragraph of this proof. The lemma follows. ☐

Proof of Theorem 2. Let μ be any Φ^t-invariant ergodic Borel probability measure on $\mathcal{Q}$ and let $q \in \mathcal{Q}$ be a density point for μ which contains itself in its α- and ω-limit set. Use q to construct the countable two-sided Markov shift (Σ, T) with roof function φ and suspension flow (Y, Ψ^t).

By Theorem 1, there exists a Θ^t-invariant Borel probability measure $\tilde{\mu}$ on X whose image under the map Ξ_* equals μ. Since for μ-almost every $z \in \mathcal{Q}$, the orbit of Φ^t through z contains q in its α and ω limit set, Lemma 5.3 shows that $\tilde{\mu}$ gives full mass to $\Upsilon(Y)$. In particular, as Υ is bounded to one, there exists a Ψ^t-invariant Borel probability measure $\hat{\mu}$ on Y which is mapped by the partial semi-conjugacy Υ to μ. By disintegration, the measure $\hat{\mu}$ induces a T-invariant measure on (Σ, T).

Let Σ^+ be the set of all one-sided infinite admissible sequences $(x_i) \subset \mathcal{S}^{\mathbb{N}}$ with $a_{x_i x_{i+1}} = 1$ for all i, equipped with the one-sided shift $T_+ : \Sigma^+ \to \Sigma^+$. Since the roof function φ only depends on the future, it defines a function on Σ^+ which will be denoted again by φ. Since the roof function φ is bounded from below by a positive constant, up to normalization, the measure $\hat{\mu}$ descends to a T^+-invariant Borel probability measure on Σ^+.

Theorem 1.1 of [BS03] states that given a topologically transitive countable Markov shift (Σ, T) and a function $\varphi : \Sigma \to [0, \infty)$ so that the Gurevich pressure of $-\varphi$ is finite and of finite variation, there exists at most one invariant probability measure μ which maximes the quantity $h_\mu + \int \varphi d\mu$.

By Lemma 5.2 and Lemma 5.4, the function $-h\varphi$ on Σ^+ satisfies all the assumptions in this result. In particular, since $\mathrm{pr}_G(-h\varphi) \leq 0$, the entropy h_μ of the invariant Borel probability measure μ does not exceed h. As a consequence, the Lebesgue measure λ on $\mathcal{Q}$ is a measure of maximal entropy, and by Theorem 1.1 of [BS03], it is unique with this property. In other words, if $h_\mu = h$ then $\mu = \lambda$. Since μ was an arbitrary Φ^t-invariant ergodic Borel probability measure on $\mathcal{Q}$, this completes the proof of the Theorem. □

Any finite admissible sequence $(x_0, \dots, x_{n-1}) \subset \mathcal{S}$ defines the cylinder

$$[x_0, \dots, x_{n-1}] = \{(y_i) \in \Sigma^+ \mid y_i = x_i \text{ for } 0 \leq i \leq n-1\}.$$

A *Gibbs measure* for the function $-h\varphi$ on Σ^+ is a Borel probability measure ν with the following property. There is a number $c > 0$ so that for every cylinder set $[x_0, \dots, x_\ell]$ and every $z \in [x_0, \dots, x_\ell]$ we have

$$\nu[x_0, \dots, x_\ell] \in [c^{-1} e^{-h \sum_{0 \leq i \leq \ell-1} \varphi(T^i z)}, c e^{-h \sum_{0 \leq i \leq \ell-1} \varphi(T^i z)}].$$

As a consequence of [S03] and of the above discussion, we conclude

Corollary 5.5. *The Lebesgue measure is a Gibbs measure on Σ^+.*

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