# SYMBOLIC DYNAMICS FOR THE TEICHMÜLLER FLOW 

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#### Abstract

Let $\mathcal{Q}$ be a stratum of abelian or quadratic differentials on an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3 g-3+m \geq 2$. We construct a subshift of finite type $(\Omega, \sigma)$ and Borel suspension of $(\Omega, \sigma)$ which admits a finite-to-one semi-conjugacy into the Teichmüller flow $\Phi^{t}$ on $\mathcal{Q}$. This is used to show that the $\Phi^{t}$-invariant Lebesgue measure $\lambda$ on $\mathcal{Q}$ is the unique measure of maximal entropy.


## 1. Introduction

A surface $S$ of finite type is a closed oriented surface of genus $g \geq 0$ with $m \geq 0$ marked points, so-called punctures. We assume that $3 g-3+m \geq 2$, that is, $S$ is not a sphere with at most 4 punctures or a torus with at most 1 puncture. We then call the surface $S$ nonexceptional. The Euler characteristic of $S$ is negative.

The Teichmüller space $\mathcal{T}(S)$ of $S$ is the quotient of the space of all complete finite area hyperbolic metrics on the complement of the punctures in $S$ under the action of the group of diffeomorphisms of $S$ which are isotopic to the identity. The sphere bundle

$$
\tilde{\mathcal{Q}}(S) \rightarrow \mathcal{T}(S)
$$

of all holomorphic quadratic differentials of area one can naturally be identified with the unit cotangent bundle for the Teichmüller metric. If the surface $S$ has punctures, that is, if $m>0$, then we define a holomorphic quadratic differential on $S$ to be a meromorphic quadratic differential on the closed Riemann surface obtained from $S$ by filling in the punctures, with a simple pole at each of the punctures and no other poles.

The mapping class group $\operatorname{Mod}(S)$ of all isotopy classes of orientation preserving diffeomorphisms of $S$ naturally acts on $\tilde{\mathcal{Q}}(S)$. The quotient

$$
\mathcal{Q}(S)=\tilde{\mathcal{Q}}(S) / \operatorname{Mod}(S)
$$

is the moduli space of area one quadratic differentials. It can be partitioned into so-called strata. Namely, let $1 \leq m_{1} \leq \cdots \leq m_{\ell}(\ell \geq 1)$ be a sequence of positive integers with

$$
\sum_{i} m_{i}=4 g-4+m
$$

[^0]The stratum $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ defined by the $\ell$-tuple $\left(m_{1}, \ldots, m_{\ell}\right)$ is the moduli space of pairs $(C, \varphi)$ where $C$ is a closed Riemann surface of genus $g$ and where $\varphi$ is an area one meromorphic quadratic differential on $C$ with $\ell$ zeros of order $m_{i}$ and $m$ simple poles and which is not the square of a holomorphic one-form.

A stratum $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ is a real hypersurface in a complex orbifold of complex dimension

$$
h=2 g+\ell+m-2 .
$$

Strata need not be connected, however they have at most two connected components [L08]. The closure in $\mathcal{Q}(S)$ of a component of a stratum $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ is a union of components of strata $\mathcal{Q}\left(n_{1}, \ldots, n_{s} ;-m^{\prime}\right)$ where $s \leq \ell, m^{\prime} \leq m$. Note here that it is natural to allow a simple pole to merge with a zero in the closure, thus decreasing the number $m$ of marked points.

If the surface $S$ is closed, that is, if $m=0$, then we can also consider the bundle

$$
\tilde{\mathcal{H}}(S) \rightarrow \mathcal{T}(S)
$$

of area one abelian differentials. It descends to the moduli space $\mathcal{H}(S)$ of holomorphic one-forms defining a singular euclidean metric of area one. Again this moduli space decomposes into a union of strata $\mathcal{H}\left(k_{1}, \ldots, k_{s}\right)$ corresponding to the orders of the zeros of the differentials. Strata are in general not connected, but there are at most three connected components [KZ03]. The stratum $\mathcal{H}\left(k_{1}, \ldots, k_{s}\right)$ is a real hypersurface in a complex orbifold of dimension

$$
h=2 g+s-1
$$

The Teichmüller flow $\Phi^{t}$ acts on $\mathcal{Q}(S)($ or $\mathcal{H}(S))$ preserving the strata. If $\mathcal{Q}$ is a component of a stratum of abelian differentials then Rauzy induction for interval exchange transformations can be used to construct a symbolic coding for the Teichmüller flow on $\mathcal{Q}$ ([V82], see also [AGY06] for a discussion and references). Rauzy induction has been extended to strata of quadratic differentials by Boissy and Lanneau [BL09]. It was used to identify connected components of strata of quadratic differentials.

Our main goal is to construct a new coding for the Teichmüller flow on any component $\mathcal{Q}$ of a stratum. This coding is based on the perspective on components of strata of abelian or quadratic differentials developed in [H22], and it is well suited to study the space $\mathcal{M}_{\mathrm{inv}}(\mathcal{Q})$ of all $\Phi^{t}$-invariant Borel probability measures on $\mathcal{Q}$, equipped with the weak*-topology.

For the formulation of our main result, recall that a biinfinite subshift of finite type $(\Omega, \sigma)$ is defined by a finite alphabet $\mathcal{A}=\{1, \ldots, p\}$ and a $(p, p)$-matrix $\left(a_{i j}\right)$ with entries in $\{0,1\}$ such that

$$
\Omega=\left\{\left(x_{i}\right) \in \mathcal{A}^{\mathbb{Z}} \mid a_{x_{i} x_{i+1}}=1 \text { for all } i\right\} .
$$

The shift $\sigma$ acts on $\Omega$ by $\sigma\left(x_{i}\right)=\left(x_{i+1}\right)$. The set $\Omega$ carries a natural $\sigma$-invariant topology, and for this topology, $\Omega$ is compact.

The shift is called topologically transitive if the $\sigma$-action has a dense orbit. A sequence $\left(x_{i}\right) \subset \Omega$ is called normal if every finite string $\left(y_{i}\right)_{1 \leq i \leq k}$ with $a_{y_{i} y_{i+1}}=1$
for all $0 \leq i \leq k-1$ occurs infinitely often in forward and backward direction as a substring of $\left(x_{i}\right)$.

In the statement of the following result, spaces of probability measures are equipped with the weak*-topology.
Theorem 1. Let $\mathcal{Q}$ be a component of a stratum of quadratic or abelian differentials. Then there exists

- a topologically transitive subshift of finite type $(\Omega, \sigma)$,
- a $\sigma$-invariant dense Borel set $\mathcal{U} \subset \Omega$ containing all normal sequences,
- a suspension $\left(X, \Theta^{t}\right)$ of $\sigma$ over $\mathcal{U}$, given by a positive bounded continuous roof function on $\mathcal{U}$
and a finite-to-one semi-conjugacy $\Xi:\left(X, \Theta^{t}\right) \rightarrow\left(\mathcal{Q}, \Phi^{t}\right)$ which maps the space of $\sigma$-invariant Borel probability measures on $\mathcal{U}$ continuously onto $\mathcal{M}_{\mathrm{inv}}(\mathcal{Q})$.

As is the case for Rauzy induction or, more precisely, the zippered rectangle flow considered in [V86], the coding constructed in Theorem 1 can be thought of as a finite cover of the Teichmüller flow on $\mathcal{Q}$. In particular, it maps the collection of all periodic orbits for $\sigma$ contained in $\mathcal{U}$ onto the collection of all periodic orbits in $\mathcal{Q}$. However, a periodic orbit in $\mathcal{Q}$ may have more than one preimage in the suspension flow $\left(X, \Theta^{t}\right)$, and the restriction of $\Xi$ to any such preimage may be a nontrivial finite covering of the periodic orbit.

Our construction is valid for strata of abelian differentials, but it is different from Rauzy induction. A dictionary between these two codings has yet to be established.

A specific example of a $\Phi^{t}$-invariant Borel probability measure on a component $\mathcal{Q}$ of a stratum in the Lebesgue measure class was constructed by Masur and Veech [M82, V86]. This measure $\lambda$ is ergodic [M82, V86] and of full support, and its entropy $h_{\lambda}$ coincides with the complex dimension $2 g+\ell+m-2$ (or $2 g+s-1$ ) of the complex orbifold defining the stratum (note that we use a normalization for the Teichmüller flow which is different from the one used by Masur and Veech). In particular, the entropy of the Lebesgue measure on the open connected stratum $\mathcal{Q}(1, \ldots, 1 ;-m)$ equals $6 g-6+2 m$.

Denote by $h_{\nu}$ the entropy of a measure $\nu \in \mathcal{M}_{\mathrm{inv}}(\mathcal{Q})$. Define

$$
h_{\mathrm{top}}(\mathcal{Q})=\sup \left\{h_{\nu} \mid \nu \in \mathcal{M}_{\mathrm{inv}}(\mathcal{Q})\right\} .
$$

A measure of maximal entropy for the component $\mathcal{Q}$ is a measure $\mu \in \mathcal{M}_{\mathrm{inv}}(\mathcal{Q})$ such that $h_{\mu}=h_{\text {top }}(\mathcal{Q})$. Since $\mathcal{Q}$ is non-compact, a priori such a measure need not exist. However, using Rauzy induction, Bufetov and Gurevich [BG11] showed that for components of strata of abelian differentials, the $\Phi^{t}$-invariant probability measure in the Lebesgue measure class is the unique measure of maximal entropy for the component. We use Theorem 1 and the work of Buzzi and Sarig [BS03] to extend this result to all components of strata of quadratic or abelian differentials, with a different proof.

Theorem 2. For every component $\mathcal{Q}$ of a stratum in $\mathcal{Q}(S)$ or $\mathcal{H}(S)$, the $\Phi^{t}$ invariant Borel probability measure in the Lebesgue measure class is the unique measure of maximal entropy.

In view of the groundbreaking work of Eskin and Mirzakhani [EM18] and of Eskin, Mirzakhani and Mohammadi [EMM15], we expect that the analog of Theorem 2 also holds for arbitrary affine invariant manifolds in $\mathcal{Q}(S)$. However our methods do not apply in this generality. Instead they are very well suited to study the dynamics of the Teichmüller flow near the principal boundary of a stratum as initiated in [H22]. This analysis is will be made precise in a sequel to this article.

Organization of the article and outline of the proofs. The organization of the article is as follows. In Section 2 we collect some results from [H22] relating train tracks to components of strata of abelian or quadratic differentials. This is used in Section 3 to construct for every connected component $\mathcal{Q}$ of a stratum an associated topologically transitive subshift of finite type $(\Omega, \sigma)$.

In Section 4 we define a roof function $\rho$ on a $\sigma$-invariant dense Borel subset $\mathcal{U}$ of $\Omega$ containing all normal sequences and use this roof function to define a suspension flow $\left(X, \Theta^{t}\right)$ over $\mathcal{U}$. It fairly immediately follows from the construction that there is a semi-conjugacy $\Xi:\left(X, \Theta^{t}\right) \rightarrow\left(\mathcal{Q}, \Phi^{t}\right)$. We establish that this semi-conjugacy is finite-to-one, which is the most elaborate part of the proof of Theorem 1. This is used to show that every ergodic $\Phi^{t}$-invariant Borel probability measure on $\mathcal{Q}$ is the push-forward under $\Xi$ of a finite invariant measure on $\left(X, \Theta^{t}\right)$ and hence on $(\Omega, \sigma)$, which completes the proof of Theorem 1.

Theorem 1 is not sufficient for the proof of Theorem 2. Namely, the semiconjugacy $\Xi$ is only finite-to-one but not bounded-to-one, and it is defined on a suspension over a Borel subset of $\Omega$ which is not closed, reflecting the fact that the component $\mathcal{Q}$ is not compact, and the Teichmüller flow $\Phi^{t}$ is not hyperbolic. Instead, in Section 5 we start with a point $q \in \mathcal{Q}$ which is contained in both the $\alpha$ - and $\omega$ limit set of its own orbit under $\Phi^{t}$. By the Poincarë recurrence theorem, for any invariant Borel probability measure on $\mathcal{Q}$, the set of such points has full measure. We then use the point $q$ and the subshift $(\Omega, \sigma)$ to construct a new Markov shift, now over a countably infinite alphabet. We also construct a continuous roof function which is bounded from below by a universal positive constant, but is unbounded. We then show that the corresponding suspension flow admits a bounded-to-one semi-conjugacy onto the restriction of the Teichmüller flow $\Phi^{t}$ to the invariant set of all points whose orbit under $\Phi^{t}$ contain $q$ in its $\alpha$ and $\omega$ limit set.

It follows from the work of Buzzi and Sarig [BS03] that the suspension flow over the countable alphabet admits at most one measure of maximal entropy provided that some technical conditions are fullfilled. We then verify that these technical assumptions are indeed satisfied for the flow constructed earlier, which leads to the proof of Theorem 2.

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## 2. Strata and train tracks

In this section we summarize some results from [H22] which will be used throughout the paper.
2.1. Geodesic laminations. Let $S$ be an oriented surface of genus $g \geq 0$ with $m \geq 0$ marked points and where $3 g-3+m \geq 2$. A geodesic lamination for a complete hyperbolic structure of finite volume on $S$ (which means in the sequel that the metric is defined on the complement of the marked points in $S$ ) is a compact subset of $S$ which is foliated into simple geodesics. A geodesic lamination $\lambda$ is called minimal if each of its half-leaves is dense in $\lambda$.

A geodesic lamination $\lambda$ on $S$ is said to fill up $S$ if its complementary regions are all topological discs or once punctured monogons.

Definition 2.1 (Definition 2.1 of [H22]). A geodesic lamination $\lambda$ is called large if $\lambda$ fills up $S$ and if moreover $\lambda$ can be approximated in the Hausdorff topology by simple closed geodesics.

Since every minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed geodesics [CEG87], a minimal geodesic lamination which fills up $S$ is large. However, there are large geodesic laminations with finitely many leaves.

The topological type of a large geodesic lamination $\nu$ is a tuple

$$
\left(m_{1}, \ldots, m_{\ell} ;-m\right) \text { where } 1 \leq m_{1} \leq \cdots \leq m_{\ell}, \sum_{i} m_{i}=4 g-4+m
$$

such that the complementary regions of $\nu$ which are topological discs are $m_{i}+2$-gons and the complementary regions which are once punctured discs are once punctured monogons. Let

$$
\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-m\right)
$$

be the space of all large geodesic laminations of type $\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ equipped with the restriction of the Hausdorff topology for compact subsets of $S$.

A measured geodesic lamination is a geodesic lamination $\lambda$ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in $S$ with endpoints in the complementary regions of $\lambda$ which intersects $\lambda$ nontrivially and transversely. The geodesic lamination $\lambda$ is called the support of the measured geodesic lamination; it consists of a disjoint union of minimal components. The space $\mathcal{M} \mathcal{L}$ of all measured geodesic laminations on $S$ equipped with the weak*-topology is homeomorphic to $S^{6 g-7+2 m} \times(0, \infty)$. Its projectivization is the space $\mathcal{P} \mathcal{M} \mathcal{L}$ of all projective measured geodesic laminations.

The measured geodesic lamination $\mu \in \mathcal{M} \mathcal{L}$ fills up $S$ if its support fills up $S$. This support is then necessarily connected and hence minimal, and for some tuple $\left(m_{1}, \ldots, m_{\ell} ;-m\right)$, it defines a point in the set $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$. The projectivization of a measured geodesic lamination which fills up $S$ is also said to fill up $S$. We call $\mu \in \mathcal{M L}$ strongly uniquely ergodic if the support of $\mu$ fills up $S$ and admits a unique transverse measure up to scale.

There is a continuous symmetric pairing

$$
\iota: \mathcal{M L} \times \mathcal{M L} \rightarrow[0, \infty)
$$

the so-called intersection form, which extends the geometric intersection number between simple closed curves.
2.2. Train tracks. A train track on $S$ is an embedded 1-complex $\tau \subset S$ whose edges (called branches) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a switch) the incident edges are mutually tangent. Through each switch there is a path of class $C^{1}$ which is embedded in $\tau$ and contains the switch in its interior. A simple closed curve component of $\tau$ contains a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with 0,1 or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. We always identify train tracks which are isotopic. Throughout we use the book [PH92] as the main reference for train tracks.

A train track is called generic if all switches are at most trivalent. For each switch $v$ of a generic train track $\tau$ which is not contained in a simple closed curve component, there is a unique half-branch $b$ of $\tau$ which is incident on $v$ and which is large at $v$. This means that every germ of an immersed arc of class $C^{1}$ on $\tau$ which passes through $v$ also passes through the interior of $b$. A half-branch which is not large is called small. A branch $b$ of $\tau$ is called large (or small) if each of its two half-branches is large (or small). A branch which is neither large nor small is called mixed.

Remark 2.2. As in [H09a], all train tracks are assumed to be generic. Unfortunately this leads to a small inconsistency of our terminology with the terminology found in the literature.

A generic train track $\tau$ is orientable if there is a consistent orientation of the branches of $\tau$ such that at any switch $s$ of $\tau$, the orientation of the large half-branch incident on $s$ extends to the orientation of the two small half-branches incident on $s$. If $C$ is a complementary polygon of an oriented train track then the number of sides of $C$ is even. In particular, a train track which contains a once punctured monogon component, that is, a once punctured disc with one cusp at the boundary, is not orientable (see p. 31 of [PH92] for a more detailed discussion).

A train track or a geodesic lamination $\eta$ is carried by a train track $\tau$ if there is a map $F: S \rightarrow S$ of class $C^{1}$ which is homotopic to the identity and maps $\eta$ into $\tau$ in such a way that the restriction of the differential of $F$ to the tangent space of $\eta$ vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of $F$ to $\eta$ a carrying map for $\eta$. Write $\eta \prec \tau$ if the train track $\eta$ is carried by the train track $\tau$. Then every geodesic lamination $\nu$ which is carried by $\eta$ is also carried by $\tau$.

A train track fills up $S$ if its complementary components are topological discs or once punctured monogons. Note that such a train track $\tau$ is connected. Let $\ell \geq 1$ be the number of those complementary components of $\tau$ which are topological
discs. Each of these discs is an $m_{i}+2$-gon for some $m_{i} \geq 1(i=1, \ldots, \ell)$. The topological type of $\tau$ is defined to be the ordered tuple ( $m_{1}, \ldots, m_{\ell} ;-m$ ) where $1 \leq m_{1} \leq \cdots \leq m_{\ell}$; then $\sum_{i} m_{i}=4 g-4+m$. If $\tau$ is orientable then $m=0$ and $m_{i}$ is even for all $i$. A train track of topological type $\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ is called fully recurrent if it carries a large geodesic lamination of type $\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ [H22].

A transverse measure on a generic train track $\tau$ is a nonnegative weight function $\mu$ on the branches of $\tau$ satisfying the switch condition: for every trivalent switch $s$ of $\tau$, the sum of the weights of the two small half-branches incident on $s$ equals the weight of the large half-branch. The space

$$
\mathcal{V}(\tau)
$$

of all measured geodesic laminations whose supports are carried by $\tau$ can naturally be identified with the space of all transverse measures on $\tau$. Thus $\mathcal{V}(\tau)$ has the structure of a cone in a finite dimensional real vector space. and The train track is called recurrent if it admits a transverse measure which is positive on every branch. A fully recurrent train track is recurrent [H22].

There are two simple ways to modify a fully recurrent train track $\tau$ to another fully recurrent train track. Namely, if $b$ is a mixed branch of $\tau$ then we can shift $\tau$ along $b$ to a new train track $\tau^{\prime}$. This new train track carries $\tau$ and hence it is fully recurrent since it carries every geodesic lamination which is carried by $\tau$ [PH92, H09a].

Similarly, if $e$ is a large branch of $\tau$ then we can perform a right or left split of $\tau$ at $e$ as shown in Figure A below. A (right or left) split $\tau^{\prime}$ of a train track $\tau$ is carried by $\tau$. If $\tau$ is of topological type $\left(m_{1}, \ldots, m_{\ell} ;-m\right)$, if $\nu \in \mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ is carried by $\tau$ and if $e$ is a large branch of $\tau$, then there is a unique choice of a right or left split of $\tau$ at $e$ such that the split track $\eta$ carries $\nu$. In particular, $\eta$ is fully recurrent. Note however that there may be a split of $\tau$ at $e$ such that the split track is not fully recurrent any more (see Section 2 of [H09a] for details).

Figure A


To each train track $\tau$ which fills up $S$ one can associate a dual bigon track $\tau^{*}$ (Section 3.4 of [PH92]). There is a bijection between the complementary components of $\tau$ and those complementary components of $\tau^{*}$ which are not bigons, i.e. discs with two cusps at the boundary. This bijection maps a component $C$ of $\tau$ which is an $n$-gon for some $n \geq 3$ to an $n$-gon component of $\tau^{*}$ contained in $C$, and it maps a once punctured monogon $C$ to a once punctured monogon contained in $C$. If $\tau$ is orientable then the orientation of $S$ and an orientation of $\tau$ induce an orientation on $\tau^{*}$, that is, $\tau^{*}$ is orientable.

There is a notion of carrying for bigon tracks which is analogous to the notion of carrying for train tracks.

Definition 2.3 (Definition 2.8 of [H22]). A train track $\tau$ of topological type $\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ is called large both $\tau, \tau^{*}$ carry a large geodesic lamination of the same topological type as $\tau$.

For a large train track $\tau$ let $\mathcal{V}^{*}(\tau) \subset \mathcal{M \mathcal { L }}$ be the set of all measured geodesic laminations whose support is carried by $\tau^{*}$. Then $\mathcal{V}^{*}(\tau)$ is naturally homeomorphic to a convex cone in a real vector space. The dimension of this cone coincides with the dimension of $\mathcal{V}(\tau)$.

Denote by

$$
\mathcal{L T}\left(m_{1}, \ldots, m_{\ell} ;-m\right)
$$

the set of all isotopy classes of large train tracks on $S$ of type $\left(m_{1}, \ldots, m_{\ell} ;-m\right)$.
2.3. Strata. For a closed oriented surface $S$ of genus $g \geq 0$ with $m \geq 0$ punctures let $\tilde{\mathcal{Q}}(S)$ be the bundle of marked area one holomorphic quadratic differentials with a simple pole at each puncture over the Teichmüller space $\mathcal{T}(S)$ of marked complex structures on $S$. For a complete hyperbolic metric on $S$ of finite area, an area one quadratic differential $q \in \tilde{\mathcal{Q}}(S)$ is determined by a pair $\left(\lambda^{+}, \lambda^{-}\right)$of measured geodesic laminations which bind $S$, that is, we have

$$
\iota\left(\lambda^{+}, \mu\right)+\iota\left(\lambda^{-}, \mu\right)>0
$$

for every measured geodesic lamination $\mu$, moreover $\iota\left(\lambda^{+}, \lambda^{-}\right)=1$ as the area of $q$ equals one. The vertical measured geodesic lamination $\lambda^{+}$for $q$ corresponds to the equivalence class of the vertical measured foliation of $q$. The horizontal measured geodesic lamination $\lambda^{-}$for $q$ corresponds to the equivalence class of the horizontal measured foliation of $q$.

A tuple $\left(m_{1}, \ldots, m_{\ell}\right)$ of positive integers $1 \leq m_{1} \leq \cdots \leq m_{\ell}$ with $\sum_{i} m_{i}=$ $4 g-4+m$ defines a stratum $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ in $\tilde{\mathcal{Q}}(S)$. This stratum consists of all marked area one quadratic differentials with $m$ simple poles and $\ell$ zeros of order $m_{1}, \ldots, m_{\ell}$ which are not squares of holomorphic one-forms. The stratum is a real hypersurface in a complex manifold of dimension

$$
h=2 g-2+m+\ell .
$$

The closure in $\tilde{\mathcal{Q}}(S)$ of a stratum is a union of components of strata. Strata are invariant under the action of the mapping class $\operatorname{group} \operatorname{Mod}(S)$ of $S$ and hence they project to strata in the moduli space $\mathcal{Q}(S)=\tilde{\mathcal{Q}}(S) / \operatorname{Mod}(S)$ of quadratic differentials on $S$ with a simple pole at each puncture. We denote the projection of the stratum $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ by $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$. The strata in moduli space need not be connected, but their connected components have been identified by Lanneau [L08]. A stratum in $\mathcal{Q}(S)$ has at most two connected components.

Similarly, if $m=0$ then we let $\tilde{\mathcal{H}}(S)$ be the bundle of marked area one holomorphic one-forms over Teichmüller space $\mathcal{T}(S)$ of $S$. For a tuple $k_{1} \leq \cdots \leq k_{\ell}$ of positive integers with $\sum_{i} k_{i}=2 g-2$, the stratum $\tilde{\mathcal{H}}\left(k_{1}, \ldots, k_{\ell}\right)$ of marked area
one holomorphic one-forms on $S$ with $\ell$ zeros of order $k_{i}(i=1, \ldots, \ell)$ is a real hypersurface in a complex manifold of dimension

$$
h=2 g-1+\ell
$$

It projects to a stratum $\mathcal{H}\left(k_{1}, \ldots, k_{\ell}\right)$ in the moduli space $\mathcal{H}(S)$ of area one holomorphic one-forms on $S$. Strata of holomorphic one-forms in moduli space need not be connected, but there are at most three connected components [KZ03].

For a large train track $\tau \in \mathcal{L T}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ let

$$
\mathcal{Q}(\tau) \subset \tilde{\mathcal{Q}}(S)
$$

be the set of all marked area one quadratic differentials whose vertical measured geodesic lamination is carried by $\tau$ and whose horizontal measured geodesic lamination is carried by the dual bigon track $\tau^{*}$ of $\tau$. By definition of a large train track, we have $\mathcal{Q}(\tau) \neq \emptyset$.

The next proposition relates $\mathcal{Q}(\tau)$ to components of strata.
Proposition 2.4 (Proposition 3.2 and Proposition 3.3 of [H22]). (1) For any large non-orientable train track $\tau \in \mathcal{L} \mathcal{T}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ there is a component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ such that $\mathcal{Q}(\tau)$ is the closure in $\tilde{\mathcal{Q}}(S)$ of an open path connected subset of $\tilde{\mathcal{Q}}$.
(2) For every large orientable train track $\tau \in \mathcal{L} \mathcal{T}\left(m_{1}, \ldots, m_{\ell} ; 0\right)$ there is a component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{H}}\left(m_{1} / 2, \ldots, m_{\ell} / 2\right)$ such that $\mathcal{Q}(\tau)$ is the closure in $\tilde{\mathcal{H}}(S)$ of an open path connected subset of $\tilde{\mathcal{Q}}$.
(3) For every component $\tilde{\mathcal{Q}}$ of a stratum of $\tilde{\mathcal{Q}}(S)$ (or of a stratum of $\tilde{\mathcal{H}}(S)$ ) and for every $q \in \tilde{\mathcal{Q}}$ there is a large train track $\tau$ such that $q \in \underset{\tilde{\mathcal{Q}}}{\mathcal{Q}}(\tau)$ and that $Q(\tau)$ is the closure of the open dense path connected subset $\tilde{\mathcal{Q}} \cap \mathcal{Q}(\tau)$.

## 3. A SYMBOLIC SYSTEM

In this section we construct a subshift of finite type which is used in the following sections to construct a symbolic coding for the Teichmüller flow on a component of a stratum in the moduli space of quadratic or abelian differentials. We continue to use the assumptions and notations from Section 2.

Thus let $\mathcal{Q}$ be a connected component of a stratum $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ of $\mathcal{Q}(S)$ (or of a stratum $\mathcal{H}\left(m_{1} / 2, \ldots, m_{\ell} / 2\right)$ of $\left.\mathcal{H}(S)\right)$. Let $\tilde{\mathcal{Q}}$ be the preimage of $\mathcal{Q}$ in $\tilde{\mathcal{Q}}(S)$ (or in $\tilde{\mathcal{H}}(S)$ ). Let

$$
\mathcal{L T}(\mathcal{Q}) \subset \mathcal{L T}\left(m_{1}, \ldots, m_{\ell} ;-m\right)
$$

be the set of all large train tracks $\tau$ of the same topological type as $\mathcal{Q}$ such that $\mathcal{Q}(\tau)$ is the closure of the open dense subset $\mathcal{Q}(\tau) \cap \tilde{\mathcal{Q}}$. Proposition 2.4 shows that this is well defined and that furthermore

$$
\tilde{\mathcal{Q}}=\cup_{\tau \in \mathcal{L} \mathcal{T}(\mathcal{Q})}(\mathcal{Q}(\tau) \cap \tilde{\mathcal{Q}})
$$

The set $\mathcal{L T}(\mathcal{Q})$ is invariant under the action of the mapping class group.
For ease of notation, define

$$
\mathcal{L} \mathcal{L}(\mathcal{Q}) \subset \mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-m\right)
$$

to be the closure (in the restriction of the Hausdorff topology) of the set of all minimal large geodesic laminations which can be represented as the support of the vertical measured geodesic lamination of some quadratic differential $q \in \tilde{\mathcal{Q}}$. Then $\mathcal{L} \mathcal{L}(\mathcal{Q})$ is invariant under the action of $\operatorname{Mod}(S)$, and for every $\tau \in \mathcal{L} \mathcal{T}(\mathcal{Q})$ it contains the set of all large geodesic laminations of topological type $\left(m_{1}, \ldots, m_{\ell} ;-m\right)$ carried by $\tau$.

Our goal is to use train tracks for a symbolic coding of the Teichmüller flow on $\mathcal{Q}$. However, the mapping class group $\operatorname{Mod}(S)$ does not act freely on $\mathcal{L T}(\mathcal{Q})$. To overcome this difficulty we extend the definition of a large train track as follows.

Definition 3.1. A numbered marked large train track is a marked large train track $\tau$ together with a numbering of the branches of $\tau$.

The set

$$
\begin{equation*}
\mathcal{N} \mathcal{T}(\mathcal{Q}) \tag{1}
\end{equation*}
$$

of all isotopy classes of numbered marked large train tracks on $S$ whose underlying unnumbered large train track is contained in $\mathcal{L T}(\mathcal{Q})$ is invariant under the natural action of the mapping class group.

A mapping class which preserves a large train track $\tau$ as well as each of its branches is the identity. Namely, such a mapping class can be represented by a homeomorphism of $S$ whose restriction to $\tau$ is the identity. Since all complementary components of $\tau$ are discs or once punctured discs, such a homeomorphism is homotopic to the identity. Thus the action of the mapping class group on $\mathcal{N} \mathcal{T}(\mathcal{Q})$ is free.

Define a (numbered) combinatorial type to be an orbit of a (numbered) large train track under the action of the mapping class group. Thus the set of numbered combinatorial types is the quotient of the set of all numbered large train tracks by the action of the mapping class group. Let

$$
\mathcal{E}_{0}(\mathcal{Q})
$$

be the set of all numbered combinatorial types which are $\operatorname{Mod}(S)$-orbits of numbered train tracks in $\mathcal{N} \mathcal{T}(\mathcal{Q})$.

Note that if the large train track $\tau^{\prime}$ can be obtained from a large train track $\tau$ by a single split, then a numbering of the branches of $\tau$ naturally induces a numbering of the branches of $\tau^{\prime}$ and therefore such a numbering defines a numbered split.

Definition 3.2. A full split of a (numbered) large train track $\tau$ is a (numbered) large train track $\tau^{\prime}$ which can be obtained from $\tau$ by splitting $\tau$ at each large branch precisely once.

A full (numbered) splitting sequence is a sequence $\left(\tau_{i}\right)$ of (numbered) large train tracks such that for each $i$, the (numbered) large train track $\tau_{i+1}$ can be obtained from $\tau_{i}$ by a full (numbered) split.

Definition 3.3. A numbered combinatorial type $x \in \mathcal{E}_{0}(\mathcal{Q})$ is splittable to a numbered combinatorial type $x^{\prime}$ if there is a numbered large train track $\tau$ contained in $x$ which can be connected to a numbered large train track $\tau^{\prime}$ contained in $x^{\prime}$ by a full numbered splitting sequence.

In general it is unclear whether a given numbered combinatorial type is splittable to another type. This issue is addressed in Lemma 3.5 below which is a main technical ingredient towards the construction of a subshift of finite type with the properties stated in Theorem 1. For the purpose of its proof and later use, we first establish a simple fact (which is well known, see [M82]) about the structure of quadratic differentials $q \in \mathcal{Q}$ which are recurrent under the Teichmüller flow.

Lemma 3.4. Let $q \in \mathcal{Q}$ be a point whose forward orbit $\Phi^{t} q(t \geq 0)$ under the Teichmüller flow returns to a compact set $K \subset \mathcal{Q}$ for arbitrarily large times. Then the vertical measured geodesic lamination of a lift of $q$ to $\tilde{\mathcal{Q}}$ is uniquely ergodic, with support in $\mathcal{L} \mathcal{L}(\mathcal{Q})$.

Proof. Let $q \in \mathcal{Q}$, let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of $q$ and assume that the support of the vertical measured geodesic lamination of $\tilde{q}$ is not contained in $\mathcal{L} \mathcal{L}(\mathcal{Q})$. Then $q$ admits at least one vertical saddle connection. This saddle connection has finite length.

By continuity, for every compact set $K \subset \mathcal{Q}$ there exists a number $\kappa(K)>$ 0 which bounds from below the minimal length of any saddle connection for a differential $u \in K$. Now the length of a vertical saddle connection is exponentially decreasing under the Teichmüller flow and therefore the orbit of $q$ does not return to $K$ for arbitrarily large times. This is the contrapositive of the second claim in the lemma.

Unique ergodicity of the vertical measured geodesic lamination of a forward recurrent quadratic differential is a well known result of Masur [M82] (which also covers the statement shown in the beginning of this proof). The lemma follows.

Given a numbered marked train track $\tau$ and a subset $\mathcal{W}$ of $\mathcal{E}_{0}(\mathcal{Q})$, we write $[\tau] \in \mathcal{W}$ if the $\operatorname{Mod}(S)$-orbit of $\tau$ is contained in $\mathcal{W}$.

Lemma 3.5. For every connected component $\mathcal{Q}$ of a stratum of $\mathcal{Q}(S)$ (or of a stratum of $\mathcal{H}(S)$ ) there is a set $\mathcal{E}(\mathcal{Q}) \subset \mathcal{E}_{0}(\mathcal{Q})$ of numbered combinatorial types with the following properties.
(1) For all $x, x^{\prime} \in \mathcal{E}(\mathcal{Q}), x$ is splittable to $x^{\prime}$.
(2) If $\tau$ is contained in $\mathcal{E}(\mathcal{Q})$ and if $\left(\tau_{i}\right)$ is any full numbered splitting sequence issuing from $\tau_{0}=\tau$ then $\tau_{i}$ is contained in $\mathcal{E}(\mathcal{Q})$ for all $i \geq 0$.

Proof. For $[\beta] \in \mathcal{E}_{0}(\mathcal{Q})$ let $\mathcal{A}([\beta]) \subset \mathcal{E}_{0}(\mathcal{Q})$ be the set of all combinatorial types of numbered large train tracks $\xi$ with the following additional property. There is a representative $\beta \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ of $[\beta]$ which can be connected to $\xi$ by a (possibly trivial) full numbered splitting sequence. Since the concatenation of two full splitting sequences is a full splitting sequence, if $[\xi] \in \mathcal{A}([\beta])$ then $\mathcal{A}([\xi]) \subset \mathcal{A}([\beta])$.

Since $\mathcal{E}_{0}(\mathcal{Q})$ is a finite set, there exists some $[\sigma] \in \mathcal{E}_{0}(\mathcal{Q})$ such that the cardinality of $\mathcal{A}([\sigma \mid)$ is minimal among the cardinalities of the sets $\mathcal{A}([\beta])$ where $[\beta]$ ranges through $\mathcal{E}_{0}(\mathcal{Q})$. Let $[\xi] \in \mathcal{A}([\sigma])$. Since $\mathcal{A}([\xi]) \subset \mathcal{A}([\sigma])$ and the cardinality of $\mathcal{A}([\sigma \mid)$ is minimal, we know that $\mathcal{A}([\xi])=\mathcal{A}([\sigma])$ and, in particular, $[\sigma] \in \mathcal{A}([\xi])$. As $[\xi] \in \mathcal{A}([\sigma)])$ was arbitrary, we conclude that for all $\xi, \xi^{\prime} \in \mathcal{L} \mathcal{T}(\mathcal{Q})$ which are contained in $\mathcal{A}([\sigma])$, there is a full numbered splitting sequence connecting $\xi$ to a train track $g \xi^{\prime}$ for some $g \in \operatorname{Mod}(S)$.

Define

$$
\mathcal{E}(\mathcal{Q})=\mathcal{A}([\sigma]) \subset \mathcal{E}_{0}(\mathcal{Q})
$$

Let $\theta \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ with $[\theta] \in \mathcal{E}(\mathcal{Q})$. By the above discussion, the train track $\theta$ can be connected to a numbered train track $\sigma^{\prime}$ in the $\operatorname{Mod}(S)$-orbit of $\sigma$ by a full numbered splitting sequence. Thus the first property in the lemma holds true for $\mathcal{E}(\mathcal{Q})$, and the second is true by the definition of $\mathcal{E}(\mathcal{Q})$. This completes the proof of the lemma.

The union $\cup_{[\tau] \in \mathcal{E}(\mathcal{Q})} \mathcal{Q}(\tau) \cap \tilde{\mathcal{Q}}$ is clearly invariant under the Teichmüller flow and under the action of the mapping class group, and it contains a non-empty open invariant subset of $\tilde{\mathcal{Q}}$. However, this does not imply that it projects to a subset of $\mathcal{Q}$ which contains the support of every $\Phi^{t}$-invariant Borel probability measure. The next lemma provides the relevant additional information we need,

Lemma 3.6. For every $q \in \mathcal{Q}$ without vertical saddle connections and every lift $\tilde{q}$ of $q$ to $\tilde{\mathcal{Q}}$ there is some $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ such that $[\tau] \in \mathcal{E}(\mathcal{Q})$ and $\tilde{q} \in \mathcal{Q}(\tau)$.

Proof. Let $q \in \mathcal{Q}$ and assume that $q$ does not have vertical saddle connections. Let $\tilde{q} \in \mathcal{Q}$ be a lift of $q$. By Proposition 2.4, there is a train $\operatorname{track} \eta \in \mathcal{L} \mathcal{T}(\mathcal{Q})$ so that $\tilde{q} \in \mathcal{Q}(\eta)$. Recall that $\mathcal{Q}(\eta) \cap \tilde{\mathcal{Q}}$ contains an open subset of $\tilde{\mathcal{Q}}$ which projects to an open subset of $\mathcal{Q}$. Since the set of all points in $\mathcal{Q}$ whose $\Phi^{t}$-orbits are dense in $\mathcal{Q}$ has full Lebesgue measure and hence is dense, there is a point $z \in \mathcal{Q}$ whose $\Phi^{t}$-orbit is dense in $\mathcal{Q}$ in forward and backward direction and which admits a lift $\tilde{z} \in \mathcal{Q}(\eta)$.

By the definition of $\mathcal{Q}(\eta)$, the support of the horizontal measured geodesic lamination $\tilde{z}^{h}$ of $\tilde{z}$ is carried by the dual $\eta^{*}$ of $\eta$. Furthermore, by Lemma 3.4, the support of $\tilde{z}^{h}$ fills $S$. Thus since the support of the vertical measured geodesic lamination $\tilde{q}^{v}$ of $\tilde{q}$ is carried by $\eta$, the measured geodesic laminations $\tilde{q}^{v}, \tilde{z}^{h}$ bind $S$. Hence there is a quadratic differential $\tilde{u} \in \mathcal{Q}(\eta)$ whose horizontal measured geodesic lamination equals $\tilde{z}^{h}$ and whose vertical measured geodesic lamination equals $c \tilde{q}^{v}$ for a number $c>0$. Proposition 2.4 implies that $\tilde{u} \in \tilde{\mathcal{Q}}$.

Since the backward orbit of $z$ under the Teichmüller flow $\Phi^{t}$ is dense in $\mathcal{Q}$, we have $d\left(\Phi^{-t} \tilde{u}, \Phi^{-t} \tilde{z}\right) \rightarrow 0(t \rightarrow \infty)$ for any distance function $d$ on $\tilde{\mathcal{Q}}$ which is induced by a complete $\operatorname{Mod}(S)$-invariant Riemannian metric on $\tilde{\mathcal{Q}}$ [M80].

On the other hand, let $\sigma \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ be contained in $\mathcal{E}(\mathcal{Q})$. Then $\mathcal{Q}(\sigma)$ contains an open subset of $\tilde{\mathcal{Q}}$. Using once more the fact that the backward orbit of $z$ is dense in $\mathcal{Q}$, there is some $g \in \operatorname{Mod}(S)$ and some $T>0$ so that $g \Phi^{-T} \tilde{u} \in \mathcal{Q}(\sigma)$ and hence $g \tilde{u} \in \mathcal{Q}(\sigma)$ by invariance. In particular, the vertical measured geodesic lamination $g\left(c \tilde{q}^{v}\right)$ of $g \tilde{u}$ is carried by $\sigma$. Equivalently, $\tilde{q}^{v}$ is carried by $g^{-1} \sigma$.

Now $\tilde{q}$ does not have any vertical saddle connection by assumption and hence using again Lemma 3.5 or [M82], the vertical measured geodesic lamination $\tilde{q}^{v}$ of $\tilde{q}$ is minimal, with support in $\mathcal{L} \mathcal{L}(\mathcal{Q})$. As a consequence, there is a unique full numbered splitting sequence $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ beginning at $\tau_{0}=g^{-1} \sigma$ which consists of train tracks carrying $\tilde{q}^{v}$. By the second statement in Lemma 3.5, each of the train tracks $\tau_{i}$ is contained in $\mathcal{E}(\mathcal{Q})$.

We claim that for sufficiently large $i$ we have $\tilde{q} \in \mathcal{Q}\left(\tau_{i}\right)$. This then completes the proof of the lemma. To this end note that $\tilde{q}^{v}$ is carried by both $\eta, \tau_{0}$, and the topological type of its support coincides with the topological type of $\eta, \tau_{0}$. Thus by Corollary 2.4.3 of [PH92], there exists a train track $\nu$ which can be obtained from both $\eta, \tau_{0}$ by a splitting and collision sequence (where a collision is a split followed by the removal of the diagonal) and which carries $\tilde{q}^{v}$. Since the topological type of $\tilde{q}^{v}$ coincides with the topological type of $\eta$ and $\tilde{q}^{v}$ is carried by $\nu$, the topological type of $\nu$ coincides with the topological type of $\eta, \tau_{0}$ and hence there is no collision in the transformation of $\tau_{0}$ to $\nu$.

Now by uniqueness of splitting sequences as established in Lemma 5.1 of [H09a] and the fact that the (numbered) splitting sequence $\left(\tau_{i}\right)$ is full, there exists some $j>0$ such that $\nu$ is splittable to $\tau_{j}$ and hence $\eta$ is splittable to $\tau_{j}$. As $\tilde{q} \in \mathcal{Q}(\eta)$, the horizontal measured geodesic lamination $\tilde{q}^{h}$ of $\tilde{q}$ is carried by $\eta^{*}$ and hence by $\tau_{j}^{*}$. But this means that $\tilde{q} \in \mathcal{Q}\left(\tau_{j}\right)$. As $\left[\tau_{j}\right] \in \mathcal{E}(\mathcal{Q})$, this completes the proof of the lemma.

Remark 3.7. We do not know whether there are components $\mathcal{Q}$ with $\mathcal{E}(\mathcal{Q})=$ $\mathcal{E}_{0}(\mathcal{Q})$. It seems likely that such components do not exist as we expect that for $\eta \in \mathcal{E}(\mathcal{Q})$, not all permutations of the numberings of the branches of $\eta$ are contained in $\mathcal{E}(\mathcal{Q})$.

Let $k>0$ be the cardinality of the set $\mathcal{E}(\mathcal{Q}) \subset \mathcal{E}_{0}(\mathcal{Q})$ as in Lemma 3.5. Number the $k$ elements of $\mathcal{E}(\mathcal{Q})$ in an arbitrary way. Identify each element of $\mathcal{E}(\mathcal{Q})$ with its number. Define $a_{i j}=1$ if the numbered combinatorial type $i$ can be split with a single full numbered split to the numbered combinatorial type $j$ and define $a_{i j}=0$ otherwise. The matrix $A=\left(a_{i j}\right)$ defines a subshift of finite type. Its phase space is the set of biinfinite sequences

$$
\Omega=\left\{\left(x_{i}\right) \in \prod_{i=-\infty}^{\infty}\{1, \ldots, k\} \mid a_{x_{i} x_{i+1}}=1 \text { for all } i\right\}
$$

Every biinfinite full numbered splitting sequence $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ contained in $\mathcal{E}(\mathcal{Q})$ defines a point in $\Omega$. Vice versa, since the action of $\operatorname{Mod}(S)$ on the set of numbered large train tracks is free, a point in $\Omega$ determines a $\operatorname{Mod}(S)$-orbit of biinfinite full numbered splitting sequences. We say that such a full numbered splitting sequence realizes $\left(x_{i}\right)$.

The shift map $\sigma: \Omega \rightarrow \Omega, \sigma\left(x_{i}\right)=\left(x_{i+1}\right)$ acts on $\Omega$. For $n>0$ write $A^{n}=\left(a_{i j}^{(n)}\right)$; the shift $\sigma$ is topologically transitive if for all $i, j$ there is some $n>0$ such that $a_{i j}^{(n)}>$ 0 . Define a finite sequence $\left(x_{i}\right)_{0 \leq i \leq n}$ of points $x_{i} \in\{1, \ldots, k\}$ to be admissible if $a_{x_{i} x_{i+1}}=1$ for all $i \leq n-1$. Then $a_{i j}^{(n)}$ equals the number of all admissible sequences
of length $n$ connecting $i$ to $j$ [Mn87]. The following observation is immediate from the definitions.
Lemma 3.8. The shift $(\Omega, \sigma)$ is topologically transitive.

Proof. Let $i, j \in\{1, \ldots, k\}$ be arbitrary. By Lemma 3.5, there is a nontrivial finite full numbered splitting sequence $\left\{\tau_{i}\right\}_{0 \leq i \leq n} \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ connecting a train track $\tau_{0}$ of numbered combinatorial type $i$ to a train track $\tau_{n}$ of numbered combinatorial type $j$. This splitting sequence then defines an admissible sequence $\left(x_{i}\right)_{0 \leq i \leq n}$ connecting $i$ to $j$.

Remark 3.9. Without loss of generality, we can in fact assume that the shift $(\Omega, \sigma)$ is topologically mixing. Namely, by the discussion on p. 55 of [HK95], otherwise there are numbers $\ell, n>0$ such that $\ell n=k$ and that the following holds true. The elements of $\mathcal{E}(\mathcal{Q})$ are divided into $n$ disjoint sets $C_{1}, \ldots, C_{n}$ of $\ell$ elements each so that for $x_{i} \in C_{j}$ we have $a_{x_{i} x_{i+1}}=1$ only if $x_{i+1} \in C_{j+1}$ (indices are taken modulo $n$ ). Moreover, the restriction of $\sigma^{n}$ to $C_{1}$ is topologically mixing. However, in this case we can repeat the argument in the proof of Lemma 3.5 with a single full numbered split replaced by a full numbered splitting sequence of length $n$. This amounts to replacing $(\Omega, \sigma)$ by the topologically mixing subshift $\left(C_{1}, \sigma^{n}\right)$ which has all properties stated above.

## 4. Symbolic dynamics for the Teichmüller flow

In this section we relate the subshift of finite type $(\Omega, \sigma)$ constructed in Section 3 to the Teichmüller flow $\Phi^{t}$ on the component $\mathcal{Q}$ of a stratum. We continue to use the assumptions and notations from Section 2 and Section 3.

Let as before $\mathcal{Q}$ be a component of a stratum. Let $\mathcal{N} \mathcal{T}(\mathcal{Q})$ be as in (1) and let $p>0$ be the number of branches of a train track in $\mathcal{N T}(\mathcal{Q})$. For $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ let as before $\mathcal{V}(\tau)$ be the space of all measured geodesic laminations carried by $\tau$, equipped with the topology as the subcone of $\mathbb{R}^{p}$ of nonnegative functions on the branches of $\tau$ which satisfy the switch condition. For each $\mu \in \mathcal{V}(\tau)$ we denote by $\mu(\tau)$ the total mass of $\mu$, that is, the sum of the weights of $\mu$ over all branches of $\tau$. Define

$$
\mathcal{V}_{0}(\tau) \subset \mathcal{V}(\tau)
$$

to be the subspace of transverse measures of total mass 1 .
Denote by $P \mathcal{V}(\tau)$ the space of all projective measured geodesic laminations which are carried by $\tau$. Note that $P \mathcal{V}(\tau)$ is a compact subset of the compact space $\mathcal{P} \mathcal{M} \mathcal{L}$ of all projective measured geodesic laminations on $S$.

Let $\left(\tau_{i}\right)_{0 \leq i} \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ be any full numbered splitting sequence. Then we have $\emptyset \neq P \mathcal{V}\left(\tau_{i+1}\right) \subset P \mathcal{V}\left(\tau_{i}\right)$ and hence $\cap_{i} P \mathcal{V}\left(\tau_{i}\right)$ is a non-empty compact subset of $\mathcal{P} \mathcal{M} \mathcal{L}$. If $\cap_{i} P \mathcal{V}\left(\tau_{i}\right)$ consists of a unique point then we call $\left(\tau_{i}\right)$ uniquely ergodic.

Call the sequence $\left(x_{i}\right) \in \Omega$ uniquely ergodic if some (and hence every) full numbered splitting sequence $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ which realizes $\left(x_{i}\right)$ is uniquely ergodic, and the support of $\cap_{i} P \mathcal{V}\left(\tau_{i}\right)$ is of the same topological type as $\tau_{0}$. This implies in particular that for every $i$, a transverse measure on $\tau_{i}$ defined by a point $\zeta \in \cap_{i} \mathcal{V}\left(\tau_{i}\right)$ is
positive on every branch of $\tau_{i}$. Moreover, the sequence $\left(\tau_{i}\right)$ is uniquely determined by $\tau_{0}$ and $\zeta$ [H09a].

Let

$$
\mathcal{U} \subset \Omega
$$

be the set of all uniquely ergodic sequences. We define a function $\rho: \mathcal{U} \rightarrow \mathbb{R}$ as follows. For $\left(x_{i}\right) \in \mathcal{U}$ choose a full numbered splitting sequence $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ which realizes $\left(x_{i}\right)$. Let $\mu \in \mathcal{V}_{0}\left(\tau_{0}\right) \cap \bigcap_{i \geq 0} \mathcal{V}\left(\tau_{i}\right)$ be carried by each of the train tracks $\tau_{i}$ and note that $\mu$ is uniquely determined by these properties. Define

$$
\rho\left(x_{i}\right)=-\log \mu\left(\tau_{1}\right)
$$

By equivariance under the action of the mapping class group, the number $\rho\left(x_{i}\right) \in$ $(0, \infty)$ only depends on the sequence $\left(x_{i}\right) \in \mathcal{U}$. In other words, $\rho$ is a function defined on $\mathcal{U}$. We have

Lemma 4.1. The function $\rho$ maps $\mathcal{U}$ to $(0, p \log 2]$.

Proof. Let $\left(x_{i}\right) \in \mathcal{U}$ and choose a full numbered splitting sequence $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ which realizes $\left(x_{i}\right)$. Let $\mu \in \mathcal{V}_{0}\left(\tau_{0}\right) \cap \bigcap_{i>0} \mathcal{V}\left(\tau_{i}\right)$. Thus $\mu$, viewed as a measured geodesic lamination, is carried by each of the train tracks $\tau_{i}$, and it defines a transverse measure $\mu$ on $\tau_{0}$ of total weight one.

Let $e$ be a large branch of $\tau_{0}$ and let $\tau^{\prime}$ be the large train track which is obtained from $\tau_{0}$ by a single split at $e$ and which is splittable to $\tau_{1}$ (compare [H09a] for details). Let $e^{\prime}$ be the branch in $\tau^{\prime}$ which is the diagonal of the split of $\tau_{0}$ at $e$. This means that $e^{\prime}$ is the small branch in $\tau^{\prime}$ which is the image of $e$ under the natural bijection $\Lambda$ of the branches of $\tau_{0}$ onto the branches of $\tau^{\prime}$. Let $\mu^{\prime}$ be the transverse measure on $\tau^{\prime}$ defined by the measured geodesic lamination $\mu$. There are two branches $b, d$ in $\tau$ incident on the two endpoints of $e$ such that

$$
\mu(e)=\mu^{\prime}\left(e^{\prime}\right)+\mu^{\prime}(\Lambda(b))+\mu^{\prime}(\Lambda(d))
$$

Moreover, we have $\mu(a)=\mu^{\prime}(\Lambda(a))$ for every branch $a \neq e$ of $\tau_{0}$ and hence $\mu^{\prime}\left(\tau^{\prime}\right) \in$ $[1 / 2,1]$. Since $\tau_{1}$ can be obtained from $\tau$ by at most $p$ splits this immediately implies that $\rho$ is nonnegative and bounded from above by $p \log 2$.

On the other hand, by the definition of $\mathcal{U}$, the measure $\mu^{\prime}$ is positive on every branch of $\tau^{\prime}$. Then the same consideration as above shows that $\rho>0$ on $\mathcal{U}$.

Remark 4.2. It is not hard to see that the function $\rho$ is not bounded from below by a universal positive constant. Since this fact is not relevant for us, we omit to give an example.
Lemma 4.3. The function $\rho: \mathcal{U} \rightarrow \mathbb{R}$ is continuous and only depends on the future.

Proof. By construction, if $x_{i}=y_{i}$ for all $i \geq 0$, then we have $\rho\left(x_{i}\right)=\rho\left(y_{i}\right)$, that is, $\rho$ only depends on the future.

To show continuity of $\rho$ let $\left(x_{i}\right) \in \mathcal{U}$. By the definition of the topology on the shift space $(\Omega, \sigma)$ with basis the cylinder sets, it suffices to show that for every $\epsilon>0$ there is some $j \geq 0$ (depending on $\left.\left(x_{i}\right)\right)$ such that

$$
\left|\rho\left(y_{i}\right)-\rho\left(x_{i}\right)\right| \leq \epsilon
$$

whenever $\left(y_{i}\right) \in \mathcal{U}$ is such that $x_{i}=y_{i}$ for $0 \leq i \leq j$.
For this let $\left(\tau_{i}\right)$ be a full numbered splitting sequence which realizes $\left(x_{i}\right)$. Then $\left(\tau_{i}\right)$ determines a measured geodesic lamination

$$
\mu=\mathcal{V}_{0}\left(\tau_{1}\right) \cap \bigcap_{i \geq 0} \mathcal{V}\left(\tau_{i}\right)
$$

By definition, $\rho\left(x_{i}\right)=\log \mu\left(\tau_{0}\right)$ where $\mu$ is viewed as a weight function on $\tau_{0}$ via a carrying map $\tau_{1} \rightarrow \tau_{0}$. Note that here we normalize $\mu$ at $\tau_{1}$ for ease of exposition.

Let as before $p>0$ be the number of branches of a train track in $\mathcal{N} \mathcal{T}(\mathcal{Q})$. The set $\mathcal{V}_{0}\left(\tau_{1}\right)$ of all transverse measures on $\tau_{1}$ of weight one can be identified with a compact convex subset of $\mathbb{R}^{p}$. The natural projection $\pi: \mathcal{V}_{0}\left(\tau_{1}\right) \rightarrow \mathcal{P} \mathcal{M} \mathcal{L}$ is a homeomorphism onto its image with respect to the weak*-topology on $\mathcal{P M} \mathcal{L}$ [PH92]. Since a carrying map $\tau_{1} \rightarrow \tau_{0}$ induces a linear and hence continuous map $\mathcal{V}_{0}\left(\tau_{1}\right) \rightarrow \mathcal{V}\left(\tau_{0}\right)$ (see the proof of Lemma 4.1 for details on this well known fact), there is an open neighborhood $V$ of $\pi(\mu)$ in $\mathcal{P} \mathcal{M} \mathcal{L}$ with the following property. Every $\nu \in \mathcal{V}_{0}\left(\tau_{1}\right)$ with $\pi(\nu) \in V$ defines a transverse measure on $\tau_{0}$ whose total weight is contained in the interval $\left(e^{\rho\left(x_{i}\right)-\epsilon}, e^{\rho\left(x_{i}\right)+\epsilon}\right)$.

Now for every $j>0$ the set $P \mathcal{V}\left(\tau_{j}\right)$ of all projective measured geodesic laminations which are carried by $\tau_{j}$ is a compact subset of $\mathcal{P} \mathcal{M} \mathcal{L}$ containing $\pi(\mu)$, and we have $P \mathcal{V}\left(\tau_{j}\right) \subset P \mathcal{V}\left(\tau_{i}\right)$ for $j \geq i$ and $\cap_{j} P \mathcal{V}\left(\tau_{j}\right)=\pi(\mu)$. As a consequence, there is some $j_{0}>0$ such that $P \mathcal{V}\left(\tau_{j_{0}}\right) \subset V$. By the definition of $\rho$, this implies that the value of $\rho$ on the intersection with $\mathcal{U}$ of the cylinder $\left\{\left(y_{i}\right) \mid y_{j}=x_{j}\right.$ for $\left.0 \leq j \leq j_{0}\right\}$ is contained in the interval $\left(\rho\left(x_{i}\right)-\epsilon, \rho\left(x_{i}\right)+\epsilon\right)$. This shows the lemma.

Our next goal is to obtain a better understanding of the set $\mathcal{U} \subset \Omega$ of uniquely ergodic sequences. It follows from the definitions that $\mathcal{U}$ is a Borel subset of $\Omega$. We begin with invoking some results from [H09b].

A marking of the finite type surface $S$ consists of a pants decomposition $P$ for $S$ and a system of simple closed spanning curves [MM99]. For each curve $\gamma \in P$ there is a unique spanning curve which is contained in $S-(P-\gamma)$ and which intersects $\gamma$ in the minimal number of points. The spanning curves may intersect.

The mapping class group $\operatorname{Mod}(S)$ naturally acts on the set of all markings of $S$. By equivariance and the fact that $\operatorname{Mod}(S)$ acts on $\mathcal{N} \mathcal{T}(\mathcal{Q})$ with finitely many orbits, there is a number $b>0$, and for every train $\operatorname{track} \tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ there is a marking $F$ of $S$ which consists of simple closed curves carried by $\tau$ and such that the sum of the masses of the counting measures on $\tau$ defined by these curves does not exceed $b$ (compare the discussion in [MM99]). We call such a marking short for $\tau$.

The intersection number $\iota\left(c, c^{\prime}\right)$ between any two simple closed curves $c, c^{\prime}$ which are carried by some $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ and which define counting measures on $\tau$ of total mass at most $b$ is bounded from above by a universal constant $\beta>0$ (Corollary 2.3 of [H06]). Thus if $F$ is a short marking for $\tau$, with pants decomposition $P$, then $\iota\left(c, c^{\prime}\right) \leq \beta$ for any two spanning curves $c, c^{\prime}$ for $P$. In the sequel, we call such a marking good. Thus short markings for a train track $\tau$ are good.

Let $\mathcal{T}(S)$ be the Teichmüller space of all complete finite volume hyperbolic metrics on $S$. By equivariance under the action of the mapping class group, there is a number

$$
\chi_{0}>0
$$

and for every good marking $F$ of $S$ there is a complete finite volume hyperbolic metric $h \in \mathcal{T}(S)$ such that the $h$-length of each marking curve from $F$ is at most $\chi_{0}$. We call such a hyperbolic metric short for $F$.

By standard hyperbolic trigonometry, there is a number $\epsilon>0$ such that every hyperbolic metric which is short for some good marking $F$ of $S$ is contained in the set $\mathcal{T}(S)_{\epsilon}$ of all hyperbolic metrics whose systole, that is, the shortest length of a closed geodesic, is at least $\epsilon$. Moreover, the diameter in $\mathcal{T}(S)$ with respect to the Teichmüller metric $d_{\mathcal{T}}$ of the set of all hyperbolic metrics which are short for a fixed good marking $F$ is bounded from above by a universal constant.

Define a map

$$
\Lambda: \mathcal{N} \mathcal{T}(\mathcal{Q}) \rightarrow \mathcal{T}(S)
$$

by associating to a large train track $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ a hyperbolic metric $\Lambda(\tau) \in \mathcal{T}(S)$ which is short for a short marking for $\tau$. This map depends on choices, but by the above discussion, there is a number $\chi_{1}>0$ only depending on the topological type of $S$ (and not on $\mathcal{Q}$ ) such that if $\Lambda^{\prime}$ is another choice of such a map then we have $d_{\mathcal{T}}\left(\Lambda(\tau), \Lambda^{\prime}(\tau)\right) \leq \chi_{1}$ for every $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$. In particular, the map $\Lambda$ is coarsely $\operatorname{Mod}(S)$-equivariant: For every $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ and every $g \in \operatorname{Mod}(S)$ we have

$$
d_{\mathcal{T}}(\Lambda(g \tau), g \Lambda(\tau)) \leq \chi_{1}
$$

By properness of the action of $\operatorname{Mod}(S)$ on $\mathcal{T}(S)_{\epsilon}$ and on $\mathcal{N} \mathcal{T}(\mathcal{Q})$, this implies that for every $x \in \mathcal{T}(S)$ and every $R>0$ there are only finitely many large numbered train tracks $\eta \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ with $d_{\mathcal{T}}(\Lambda(\eta), x) \leq R$.

Since a quadratic differential $q \in \tilde{\mathcal{Q}}$ defines a singular euclidean metric on $S$ of area one, the injectivity radius of this metric is uniformly bounded from above. Thus up to increasing the above constant $\chi_{0}>0$, we may assume that for every quadratic differential $z \in \tilde{\mathcal{Q}}$ there is an essential simple closed curve on $S$ whose $q$-length, that is, the length with respect to the singular euclidean metric defined by $q$, is at most $\chi_{0}$.

The curve graph $\mathcal{C}(S)$ of $S$ is the metric graph whose vertices are the essential simple closed curves on $S$ and where two such vertices are connected by an edge of length one if and only if they can be realized disjointly (see [MM99]). Define a map

$$
\Upsilon_{\tilde{\mathcal{Q}}}: \tilde{\mathcal{Q}} \rightarrow \mathcal{C}(S)
$$

by associating to a quadratic differential $\tilde{q}$ a simple closed curve $\Upsilon_{\tilde{\mathcal{Q}}}(\tilde{q})$ of $\tilde{q}$-length at most $\chi_{0}$. Then there is a number $L>1$ such that the image under $\Upsilon_{\tilde{\mathcal{Q}}}$ of every flow $\operatorname{line}_{\tilde{\mathcal{Q}}}$ of the Teichmüller flow is an unparameterized L-quasi-geodesic: For every $z \in \tilde{\mathcal{Q}}$ there is an increasing surjective homeomorphism $\sigma:(a, b) \rightarrow \mathbb{R}$ such that the curve $t \rightarrow \Upsilon_{\tilde{\mathcal{Q}}}\left(\Phi^{\sigma(t)} z\right)$ is an $L$-quasi-geodesic in $\mathcal{C}(S)$ [MM99]. Here $(a, b)$ is an open subinterval of $\mathbb{R}$ which may be finite, one-sided infinite or may coincide with $\mathbb{R}$.

A vertex cycle for a large train track $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ is a simple closed curve carried by $\tau$ whose counting measure defines an extreme point for the space of all transverse measures on $\tau$. The distance in $\mathcal{C}(S)$ between a vertex cycle of $\tau$ and a curve of $\Lambda(\tau)$-length at most $\chi_{0}$ is bounded from above by a universal constant $b>0$ (see the discussion in [H09b]). Moreover, up to increasing the number $b$, for every $\tilde{q} \in \mathcal{Q}(\tau)$ with $\tilde{q}^{v} \in \mathcal{V}_{0}(\tau)$, the distance in $\mathcal{C}(S)$ between a vertex cycle for $\tau$ and the curve $\Upsilon_{\tilde{\mathcal{Q}}}(\tilde{q})$ does not exceed $b$ (see [H22] for details).

We use these facts to establish a more precise relationship between the subshift $(\Omega, \sigma)$ and the Teichmüller flow $\Phi^{t}$ on $\mathcal{Q}$. To this end call as before a (finite or infinite) sequence $\left(x_{i}\right) \subset\{1, \ldots, k\}$ admissible if $a_{x_{i} x_{i+1}}=1$ for all $i$ where $\left(a_{i j}\right)$ is the transition matrix defining the subshift of finite type $(\Omega, \sigma)$.

## Let

$$
P: \tilde{\mathcal{Q}}(S) \rightarrow \mathcal{T}(S)(\text { or } P: \tilde{\mathcal{H}}(S) \rightarrow \mathcal{T}(S))
$$

be the canonical projection. Using the above discussion, the following observation can now be deduced from Lemma 4.3 of [H09b].

Lemma 4.4. There exists a finite admissible sequence $\left(y_{i}\right)_{0 \leq i \leq n}$ and a compact subset $K$ of $\mathcal{Q}$ with the following property. Let $\left(\tau_{i}\right)_{0 \leq i \leq n} \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ be a full numbered splitting sequence realizing $\left(y_{i}\right)_{0 \leq i \leq n}$ and let $\overline{\tilde{q}} \in \mathcal{Q}\left(\tau_{0}\right)$ be such that the vertical measured geodesic lamination $\tilde{q}^{v}$ of $\overline{\tilde{q}}$ is carried by $\tau_{n}$ and is contained in $\mathcal{V}_{0}\left(\tau_{0}\right)$. Then the projection of $\tilde{q}$ to $\mathcal{Q}(S)$ is contained in $K$. Furthermore, there exists a number $\chi>0$ such that $\rho\left(x_{i}\right) \geq \chi$ whenever $x_{i}=y_{i}$ for $0 \leq i \leq n$.

Proof. Since periodic orbits for $\Phi^{t}$ are dense in $\mathcal{Q}$, for $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ there exist a quadratic differential $\tilde{q} \in \mathcal{Q}(\tau) \cap \tilde{\mathcal{Q}}$ whose projection $q$ to $\mathcal{Q}$ is contained in a periodic orbit of $\Phi^{t}$. We may assume that the vertical measured geodesic lamination $\tilde{q}^{v}$ of $\tilde{q}$ is contained in $\mathcal{V}_{0}\left(\tau_{0}\right)$. It follows from [MM99, H10b] that the restriction of the $\operatorname{map} \Upsilon_{\tilde{\mathcal{Q}}}$ to the orbit $t \rightarrow \Phi^{t} \tilde{q}$ is a quasi-geodesic, that is, it is a quasi-geodesic without changing the parameterization (but perhaps for a constant which is much larger than the constant $L>1$ above).

Let $\left(\tau_{i}\right)$ be the full splitting sequence issuing from $\tau_{0}=\tau$ consisting of train tracks which carry $\tilde{q}^{v}$. We claim that for a given number $\ell>0$ there exists a number $n(\ell)>0$ such that for $n \geq n(\ell)$, the distance in the curve graph between a vertex cycle of $\tau_{n}$ and a vertex cycle of $\tau_{0}$ is at least $\ell$. Namely, the coarsely well defined map which associates to $\tau_{i}$ one of its vertex cycles in $\mathcal{C}(S)$ is an unparameterized quasi-geodesic in $\mathcal{C}(S)$ [H06] (in fact, in the situation at hand, it is a parameterized quasi-geodesic), and this quasi-geodesic has infinite diameter by the assumption on $\tilde{q}^{v}$.

As a consequence, for a suitably chosen $\ell$ and $n=n(\ell)$, the full splitting sequence $\left(\tau_{i}\right)_{0 \leq i \leq n}$ satisfies the assumptions in Lemma 4.3 of [H09b]. Furthermore, by compactness, there exists a number $\epsilon>0$ such that whenever $\xi \in \mathcal{V}_{0}(\tau)$ is carried by $\tau_{n}$, then $\xi\left(\tau_{n}\right) \geq \epsilon$.

Let $A(n) \subset \tilde{\mathcal{Q}}(S)$ be the closed set of all quadratic differentials $\tilde{z} \in \mathcal{Q}\left(\tau_{0}\right)$ whose vertical measured geodesic lamination $\tilde{z}^{v}$ is contained in $\mathcal{V}_{0}\left(\tau_{0}\right) \cap \mathcal{V}\left(\tau_{n}\right)$. An application of Lemma 4.3 of [H09b] shows that we have $d_{\mathcal{T}}\left(P \tilde{q}, \Lambda\left(\tau_{0}\right)\right) \leq b$
for a universal constant $b>0$. Since the projection $P$ is a closed map, the set $P(A(n)) \subset \mathcal{T}(S)$ is compact. But then $A(n)$ is compact and hence $A(n)$ is a compact subset of the closure of $\tilde{\mathcal{Q}}$. Its intersection with $\tilde{\mathcal{Q}}$ dense in $A(n)$ and contains an open set.

We are left with showing that up to increasing $n$, we may assume that $A(n)$ is contained in $\tilde{\mathcal{Q}}$. However by construction, we have $A(n) \supset A(n+1)$ and moreover $\cap_{n} A(n) \subset \tilde{\mathcal{Q}}$ since $\mathcal{V}_{0}\left(\tau_{0}\right) \cap \bigcap_{i \geq 1} \mathcal{V}\left(\tau_{i}\right)=\tilde{q}^{v}$ and the support of $\tilde{q}^{v}$ is contained in $\mathcal{L} \mathcal{L}(\mathcal{Q})$. Since $A(n)$ is a compact subset of the closure of $\tilde{\mathcal{Q}}$ for all large $n$ and $\cap_{n} A(n) \subset \tilde{\mathcal{Q}}$, we conclude that indeed, $A(n) \subset \tilde{\mathcal{Q}}$ for sufficiently large $n$.

Fix such a number $n>1$. For $0 \leq i \leq n$ put $y_{i}=\left[\tau_{i}\right] \in \mathcal{E}(\mathcal{Q})$. We are left with establishing a positive lower bound for the restriction of the function $\rho$ to the cylinder set

$$
\mathcal{U} \cap\left\{\left(x_{i}\right) \in \Omega \mid x_{i}=y_{i} \text { for } 0 \leq i \leq n\right\}
$$

However, such a lower bound follows from the definition of $\rho$ and from compactness of $A(n)$, completing the proof.

Call a biinfinite sequence $\left(x_{j}\right) \in \Omega$ normal if every finite admissible sequence occurs in $\left(x_{j}\right)$ infinitely often in forward and backward direction. We use Lemma 4.4 to show

Lemma 4.5. Let $\left(x_{i}\right) \in \Omega$ be normal and let $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ be a full numbered splitting sequence which realizes $\left(x_{i}\right)$. Then $\cap_{i \geq 0} P \mathcal{V}\left(\tau_{i}\right)$ consists of a single uniquely ergodic projective measured geodesic lamination whose support is contained in $\mathcal{L} \mathcal{L}(\mathcal{Q})$.

Proof. Let $\left(y_{i}\right)_{0 \leq i \leq n}$ be an admissible sequence as in Lemma 4.4. Let $\left(x_{i}\right) \in \Omega$ be a normal sequence, let $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ be a full numbered splitting sequence which realizes $\left(x_{i}\right)$ and let $\tilde{q} \in \mathcal{Q}\left(\tau_{0}\right)$ be a differential whose vertical measured geodesic lamination $\tilde{q}^{v}$ satisfies $\tilde{q}^{v} \in \mathcal{V}_{0}\left(\tau_{0}\right) \cap \bigcap_{i \geq 0} \mathcal{V}\left(\tau_{i}\right)$. As the sets $P \mathcal{V}\left(\tau_{i}\right) \supset P \mathcal{V}\left(\tau_{i+1}\right)$ are compact and non-empty, such a differential exists. Let $q$ be the projection of $\tilde{q}$ to $\mathcal{Q}$.

Since the horizontal measured geodesic lamination $\tilde{q}^{h}$ of $\tilde{q}$ is carried by $\tau_{0}^{*}$, it is carried by $\tau_{i}^{*}$ for each $i \geq 0$. Thus if $j>0$ is such that $x_{j+i}=y_{i}$ for $0 \leq i \leq n$ and if $t_{j} \geq 0$ is such that $e^{t_{j}} \tilde{q}^{v} \in \mathcal{V}_{0}\left(\tau_{j}\right)$ then by Lemma 4.4 , the projection to $\mathcal{Q}(S)$ of the differential $\Phi^{t_{j}} \tilde{q}$ is contained in a fixed compact subset $K$ of $\mathcal{Q}$. Furthermore, we have $t_{j+1} \geq t_{j}+\chi$ for all $j$ where $\chi>0$ is as in Lemma 4.4.

As a consequence, each occurrence of the sequence $\left(y_{i}\right)_{0 \leq i \leq n}$ in the sequence $\left(x_{i}\right)$ corresponds to a subsegment of the orbit of $q$ under the Teichmüller flow of length at least $\chi$ which is contained in a fixed compact subset $K^{\prime}$ of $\mathcal{Q}(S)$, and segments corresponding to disjoint subsequences $\left(y_{i}\right)_{0 \leq i \leq n}$ correspond to disjoint orbit segments. Thus $\Phi^{t} q$ recurs to $K$ for arbitrarily large times. Invoking Lemma 3.4 then completes the proof of the lemma.

By Lemma 4.5, the set of normal points in $\Omega$ is contained in the set $\mathcal{U}$ of uniquely ergodic points. Since normal points are dense in $\Omega$, the same is true for uniquely ergodic points.

As in Section 2, for $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ let $\mathcal{V}^{*}(\tau)$ be the set of all measured geodesic laminations carried by $\tau^{*}$ and denote by $P \mathcal{V}^{*}(\tau)$ the projectivization of $\mathcal{V}^{*}(\tau)$. If $\tau^{\prime} \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ is obtained from $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ by a single split at a large branch $e$ and if $C$ is the matrix which describes the transformation $\mathcal{V}\left(\tau^{\prime}\right) \rightarrow \mathcal{V}(\tau)$ then the dual transformation $\mathcal{V}^{*}(\tau) \rightarrow \mathcal{V}^{*}\left(\tau^{\prime}\right)$ is given by the transposed matrix $C^{t}$ [PH92]. Thus as in the proof of Lemma 4.5 we observe
Lemma 4.6. Let $\left(x_{i}\right) \in \Omega$ be normal and let $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ be a full numbered splitting sequence which realizes $\left(x_{i}\right)$; then $\cap_{i<0} P \mathcal{V}^{*}\left(\tau_{i}\right)$ consists of a single uniquely ergodic projective measured geodesic lamination whose support is contained in $\mathcal{L} \mathcal{L}(\mathcal{Q})$.

We call the sequence $\left(x_{i}\right) \in \Omega$ doubly uniquely ergodic if $\left(x_{i}\right)$ is uniquely ergodic as defined above and if moreover for one (and hence every) full numbered splitting sequence $\left(\tau_{i}\right) \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ which realizes $\left(x_{i}\right)$ the intersection $\cap_{i<0} \mathcal{P} \mathcal{V}^{*}\left(\tau_{i}\right)$ consists of a unique projective measured geodesic lamination whose support is contained in $\mathcal{L} \mathcal{L}(\mathcal{Q})$. By Lemma 4.5 and Lemma 4.6, every normal sequence is doubly uniquely ergodic and hence the Borel set

$$
\mathcal{D U} \subset \Omega
$$

of all doubly uniquely ergodic sequences $\left(x_{i}\right) \in \Omega$ is dense.
Now let $\left(x_{i}\right) \in \mathcal{D U}$ and let $\left(\tau_{i}\right)$ be a full numbered splitting sequence which realizes $\left(x_{i}\right)$. By the above discussion, $\left(\tau_{i}\right)$ determines a pair $(\mu, \nu)$ of measured geodesic laminations by the requirement that $\mu \in \mathcal{V}_{0}\left(\tau_{0}\right) \cap \bigcap_{i \geq 0} \mathcal{V}\left(\tau_{i}\right)$, that $\nu \in$ $\cap_{i \leq 0} \mathcal{V}^{*}\left(\tau_{i}\right)$ and that $\iota(\mu, \nu)=1$. By equivariance under the action of the mapping class group, this implies that every sequence $\left(x_{i}\right) \in \mathcal{D} \mathcal{U}$ determines a quadratic differential

$$
\begin{equation*}
\Xi\left(x_{i}\right) \in \mathcal{Q} \subset \mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-m\right) \tag{2}
\end{equation*}
$$

By the definition of $\mathcal{D} \mathcal{U}$, this quadratic differential is contained in the subset

$$
\mathcal{U} \mathcal{Q} \subset \mathcal{Q}
$$

of all area one quadratic differentials in $\mathcal{Q}$ whose vertical and horizontal measured geodesic laminations are uniquely ergodic, with support in $\mathcal{L} \mathcal{L}(\mathcal{Q})$. Note that $\mathcal{U} \mathcal{Q}$ is a $\Phi^{t}$-invariant Borel subset of $\mathcal{Q}$. Its preimage $\mathcal{U} \tilde{\mathcal{Q}}$ in $\tilde{\mathcal{Q}}(S)$ (or in $\tilde{\mathcal{H}}(S)$ ) is a $\Phi^{t}$-invariant Borel subset of $\tilde{\mathcal{Q}}$.

Our next goal is to show that the map

$$
\Xi: \mathcal{D U} \rightarrow \mathcal{Q}
$$

is finite-to-one. To this end let $\mathcal{Q}_{0}(\tau)$ be the set of all $\tilde{q} \in \mathcal{Q}(\tau)$ with the property that the vertical measured geodesic lamination of $\tilde{q}$ is contained in $\mathcal{V}_{0}(\tau)$. Recall Lemma 4.1.

Lemma 4.7. For $\tilde{q} \in \mathcal{U} \tilde{\mathcal{Q}}$ there is a neighborhood $V$ of $\tilde{q}$ in $\tilde{\mathcal{Q}}$, and there are finitely many train tracks $\tau_{1}, \ldots, \tau_{n} \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ (where $n \geq 1$ depends on $q$ ) with the following property. If $\eta \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ is such that $\Phi^{t} \tilde{z} \in \mathcal{Q}_{0}(\eta)$ for some $\tilde{z} \in V$ and some $t \in[0, p \log 2]$ then $\eta \in\left\{\tau_{1}, \ldots, \tau_{n}\right\}$.

Proof. Recall the definition of the hyperbolic metric $\Lambda(\tau) \in \mathcal{T}(S)$ for a large train track $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$. For the proof of the lemma it suffices to show that for $\tilde{q} \in \mathcal{U} \tilde{\mathcal{Q}}$ there is a neighborhood $V$ of $\tilde{q}$ in $\tilde{\mathcal{Q}}$ and a number $R>0$ with the following property. If $\tilde{z} \in V$, if $t \in[0, p \log 2]$ and $\eta \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ are such that $\Phi^{t} \tilde{z} \in \mathcal{Q}_{0}(\eta)$ then $d_{\mathcal{T}}(\Lambda(\eta), P \tilde{q}) \leq R$.

To show that this indeed holds true we follow the reasoning in Section 4 and Section 5 of [H09b]. Namely, note first that there exists a number $b>0$ with the following property. Let $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ and let $\tilde{q} \in \mathcal{Q}_{0}(\tau)$; then the distance in the curve graph between a vertex cycle of $\tau$ and the curve $\Upsilon_{\tilde{\mathcal{Q}}}(\tilde{q})$ is at most $b$.

By Lemma 4.2 of [H09b], there is a number $\ell>0$, and for every $\epsilon>0$ there is a number $n(\epsilon)>0$ with the following property. Let $\sigma, \tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ and assume that $\sigma$ is carried by $\tau$ and that the distance in $\mathcal{C}(S)$ between any vertex cycle of $\sigma$ and any vertex cycle of $\tau$ is at least $\ell$. Let $\tilde{q} \in \mathcal{Q}_{0}(\tau)$ be a differential whose vertical measured geodesic lamination $\tilde{q}^{v}$ is carried by $\sigma$. If the total weight of the transverse measure on $\sigma$ defined by $\tilde{q}^{v}$ is not smaller than $\epsilon$, then $d_{\mathcal{T}}(\Lambda(\tau), P \tilde{q}) \leq n(\epsilon)$.

Now let $\tilde{q} \in \mathcal{U} \tilde{\mathcal{Q}}$. Then $d\left(\Upsilon_{\tilde{\mathcal{Q}}}\left(\Phi^{t} \tilde{q}\right), \Upsilon_{\tilde{\mathcal{Q}}}(\tilde{q})\right) \rightarrow \infty(t \rightarrow \infty)$ (compare [MM99, H06]) and therefore by continuity, there is a neighborhood $V$ of $\tilde{q}$ in $\tilde{\mathcal{Q}}$ and there is a number $T>0$ such that

$$
d\left(\Upsilon_{\tilde{\mathcal{Q}}}\left(\Phi^{t} \tilde{z}\right), \Upsilon_{\tilde{\mathcal{Q}}}\left(\Phi^{s} \tilde{z}\right)\right) \geq \ell+2 b+2 p \log 2
$$

for all $t \geq T$, all $s \in[0, p \log 2]$ and for all $\tilde{z} \in V$.
Let $\tilde{z} \in V$, let $s \in[0, p \log 2]$ and let $\eta \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ be such that $\Phi^{s} \tilde{z} \in \mathcal{Q}_{0}(\eta)$. If $\sigma \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ is such that $\Phi^{t} \tilde{z} \in \mathcal{Q}_{0}(\sigma)$ for some $t \in[T, T+p \log 2]$ and if $\sigma$ can be obtained from $\eta$ by a full splitting sequence, then by the choice of the constants $b>0, T>0$ the distance in $\mathcal{C}(S)$ between a vertex cycle of $\eta$ and a vertex cycle of $\sigma$ is at least $\ell$. Thus $\sigma, \eta$ satisfy the hypothesis in Lemma 4.2 of [H09b] as stated above with a number $\epsilon \geq e^{-T-p \log 2}$. This implies that $d_{\mathcal{T}}(P \tilde{z}, \Lambda(\eta)) \leq n(\epsilon)$ which completes the proof of the lemma.

Recall that the function $\rho$ on the dense shift invariant Borel set $\mathcal{D U} \subset \Omega$ is continuous, positive and bounded from above by $p \log 2$. The suspension for the shift $\sigma$ on the invariant subspace $\mathcal{D U}$ with roof function $\rho$ is the space

$$
X=\left\{\left(x_{i}\right) \times\left[0, \rho\left(x_{i}\right)\right] \mid\left(x_{i}\right) \in \mathcal{D} \mathcal{U}\right\} / \sim
$$

where the equivalence relation $\sim$ identifies the point $\left(\left(x_{i}\right), \rho\left(x_{i}\right)\right)$ with the point $\left(\sigma\left(x_{i}\right), 0\right)$. Note that $\sim$ is a closed equivalence relation on $\mathcal{D U}$ since the function $\rho$ is continuous. There is a natural flow $\Theta^{t}$ on $X$ defined by $\Theta^{t}(x, s)=\left(\sigma^{j} x, \tilde{s}\right)$ (for $t \geq 0)$ where $j \geq 0$ is such that $0 \leq \tilde{s}=t+s-\sum_{i=0}^{j-1} \rho\left(\sigma^{i} x\right)<\rho\left(\sigma^{j} x\right)$.

A semi-conjugacy of $\left(X, \Theta^{t}\right)$ into a flow space $\left(Y, \Phi^{s}\right)$ is a continuous map $\Psi$ : $X \rightarrow Y$ such that $\Phi^{t} \Psi(x)=\Psi\left(\Theta^{t} x\right)$ for all $x \in X$ and all $t \in \mathbb{R}$. We call a semi-conjugacy $\Psi$ finite-to-one if the number of preimages of any point is finite.

By construction, there is a natural extension of the map $\Xi$ defined in equation (2) to the suspension flow $\left(X, \Theta^{t}\right)$, again denoted by $\Xi$, which is a semi-conjugacy of $\left(X, \Theta^{t}\right)$ into $\left(\mathcal{Q}, \Phi^{t}\right)$. The next lemma gives additional information on $\Xi$.

Corollary 4.8. The semi-conjugacy $\Xi:\left(X, \Theta^{t}\right) \rightarrow\left(\mathcal{Q}, \Phi^{t}\right)$ is finite-to-one. Its image equals the $\Phi^{t}$-invariant subset $\mathcal{U} \mathcal{Q} \subset \mathcal{Q}$.

Proof. For every $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ and every $\tilde{q} \in \mathcal{Q}(\tau)$ whose vertical measured geodesic lamination $\tilde{q}^{v}$ has support $\nu \in \mathcal{L} \mathcal{L}(\mathcal{Q})$ there is a unique full numbered splitting sequence $\left(\tau_{i}\right)_{i>0}$ issuing from $\tau_{0}=\tau$ which consists of train tracks carrying $\tilde{q}^{v}$ [H09a]. Thus Lemma 4.7 implies that for every $x \in X$ the cardinality of $\Xi^{-1}(\Xi(x))$ is finite. The map $\Xi$ is clearly a semi-conjugacy of $X$ into $\mathcal{U} \mathcal{Q}$. Continuity follows as in the proof of Lemma 4.3. Consequently we are left with showing that $\Xi(X)$ is all of $\mathcal{U Q}$.

For this let $q \in \mathcal{U Q}$ and let $\tilde{q}$ be a lift of $q$ to $\tilde{\mathcal{Q}}$. By Lemma 3.6, there is some $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ which is contained in $\mathcal{E}(\mathcal{Q})$ and such that $\tilde{q} \in \mathcal{Q}(\tau)$. If $\tilde{q}^{v}, \tilde{q}^{h}$ are the vertical and horizontal measured geodesic laminations of $\tilde{q}$, respectively, then there is a biinfinite full numbered splitting sequence $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ issuing from $\tau$ such that the intersection $\cap_{i>0} P \mathcal{V}\left(\tau_{i}\right)$ consists of a unique point which is just the class of $\tilde{q}^{v}$, and that the intersection $\cap_{i<0} P \mathcal{V}^{*}(\tau)$ consists of a unique point which is the class of $\tilde{q}^{h}$. Since the suspension of the orbit of a point in $\mathcal{D U}$ under the shift is mapped to a biinfinite flow line of the Teichmüller flow, this implies that $q \in \Xi(X)$.

Remark 4.9. The Teichmüller flow is not hyperbolic, and the map $\Xi$ is not bounded-to-one.

Since the roof function $\rho$ on $\mathcal{D U} \subset \Omega$ is continuous, uniformly bounded and positive, every $\sigma$-invariant Borel probability measure $\nu$ on $\Omega$ which gives full mass to $\mathcal{D U}$ induces an invariant measure $\tilde{\nu}$ for the suspension flow $\left(X, \Theta^{t}\right)$ of total mass $\int \rho d \nu<\infty$. The measure $\tilde{\nu}$ is defined by $d \tilde{\nu}=d \nu \times d t$ where $d t$ is the Lebesgue measure on the flow lines of the suspension flow.

Since by Lemma 4.7, every $q \in \mathcal{Q}$ has a neighborhood $V$ so that the cardinality of the preimage of any point $q \in V$ in bounded from above by a constant not depending on $q$, we can push forward the measure $\tilde{\nu}$ with the semi-conjugacy $\Xi$ and obtain a finite $\Phi^{t}$-invariant Borel measure on $\mathcal{Q}$ which we may normalize to have total mass one. Thus if $\mathcal{M}_{\sigma}(\mathcal{D} \mathcal{U})$ denotes the space of all $\sigma$-invariant Borel probability measures on $\Omega$ which give full measure to $\mathcal{D U}$ then $\Xi$ induces a map

$$
\Xi_{*}: \mathcal{M}_{\sigma}(\mathcal{D U}) \rightarrow \mathcal{M}_{\mathrm{inv}}(\mathcal{Q})
$$

where $\mathcal{M}_{\text {inv }}(\mathcal{Q})$ is the space of $\Phi^{t}$-invariant Borel probability measures on $\mathcal{Q}$. We equip both spaces with the weak*-topology. We have

Lemma 4.10. The map $\Xi_{*}$ is continuous.

Proof. Since $\Omega$ is a compact metrizable space, the space of all Borel probability measures on $\Omega$ equipped with the weak*-topology is compact and metrizable. Thus we only have to show that whenever $\mu_{i} \rightarrow \mu$ in $\mathcal{M}_{\sigma}(\mathcal{D U})$ then $\Xi_{*}\left(\mu_{i}\right) \rightarrow \Xi_{*}(\mu)$.

Now by Lemma 4.1, the function $\rho$ is continuous on $\mathcal{D U}$, bounded and positive and hence if $\mu_{i} \rightarrow \mu$ in $\mathcal{M}_{\sigma}(\mathcal{D} \mathcal{U})$ then $\int \rho d \mu_{i} \rightarrow \int \rho d \mu>0$. In particular, we have $\tilde{\mu}_{i}(X) \rightarrow \tilde{\mu}(X)$ where $\tilde{\mu}_{i}, \tilde{\mu}$ are the finite Borel measures on the suspension space
$\left(X, \Theta^{t}\right)$ defined by the measures $\mu_{i}, \mu$. Therefore $\Xi_{*}\left(\mu_{i}\right) \rightarrow \Xi_{*}(\mu)$ if and only if for every continuous function $f$ on $\mathcal{Q}$ with compact support we have $\int f \circ \Xi d \tilde{\mu}_{i} \rightarrow$ $\int f \circ \Xi d \tilde{\mu}$. However, since $\Xi$ is continuous this is immediate.

The next result completes the proof of Theorem 1 from the introduction.
Lemma 4.11. The map $\Xi_{*}: \mathcal{M}_{\sigma}(\mathcal{D U}) \rightarrow \mathcal{M}_{\mathrm{inv}}(\mathcal{Q})$ is surjective.

Proof. It suffices to show that every ergodic $\Phi^{t}$-invariant Borel probability measure on $\mathcal{Q}$ is contained in the image of $\Xi_{*}$.

Thus let $\nu$ be an ergodic $\Phi^{t}$-invariant Borel probability measure on $\mathcal{Q}$. By the Birkhoff ergodic theorem, there is a density point $q \in \mathcal{Q}$ for $\nu$ such that the Borel probability measures

$$
\nu_{T}=\frac{1}{T} \int_{0}^{T} \delta_{\Phi^{t} q} d t
$$

converge weakly to $\nu$ as $T \rightarrow \infty$ where $\delta_{x}$ denotes the Dirac mass at $x$. By Lemma 3.4 and the Poincaré recurrence theorem, we have $q \in \mathcal{U Q}$. Hence by Corollary 4.8, up to possibly replacing $q$ by $\Phi^{t} q$ for some $t \in \mathbb{R}$ there is some $\left(x_{i}\right) \in \mathcal{D U}$ with $\Xi\left(x_{i}\right)=q$.

By Lemma 4.7, there is a neighborhood $V$ of $q$ in $\mathcal{Q}$ such that the preimage of $V \cap \mathcal{U Q}$ under the map $\Xi$ is a finite union of Borel sets $W_{i} \subset X(i=1, \ldots, n)$ with the property that the restriction of $\Xi$ to each of the sets $W_{i}$ is injective. As $\nu_{T}|V \rightarrow \nu| V$ weakly as $T \rightarrow \infty$ and as the map $\Xi$ is equivariant with respect to the suspension flow and the Teichmüller flow, we conclude that the restriction to $\cup_{i=1}^{k} W_{i}$ of the Borel probability measures

$$
\tilde{\nu}_{T}=\frac{1}{T} \int_{0}^{T} \delta_{\Theta^{t}\left(x_{i}\right)} d t
$$

converge weakly to a measure on $\cup_{i} W_{i}$ which projects to the measure $\nu$ on $V$.

Since $q$ was an arbitrary density point for $\nu$, we deduce that the measures $\tilde{\nu}_{T}$ converge weakly to a $\Theta^{t}$-invariant Borel probability measure on $X$ whose image under the map $\Xi_{*}$ equals $\nu$. Thus $\Xi_{*}$ is surjective.

Consider again the shift space $(\Omega, \sigma)$. Let $f: \Omega \rightarrow \mathbb{R}$ be a Hölder continuous function. Then $f$ defines an equilibrium state $\mu_{f}$ which is an ergodic mixing $\sigma$ invariant Borel probability measure on $\Omega$. This measure gives full mass to normal sequences and hence it gives full support to the domain of the map $\Xi$. Since $\Xi$ is finite-to-one we conclude

Theorem 4.12. Any Gibbs equilibrium state on $\Omega$ induces via the map $\Xi_{*} a \Phi^{t}$ invariant mixing Borel probability measure on $\mathcal{Q}$.

## 5. The measure of maximal entropy

In this section we use the subshift of finite type constructed in Sections 2-4 to show that for every component $\mathcal{Q}$ of a stratum, the $\Phi^{t}$-invariant probability measure $\lambda$ in the Lebesgue measure class is the unique measure of maximal entropy. For strata of abelian differentials, this was earlier shown by Bufetov and Gurevich [BG11].

The strategy is as follows. Let $q \in \mathcal{Q}$ be any birecurrent point which is contained in its own $\alpha$-and $\omega$-limit set for the flow $\Phi^{t}$. By the Poincaré recurrence theorem, for every $\Phi^{t}$-invariant Borel probability measure $\mu$ the set of such points is of full $\mu$-mass. Starting from the symbolic system constructed in Section 3, we construct a topological Markov shift on a countable set $\mathcal{S}$ of symbols, given by a transition $\operatorname{matrix} A=\left(a_{i j}\right)_{\mathcal{S} \times \mathcal{S}}$. The phase space of this shift is the space

$$
\Sigma=\left\{\left(y_{i}\right) \in \mathcal{S}^{\mathbb{Z}} \mid a_{y_{i} y_{i+1}}=1 \text { for all } i\right\} .
$$

We find a positive roof function $\varphi: \Sigma \rightarrow(0, \infty)$ of bounded variation and only depending on the future such that the suspension of the shift $T: \Sigma \rightarrow \Sigma$ with roof function $\varphi$ admits a bounded-to-one semi-conjugacy into ( $\mathcal{Q}, \Phi^{t}$ ). Its image $\mathcal{D}$ is $\Phi^{t}$-invariant and contains all points $z \in \mathcal{Q}$ which contain the fixed quadratic differential $q$ in their $\alpha$-and $\omega$-limit set. Since the Lebesgue measure $\lambda$ on $\mathcal{Q}$ has full support and is ergodic under the Teichmüller flow, $\lambda$ - almost every orbit for $\Phi^{t}$ is dense in $\mathcal{Q}$. Thus the set $\mathcal{D}$ is of full Lebesgue measure.

We use this coding and a result of Sarig [S99] to show that the supremum of the entropies of all $\Phi^{t}$-invariant Borel probability measures on $\mathcal{Q}$ which give full mass to $\mathcal{D}$ is the supremum of the entropies of all such measures which are supported in some compact invariant subset of $\mathcal{Q}$. That the supremum of the entropies of all invariant probability measures on $\mathcal{Q}$ supported in a compact subset of $\mathcal{Q}$ equals the entropy $h$ of the Lebesgue measure $\lambda$ was established in [H10a]. Thus $\lambda$ is a measure of maximal entropy for the Teichmüller flow on $\mathcal{Q}$. We then apply a result of Buzzi and Sarig [BS03] to conclude that there exists at most one such measure. Together this shows Theorem 2 from the introduction.

We need the following simple observation.
Lemma 5.1. Let $\left(\tau_{i}\right)_{0 \leq i \leq n} \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ be any finite full splitting sequence, let $a_{0}>0$ and suppose that $\mu, \nu \in \mathcal{V}\left(\tau_{n}\right)$ fulfill $\mu(b) \leq a_{0} \nu(b)$ for every branch $b$ of $\tau_{n}$. Then the transverse measures $\mu_{0}, \nu_{0}$ on $\tau_{0}$ defined by $\mu, \nu$ via a carrying map $\tau_{n} \rightarrow \tau_{0}$ satisfy $\mu_{0}(e) \leq a_{0} \nu_{0}(e)$ for every branch $e$ of $\tau_{0}$.

Proof. The lemma follows immediately from the fact that the natural map $\mathcal{V}\left(\tau_{n}\right) \rightarrow$ $\mathcal{V}\left(\tau_{0}\right)$ induced by a carrying map $\tau_{n} \rightarrow \tau_{0}$ is the restriction of a linear map from the finite dimensional vector space of weight functions on the branches of $\tau_{n}$ to the vector space of weight functions on the branches of $\tau_{0}$ which preserves positivity.

We say that a simple closed curve $c$ on $S$ fills a large train track $\tau$ if $c$ is carried by $\tau$ and if the transverse counting measure on $\tau$ defined by $c$ which counts the number of transitions of $c$ through each branch of $\tau$ is positive on every branch.

For technical reasons we define now a norm-like quantity which measures the difference between two projective measured geodesic laminations $[\mu],[\nu]$ which are carried by a train track $\tau$ and define positive weight functions on $\tau$. Namely, choose representatives $\mu, \nu \in \mathcal{V}(\tau)$ of $[\mu],[\nu]$ and put

$$
\begin{aligned}
(\mu \mid \nu)_{\tau}^{0} & =\max \{\mu(b) / \nu(b), \nu(b) / \mu(b) \mid b \text { is a branch of } \tau\} \text { and } \\
([\mu] \mid[\nu])_{\tau} & =\min \left\{(\mu \mid a \nu)_{\tau}^{0} \mid a>0\right\} .
\end{aligned}
$$

Note that $([\mu] \mid[\nu])_{\tau}$ indeed only depends on the projective classes of $\mu, \nu$. Moreover, we have $([\mu] \mid[\nu])_{\tau}=([\nu] \mid[\mu])_{\tau} \geq 1$ for all $[\mu],[\nu]$, with equality if and only if $[\mu]=[\nu]$.

Using the assumptions and notations from Sections 2-4 we observe
Lemma 5.2. Let $\tau_{0} \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ and let $\zeta \in \mathcal{V}_{0}(\tau)$ be a uniquely ergodic geodesic lamination with support in $\mathcal{L} \mathcal{L}(\mathcal{Q})$. Let $\left(\tau_{i}\right) \subset \mathcal{N} \mathcal{T}(\mathcal{Q})$ be the full splitting sequence starting at $\tau_{0}$ with $\cap_{i} \mathcal{V}\left(\tau_{i}\right)=(0, \infty) \zeta$. Then there is some $n>0$ with the following properties.
(1) There exists a number $\beta>0$ such that $\mu(b) / \mu\left(b^{\prime}\right) \geq \beta$ for every $\mu \in \mathcal{V}\left(\tau_{0}\right)$ which is carried by $\tau_{n}$ and all branches $b, b^{\prime}$ of $\tau_{0}$. In particular, every vertex cycle of $\tau_{n}$ fills $\tau_{0}$.
(2) There is a number $\delta>0$ with the following property. Let $\mu, \nu \in \mathcal{V}\left(\tau_{n}\right)$ be positive transverse measures; then

$$
([\mu] \mid[\nu])_{\tau_{0}}^{-1} \geq([\mu] \mid[\nu])_{\tau_{n}}^{-1}(1-\delta)+\delta
$$

Proof. Let $\tau_{0} \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ and let $\zeta \in \mathcal{V}_{0}\left(\tau_{0}\right)$ be a uniquely ergodic measured geodesic lamination whose support is contained in $\mathcal{L} \mathcal{L}(\mathcal{Q})$. Then the $\zeta$-weight of every branch of $\tau_{0}$ is positive.

Let $\left(\tau_{i}\right)$ be the full (numbered) splitting sequence such that for every $i$, the train track $\tau_{i}$ carries $\zeta$. For each $i$ let

$$
A_{i}=\mathcal{V}_{0}\left(\tau_{0}\right) \cap \mathcal{V}\left(\tau_{i}\right)
$$

be the set of all normalized transverse measures on $\tau_{0}$ defined by measured geodesic laminations which are carried by $\tau_{i}$. Then $A_{i+1} \subset A_{i}$ and moreover $\cap_{i} A_{i}=\{\zeta\}$. Since the $\zeta$-weight of every branch of $\tau_{0}$ is positive, there is a number $\kappa>0$ so that for sufficiently large $i$, say for all $i \geq i_{0}$, and for every $\nu \in A_{i}$ we have

$$
\begin{equation*}
\min \left\{\nu(b) / \nu\left(b^{\prime}\right) \mid b, b^{\prime} \text { are branches of } \tau_{0}\right\} \geq \kappa \tag{3}
\end{equation*}
$$

In particular, every vertex cycle of $\tau_{i}$ fills $\tau_{0}$. This shows the first part of the lemma.
Choose a carrying map $F: \tau_{i_{0}} \rightarrow \tau_{0}$ which maps switches to switches and branches to trainpaths on $\tau_{0}$. Using the fact that every leaf of $\zeta$ is dense in $\zeta$ and the unzipping procedure introduced in [PH92], by possibly enlarging $i_{0}$ we may assume that the image under $F$ of any branch $z$ of $\tau_{i_{0}}$ is all of $\tau_{0}$.

Apply the first part of the lemma to the full splitting sequence $\left(\tau_{j}\right)_{j \geq i_{0}}$ starting at $\tau_{i_{0}}$. Let $n>i_{0}$ be such that the estimate (3) holds for all $\nu \in \mathcal{V}\left(\tau_{i_{0}}\right)$ which are carried by $\tau_{n}$, with perhaps a different constant $\kappa$. By Lemma 5.1 we have
$a_{0}=([\mu] \mid[\nu])_{\tau_{i_{0}}} \leq([\mu] \mid[\nu])_{\tau_{n}}$ and hence it suffices to find a number $\delta>0$ such that $([\mu] \mid[\nu])_{\tau_{0}}^{-1} \geq(1-\delta) a_{0}^{-1}+\delta$.

Assume that $\mu, \nu \in \mathcal{V}\left(\tau_{n}\right)$ are positive and chosen so that

$$
([\mu] \mid[\nu])_{\tau_{i_{0}}}=\max \left\{\mu(b) / \nu(b), \nu(b) / \mu(b) \mid b \text { is a branch of } \tau_{i_{0}}\right\}=a_{0}>1
$$

By the definition of $([\mu] \mid[\nu])_{\tau_{i_{0}}}$, this implies that there exists a branch $b$ of $\tau_{i_{0}}$ with $\mu(b) / \nu(b)=a_{0}$, and there exists a branch $e$ with $\nu(e) / \mu(e)=a_{0}$.

For a branch $z$ of $\tau_{i_{0}}$ and a branch $u$ of $\tau_{0}$ let $\beta(z)(u)$ be the number of times the trainpath $F(z)$ passes through $u$. Let

$$
m_{1}=\max \{\beta(z)(u) \mid z, u\} \text { and } m_{0}=\min \{\beta(z)(u) \mid z, u\} \geq 1
$$

Denote as before by $p>2$ the number of branches of $\tau_{0}$. Then for every branch $z$ of $\tau_{i_{0}}$ and $u$ of $\tau_{0}$ we have

$$
\begin{equation*}
\beta(z)(u) \geq \frac{m_{0}}{m_{1} p} \cdot \sum_{s} \beta(s)(u) \tag{4}
\end{equation*}
$$

By linearity, for any branch $u$ of $\tau_{0}$ we have

$$
\begin{aligned}
\nu(u) & =\sum_{s \neq e} \nu(s) \beta(s)(u)+\nu(e) \beta(e)(u) \geq a_{0}^{-1} \sum_{s \neq e} \mu(s) \beta(s)(u)+a_{0} \mu(e) \beta(e)(u) \\
& =a_{0}^{-1} \mu(u)+\left(a_{0}-a_{0}^{-1}\right) \mu(e) \beta(e)(u)
\end{aligned}
$$

Since by the estimate (3) we have $\mu(e) \geq \kappa \max \{\mu(z) \mid z\}$, together with the estimate (4) we conclude that

$$
\mu(e) \beta(e)(u) \geq \frac{m_{0} \kappa}{m_{1} p} \mu(u)
$$

Now for $\delta=m_{0} \kappa / m_{1} p$ we obtain

$$
\nu(u) \geq a_{0}^{-1} \mu(u)+\delta\left(a_{0}-a_{0}^{-1}\right) \mu(u)=\left(a_{0}^{-1}+\delta\left(a_{0}-a_{0}^{-1}\right)\right) \mu(u)
$$

Since $1 \leq a_{0}$, the branch $u$ of $\tau_{0}$ was arbitrary and we can exchange the roles of $\mu, \nu$, this yields the desired estimate.

We call a finite full splitting sequence $\left(\tau_{i}\right)_{0 \leq i \leq n}$ weakly tight if the train tracks $\tau_{n} \prec \tau_{0}$ have properties (1) and (2) stated in Lemma 5.2. It follows from the definitions and Lemma 5.2 that if $\left(\tau_{i}\right)_{0 \leq i \leq n}$ is weakly tight, then the same holds true for the sequence $\left(\tau_{i}\right)_{0 \leq i \leq n+1}$ where $\tau_{n+1}$ is obtained from $\tau_{n}$ by a full split.

Let $q \in \mathcal{Q}$ be any point which is contained in the both the $\alpha$ - and $\omega$-limit set of its orbit under the flow $\Phi^{t}$. Let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of $q$. By Lemma 3.4, the vertical and horizontal measured geodesic lamination of $\tilde{q}$ is strongly uniquely ergodic, with support $\zeta \in \mathcal{L} \mathcal{L}(\mathcal{Q})$. Let $p>0$ be as in Lemma 4.1. By Lemma 3.5 and Lemma 4.7, there is a number $\ell \geq 1$, and there are $\ell$ large numbered train tracks $\tau_{1}, \ldots, \tau_{\ell} \in \mathcal{E}(\mathcal{Q})$ and such that $\Phi^{t} \tilde{q} \in \mathcal{Q}_{0}(\eta)$ for some $t \in[0, p \log 2]$ and some $\eta \in \mathcal{E}(\mathcal{Q})$ if and only if $\eta \in\left\{\tau_{1}, \ldots, \tau_{\ell}\right\}$.

By Lemma 5.2, there is a number $n>0$ such that the following holds true. Let $i \leq \ell$ and let $\left(\sigma_{j}^{i}\right)_{0 \leq j \leq n}$ be a full numbered splitting sequence of length $n$
issuing from $\sigma_{0}^{i}=\tau_{i}$ with the property that $\sigma_{n}^{i}$ carries the support $\zeta$ of the vertical measured geodesic lamination of $\tilde{q}$. Then the sequence $\left(\sigma_{j}^{i}\right)_{0 \leq j \leq n}$ is weakly tight.

Recall the definition of an admissible sequence. Define $\mathcal{S}$ to be the set of all finite admissible sequences $\left(x_{i}\right)_{0 \leq i \leq s}$ with the following additional properties.
(1) $s \geq 2 n$ and the sequences $\left(x_{j}\right)_{0 \leq j \leq n}$ and $\left(x_{j}\right)_{s-n \leq j \leq s}$ are realized by one of the full splitting sequences $\left(\sigma_{j}^{i}\right)_{0 \leq j \leq n}(i \leq n)$.
(2) There is no number $t \in[n, s-n)$ such that the sequence $\left(x_{j}\right)_{t \leq j \leq t+n}$ is realized by one of the full splitting sequences $\left(\sigma_{j}^{i}\right)_{0 \leq j \leq n}$.

Note that $\mathcal{S}$ is a countable set.
Define a transition matrix $A=\left(a_{i j}\right)_{\mathcal{S} \times \mathcal{S}}$ by requiring that $a_{i j}=1$ if and only if the sequence $\left(x_{\ell}\right)_{0 \leq \ell \leq s}$ representing the symbol $i$ and the sequence $\left(y_{t}\right)_{0 \leq t \leq u}$ representing the symbol $j$ satisfy $y_{t}=x_{s-n+t}$ for every $t \in\{0, \ldots, n\}$. By construction and the properties of the set $\mathcal{E}(\mathcal{Q})$ established in Lemma 3.5,
(5) there are $i_{1}, \ldots, i_{N} \in \mathcal{S}$ such that for every $u \in \mathcal{S}, \exists j, v$ with $a_{i_{j} u} a_{u i_{v}}=1$.

In other words, the transition matrix has the big images and preimages (BIP) property as defined in [S03].

Let $\Sigma$ be the set of all biinfinite sequences $\left(y_{i}\right) \subset \mathcal{S}^{\mathbb{Z}}$ with $a_{y_{i} y_{i+1}}=1$ for all $i$, equipped with the (biinfinite) shift $T: \Sigma \rightarrow \Sigma$. There is a natural continuous injective map

$$
G: \Sigma \rightarrow \Omega
$$

whose image contains the set of all normal sequences. Here $\Omega$ is as in Lemma 3.8.
We claim that

$$
G(\Sigma) \subset \mathcal{D U}
$$

Namely, by the definition of a weakly tight sequence and by Lemma 5.1, if $\left(\tau_{i}\right)$ is a full splitting sequence which realizes $\left(y_{i}\right) \in \Sigma$ then $\cap_{i} \mathcal{V}\left(\tau_{i}\right) \cap \mathcal{V}$ ( $\tau_{0}$ ) consists of a unique positive transverse measure, and similarly for $\cap \mathcal{V}^{*}\left(\tau_{i}\right)$.

By the BIP-property (5) and the discussion at the end of Section 3, we may assume that the topological Markov chain $(\Sigma, T)$ is topologically mixing.

Define a roof function $\varphi$ on $\Sigma$ by associating to an infinite sequence $\left(y_{i}\right) \in \Sigma$ with $y_{0}=\left(x_{i}\right)_{0 \leq i \leq s}$ the value

$$
\varphi\left(y_{i}\right)=\sum_{i=0}^{s-n-1} \rho\left(\sigma^{i}\left(G\left(y_{i}\right)\right)\right)
$$

By the second part of Lemma 5.2 , the function $\varphi$ is bounded from below by a positive constant $\log C>0$, is unbounded and only depends on the future.

For $m \geq 1$ define the $m$-th variation of $\varphi$ by

$$
\operatorname{var}_{m}(\varphi)=\sup \left\{\varphi(y)-\varphi(z) \mid y_{i}=z_{i} \text { for } i=0, \ldots, m-1\right\}
$$

The following is the main technical tool towards the proof of Theorem 2.

Lemma 5.3. There are numbers $\theta \in(0,1)$ and $L>0$ such that $\operatorname{var}_{m}(\varphi) \leq L \theta^{m}$ for all $m \geq 1$. In particular,

$$
\sum_{m \geq 1} \operatorname{var}_{m}(\varphi)<\infty
$$

Proof. Let $m \geq 1$ and let $\left(y_{i}\right),\left(z_{i}\right) \in \Sigma$ be such that $y_{i}=z_{i}$ for $i=0, \ldots, m$. By definition, there is a finite full numbered splitting sequence $\left(\tau_{i}\right)_{0 \leq i \leq u} \subset \mathcal{E}(\mathcal{Q})$, there are numbers $k \leq \ell_{1}-k<\ell_{1} \leq \cdots<\ell_{m}=u$, and there are two uniquely ergodic measured geodesic laminations $\mu, \nu \in \mathcal{V}_{0}\left(\tau_{\ell_{1}-k}\right)$ with support in $\mathcal{L} \mathcal{L}(\mathcal{Q})$ such that the following holds.
(1) The measured geodesic lamination $\mu, \nu$ projects to the vertical measured geodesic lamination of $\Xi G\left(y_{i}\right), \Xi G\left(z_{i}\right)$, respectively (where $\Xi$ is as in Section 4).
(2) $\mu, \nu$ are carried by $\tau_{u}$.
(3) The sequence $\left(\tau_{i}\right)_{0 \leq i \leq n}$ and each of the sequences $\left(\tau_{i}\right)_{\ell_{j}-n \leq i \leq \ell_{j}}(j \leq m)$ is weakly tight.
(4) $\varphi\left(y_{i}\right)=\sum_{j=0}^{\ell_{1}-n-1} \rho\left(\sigma^{j} G\left(y_{i}\right)\right)$ and $\varphi\left(z_{i}\right)=\sum_{j=0}^{\ell_{1}-n-1} \rho\left(\sigma^{j} G\left(z_{i}\right)\right)$.

Let $\beta \in(0,1 / 2)$ be as in the first part of Lemma 5.2 and let $\delta \in\left(0, \beta^{2}\right)$ be as in Lemma 5.2 for the sequences arising in the definition of the alphabet $\mathcal{S}$. We show by induction on $m$ that for

$$
\kappa=1-\delta \in(1 / 2,1)
$$

we have

$$
\begin{equation*}
1-\kappa^{m} \geq([\mu] \mid[\nu])_{\tau_{\ell_{1}-n}}^{-1} \tag{6}
\end{equation*}
$$

For this note first that the claim holds true for $m=1$. Namely, by assumption, $\mu, \nu$ are carried by $\tau_{\ell_{1}}$. Thus by the properties of the sequence $\left(\tau_{i}\right)_{\ell_{1}-n \leq i \leq \ell_{1}}$ we obtain from the first part of Lemma 5.2 that $\mu(b) / \mu\left(b^{\prime}\right) \geq \beta$ for all branches $\bar{b}, b^{\prime}$ of $\tau_{\ell_{1}-n}$, and similarly for $\nu$.

Let $b_{0}$ be any branch of $\tau_{\ell_{1}-n}$ and let $\hat{\nu}$ be the multiple of $\nu$ so that $\hat{\nu}\left(b_{0}\right)=\mu\left(b_{0}\right)$. Then for all branches $b$ of $\tau_{\ell_{1}-n}$ we have

$$
\mu(b) / \hat{\nu}(b) \geq \beta^{2} \mu\left(b_{0}\right) / \hat{\nu}\left(b_{0}\right)=\beta^{2}
$$

and similarly $\hat{\nu}(b) / \mu(b) \geq \beta^{2}$. This implies that indeed, $([\mu] \mid[\nu])_{\tau_{\ell_{1}-n}}^{-1} \geq \beta^{2}$.
By induction, assume that for some $m \geq 2$ the claim holds true for $m-1$. Let $\left(y_{i}\right),\left(z_{i}\right) \in \Sigma$ be such that $y_{i}=z_{i}$ for $0 \leq i \leq m$. By the induction hypothesis, applied to the sequences $T\left(y_{i}\right), T\left(z_{i}\right)$ where as before, $T$ is the shift, we have

$$
1-\kappa^{m-1} \leq([\mu] \mid[\nu])_{\tau_{\ell_{2}-n}}^{-1}
$$

and hence $([\mu] \mid[\nu])_{\tau_{\ell_{1}}}^{-1} \geq 1-\kappa^{m-1}$ by Lemma 5.1. Lemma 5.2 now shows that

$$
([\mu] \mid[\nu])_{\tau_{\ell_{1}-n}}^{-1} \geq\left(1-\kappa^{m-1}\right)(1-\delta)+\delta=1-\kappa^{m}
$$

as claimed.

Let for the moment $\tau \in \mathcal{N} \mathcal{T}(\mathcal{Q})$ be arbitrary and let $\zeta, \xi \in \mathcal{V}_{0}(\tau)$ be positive. Assume that $([\zeta] \mid[\xi])_{\tau}=a \geq 1$. Let $\hat{\xi}=c \xi \in \mathcal{V}(\tau)$ be such that $\max \{\zeta(b) / \hat{\xi}(b), \hat{\xi}(b) / \zeta(b) \mid b\}=a$. Then we have

$$
a^{-1} \leq \hat{\xi}(\tau) \leq a
$$

and hence

$$
\max \{\zeta(b) / \xi(b), \xi(b) / \zeta(b) \mid b\} \leq a^{2}
$$

An application of this simple estimate to normalized positive transverse measures $\mu, \nu \in \mathcal{V}_{0}\left(\tau_{\ell_{1}-n}\right)$ on $\tau_{\ell_{1}-n}$ with $([\mu] \mid[\nu])_{\tau_{\ell_{1}-n}}=a \geq 1$ together with Lemma 5.1 yields $\mu\left(\tau_{0}\right) / \nu\left(\tau_{0}\right) \in\left[a^{2}, a^{-2}\right]$ and therefore if $\mu_{0}, \nu_{0} \in \mathcal{V}_{0}\left(\tau_{0}\right)$ are the normalizations of $\mu, \nu$ then

$$
\left|\varphi\left(\mu_{0}\right)-\varphi\left(\nu_{0}\right)\right|=\left|\log \mu\left(\tau_{0}\right)-\log \nu\left(\tau_{0}\right)\right| \leq-2 \log a
$$

As a consequence, the estimate (6) implies that

$$
\left|\varphi\left(y_{i}\right)-\varphi\left(z_{i}\right)\right| \leq-2 \log \left(1-\kappa^{m}\right) \text { if } y_{i}=z_{i} \text { for } 0 \leq i \leq m
$$

As $\frac{-\log (1-t)}{t} \rightarrow 1(t \rightarrow 0)$, from this the lemma follows.

By Lemma 5.3, the function $\varphi$ is defined on the entire space $\Sigma$. In particular, we can construct the suspension $\left(Y, \Psi^{t}\right)$ over $\Sigma$ with roof function $\varphi$. We have

Lemma 5.4. There is a bounded-to-one semiconjugacy $\Upsilon:\left(Y, \Psi^{t}\right) \rightarrow\left(\mathcal{Q}, \Phi^{t}\right)$. Its image contains the set of all points $z$ whose $\alpha$ - and $\omega$ limit set contains $q$.

Proof. There is an obvious semi-conjugacy $\left(Y, \Psi^{t}\right) \rightarrow\left(X, \Theta^{t}\right)$ which composes with the semi-conjugacy $\Xi$ defined in Section 3 to a semi-conjugacy

$$
\Upsilon:\left(Y, \Psi^{t}\right) \rightarrow\left(\mathcal{Q}, \Phi^{t}\right)
$$

Thus we only have to show that $\Upsilon$ is bounded-to-one. However this is a consequence of Lemma 4.7 as discussed after Lemma 5.2.

As before, let $h$ be the entropy of the $\Phi^{t}$-invariant Lebesgue measure on $\mathcal{Q}$. Let $\mathcal{M}_{T}(\Sigma)$ be the space of all $T$-invariant Borel probability measures on $\Sigma$. For $\mu \in \mathcal{M}_{T}(\Sigma)$ let

$$
\operatorname{pr}_{\mu}(-h \varphi)=h_{\mu}-h \int \varphi d \mu
$$

where $h_{\mu}$ is the entropy of $\mu$. By [S99], under the assumptions at hand, the Gurevich pressure of the function $-h \varphi$ is given by

$$
\begin{equation*}
\operatorname{pr}_{G}(-h \varphi)=\sup \left\{\operatorname{pr}_{\mu}(-h \varphi) \mid \mu \in \mathcal{M}_{T}(\Sigma), \operatorname{pr}_{\mu}(-h \varphi) \text { is well-defined }\right\} \tag{7}
\end{equation*}
$$

The following observation relies on the results of Sarig [S99] and on [H10a].
Lemma 5.5. $\operatorname{pr}_{G}(-h \varphi) \leq 0$.

Proof. By Theorem 2 of [S99], $\operatorname{pr}_{G}(-h \varphi)$ equals the supremum of the quantity defined in (7) but restricted to invariant measures $\mu$ supported in compact subsets of $\Sigma$.

By Abramov's formula, if $A \subset \Sigma$ is any compact invariant set and if $\mu$ is a $T$ invariant Borel probability measure supported in $A$ then the entropy of the induced invariant measure for the suspension flow $\left(Y, \Psi^{t}\right)$ equals

$$
h_{\mu} / \int \varphi d \mu
$$

As a consequence, the Gurevich pressure of $-h \varphi$ is nonpositive if the entropy of every $\Psi^{t}$-invariant Borel probability measure on $Y$ which is supported in a compact set does not exceed $h$.

The semi-conjugacy $\Upsilon:\left(Y, \Psi^{t}\right) \rightarrow\left(\mathcal{Q}, \Phi^{t}\right)$ is bounded-to-one, and it maps the suspension of a $T$-invariant compact set $A \subset \Sigma$ to a compact $\Phi^{t}$-invariant subset of $\mathcal{Q}$. This implies that the entropy of a $\Psi^{t}$-invariant Borel probability measure on $Y$ supported in a compact set is bounded from above by the supremum of the topological entropies of the restriction of the Teichmüller flow to compact invariant subsets of $\mathcal{Q}$. That this quantity equals $h$ was shown in [H10a]. The lemma follows.

Now we are ready to complete the proof of Theorem 2 from the introduction. Namely, let $\mu$ be any $\Phi^{t}$-invariant ergodic Borel probability measure on $\mathcal{Q}$ and let $q \in \mathcal{Q}$ be a density point for $\mu$ which contains itself in its $\alpha$ - and $\omega$-limit set. Use $q$ to construct the countable two-sided Markov shift $(\Sigma, T)$ with roof function $\varphi$ and suspension flow $\left(Y, \Psi^{t}\right)$.

By Theorem 1, there exists a $\sigma$-invariant Borel probability measure $\tilde{\mu}$ on $\Omega$ whose image under the map $\Xi_{*}$ equals $\mu$. Then $\tilde{\mu}$ gives full mass to $G(\Sigma)$. Since $G$ is injective, the measure $\tilde{\mu}$ defines a shift invariant Borel probability measures $\tilde{\mu}$ on $\Sigma$ which is mapped by the semi-conjugacy $\Upsilon$ to $\mu$.

Let $\Sigma^{+}$be the set of all one-sided infinite admissible sequences $\left(x_{i}\right) \subset \mathcal{S}^{\mathbb{N}}$ with $a_{x_{i} x_{i+1}}=1$ for all $i$, equipped with the one-sided shift $T_{+}: \Sigma^{+} \rightarrow \Sigma^{+}$. Since the roof function $\varphi$ only depends on the future, it defines a function on $\Sigma^{+}$which will be denoted again by $\varphi$. Up to normalization, the measure $\mu$ lifts to a $T^{+}$-invariant Borel probability measure on $\Sigma^{+}$.

By Lemma 5.3 and Lemma 5.5, the function $-h \varphi$ on $\Sigma^{+}$satisfies all the assumptions in Theorem 1.1 of [BS03]. In particular, since $\operatorname{pr}_{G}(-h \varphi) \leq 0$, the entropy $h_{\mu}$ of the invariant Borel probability measure $\mu$ does not exceed $h$. As a consequence, the Lebesgue measure $\lambda$ on $\mathcal{Q}$ is a measure of maximal entropy. Moreover, by Theorem 1.1 of [BS03], it is unique with this property. In other words, if $h_{\mu}=h$ then $\mu=\lambda$. Since $\mu$ was an arbitrary $\Phi^{t}$-invariant ergodic Borel probability on $\mathcal{Q}$, this completes the proof of Theorem 2 from the introduction.

Any finite admissible sequence $\left(x_{0}, \ldots, x_{n-1}\right) \subset \mathcal{S}$ defines the cylinder

$$
\left[x_{0}, \ldots, x_{n-1}\right]=\left\{\left(y_{i}\right) \in \Sigma^{+} \mid y_{i}=x_{i} \text { for } 0 \leq i \leq n-1\right\}
$$

A Gibbs measure for the function $-h \varphi$ on $\Sigma^{+}$is a Borel probability measure $\nu$ with the following property. There is a number $c>0$ so that for every cylinder set $\left[x_{0}, \ldots, x_{\ell}\right]$ and every $z \in\left[x_{0}, \ldots, x_{\ell}\right]$ we have

$$
\nu\left[x_{0}, \ldots, x_{\ell}\right] \in\left[c^{-1} e^{-h \sum_{0 \leq i \leq \ell-1} \varphi\left(T^{i} z\right)}, c e^{-h \sum_{0 \leq i \leq \ell-1} \varphi\left(T^{i} z\right)}\right]
$$

As a consequence of [S03] and of the above discussion, we conclude
Corollary 5.6. The Lebesgue measure is a Gibbs measure on $\Sigma^{+}$.

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