EXPONENTIAL MIXING OF THE TEICHMÜLLER FLOW ON AFFINE INVARIANT MANIFOLDS

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Abstract. Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$. Extending earlier work of Avila, Gouezel and Yoccoz, we show that the Teichmüller flow on the moduli space of abelian or quadratic differentials of S is exponentially mixing with respect to any $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic Borel probability measure.

1. Introduction

An oriented surface S of finite type is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g-3+m \geq 2$, i.e. that S is not a sphere with at most 4 punctures or a torus with at most 1 puncture. We then call the surface S *nonexceptional*. The Euler characteristic of S is negative.

The *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all complete finite area hyperbolic metrics on S under the action of the group of diffeomorphisms of S which are isotopic to the identity. The sphere bundle

$$\tilde{\mathcal{Q}}(S) \to \mathcal{T}(S)$$

of all *holomorphic quadratic differentials* of area one can naturally be identified with the unit cotangent bundle for the *Teichmüller metric*. If the surface S has punctures, that is, if $m > 0$, then we define a holomorphic quadratic differential on S to be a meromorphic quadratic differential on the closed Riemann surface obtained from S by filling in the punctures, with a simple pole at each of the punctures and no other poles.

The *mapping class group* $\mathrm{Mod}(S)$ of all isotopy classes of orientation preserving self-diffeomorphisms of S naturally acts on $\tilde{\mathcal{Q}}(S)$. The quotient

$$\mathcal{Q}(S) = \tilde{\mathcal{Q}}(S)/\mathrm{Mod}(S)$$

is the *moduli space of area one quadratic differentials*. It can be partitioned into so-called *strata* consisting of differentials with the same number and multiplicity of zeros and poles. The strata need not be connected, however they have at most two connected components [L08].

Date: August 20, 2025.
AMS subject classification: 37A25,37D40,30F60.

If the surface S is closed, that is, if $m = 0$, then we can also consider the bundle $\tilde{\mathcal{H}}(S) \to \mathcal{T}(S)$ of area one *abelian differentials*. It descends to the moduli space

$$\mathcal{H}(S) = \tilde{\mathcal{H}}(S)/\mathrm{Mod}(S)$$

of holomorphic one-forms defining a singular euclidean metric of area one on S. Again this moduli space decomposes into strata with the same number and multiplicity of zeros. Strata are in general not connected, but the number of their connected components is at most 3 [KZ03].

The group $\mathrm{SL}(2,\mathbb{R})$ admits a natural action on a component Q of a stratum of abelian or quadratic differentials. The action of the diagonal subgroup is called the *Teichmüller flow* Φ^t. An *affine invariant manifold* $\mathcal{C}$ in Q is an $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold. By the groundbreaking work of Eskin, Mirzakhani and Mohammadi [EMM15], affine invariant submanifolds of Q are precisely the supports of $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measures on Q. The invariant measure is contained in the Lebesgue measure class of its support. Note that as the double orientation cover of an affine invariant manifold in a component of a stratum of quadratic differentials is an affine invariant manifold in a component of a stratum of abelian differentials, the results of [EMM15] apply to affine invariant manifold in components of strata of quadratic differentials.

The goal of this article is to extend a result of Avila, Gouezel and Yoccoz and of Avila and Resende [AGY06, AR09], see also [AG13], on dynamical properties of the Teichmüller flow on components of strata of abelian or quadratic differentials to the Teichmüller flow on affine invariant manifolds, with a unified and simplified proof.

Theorem. *An* $\mathrm{SL}(2,\mathbb{R})$*-invariant ergodic probability measure on* $\mathcal{Q}(S)$ *or* $\mathcal{H}(S)$ *is exponentially mixing for the Teichmüller flow.*

The proof follows the strategy laid out in [AGY06]. It is based on a symbolic coding for the Teichmüller flow on components Q of strata of abelian differentials. While the coding used in [AGY06] is Rauzy induction, we embark from the coding introduced in [H11]. Using the fact that in period coordinates for components of strata of abelian differentials, an affine invariant manifold is cut out by linear equations with real coefficients, and a localization procedure similar to the construction in Section 5 of [H11], we construct a coding for the Teichmüller flow on an affine invariant manifold by the suspension of a Markov shift on countably many symbols. This construction is carried out in Section 3.

In Section 4 we adapt the Markov shift so that the criterion for exponential mixing in Section 2 of [AGY06] can directly be applied. The necessary control of the roof function defining the suspension flow is established in Section 5. It is based on an entropy argument inspired by [S03].

In the short Section 6, we verify that the conditions for an application of the results of Section 2 of [AGY06] are satisfied for the suspension flow. The preliminary Section 2 collects some technical results from the literature and introduces some notations used throughout the article.

Acknowledement: The major part of this work was carried out during two visits of the MSRI in Berkeley (in fall 2007 and in fall 2011) and at the Hausdorff Institut in Bonn in spring 2010. I am very grateful to these two institutions for their hospitality.

2. Train tracks and geodesic laminations

In this section we summarize some constructions from [PH92, H11, H24] which will be used throughout the paper.

2.1. **Geodesic laminations.** Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and where $3g-3+m \geq 2$. A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ. A geodesic lamination λ on S is said to *fill up* S if its complementary regions are all topological discs or once punctured monogons.

Definition 2.1 (Definition 2.1 of [H24])**.** A geodesic lamination λ is called *large* if λ fills up S and if moreover λ can be approximated in the *Hausdorff topology* by simple closed geodesics.

Since every minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed geodesics, a minimal geodesic lamination which fills up S is large. However, there are large geodesic laminations with finitely many leaves.

The *topological type* of a large geodesic lamination ν is a tuple

$$(m_1,\dots,m_\ell;-m) \text{ where } 1\leq m_1\leq \cdots \leq m_\ell,\ \sum_i m_i = 4g-4+m$$

such that the complementary regions of ν which are topological discs are m_i+2-gons (Section 2.1 of [H24]).

A *measured geodesic lamination* is a geodesic lamination λ equipped with a translation invariant transverse measure. As it is customary, we denote by $\mathcal{ML}$ the space of measured geodesic laminations equipped with the weak*-topology, and by $\mathcal{PML}$ the space of *projective measured geodesic laminations*. The measured geodesic lamination $\mu\in\mathcal{ML}$ *fills up* S if its support fills up S. This support is then necessarily connected and hence large. The *projectivization* of a measured geodesic lamination which fills up S is also said to fill up S. We call $\mu\in\mathcal{ML}$ *strongly uniquely ergodic* if the support of μ fills up S and admits a unique transverse measure up to scale. There is a distinguished symmetric continuous function $\iota:\mathcal{ML}\times\mathcal{ML}\to[0,\infty)$, the so-called *intersection form*, which extends the geometric intersection number of simple closed curves.

2.2. **Train tracks.** A *train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with $0, 1$ or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. We always identify train tracks which are isotopic. A train track is called *generic* if all switches are at most trivalent. Throughout we use the book [PH92] as the main reference for train tracks.

A train track or a geodesic lamination η is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is homotopic to the identity and maps η into τ in such a way that the restriction of the differential of F to the tangent space of η vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of F to η a *carrying map* for η. Write $\eta \prec \tau$ if the train track η is carried by the train track τ. Then every geodesic lamination ν which is carried by η is also carried by τ.

A train track *fills up* S if its complementary components are topological discs or once punctured monogons. Note that such a train track τ is connected. Let $\ell \geq 1$ be the number of those complementary components of τ which are topological discs. Each of these discs is an $m_i + 2$-gon for some $m_i \geq 1$ $(i = 1, \dots, \ell)$. The *topological type* of τ is defined to be the ordered tuple $(m_1, \dots, m_\ell; -m)$ where $1 \leq m_1 \leq \cdots \leq m_\ell$; then $\sum_i m_i = 4g - 4 + m$. If τ is *orientable*, that is, there exists a consistent orientation of the branches of τ, then $m = 0$ and m_i is even for all i (Section 2.2 of [H24]).

A *transverse measure* on a generic train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: for every trivalent switch s of τ, the sum of the weights of the two *small half-branches* incident on s (which have the same inward pointing tangent at s) equals the weight of the *large half-branch* (see [PH92] for more information). The space $\mathcal{V}(\tau)$ of all transverse measures on τ has the structure of a cone in a finite dimensional real vector space, and it is naturally homeomorphic to the space of all measured geodesic laminations whose support is carried by τ. The train track is called *recurrent* if it admits a transverse measure which is positive on every branch (see [PH92] for more details). A train track τ of topological type $(m_1, \dots, m_\ell; -m)$ is *fully recurrent* if τ carries a large minimal geodesic lamination of the same topological type (Definition 2.6 of [H24]).

A *vertex cycle* of a traintrack τ is an extreme point for the space of transverse measures on τ. The support of a vertex cycle is a simple closed curve, and this curve is either embedded in τ or a *dumbbell*, that is, it consists of two embedded simple loops in τ with one cusp at the boundary, connected by an embedded arc connecting the cusps.

If e is a *large* branch of τ then we can perform a right or left *split* of τ at e (see [PH92]). If τ is of topological type $(m_1, \dots, m_\ell; -m)$, if ν is a minimal geodesic lamination of the same topological type as τ which is carried by τ and if e is a large branch of τ, then there is a unique choice of a right or left split of τ at e such that the split track η carries ν. In particular, η is fully recurrent [H24].

To each train track τ which fills up S one can associate a *dual bigon track* τ^* (Section 3.4 of [PH92]). There is a bijection between the complementary components of τ and those complementary components of τ^* which are not *bigons*, i.e. discs with two cusps at the boundary. This bijection maps a component C of τ which is an n-gon for some $n \geq 3$ to an n-gon component of τ^* contained in C, and it maps a once punctured monogon C to a once punctured monogon contained in C. If τ is orientable then the orientation of S and an orientation of τ induce an orientation on τ^*, i.e. τ^* is orientable.

There is a notion of carrying for bigon tracks which is analogous to the notion of carrying for train tracks. A train track τ is called *fully transversely recurrent* if its dual bigon track τ^* carries a large minimal geodesic lamination ν of the same topological type as τ.

Definition 2.2 (Definition 2.8 of [H24])**.** A train track τ of topological type $(m_1, \ldots, m_\ell; -m)$ is called *large* if τ is fully recurrent and fully transversely recurrent.

For a large train track τ let $\mathcal{V}^*(\tau) \subset \mathcal{ML}$ be the set of all measured geodesic laminations whose support is carried by τ^*. Each of these measured geodesic laminations corresponds to a *tangential measure* on τ (see [PH92, H24] for more information). With this identification, the pairing

$$(\nu, \mu) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau) \to \sum_b \nu(b)\mu(b) \tag{1}$$

is just the restriction of the intersection form ι on measured lamination space (Section 3.4 of [PH92]). Moreover, $\mathcal{V}^*(\tau)$ is naturally homeomorphic to a convex cone in a real vector space. The dimension of this cone coincides with the dimension of $\mathcal{V}(\tau)$.

2.3. **Components of strata and affine invariant manifolds.** To each component Q of a stratum of abelian or quadratic differentials one can associate a collection $\mathcal{LT}(Q)$ of large train tracks on S (Section 3 of [H24]). A train track $\tau \in \mathcal{LT}(Q)$ is characterized by the following properties. First, if Q consists of differentials with zeros of order m_i and m poles, then τ is of topological type $(m_1, \ldots, m_\ell; -m)$. Moreover, let us assume that μ, ν are measured geodesic laminations whose supports are of the same topological type as τ and that μ is carried by τ, ν is carried by τ^*. Then the pair (ξ, ζ) *binds* S, that is, for any geodesic lamination β, we have $\iota(\xi + \zeta, \beta) > 0$. Hence there exists a quadratic differential with *vertical measured lamination* μ and *horizontal measured lamination* ν (which is of area one if and only if we have $\iota(\mu, \nu) = 1$). This differential is contained in $\mathbb{R}_+ Q$, that is, in the space of differentials whose area normalization is contained in Q. As in [H24] we denote by $Q(\tau) \subset Q$ the set of all quadratic (or abelian) differentials of this form.

In view of this, the set $Q(\tau)$ can be thought of as an enlargement of a subset of Q with a *local product structure*. Such a set is homeomorphic to a product $A \times B \times (-\epsilon, \epsilon)$ where A, B are sets of projective measured geodesic laminations so that for all $[\xi] \in A, [\zeta] \in B$ the pair $([\xi], [\zeta])$ binds S (this makes sense for projective measured laminations). The identification is via a map which associates to a (marked) quadratic or abelian differential z the pair $([z^h], [z^v])$ of its projective horizontal and vertical measured geodesic laminations as well as a scaling parameter with respect to a local section $[z^v] \to z^v$ of the fibration $\mathcal{ML} \to \mathcal{PML}$. Note that $Q(\tau)$ is not literally a product subset of Q as there may be pairs of projective measured laminations $([v],[w])$ so that $[v]$ is carried by τ, $[w]$ is carried by τ^* but that $[v],[w]$ do not bind S, or the pair binds but the differential it defines is contained in a boundary stratum of Q.

For each $q \in Q$ there exists a train track $\tau \in \mathcal{LT}(Q)$ so that $q \in Q(\tau)$ (Proposition 3.3 of [H24]). Motivated by this correspondence, one defines a set $\mathcal{LL}(Q)$ of large geodesic laminations as the closure with respect to the Hausdorff topology of the set of minimal large geodesic laminations which are carried by some train track $\tau \in \mathcal{LT}(Q)$ and are of the same topological type as τ.

An *affine invariant manifold* in a component Q of a stratum of abelian differentials is a closed $\mathrm{SL}(2,\mathbb{R})$-invariant subset $\mathcal{C}$ of Q which can be cut out by linear equations with real coefficients in complex *period coordinates* of Q. Equivalently, by the groundbreaking results of Eskin and Mirzakhani and Mohammadi, it is the support of an ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure on Q, and every ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure on Q arises in this way [EM18, EMM15].

Up to scaling, the $\mathrm{SL}(2,\mathbb{R})$-invariant measure is the standard Lebesgue measure defined by the restriction of the period coordinates. This measure is absolutely continuous with respect to the *stable* and *unstable foliation*, locally defined by differentials with the same vertical and horizontal measured geodesic lamination, respectively. Furthermore, for any $q \in \mathcal{C}$ and $\tau \in \mathcal{LT}(Q)$ so that $q \in \mathcal{Q}(\tau)$, the intersection of $\mathcal{C}$ with $Q(\tau)$ consists of at most countably many connected components which do not accumulate in the interior of Q. Any such component can locally be described by a pair of piecewise linear submanifolds of $\mathcal{PML}$; to this end observe that $\mathcal{PML}$ has a natural piecewise linear structure, and this is compatible with the local structure given by period coordinates. We provide a more detailed and rigorous discussion of this viewpoint in Section 3.

3. Symbolic coding for the Teichmüller flow on affine invariant manifolds

Consider an affine invariant manifold $\mathcal{C}$ contained in a component $\mathcal{Q}$ of a stratum of abelian differentials. Note that an affine invariant manifold in a component of a stratum of quadratic differentials admits a double orientation cover which is contained in a stratum of abelian differentials. This construction commutes with the $\mathrm{SL}(2,\mathbb{R})$-action and hence the orientation cover is an affine invariant manifold. Thus for the purpose of symbolic coding, it suffices to assume that $\mathcal{Q}$ is a component of a stratum of abelian differentials.

Denote by λ the unique $\mathrm{SL}(2,\mathbb{R})$-invariant Borel probability measure on Q whose support equals $\mathcal{C}$. The goal of this section is to summarize and extend some results from [H11] and construct a Markov shift with countably many symbols which defines a symbolic coding of the restriction of the Teichmüller flow Φ^t to an invariant Borel set of $\mathcal{C}$ of full λ-measure. The starting point is the following main result of [H11].

Theorem 3.1 (Theorem 1 of [H11])**.** *Let Q be a component of a stratum of quadratic or abelian differentials. Then there exists*

- *a topologically transitive subshift of finite type (Ω, σ),*
- *a σ- invariant dense Borel set $\mathcal{U} \subset \Omega$ containing all normal sequences,*
- *a suspension (X, Θ^t) of σ over $\mathcal{U}$, given by a positive bounded continuous roof function ρ on $\mathcal{U}$*

and a finite-to-one semi-conjugacy $\Xi : (X, \Theta^t) \to (\mathcal{Q}, \Phi^t)$ which maps the space of Θ^t-invariant ergodic Borel probability measures on X continuously onto the space of Φ^t-invariant ergodic Borel probability measures on Q.

We shall use some additional specific information on this coding. Namely, there is a collection $\mathcal{E}(Q)$ of *numbered* train tracks (train tracks with a numbering of the branches) whose unnumbered tracks are contained in the set $\mathcal{LT}(Q)$ introduced in Section 2.3. The finite alphabet for the shift (Ω, σ) is the set $\mathcal{E}(Q)$ [H11]. Here we view a numbered train track as a graph on the surface S up to homeomorphisms of S. Equivalently, a train track in $\mathcal{E}(Q)$ is a $\mathrm{Mod}(S)$-orbit of isotopy classes of *marked* train tracks on S.

By Section 3 of [H11], the set $\mathcal{E}(Q) \subset \mathcal{LT}(Q)$ has the following properties (Lemma 3.4 and Lemma 3.8 of [H11], note however that in this article, we do not work with marked train tracks).

(1) If $\tau \in \mathcal{E}(Q)$ and if $\eta \in \mathcal{LT}(Q)$ is obtained from τ by a *full numbered split*, that is, by splitting at each large branch of τ and keeping track of the numbers, then $\eta \in \mathcal{E}(Q)$.
(2) For every $q \in Q$ without vertical saddle connections there exists some $\tau \in \mathcal{E}(Q)$ such that $q \in Q(\tau)$.

The *shift* σ on Ω corresponds to the transition which alters $\tau \in \mathcal{E}(Q)$ by a full numbered split. This construction gives rise to a transition matrix in the following sense. Number the elements of $\mathcal{E}(Q)$ in an arbitrary way. Define a transition matrix $(a_{i,j})$ by the requirement that $a_{i,j} = 1$ if and only if the numbered train track with number j can be obtained from the numbered train track with number i by a full numbered split.

Any one-sided *admissible* sequence, that is, a sequence $(x_i)_{i \geq 0} \subset \mathcal{E}(Q)$ so that $a_{i,i+1} = 1$ for all i, defines a one-sided infinite splitting sequence of large numbered train tracks. If we let $\mathcal{V}(x_i)$ be the set of all measured geodesic laminations carried by x_i, then this sequence determines the non-empty set $\cap_{i \geq 0} \mathcal{V}(x_i)$. If this set consists of a unique point up to scaling with a positive real whose support is contained in $\mathcal{LL}(Q)$ then (x_i) is called *uniquely ergodic.* Any strongly uniquely ergodic measured geodesic lamination carried by x_0 with support in $\mathcal{LL}(Q)$ determines such a sequence (Section 4 of [H24]).

The set $\mathcal{U}$ consists precisely of the set of *doubly uniquely ergodic sequences* whose positive half-sequence is uniquely ergodic and such that moreover the intersection $\cap_i \mathcal{V}^*(x_{-i})$ consists of a unique point up to scaling, with support contained in $\mathcal{LL}(Q)$.

The roof function ρ is given as follows. By the discussion in the previous paragraph, a sequence $(x_i) \in \mathcal{U} \subset \Omega$ defines up to scaling a measured geodesic lamination μ with support in $\mathcal{LL}(Q)$ which is carried by each of the train tracks x_i. We then define $\rho(x_i) = \log(\mu(x_1)^{-1}\mu(x_0))$ where $\mu(x_i) = \sum_{b \text{ branch of } x_i} \mu(b)$ is the total mass deposited by μ on x_i. Note that the roof function only depends on the future, and its values are contained in an interval $(0, r_0]$ for a number $r_0 > 0$ (Lemma 4.2 of [H11]).

As in Section 5 of [H11], we have to localize this construction near a typical point for λ to construct a symbolic coding of the Teichmüller flow on $\mathcal{C}$ whose roof function is better behaved. The construction is based on Lemma 4.6 of [H11]. For its formulation, we denote by $\mathcal{V}_0(\tau)$ the subset of $\mathcal{V}(\tau)$ of all transverse measures on τ of total mass one, that is, measures μ with $\mu(\tau) = 1$. Call a transverse measure *positive* if it gives a positive mass to every branch.

Lemma 3.2 (Lemma 4.6 of [H11])**.** *Let $\tau_0 \in \mathcal{LT}(Q)$ and let $q \in Q(\tau_0)$ be such that the vertical measured geodesic lamination ζ of q is uniquely ergodic, with support in $\mathcal{LL}(Q)$. Let $(\tau_i) \subset \mathcal{LT}(Q)$ be a full splitting sequence with $\cap_{i\geq 0}\mathcal{V}(\tau_i) = (0,\infty)\zeta$. Then there is some $k > 0$ with the following properties.*

1. *Via a carrying map $\tau_k \to \tau_0$, every vertex cycle of τ_k is mapped onto τ_0.*
2. *There exists a number $\beta > 0$ such that $\nu(b)/\nu(b') \geq \beta$ for every $\nu \in \mathcal{V}(\tau_0)$ which is carried by τ_k and all branches b, b' of τ_0.*
3. *There is a number $\delta > 0$ with the following property. Let $\mu, \nu \in \mathcal{V}_0(\tau_k)$ be positive normalized measures and let $a_0 = \min\{\mu(b)/\nu(b), \nu(b)/\mu(b) \mid b\}$ where b ranges through the branches of τ_k. Then the normalized measures $\mu_0, \nu_0 \in \mathcal{V}_0(\tau_0)$ induced by μ, ν via a carrying map $\tau_k \to \tau_0$ and scaling satisfy*
$$\min\{\mu_0(b)/\nu_0(b), \nu_0(b)/\mu_0(b) \mid b\} \geq a_0 + \delta(1 - a_0).$$

Denote by $Q_0(\tau)$ the set of all $q \in Q(\tau)$ whose vertical measured geodesic lamination is contained in $\mathcal{V}_0(\tau)$. As the roof function ρ is bounded from above by $r_0 > 0$, for any $q \in Q$ without vertical saddle connection there exists a number $t \in [0, r_0)$ so that $\Phi^t q \in Q_0(\tau)$ for some $\tau \in \mathcal{E}(Q)$.

Consider a point $q \in \mathcal{C} \subset Q$ which is generic for the $\mathrm{SL}(2,\mathbb{R})$-invariant measure λ, that is, a Birkhoff regular point for the Teichmüller flow Φ^t. Such a point is contained in its own α-and ω-limit set for Φ^t, and it does not have vertical or horizontal saddle connections. More precisely, by Lemma 3.7 of [H11], the horizontal and vertical measured geodesic laminations of q are uniquely ergodic, with support in $\mathcal{LL}(Q)$.

There are finitely many large numbered train tracks $\tau_1, \dots, \tau_n \in \mathcal{E}(\mathcal{Q})$ such that $\Phi^t q \in \mathcal{Q}_0(\eta)$ for some $t \in [0, r_0)$ if and only if $\eta \in \{\tau_1, \dots, \tau_n\}$. In particular, Lemma 3.2 can be applied to q and any of the train tracks τ_i. We refer once more to [H11] for more information.

As in [H11], we use Lemma 3.2 and the point q to construct a topological Markov shift on a countable set $\mathcal{S}$ of symbols, given by a transition matrix $(a_{i,j})_{\mathcal{S}\times\mathcal{S}}$. The phase space of this shift is the space

$$\Sigma = \{(y_i) \in \mathcal{S}^{\mathbb{Z}} \mid a_{y_i,y_{i+1}} = 1 \text{ for all } i\}.$$

The construction is as follows.

By Lemma 3.2 there is a number $k > 0$ such that the following holds. Let $i \leq n$ and let $(\sigma_j^i)_{0\leq j\leq k}$ be a full numbered splitting sequence of length k issuing from $\sigma_0^i = \tau_i$ with the property that σ_k^i carries the support ζ of the vertical measured geodesic lamination of q. Then the sequence $(\sigma_j^i)_{0\leq j\leq k}$ has the properties stated in Lemma 3.2. In particular, by part (1) of Lemma 3.2, the image under a carrying map $\sigma_i^i \to \sigma_0^i$ of any measured geodesic lamination $\xi \in \mathcal{V}(\sigma_k^i)$ defines a positive transverse measure on $\tau_i = \sigma_0^i$.

We shall require a slightly stronger property. Denote by $\mathbb{R}_+Q \supset Q$ the set of differentials whose area one normalizations are contained in Q. We can split τ_i backwards and construct for each m an admissible sequence $(\eta_j^i)_{-m\leq j\leq m} \subset \mathcal{E}(Q)$ so that the vertical measured geodesic lamination of q is carried by η_m^i and the horizontal measured geodesic lamination of q is carried by $(\eta_{-m}^i)^*$. We refer to [PH92] for this construction. Then $\cap_{m\geq 0}\mathcal{V}(\eta_m^i)$ consists of the line spanned by the vertical measured geodesic lamination of q and $\cap_{m\geq 0}\mathcal{V}^*(\eta_{-m})$ equals the line spanned by the horizontal measured geodesic lamination of q.

Since for any $\eta \in \mathcal{LL}(Q)$ the set of pairs (z^v, z^h) of measured laminations so that z^v is carried by η, z^h is carried by η^* and such that (z^v, z^h) determines a differential in $\mathbb{R}_+Q$ is an open subset of $\mathcal{V}(\eta) \times \mathcal{V}^*(\eta)$ (Section 3 of [H24]), for large enough m any pair (z^v, z^h) so that z^v is carried by η_m^i and z^h is carried by $(\eta_{-m}^i)^*$ defines a differential in Q. Thus by possibly replacing q by $\Phi^{-u}q$ for some $u > 0$ we may assume that this holds true for pairs of differentials (ξ, ζ) so that ξ is carried by σ_k^i and ζ is carried by $(\sigma_0^i)^* = \tau_i^*$. The next lemma formalizes this idea.

Lemma 3.3. *Let $(x_j)_{0\leq j\leq s}$ $(s \geq k)$ be an admissible sequence in $\mathcal{E}(\mathcal{Q})$ so that there exists some $i \in \{1, \dots, n\}$ with $x_j = \sigma_j^i$ for $0 \leq j \leq k$. Then for any $\mu \in \mathcal{V}(x_s)$ and any $\nu \in \mathcal{V}^*(x_0)$ there exists a differential $q(\mu, \nu)$ in $\mathbb{R}_+Q$ with vertical measured lamination μ and horizontal measured lamination ν. Moreover, the map $(\mu, \nu) \in \mathcal{V}(x_s) \times \mathcal{V}^*(x_0) \to q(\mu, \nu) \in \mathbb{R}_+Q$ determines period coordinates on a closed $\mathbb{R}_+$-invariant subset of $\mathbb{R}_+Q$ with dense interior.*

Proof. Since by assumption, Q is a component of a stratum of abelian differentials, the train tracks x_j are equipped with an orientation. This orientation induces an orientation for any geodesic lamination carried by x_j, and there is similarly an orientation for any geodesic lamination carried by x_j^*.

It follows from the construction of the sequences $(\sigma_j^i)_{0\leq j\leq k}$ that the following holds true. Let μ be a measured geodesic lamination carried by x_s and let ν be a measured geodesic lamination carried by x_0^*. Then the pair (μ, ν) determines a differential in $\mathbb{R}_+Q$.

Period coordinates are defined as follows. Choose a basis of the relative homology $H_1(S, \Sigma; \mathbb{Z})$ where Σ is the set of singular points of $q(\mu, \nu)$. We may assume that each point of Σ is contained in the interior of a polygonal component of $S \setminus x_0^*$

which is a subset of a polygonal component of $S \setminus x_0$ (see [PH92] for details). We may moreover assume that each such basis element can be represented by a smooth embedded arc in S with endpoints in Σ. Furthermore, these arcs can be chosen in such a way that they intersect the oriented train tracks x_0 and x_0^* transversely in interior points of branches. We then can evaluate the transverse measures defined by μ, ν on this collection of oriented paths as follows.

Let $\alpha : [0,1] \to S$ be a smooth embedding defining an element of the fixed homology basis. To each intersection p of α with the oriented train track x_0 is associated a sign $\sigma(p)$ corresponding to the orientation of T_pS determined by the ordered pair of vectors consisting of the oriented tangent of α at p and the oriented tangent of x_0 at p. Weight each such intersection point $p \in b$, b a branch of x_0, by $\sigma(p)\mu(b)$ and put $q^v(\alpha) = \sum_p \sigma(p)\mu(p)$. Proceed in the same way with the intersections between α and x_0^* to define $q^h(\alpha)$. This construction does not depend on choices, and it defines a pair of classes in $H^1(S, \Sigma; \mathbb{R})$ which correspond to the imaginary and real parts of the holomorphic one-form $q(\mu, \nu)$. Namely, via the correspondence between oriented measured geodesic laminations on S and oriented measured foliations, the value of $q^v(\alpha)$ defined above is just the value of the integral over α of the closed real one-form which vanishes on the leaves of the measured foliation corresponding to μ and whose value on an arc connecting two zeros which up to homotopy intersects a single branch b of τ transversely is just $\mu(b)$.

As a consequence, the sequence $(x_p)_{0 \le p \le s}$ determines a domain for period coordinates on $\mathbb{R}_+ Q$. Furthermore, it is immediate from the construction that the linear structure on the pairs (μ, ν) defined by these period coordinates coincides with the linear structure as weight functions on the oriented train track x_s and x_0^*, respectively. This simply means that the imaginary part of the period coordinates for $q(\mu_1 + \mu_2, \nu)$ where μ_i are viewed as transverse measures on x_s, coincides with the sum of the imaginary parts of the period coordinates for $q(\mu_1, \nu)$ and $q(\mu_2, \nu)$, and naturality with respect to scaling also holds true. The same is also valid for the real parts of the coordinates. □

The affine invariant manifold $\mathcal{C}$ is a locally closed subset of Q cut out from Q by linear equations in period coordinates with real coefficients. By Lemma 3.3, any of the sequences $(\sigma_j^i)_{0 \le j \le k} \subset \mathcal{E}(Q)$ and the choice of a basis of $H_1(S, \Sigma; \mathbb{Z})$ determines period coordinates on a neighborhood in $\mathcal{Q}$ of the Birkhoff regular point $q \in \mathcal{C}$ which was used for the construction. Although $\mathcal{C}$ is a closed subset of Q, the domain of these period coordinates may intersect $\mathcal{C}$ in more than one connected component. To overcome this difficulty we decrease the size of this domain by replacing once more the defining admissible sequences $(\sigma_\ell^i)_{0 \le \ell \le k} \subset \mathcal{E}(Q)$ $(i = 1, \dots, n)$ by longer sequences which define period coordinates on a smaller set.

Using this construction, we may assume that for each i there exists a connected affine cone $A(x_s) \subset \mathcal{V}(\sigma_k^i)$ and a connected affine cone $A(x_0^*) \subset \mathcal{V}^*(\sigma_0^i)$ so that the differentials in $\mathbb{R}_+\mathcal{C}$ intersecting the domain of the period coordinates defined by the sequence $(\sigma_j^i)_{0 \le j \le k}$ are precisely those with vertical measured geodesic lamination contained in the set $A(x_s)$ and horizontal measured geodesic lamination contained in the set $A(x_0^*)$. More precisely, $A(x_s)$ is the intersection with $\mathcal{V}(x_s)$ of an affine subspace of the vector space of all weight functions on the branches of x_s which satisfy the switch condition, and similarly for $A(x_0^*)$.

Definition 3.4. Define $\mathcal{S}$ to be the set of all finite admissible sequences $(x_j)_{0\leq j\leq s} \subset \mathcal{E}(Q)$ with the following additional properties.

(1) $s \geq 2k$ and there are $i, \ell \in \{1, \dots, n\}$ so that the sequences $(x_j)_{0\leq j\leq k}$ and $(x_j)_{s-k\leq j\leq s}$ are realized by the full numbered splitting sequences $(\sigma_j^i)_{0\leq j\leq k}$ and $(\sigma_j^\ell)_{0\leq j\leq k}$, respectively.
(2) There is no number $t \in [k, s-k)$ such that the sequence $(x_j)_{t\leq j\leq t+k}$ is realized by any of the full numbered splitting sequences $(\sigma_j^i)_{0\leq j\leq k}$.
(3) There exists some $z \in \mathcal{C}$ whose vertical measured geodesic lamination is carried by x_s and whose horizontal measured geodesic lamination is carried by x_0^*.

By the choice of the sequences $(\sigma_j^i)_{0\leq j\leq k}$ the third property means the following. Consider a *marked* train track τ_0 representing x_0. By assumption, there exists a lift $\tilde{q}$ of q to the Teichmüller space of marked abelian differentials so that $\tilde{q} \in Q(\tau_0)$ (with a straightforward extension of notations). Let $\tilde{\mathcal{C}}$ be the component of the preimage of $\mathcal{C}$ in the Teichmüller space of marked abelian differentials containing $\tilde{q}$. Let τ_s be the endpoint of the splitting sequence of marked numbered train tracks which is represented by the sequence $(x_i)_{0\leq i\leq s}$. Then there exists some $z \in \tilde{\mathcal{C}} \cap Q(\tau_0) \cap Q(\tau_s)$.

Note that $\mathcal{S}$ is at most countable. Furthermore. $\mathcal{S}$ is not empty as by recurrence, the Φ^t-orbit of the Birkhoff regular differential q recurs for arbitrarily large times to an arbitrarily prescribed neighborhood of q, see [H11] for details.

Define a transition matrix $(a_{i,j})_{\mathcal{S}\times\mathcal{S}}$ by requiring that $a_{i,j} = 1$ if and only if the sequence $(x_p)_{0\leq p\leq s}$ representing the symbol i and the sequence $(y_t)_{0\leq t\leq u}$ representing the symbol j satisfy $y_t = x_{s-k+t}$ for every $t \in \{0, \dots, k\}$ and that furthermore the following holds true. Consider the affine subcone $A(x_s)$ of $\mathcal{V}(x_s)$ which is the set of horizontal measured geodesic laminations of points in $\mathcal{C}$ cut out by linear equations from the period coordinates defined by the sequence $(x_p)_{0\leq p\leq s}$. Using the period coordinates determined by $(y_t)_{0\leq t\leq u}$, we require that via the carrying map $y_u \to y_0 = x_s$, we have $A(y_u) \subset A(x_s)$ and similarly we require that $A(x_0^*) \subset A(y_u^*)$.

Let Σ be the set of all biinfinite sequences $(y_i) \subset \mathcal{S}^{\mathbb{Z}}$ with $a_{y_i,y_{i+1}} = 1$ for all i, equipped with the (biinfinite) shift $T : \Sigma \to \Sigma$. There is a natural continuous injective map

$$G : \Sigma \to \Omega$$

where Ω is as in Theorem 3.1. This map is constructed as follows. Note that two letters $x, y \in \mathcal{S}$ are given by finite admissible sequences in $\mathcal{E}(Q)$, say the sequences $(x_p)_{0\leq p\leq s}$ and $(y_\ell)_{0\leq \ell\leq u}$. By construction, if $a_{x,y} = 1$ then we have $y_\ell = x_{s-k+\ell}$ for $0 \leq \ell \leq k$. Thus the concatenation $x\cdot y$ of the sequence $(x_p)_{0\leq p\leq s}$ and the sequence $(y_\ell)_{k+1\leq \ell\leq u}$ defines an admissible sequence of length $s+u-k$ in $\mathcal{E}(Q)$. With this recipe, we can construct from any biinfinite sequence $(y_i) \subset \Sigma$ an admissible sequence $G(y_i) \in \Omega$.

Lemma 3.5. *The image of G is contained in the set of doubly uniquely ergodic sequences which define points in $\mathcal{C}$.*

Proof. By Lemma 3.2 and the properties of the alphabet $\mathcal{S}$, if (τ_i) is a full splitting sequence which realizes $(y_i) \in \Sigma$ then $\cap_{i\geq 0}\mathcal{V}(\tau_i)\cap\mathcal{V}_0(\tau_0)$ consists of a unique positive transverse measure μ, and there is a unique positive transverse measure ν contained in $\cap_{i\geq 0}\mathcal{V}^*(\tau_{-i})$ with intersection $\iota(\mu,\nu)=1$. We refer once more to [H11] for details of this construction which relies on the fact that differentials in Q whose orbits under Φ^t recur to a fixed compact set of arbitrarily large positive and negative times have uniquely ergodic vertical and horizontal measured laminations, with support in $\mathcal{LL}(Q)$.

That the differential $q(\mu,\nu)$ determined by the pair (μ,ν) of measured geodesic laminations indeed is contained in $\mathcal{C}$ can be seen as follows. By construction, for any sequence $(y_i) \in \Sigma$, identified with a full numbered splitting sequence of train tracks in $\mathcal{LT}(Q)$, and for any $j \geq 0$, there exists some point $z \in \mathcal{C}$ whose vertical measured geodesic lamination is carried by $\mathcal{V}(y_j)$ and whose horizontal measured geodesic lamination is carried by $\mathcal{V}^*(y_{-j})$. Since $\mathcal{C}$ is a closed subset of Q, the same applies to the infinite intersections by compactness. But this is equivalent to stating that $q(\mu,\nu) \in \mathcal{C}$. □

It follows from the results in Section 5 of [H11] that we may assume that the topological Markov chain (Σ, T) is topologically mixing.

Define a roof function φ on Σ by associating to an infinite sequence $(y_i) \in \Sigma$ with $y_0 = (x_i)_{0\leq i\leq s}$ the value

$$\varphi(y_i) = \sum_{i=0}^{s-k} \rho(\sigma^i(G(y_i))).$$

By Lemma 3.2, the function φ is bounded from below by a positive constant, is unbounded and only depends on the future. We refer once more to Section 5 of [H11] for more detailed information on this construction.

Lemma 3.6. *Let (Y, Θ^t) be the suspension of the Markov shift (Σ, T) by the roof function φ. Then there exists a finite-to-one semi-conjugacy of (Y, Θ^t) onto a Φ^t-invariant Borel subset of $(\mathcal{C}, \Phi^t)$ of full λ-measure.*

Proof. It follows from the results in Section 5 of [H11] that there exists a finite-to-one semi-conjugacy H of (Y, Θ^t) into the Teichmüller flow on Q whose image is contained in the set of all orbits which recur to a fixed compact neighborhood of q for arbitrarily large times. Observe to this end that the alphabet $\mathcal{S}$ is a subset of the alphabet constructed in Section 5 of [H11]. Thus the shift (Σ, T) is an invariant subset of the shift space constructed in Section 5 of [H11].

By Lemma 3.5, the image of the map H is a Φ^t-invariant Borel subset of the affine invariant manifold $\mathcal{C}$. Moreover, by construction, it contains a subset of $\mathcal{C}$ of positive λ-measure. By invariance and ergodicity of λ, it follows that the image contains a subset of full λ-measure. This completes the proof of the lemma. □

Define the n-th variation of φ by

$$\mathrm{var}_n(\varphi) = \sup\{\varphi(y) - \varphi(z) \mid y_i = z_i \text{ for } i = 0, \dots, n-1\}.$$

The following is Lemma 5.2 of [H11]. Although Lemma 5.2 uses a variation of the construction of the suspension flow (Σ, T) where the definition of the alphabet $\mathcal{S}$ is less restrictive, its proof only uses Lemma 3.2 and hence it carries over without change.

Lemma 3.7. *There is a number $\theta \in (0,1)$ and a number $L > 0$ such that $\mathrm{var}_n(\varphi) \leq L\theta^n$ for all $n \geq 1$. In particular,*

$$\sum_{n\geq 1} \mathrm{var}_n(\varphi) < \infty.$$

4. One-sided Markov shifts

In this section we construct from the coding of the Teichmüller flow on the affine invariant manifold $\mathcal{C}$ established in Section 3 a one-sided Markov shift $(\Xi^+, \beth)$, and we show that it defines an expanding Markov map in the sense of Definition 2.2 of [AGY06]. The Teichmüller flow on $\mathcal{C}$ determines a roof function ω for the shift. We shall see in Section 5 that this roof function satisfies the conditions formulated in Section 2 of [AGY06]. As in [AGY06], this is then used in Section 6 to complete the proof of the main result from the introduction.

Recall from Section 3 the definition of the two-sided Markov shift (Σ, T) with countable alphabet $\mathcal{S}$ and transition matrix $(a_{i,j})$. The alphabet $\mathcal{S}$ was constructed from a collection of finite admissible sequences $(\sigma_j^i)_{0\leq j\leq k} \subset \mathcal{E}(Q)$ $(i = 1, \dots, n)$. To facilitate the application of a technical result [Aa97] which guarantees exponential mixing as in [AGY06], we merge letters from the alphabet $\mathcal{S}$ to construct a new shift space which represents a finite cover of the shift (Σ, T). To this end partition the alphabet $\mathcal{S}$ as $\mathcal{S} = \cup_{i,m}\mathcal{S}^{i,m}$ $(i, m \in \{1, \dots, n\})$ where we have $(x_p)_{0\leq p\leq s} \in \mathcal{S}^{i,m}$ if $x_0 = \sigma_0^i$ and $x_{s-k+\ell} = \sigma_\ell^m$ for $0 \leq \ell \leq k$ (notations are as in Section 3). That this makes sense is immediate from the construction of $\mathcal{S}$.

Consider again the Birkhoff generic point $q \in \mathcal{C}$ which was used in the construction of $\mathcal{S}$. For $i \in \{1, \dots, n\}$, up to replacing q by $\Phi^{t_i}q$ for some $t_i \in [0, r_0)$ where as before, $r_0 > 0$ is an upper bound for the roof function ρ of the subshift of finite type (Ω, σ), we may assume that there exists a biinfinite admissible sequence $(y_j^i) \in \Sigma$ with $y_0^i = \sigma_0^i$ which is mapped to q by the semi-conjugacy $H : (Y, \Theta^t) \to (\mathcal{C}, \Phi^t)$ from Lemma 3.6 (we can use this property as a definition of the sequences $(\sigma_j^i)_{0\leq j\leq k}, i = 1, \dots, n$). Note however that H is not onto and hence it only is a semi-conjugacy onto an invariant Borel subset of $\mathcal{C}$. As the map H is a semi-conjugacy, the set $\{1, \dots, n\}$ is finite, q is a Birkhoff regular point for λ, and the measure λ is ergodic under the flow Φ^t, there exists some $i \in \{1, \dots, n\}$ and a Borel subset Z of $\mathcal{C}$ of positive λ-measure with the following property. For every $z \in Z$ there is some $t(z) \in [0, r_0)$ so that $\Phi^{t(z)}z$ is the image under H of a biinfinite admissible sequence $(y_i(z)) \in \Sigma$ with $y_0(z) \in \cup_m\mathcal{S}^{i,m}$, moreover there are infinitely many $j > 0$ so that $y_j(z) \in \cup_m\mathcal{S}^{i,m}$.

We now define a new countable alphabet $\mathcal{A}$ as follows. Letters in $\mathcal{A}$ are finite admissible sequences $(y_1 \cdots y_s)$ in the letters of the alphabet $\mathcal{S}$ such that the following properties are satisfied. We have $y_1 \in \cup_j \mathcal{S}^{i,j}$, $y_s \in \cup_j \mathcal{S}^{j,i}$ and $y_u \in \mathcal{S} \setminus \cup_j \mathcal{S}^{j,i}$ for $2 \leq u < s$. Here we call a sequence $(y_1 \cdots y_s)$ admissible if $a_{y_i,y_{i+1}} = 1$ for all i where $(a_{i,j})$ is the matrix defining the Markov shift (Σ, T).

Let $(\Xi, \beth)$ be the shift in the countable alphabet $\mathcal{A}$. Note that unlike for the Markov shift (Σ, T), there is no restriction on transition maps. As before, there exists a natural embedding $E : \Xi \to \Sigma$ mapping orbits of $\beth$ to orbits of T. Define a roof function ω on Ξ by $\omega(y_i) = \sum_{0 \leq u \leq \ell-1} \varphi(T^u(y_i))$ where $\ell \geq 1$ is such that $E(\beth(y_i)) = T^\ell(E(y_i))$. The following summarizes the properties of this construction for our purpose.

Lemma 4.1. *Let (Z, Ψ^t) be the suspension flow over $(\Xi, \beth)$ with roof function ω. There is a semi-conjugacy F of (Z, Ψ^t) onto a Φ^t-invariant subset of $\mathcal{C}$ of full λ-measure.*

Proof. Consider the embedding $E : \Xi \to \Sigma$. By the definition of the roof function ω and composing with the semi-conjugacy $H : (Y, \Theta^t) \to (\mathcal{C}, \Phi^t)$, the embedding E induces a semi-conjugacy of the suspension flow $F : (Z, \Psi^t) \to (\mathcal{C}, \Phi^t)$. As the image of the embedding $(\Xi, \beth) \to (\Sigma, T)$ is invariant under the shift T, the image of the semi-conjugacy F is a Φ^t-invariant Borel subset of $\mathcal{C}$. We have to show that this set has full λ-measure.

However, by the choice of the set $\cup_m \mathcal{S}^{i,m}$, we have $\lambda(F(Z)) > 0$ and hence $\lambda(F(Z)) = 1$ by invariance and ergodicity. □

For simplicity of notation, put $\tau = \sigma_0^i$ and $(\sigma_j)_{0 \leq j \leq k} = (\sigma_j^i)_{0 \leq j \leq k}$. Recall from Lemma 3.3 that the sequence $(\sigma_j)_{0 \leq j \leq k}$ determines a domain of period coordinates for Q and $\mathcal{C}$. There exists a closed affine subcone $A(\sigma_k) \subset \mathcal{V}(\sigma_k) \subset \mathcal{V}(\sigma_0)$ in the linear space of solutions of the switch condition which equals the cone of vertical measured geodesic laminations of the points of $\mathcal{C}$ contained in the domain of these period coordinates. Via a carrying map $\sigma_k \to \sigma_0 = \tau$ and mass normalization, the interior of this cone determines a connected C^1-submanifold $C \subset \mathcal{V}_0(\tau)$, homeomorphic to an open cell of dimension $\dim(A(\sigma_k)) - 1$.

There is a natural euclidean inner product on the real vector space of weight functions on the branches of τ for which the weight functions f_b, defined by $f_b(b) = 1$ and $f_b(e) = 0$ for $e \neq b$, are an orthonormal basis. This inner product restricts to a Riemannian metric on C which induces a Lebesgue measure vol.

Lemma 4.2. *The letters of the alphabet $\mathcal{A}$ define a partition of an open subset of the manifold C of full Lebesgue measure into subsets $C(x)$ $(x \in \mathcal{A})$. For each $x \in \mathcal{A}$ there exists a C^1-diffeomorphism $H_x : C(x) \to C$.*

Proof. We show first that the letters from the alphabet $\mathcal{A}$ define a partition of an open subset of C of full Lebesgue measure.

Thus let $x \neq y \in \mathcal{A}$. Then x, y are realized by full numbered splitting sequences $(x_\ell)_{0 \leq \ell \leq m}$ and $(y_j)_{0 \leq j \leq s}$ starting with the sequence $(\sigma_j)_{0 \leq j \leq k}$.

As the sequences $(x_\ell),(y_j)$ are distinct, there exists a largest $u\in[k,\min\{m,s\}-k-1]$ such that that $x_u=y_u$. Then there exists a large branch $e\in x_u$ so that up to exchanging $(x_\ell),(y_j)$, the transition modifying x_u to x_{u+1} contains a right split at e and the transition modifying $y_u=x_u$ to y_{u+1} contains a left split at e. Let $\hat{x}_{u+1},\hat{y}_{u+1}$ be the train tracks obtained from x_u by these splits at e; then $\hat{x}_{u+1}$ is splittable to x_{u+1} and $\hat{y}_{u+1}$ is splittable to y_{u+1}. In particular, x_{u+1} is carried by $\hat{x}_{u+1}$, and y_{u+1} is carried by $\hat{y}_{u+1}$.

By the construction of a split, we have $\mathcal{V}(x_u)=\mathcal{V}(\hat{x}_{u+1})\cup\mathcal{V}(\hat{y}_{u+1})$ where the sets $\mathcal{V}(\hat{x}_{u+1}),\mathcal{V}(\hat{y}_{u+1})$ are closed subcones of $\mathcal{V}(x_u)$. Their interiors are non-empty since x_{u+1},y_{u+1} are large by assumption and hence $\hat{x}_{u+1},\hat{y}_{u+1}$ are large as well. The intersection $\mathcal{V}(\hat{x}_{u+1})\cap\mathcal{V}(\hat{y}_{u+1})$ is a hyperplane in $\mathcal{V}(x_u)$.

The support of a measured geodesic lamination which is carried by both $\hat{x}_{u+1}$ and $\hat{y}_{u+1}$ is not contained in $\mathcal{LL}(Q)$. But by Lemma 3.7 of [H11] and the Poincaré recurrence theorem, the set of differentials in $\mathcal{C}$ whose vertical measured geodesic laminations have support in $\mathcal{LL}(Q)$ has full λ-measure. Since the measure λ is the standard Lebesgue measure in period coordinates for $\mathcal{C}$ (up to scaling), it is absolutely continuous with respect to the *local stable foliation* of $\mathcal{C}$ into leaves which consist of differentials with the same vertical measured lamination, with conditionals in the Lebesgue measure class. As the λ-measure of the set of differentials with vertical measured lamination in $\mathcal{V}(\hat{x}_{u+1})\cap\mathcal{V}(\hat{y}_{u+1})$ vanishes, this implies the following. Consider the cone $A(x_u)$ consisting of the vertical measured geodesic laminations of differentials in $\mathcal{C}$ carried by x_u in period coordinates determined by the sequence $(x_j)_{0\leq j\leq u}$. Then the Lebesgue measure of the intersection $A(x_u)\cap\mathcal{V}(\hat{x}_{u+1})\cap\mathcal{V}(\hat{y}_{u+1})\subset A(x_u)$ vanishes, where the Lebesgue measure is the restriction of the Lebesgue measure on $A(x_u)$. Equivalently, we have $\mathrm{vol}(C\cap\mathcal{V}(\hat{x}_{u+1})\cap\mathcal{V}(\hat{y}_{u+1}))=0$. Note that this statement required an argument as for general affine invariant manifolds, the submanifold $C\subset\mathcal{V}_0(\tau)$ has positive codimension and hence may a priori be entirely contained in a hyperplane of positive codimension.

Since by construction, every $\xi\in C$ is the vertical measured lamination for some $z\in\mathcal{C}$, it follows from Lemma 4.1 and absolute continuity of λ with respect to the local stable foliation that the letters from the alphabet $\mathcal{A}$ define a partition of an open subset of C of full Lebesgue measure. For $x=(x_\ell)_{0\leq\ell\leq m}\in\mathcal{A}$, the partition set $C(x)\subset C$ defined by x is the interior of the closed cell of measured laminations which are carried by x_{m-k} and which are horizontal measured laminations for differentials $q\in\mathcal{C}$ in the period coordinates determined by the defining sequence $(\sigma_j)_{0\leq j\leq k}=(x_j)_{0\leq j\leq k}$.

We are left with showing that there exists a natural C^1-diffeomorphism $H_x: C(x)\to C$. Let again $x=(x_\ell)_{0\leq\ell\leq m}\in\mathcal{A}$. By construction, the train track x_{s-k} coincides with $x_0=\tau$ as a numbered train track, that is, there is a canonical isomorphism $x_0\to x_{s-k}$. The composition of this isomorphism with a carrying map $x_{s-k}\to x_0$, followed by total mass renormalization, then defines a C^1-diffeomorphism $C\to C(x)$ whose inverse H_x has the required properties. $\square$

We next collect more information on the maps H_x ($x \in \mathcal{A}$). By Lemma 3.2 and the construction of the sequence $(\sigma_j)_{0\leq j\leq k}$, there is a number $\chi > 0$ such that the manifold $C \subset \mathcal{V}_0(\tau)$ is contained in the set $\mathcal{P}(\tau,\chi) = \mathcal{P}(\tau) \subset \mathcal{V}_0(\tau)$ of all measured geodesic laminations which give weight bigger than χ to every branch of τ.

Let V be the vector space of weight functions on the branches of τ which satisfy the switch condition. For $x = (x_j)_{0\leq j\leq \ell} \in \mathcal{A}$, the composition of the natural identification $\tau = x_0 \to x_{\ell-k}$ with a carrying map $x_{\ell-k} \to x_0 = \tau$ defines a linear isomorphism B_x of V. That B_x is of maximal rank and hence a linear isomorphism is immediate from an easy dimension count.

The linear map B_x maps the closed cone $\mathcal{V}(\tau)$ into itself and can be projectivized to an embedding

$$L : \zeta \in \mathcal{V}_0(\tau) \to B_x(\zeta)/(B_x(\zeta)(\tau)) \in \mathcal{V}_0(\tau) \tag{2}$$

where $B_x(\zeta)(\tau)$ is the total weight of the transverse measure $B_x(\zeta)$ on τ. The inverse L^{-1} of L, defined on the image of L, restricts to the C^1-diffeomorphism $H_x : C(x) \to C$. Write DH_x to denote its differential.

Define a Finsler metric $\| \|_{\sup}$ on $\mathcal{P}(\tau)$ as follows. Let $\nu \in \mathcal{P}(\tau)$. Then every nearby point $\mu \in \mathcal{V}_0(\tau)$ can be represented in the form $\mu = \nu + \alpha$ where α is a signed weight function on τ satisfying the switch conditions with $\alpha(\tau) = 0$. Thus the tangent space $T_\nu\mathcal{V}_0(\tau)$ of $\mathcal{V}_0(\tau)$ can naturally be identified with the vector space $W_0 = \ker(\varphi)$ where $\varphi : V \to \mathbb{R}$ is the linear functional defined by $\varphi(\alpha) = \alpha(\tau)$. For $\alpha \in W_0$ and $\nu \in \mathcal{P}(\tau)$ put

$$\|\alpha\|_{\sup} = \max\{|\alpha(b)|/\nu(b) \mid b\}.$$

Lemma 4.3. *There is a number $\kappa > 1$ and for all $x \in \mathcal{A}$ there is some $c_x > \kappa$ such that*

$$\kappa\|v\|_{\sup} \leq \|DH_x v\|_{\sup} \leq c_x\|v\|_{\sup} \text{ for all } v \in TC(x).$$

Proof. The lemma is a fairly immediate consequence of Part 2) of Lemma 3.2.

By construction of the alphabet $\mathcal{A}$, each $x \in \mathcal{A}$ corresponds to a finite full numbered splitting sequence $(\tau_i)_{0\leq i\leq \ell}$. The subsequences $(\tau_i)_{0\leq i\leq k}$ and $(\tau_i)_{\ell-k\leq i\leq \ell}$ satisfy the assumptions in Lemma 3.2, and the map H_x can be viewed as the inverse of the restriction of the linear carrying map $\hat{L} : \mathcal{V}_0(\tau_{\ell-k}) \to \mathcal{V}_0(\tau_0)$ to a subset of $\mathcal{P}(\tau_{\ell-k})$. Thus to show the left hand side of the inequality in the lemma it suffices to show that the restriction of $\hat{L}$ to $\mathcal{P}(\tau_{\ell-k})$ contracts the Finsler metric $\| \|_{\sup}$ by a fixed constant $c < 1$ not depending on $x \in \mathcal{A}$.

Thus let $\nu \in \mathcal{P}(\tau_{\ell-k})$ and let $0 \neq \alpha \in T_\nu\mathcal{V}_0(\tau_{\ell-k})$ be a signed weight function on $\tau_{\ell-k}$ satisfying the switch conditions with $\alpha(\tau_{\ell-k}) = 0$. As we are interested in derivatives, by rescaling α we may assume that $\mu = \nu + \alpha > 0$, that is, μ is a normalized transverse measure on $\tau_{\ell-k}$. Moreover, by perhaps replacing α by $-\alpha$ we may assume that

$$u = \|\alpha\|_{\sup} = \max\{\alpha(b)/\nu(b) \mid b\}.$$

Write
$$u+1=\max\{\frac{\alpha(b)+\nu(b)}{\nu(b)}\mid b\}=\max\{\frac{\mu(b)}{\nu(b)}\mid b\}=a_0^{-1};$$
then $a_0=\min\{\frac{\nu(b)}{\mu(b)}\mid b\}$. Let μ_0,ν_0 be the normalized transverse measures on τ_0 which are the images of μ,ν under a carrying map. Put
$$\tilde a_0=\min\{\frac{\nu_0(b)}{\mu_0(b)}\mid b\}<1.$$
The third part of Lemma 3.2 shows that $\tilde a_0\geq a_0+\delta(1-a_0)$ where $\delta>0$ is a universal constant.

Put $\tilde u=\tilde a_0^{-1}-1$; then we have
$$\frac{\tilde u}{u}=\frac{1-\tilde a_0}{1-a_0}\frac{a_0}{\tilde a_0}\leq(1-\delta)\frac{a_0}{a_0+\delta(1-a_0)}\leq 1-\delta$$
which shows the left hand side of the inequality stated in the lemma.

The map (2) can be identified with a projective linear map defined on a compact domain in a real projective space and hence the norm of its derivative is bounded away from zero by a constant depending on x. This shows the right hand side of the inequality in the lemma. □

For each $x\in\mathcal{A}$ let $J(x)$ be the inverse of the Jacobian of the map $H_x:C(x)\to C$ with respect to the volume element vol on C. The function $\log J(x)\circ H_x^{-1}$ on C is of class C^1. The final goal of this section is to control the differential of $\log J(x)\circ H_x^{-1}$. To this end we need some preparation which will be used to control the roof function ω in Section 5 as well.

Recall the linear map $B_x:V\to V$ obtained as a composition of the natural identification $x_0\to x_{\ell-k}$ with the carrying map $x_{\ell-k}\to x_0$. The linear space V admits a real basis $b_1,\dots,b_u$ consisting of vertex cycles for $x_0=\tau$. With respect to this basis, the entries of the matrix defining the linear map B_x are non-negative (in fact positive by the construction of the alphabet $\mathcal{A}$) and hence B_x is a Perron Frobenius map. In particular, there exists a simple eigenvalue $\lambda_1(B_x)>0$ of maximal absolute value, and the one-dimensional eigenspace for this eigenvalue is spanned by a vector with positive entries with respect to the basis $b_1,\dots,b_u$.

We next observe that the restriction of the roof function ω to $C(x)$ coincides with $\log\lambda_1(B_x)$ up to a universal additive constant.

Lemma 4.4. *There exists a number $\chi>1$ so that for every $x\in\mathcal{A}$ and every $\mu\in C(x)$ we have $e^{\omega(\mu)}/\lambda_1(B_x)\in[\chi^{-1},\chi]$.*

Proof. Let $x=(x_p)_{0\leq p\leq m}$. The Perron Frobenius linear map B_x maps the closed cone $\mathbb{R}_+\overline{C}\subset V$ homeomorphically onto the closed cone $\mathbb{R}_+\overline{C(x)}$ which is contained in the interior of $\mathbb{R}_+\overline{C}$. Thus by the Brower fixed point theorem, applied to the action of B_x on the projectivizations of these cones which are compact cells, a Perron Frobenius eigenvector v of B_x with positive entries in the basis $b_1,\dots,b_u$ is contained in the cone $\mathbb{R}_+C(x)$.

Choose the Perron Frobenius eigenvector v of B_x so that $v \in C(x) \subset \mathcal{V}_0(\tau)$. As $B_x(v) = \lambda_1(v)v$ we have $\omega(v) = \log \lambda_1(v)$. To complete the proof of the lemma it now suffices to show that for all $\mu, \nu \in C(x)$ it holds $\omega(\nu)/\omega(\mu) \in [\chi^{-1}, \chi]$ where $\chi > 0$ is a universal constant. But this follows from the fact that $C \subset \mathcal{P}(\tau)$ and the third part of Lemma 3.2. We refer to the proof of Lemma 4.3 for a more detailed discussion. □

Let $n \geq 1$ be the dimension of the manifold C.

Proposition 4.5. (1) *There exists a number* $\kappa > 1$ *such that*

$$\mathrm{vol}(C(x)) \in [\kappa^{-1}\lambda_1(B_x)^{-n}, \kappa\lambda_1(B_x)^{-n}] \text{ for all } x \in \mathcal{A}.$$

(2) *The differential of* $\log J(x) \circ H_x^{-1}$ *is uniformly bounded in* C^0*, independent of* $x \in \mathcal{A}$*.*

Proof. Let $x = (x_p)_{0 \leq p \leq s} \in \mathcal{A}$. By duality, the linear map $B_x^* : \mathcal{V}^*(x_{s-k}) \to \mathcal{V}^*(x_{s-k})$ defined by composition of the identification of x_{s-k} with x_0 with a carrying map $x_0^* \to x_{s-k}^*$ (see [PH92]) is Perron Frobenius. Let $\xi \in \mathcal{V}_0(x_0)$ be the normalized Perron Frobenius eigenvector of B_x and let $\zeta \in \mathcal{V}^*(x_0) \subset \mathcal{V}^*(x_{s-k})$ be the Perron Frobenius eigenvector of B_x^*, chosen so that $\iota(\xi, \zeta) = 1$. Note that the argument used to understand the Perron Frobenius eigenvector of B_x also shows that the Perron Frobenius eigenvector of B_x^* can be viewed as a positive transverse measure on x_0^*, equivalently a positive tangential measure on x_0.

As B_x defines a pseudo-Anosov mapping class Ψ (take a lift of the splitting sequence defining x to the space of marked large numbered train tracks and use the fact that as the train tracks are numbered, the endpoint identification defines a unique mapping class), the projective measured laminations $[\xi], [\zeta]$ are the attracting and repelling projective measured lamination of the action of Ψ on $\mathcal{PML}$. In particular, if we put $a = e^{\omega(\xi)}$ then we have $\Psi(\xi) = a\xi$ and $\Psi(\zeta) = a^{-1}\zeta$, whence $a = \lambda_1(B_x)$ (compare the proof of Lemma 4.4).

The manifold C is an open cell in an affine space $\xi + W$ where W is a linear subspace of the kernel of the linear functional $\varphi : v \to \varphi(v) = v(\tau)$ on V. Let $C_\zeta \subset \mathbb{R}_+ C \subset \mathcal{V}(x_0)$ be defined by

$$C_\zeta = \{\beta/\iota(\beta, \zeta) \mid \beta \in C\}.$$

By convex linearity of the function $\beta \to \iota(\beta, \zeta)$, the set C_ζ is an open cell in an affine space $\xi + W'$ where W' is a linear subspace of the kernel of the linear functional $\alpha : v \in V \to \iota(v, \zeta) \in \mathbb{R}$. Note that as ζ is fixed and is viewed as a tangential measure on $\tau = x_0$, the lack of continuity of the intersection form on signed measured laminations is not an issue here. Also, C_ζ is a graph over C of the inverse of an affine function. The restriction of the standard euclidean inner product on the vector space spanned by the weight functions on the branches of $\tau = x_0$ naturally induces a volume form ψ on C_ζ.

Let $A_\zeta : C \to C_\zeta$ be the natural graph map $\beta \to \beta/\iota(\beta, \zeta)$. We claim that its first and second derivatives are uniformly bounded in norm, independent of x. Here as before, we use the norm $\| \, \|$ induced by the inner product on V. Using convex linearity of the map $\beta \to \iota(\beta,\zeta)=\alpha(\beta)$ and the fact that $C\subset \xi+W$ for $\xi\in C$, we compute

$$\frac{d}{dt}\alpha^{-1}(\xi+tv)\cdot(\xi+tv) = -\alpha^{-2}(\xi+tv)\alpha(v)\cdot(\xi+tv)+\alpha^{-1}(\xi+tv)\cdot v \text{ and}$$
$$\frac{d^2}{dt^2}\alpha^{-1}(\xi+tv)\cdot(\xi+tv)|_{t=0} = 2\alpha^{-3}(\xi)\alpha(v)^2\cdot\xi - 2\alpha^{-2}(\xi)\alpha(v)\cdot v.$$

Since the restriction of α to C is bounded from above and below by a universal positive constant, these derivatives are pointwise uniformly bounded in norm. The same argument also shows that the first and second derivatives of the inverse A_ζ^{-1} of A_ζ are also uniformly bounded in norm.

The natural identification $x_0\to x_{s-k}$ which corresponds to the pseudo-Anosov mapping class Ψ maps ζ to $a^{-1}\zeta$ and hence maps $C_\zeta\subset\mathcal{V}(x_0)$ to $C_{a^{-1}\zeta}\subset\mathcal{V}(x_{k-s})$. The carrying map $\mathcal{V}(x_{s-k})\to\mathcal{V}(x_0)$ can be thought of as an inclusion of positive cones in a linear space. Renormalizing a point $\beta\in C_{a^{-1}\zeta}$ to be contained in C_ζ amounts to scaling the measured lamination β with the fixed constant a^{-1} by convex bilinearity of the intersection form. But this means that this renormalization scales the volume form on $C_{a^{-1}\zeta}$ with the constant $a^{-\dim(C)}=e^{-n\omega(\xi)}=\lambda_1(B_x)^{-n}$.

As the volume of C with respect to the measure vol is positive and fixed and the volume distortion of the map $C\to C_\zeta$ is bounded from above and below by a constant not depending on x, we obtain from the above discussion the existence of a number $\kappa>1$ so that

$$\mathrm{vol}(C(x))\in[\kappa^{-1}\lambda_1(B_x)^{-n},\kappa\lambda_1(B_x)^{-n}].$$

This shows the first part of the proposition.

The second part also follows from this estimate. Namely, decompose

$$J(x)=\mathrm{Jac}(A_\zeta^{-1})\cdot\mathrm{Jac}(U)\circ\mathrm{Jac}(A_{a^{-1}\zeta})$$

(read from right to left) where $U: C_{a^{-1}\zeta}\to C_\zeta$ is induced by the carrying map as described above and hence has constant Jacobian a^{-n}. Thus taking logarithms shows that $\log J(x)=\log\mathrm{Jac}(A_{a^{-1}\zeta})-n\log a+\log\mathrm{Jac}(A_\zeta)^{-1}\circ(U\circ A_{a^{-1}\zeta})$. By the beginning of this proof, we know that the differentials of $A_{a^{-1}\zeta}$ and A_ζ^{-1} are uniformly bounded in norm, independent of x, and the same holds true for $\log\mathrm{Jac}(A_\zeta^{-1})$ as a function on $C_\zeta\subset\mathcal{V}_0(\tau)$. As furthermore by Lemma 4.3 (and its proof), the differential of the map $U\circ A_{a^{-1}\zeta}$ contracts norms, this completes the proof of the proposition. □

The following corollary summarizes what we have established so far.

Corollary 4.6. *The map $H=\cup_x H_x:\cup_x C(x)\to C$ is an expanding Markov map in the sense of Definition 2.2 of* [AGY06]. *This means that the following holds true.*

1. *The interiors of the sets $C(x)$ $(x\in\mathcal{A})$ define a partition of a full measure sets of the Finsler manifold $(C,\|\,\|_{\sup})$ into open sets.*
2. *For each $x\in\mathcal{A}$, the map $H_x: C(x)\to C$ is a C^1-diffeomorphism between $C(x)$ and C, and there exists constants $\kappa>1$ and $c_x>\kappa$ such that*

$$\kappa\|v\|_{\sup}\le\|DH_x\cdot v\|_{\sup}\le c_x\|v\|_{\sup}.$$

(3) *For each* $x\in\mathcal{A}$, *the function* $\log J(x)\circ H_x^{-1}$ *on* C *is of class* C^1, *and*
$$\Vert D(\log J(x)\circ H_x^{-1})\Vert_{C^0(C)}\leq \kappa'$$
for a universal constant $\kappa'>0$.

We will use the following well known fact on expanding Markov maps (see Chapter 4 of [Aa97] for a detailed account on this line of ideas).

Proposition 4.7. *An expanding Markov map preserves a unique absolutely continuous measure* ψ *with* C^1*-density that is bounded from above and below. The measure* ψ *is ergodic.*

In the context at hand, absolutely continuous means in the Lebesgue measure class with respect to the manifold structure on C, that is, $\psi=f\mathrm{vol}$ for a C^1-function f with values in some compact interval $[a,b]\subset(0,\infty)$.

5. Control of the roof function

Our goal is to use Corollary 4.6 to establish exponential mixing for the Teichmüller flow Φ^t on the affine invariant manifold $\mathcal{C}$. To this end we have to control the roof function ω which enters the definition of the suspension (Z,Ψ^t). As this roof function only depends on the future, it induces a function on the one-sided shift $(\Xi^+,\sqsupset)=P^+(\Xi,\sqsupset)$ where $P^+:\Xi\to\Xi^+$ is the forgetful map $(y_i)_{i\in\mathbb{Z}}\to(y_i)_{i\geq 0}$. We next establish the control of this roof function we need.

As before, denote by V the vector space of weight functions on $\tau=x_0$ satisfying the switch conditions and let $(x_p)_{0\leq p\leq\ell}\in\mathcal{A}$.

For $x=(x_i)_{0\leq i\leq s}\in\mathcal{A}$ let $h_x:\mathcal{P}(x_{s-k})\to\mathbb{R}$ be the function which associates to $\mu\in\mathcal{P}(x_{s-k})\subset\mathcal{V}_0(x_{s-k})$ the logarithm $h_x(\mu)=\log\mu(x_0)$ of the mass that μ deposits on x_0. Let $p>0$ be the number of branches of a train track $\eta\in\mathcal{LL}(Q)$. The following observation is similar to Proposition 4.5.

Lemma 5.1. *The function* h_x *is* p*-Lipschitz continuous with respect to the Finsler metric* $\Vert\,\Vert_{\sup}$.

Proof. Let $\mu,\nu\in\mathcal{P}(x_{s-k})\subset\mathcal{V}_0(x_{s-k})$ and put $\alpha=\mu-\nu$. Then
$$\mu(x_0)=\sum_{e\subset x_0}\mu(e)=\nu(x_0)+\sum_{e\subset x_0}\alpha(e)$$
and hence
$$\frac{\mu(x_0)}{\nu(x_0)}\leq 1+\frac{\sum_e|\alpha(e)|}{\nu(x_0)}\leq 1+p\max_{e\subset x_0}\frac{|\alpha(e)|}{\nu(e)}.$$

However, Lemma 4.5 of [H11] shows that
$$\max_{e\subset x_0}\frac{|\alpha(e)|}{\nu(e)}\leq\max_{b\subset x_{s-k}}\frac{|\alpha(b)|}{\nu(b)}$$

and hence we have

$$|\log \mu(\tau_0) - \log \nu(\tau_0)| \leq \log(1 + p \max\{\frac{|\alpha(b)|}{\nu(b)} \mid b \subset x_{s-k}\}).$$

Since $\lim_{t\to 0} \frac{\log(1+t)}{t} = 1$ we conclude that indeed, the function h_x is p-Lipschitz for the Finsler metric $\| \|_{\sup}$. □

Using the notations from Section 4 we have

Lemma 5.2. *There are numbers $r_1 > 0, r_2 > 0$ with the following property.*

(1) $\omega \geq r_1$.
(2) *For every $x \in \mathcal{A}$, the function $\omega \circ H_x^{-1}$ is differentiable, and the norm with respect to the Finsler metric $\| \|_{\sup}$ of its derivative is pointwise bounded from above by r_2.*
(3) *It is not possible to write $\omega = \alpha + \beta \circ T_+ - \beta$ where α is locally constant and β is piecewise of class C^1.*

Proof. The first part of the lemma is immediate from the construction (see Section 3 for details).

Part (2) follows from Lemma 5.1. Namely, for $x = (x_i)_{0\leq i\leq s} \in \mathcal{A}$ the function $\omega \circ H_x^{-1}$ is just the function which associates to $\mu \in C \subset \mathcal{P}(x_{s-k}) \subset \mathcal{V}_0(x_{s-k})$ the logarithm of the total mass that μ deposits on x_0 by the carrying map $x_{s-k} \to x_0$. That this function is Lipschitz continuous was established in Lemma 5.1.

For the proof of (3) we follow the proof of Lemma 4.5 of [AGY06]. Namely, suppose that (3) is not true. Then there exists a locally constant function ψ and a piecewise C^1-function ζ so that $\omega = \psi + \zeta \circ T - \zeta$. Write $\omega^{(n)} = \sum_{j=0}^{n-1} \omega(\tau^j)$.

Denote as before by V the vector space of weight functions on τ satisfying the switch conditions, spanned by $b_1, \dots, b_u$. Then $b = \sum_j b_j$ can be thought of as a positive (by recurrence) transverse measure on τ. For $x = (x_j)_{0\leq j\leq \ell} \in \mathcal{A}$ consider as before the Perron Frobenius map $B_x : V \to V$ obtained from a composition of the natural identification $x_0 \to x_{\ell-k}$ with the carrying map $x_{\ell-k} \to x_0$ and let h be the inverse branch of the shift $\Xi^+ \to \Xi^+$, corresponding to the letter $x \in \mathcal{A}$. By assumption, we have $D(\omega^{(n)} \circ h^n) = D\omega - D(\omega \circ h^n)$, which can be rewritten as

$$\frac{\|(B_x^*)^n \cdot v\|}{\|(B_x^*)^n \cdot \mu\|} = D\omega(\mu)\cdot v - D(\omega \circ h^n)(\mu) \cdot v \quad (\mu \in [x])$$

or

$$\frac{\langle v, B_x^n \cdot b\rangle}{\langle \mu, B_x^n \cdot b\rangle} = D\varphi(\mu)\cdot v - D(\varphi \circ h^n)(\mu)\cdot v.$$

Since $Dh^n \to 0$, it follows that $[B_x^n \cdot b] \in P\mathbb{R}^p$ converges to a limit $[w] \in P\mathbb{R}^p$ independent of h. This implies that $[w] \in P\mathbb{R}^p$ is invariant by all B_x, $x \in \mathcal{A}$. Since w is a limit of positive vectors (vectors with positive coordinates), by the Perron-Frobenius theorem, w is collinear with the (unique) positive eigenvector of B_x. which is an eigenvector with respect to the Perron Frobenius eigenvalue.

Now note that as $[w]$ is independent of x, the Perron Frobenius eigenvector of B_x does not depend on x. But this violates the fact that the Perron Frobenius eigenvalue of B_x equals the vertical measured geodesic lamination of the periodic point corresponding to the periodic word only containing the letter x. Since for any two distinct periodic points in $\mathcal{Q} \supset \mathcal{C}$ these vertical measured geodesic laminations are distinct, this is a contradiction. □

In [AGY06], the notion of a good roof function was defined as a roof function with the properties stated in Lemma 5.2.

Corollary 5.3. *The one-sided Markov shift* $(\Xi^+, \sqsupset)$ *defines a uniformly expanding Markov map, and* ω *is a good roof function.*

Let ψ be the measure on C in the Lebesgue measure class which is invariant under the expanding Markov map $\cup_x C(x) \to C$ and whose existence is guaranteed by Proposition 4.7. Following [AGY06] we say that the roof function ω *has exponential tails* if there is a number $\beta > 0$ such that

$$\int_C e^{\beta\omega} d\psi < \infty.$$

The final goal of this section is to show.

Proposition 5.4. *The roof function* ω *has exponential tails.*

Proof. Consider as before the shift $(\Xi^+, \sqsupset)$ and the function $-n\omega$. It satisfies $\|\sum_{\sqsupset u = z} e^{-n\omega(u)}\|_\infty < \infty$. Namely, the set of all inverse branches of a point $z \in \Xi^+$ admits a natural identification with the alphabet $\mathcal{A}$. For each $u = xz$ where $x \in \mathcal{A}$, Lemma 4.4 and Lemma 4.5 show that up to a universal multiplicative constant, the value of $e^{-n\omega(u)}$ coincides with $\psi(C(x))$. As $\sum_{x\in\mathcal{A}} \psi(C(x)) = \psi(C) < \infty$, the estimate follows.

By Lemma 3.7, the roof function φ on the shift space (Σ, T) is of bounded variation. Since there exists a natural embedding $E : \Xi \to \Sigma$ mapping cylinder sets to cylinder sets so that $\omega(y) = \sum_{i=0}^{\ell-1} \varphi(T^i(E(y))$ where $\ell \geq 1$ only depends on the first letter of the one-sided infinite sequence y, the function ω is of bounded variation as well. Thus the *Gurevich pressure* $P_G(-n\omega)$ of $-n\omega$ is defined by $P_G(-n\omega) = \sup\{h_\nu(\sqsupset) - n\int \omega\, d\nu \mid \nu$ an invariant Borel probability measure such that $\int \omega d\nu < \infty\}$ and is finite [S03]. Put $P = P_G(-n\omega)$.

As a consequence [S03], there exists an invariant Gibbs measure for the Markov shift $(\Xi^+, \sqsupset)$ which associates to a finite cylinder set $[x_0, \dots, x_{\ell-1}] \subset \Xi^+$ $(x_i \in \mathcal{A})$ a volume contained in the interval $[\theta^{-1}e^{-n\omega^{(\ell)}(z)-\ell P}, \theta e^{-n\omega^{(\ell)}(z)-\ell P}]$ where $\theta > 1$ is a universal constant and z is an arbitrary point in $[x_0, \dots, x_{\ell-1}]$. Recall to this end from Lemma 4.4 and its immediate extensions to finite cylinder sets that there exists a constant $p > 0$ so that $|\omega^{(\ell)}(z) - \omega^{(\ell)}(u)| \leq p$ for all $z, u \in [x_0, \dots, x_{\ell-1}]$ and that furthermore the volume of a cylinder $[x_0, \dots, x_{\ell-1}]$ equals $\omega^{(\ell)}(z)$ up to a universal multiplicative constant where $z \in [x_0, \dots, x_{\ell-1}]$ is arbitrary.

On the other hand, by Proposition 4.5, the Lebesgue measure fulfills such an estimate for $P = 0$. Namely, viewing $C(x)$ as the cylinder subset of Ξ defined by the letter $x \in \mathcal{A}$ up to a set of measure zero, the proof of Proposition 4.5 which relies on Perron Frobenius eigenvectors for the Perron Frobenius maps B_x equally applies to any finite cylinder set in Ξ. Thus the invariant measure in the Lebesgue measure class equals the Gibbs equilibrium state of the function $-n\omega$, of vanishing pressure.

We use this and the fact that ω is essentially constant on any $C(x)$ to show the exponential tail property. Choose a number $R > 0$ which is sufficiently large that the finite set $\mathcal{A}_R = \{x \in \mathcal{A} \mid \omega(C(x)) \leq R\}$ is non-trivial and put $\hat{\mathcal{A}} = \mathcal{A} \setminus \mathcal{A}_R$. The one-sided shift $(\hat{\Xi}^+, \daleth)$ over $\hat{\mathcal{A}}$ is defined, and it is a $\daleth$-invariant subspace of Ξ^+. The roof function ω on Ξ^+ restricts to a roof function $\hat{\omega}$ on $\hat{\Xi}^+$. As $\hat{\omega}$ has the same properties as ω, for every $a > 0$ the Gurevich pressure of $-a\hat{\omega}$ is finite. Moreover, since the pressure of $-n\omega$ on Ξ^+ vanishes, and any $\daleth$-invariant measure on $\hat{\Xi}^+$ also is a $\daleth$-invariant measure on Ξ^+, the Gurevich pressure of $-n\hat{\omega}$ is non-positive.

We claim that this pressure is in fact negative. To see this assume otherwise. Then by [S03], $-n\hat{\omega}$ admits a unique Gibbs measure $\hat{\psi}$ on $\hat{\Xi}^+$. Via the embedding $\hat{\Xi}^+ \to \Xi^+$, this measure can be viewed as a $\daleth$-invariant measure on Ξ^+. As the Gibbs measure $\hat{\psi}$ of $-n\hat{\omega}$ is characterized by the equality $P_G(-n\hat{\omega}) = h_{\hat{\psi}}(\daleth) - n\int \hat{\omega} d\hat{\psi}$ and we have $\int \hat{\omega} d\hat{\psi} = \int \omega d\hat{\psi}$, we conclude that $\hat{\psi}$ is in fact the unique Gibbs measure for $-n\omega$. But the Gibbs measure of $-n\omega$ is of full support, a contradiction.

We established so far that the pressure of $-n\hat{\omega}$ is negative. On the other hand, the pressure of the constant function 0 on $\hat{\Xi}^+$ is positive, which is equivalent to stating that there are invariant measures with positive entropy on $\hat{\Xi}$, for example supported on an invariant compact subset. By continuity of the function $s \to P_G(-s\hat{\omega})$ there exists a number $\delta \in (0, n)$ so that the pressure of $-\delta\hat{\omega}$ on $\hat{\Xi}$ vanishes. The equilibrium state for this measure gives mass uniformly equivalent to $e^{-\delta\hat{\omega}(z)} = e^{-\delta\omega(z)}$ to a cylinder set $[\hat{x}]$ for $\hat{x} \in \hat{\mathcal{A}}$ and $z \in [\hat{x}]$. Here uniformly equivalent means that the ratio of these numbers is contained in a fixed compact subinterval of $(0, \infty)$, and we use once more the fact that the variation of ω on each such cylinder set is uniformly bounded, independent of the defining letter $\hat{x} \in \hat{\mathcal{A}}$. But this means that

$$\sum_{\hat{x} \in \hat{\mathcal{A}}} e^{-\delta\omega(z)} < \infty \tag{3}$$

where as before, $z \in [\hat{x}]$ is an arbitrarily chosen point.

On the other hand, up to a positive multiplicative constant, the value of the sum (3) can be identified with the integral of the function $e^{(n-\delta)\omega}$ over $\cup_{x \in \hat{\mathcal{A}}} C(x)$ with respect to the Lebesgue measure ψ. As $\mathcal{A} - \hat{\mathcal{A}}$ is finite, and the function ω is bounded on each cylinder set, this implies that $\int_C e^{(n-\delta)\omega} d\psi < \infty$, which is what we wanted to show. □

6. Exponential mixing

In this section we complete the proof of the main result from the introduction. Consider the suspension of the Markov shift $(\Xi^+, \daleth)$ by the roof function ω. This suspension admits an invariant measure $\tilde{\psi}$ defined by $d\tilde{\psi} = d\psi \times dt$.

Lemma 6.1. *The measure $\tilde{\psi}$ on the suspension space is finite.*

Proof. We have to show that $\int_C \omega d\psi < \infty$. However, by Proposition 5.4, there exists a number $\delta > 0$ with $\int e^{\delta\omega} d\psi < \infty$. As ω is bounded from below by a positive constant, we have $e^{\delta\omega} \geq C\omega = b|\omega|$ for a constant $b > 0$ and hence $\int \omega d\mu < \infty$ which shows the lemma. □

As a consequence, the suspension of the Markov shift $(\Xi^+, \daleth)$ with roof function ω is a semiflow preserving the measure $\psi \times dt$, which we normalize to be a probability measure. As ω is a good roof function with exponential tails, we can apply Theorem 7.3 of [AGY06]. In the statement of the following result, we do not specify the class of test functions for which exponential mixing can be established and instead refer to Section 7 of [AGY06].

Proposition 6.2 (Theorem 7.3 of [AGY06])**.** *The suspension of the one-sided shift $(\Xi^+, \daleth)$ with roof function ω is exponentially mixing with respect to the measure $\tilde{\psi}$.*

Recall that there exists a finite-to-one semi-conjugacy of the suspension (Z, Ψ^t) of the two-sided shift $(\Xi, \daleth)$ with roof function ω onto a Φ^t-invariant subset of $\mathcal{C}$ of full λ-measure. Furthermore, there exists a Ψ^t-invariant probability measure $\hat{\lambda}$ on Z which is mapped by F to a positive multiple of λ. The measure $\hat{\lambda}$ is absolutely continuous with respect to the stable and unstable foliation, with conditionals in the Lebesgue measure class. Let $P^+ : \Xi \to \Xi^+$ be the natural forgetful map. It satisfies $\daleth \circ \Pi^+ = \Pi^+ \circ \daleth$.

The shift Ξ defines a cross section for the suspension flow Ψ^t on Z. As the measure $\hat{\lambda}$ is a probability measure which is invariant under the flow Ψ^t, locally near a point $x \in \Xi$ it can be written in the form $\hat{\lambda} = \hat{\psi} \times dt$ where $\hat{\psi}$ is a measure on Ξ. Since the roof function ω is bounded from below by positive constant, the total mass of $\hat{\psi}$ is finite. This can be seen by noting that the set $W = \{(x,t) \in \Xi \times [0,\infty) \mid t < \omega(x)\}$ embeds into Z, and by construction, we have $\hat{\lambda}(W) = \int \omega d\hat{\psi}$.

Our next goal is to show that the measure ψ on Σ^+ can be thought of as a distintegration of $\hat{\psi}$ via the map $P^+ : \Xi \to \Xi^+$. To this end we invoke the Definition 2.5 of [AGY06] of a *hyperbolic skew product over* $\daleth$, adapted to our context. We shall not repeat the definition but rather verify that all the conditions in the definition are satisfied. As before, we identify the one-sided shift $(\Xi^+, \daleth)$ with the expanding Markov map it defines.

Proposition 6.3. *The map $P^+ : (\Xi, \hat{\psi}) \to (\Xi^+, \psi)$ is a hyperbolic skew product over $\daleth$.*

Proof. Follwing (3) of Definition 2.5 of [AGY06], we have to show that there exists a family of probability measures $\{\nu_u\}_{u \in C}$ on Ξ which define a disintegration of $\hat{\psi}$ over ψ in the following sense. $u \to \nu_u$ is measurable, ν_u is supported in $(P^+)^{-1}(u)$ and, for any measurable set $E \subset \Xi$, we have $\hat{\psi}(E) = \int \nu_u(E) d\psi(u)$.

Recall that any letter $x \in \mathcal{A}$ determines a domain of period coordinates for $\mathcal{C}$. The measure λ equals the Lebesgue measure in these period coordinates up to a positive constant multiplicative factor. The cylinder set $[x] \subset \Xi$ for $x \in \mathcal{A}$ can be identified with the set of all differentials in $\mathcal{C}$ with horizontal measured lamination contained in $C(x) \subset \mathcal{V}_0(\tau_0)$ and vertical measured lamination contained in the positive cone $A^* \subset \mathcal{V}^*(\tau)$ of a linear subspace of $\mathcal{V}^*(\tau)$. For $\xi \in C(x)$, the requirement $\iota(\xi, \cdot) = 1$ cuts out a compact subset C_ξ of A^* which is a graph over the subset of $\mathcal{V}^*(\tau)$ of tangential measures of total mass one, where the defining function is uniformly bounded from above and below by a positive constant and depends smoothly on ξ.

As a consequence, the imaginary parts of the period coordinates for $\mathcal{C}$ defined by x determine a family of smooth measures on the manifolds C_ξ which can be normalized to be probability measures. These measure fulfill the requirement of distintegration formulated in the beginning of this proof.

As furthermore, via Lemma 4.3 and duality between $\mathcal{V}(\tau)$, equivalently the positive shift, and $\mathcal{V}^*(\tau)$, equivalently the negative shift, there exists $\kappa > 1$ such that for $y_1, y_2 \in \Xi$ with $P^+(y_i) = P^+(y_2)$ it holds $d(\sqsupset y_1, \sqsupset y_2) \leq \kappa^{-1} d(y_1, y_2)$ where d is the distance function induced by the (dual of) the Finsler metric $\| \|_{\sup}$, all requirements in Definition 2.5 of [AGY06] are fulfilled. This completes the proof of the proposition. □

As the roof function ω has exponential tails, we can apply Theorem 2.7 of [AGY06] and obtain.

Theorem 6.4. *The Teichmüller flow on affine invariant manifolds is exponentially mixing for the* $\mathrm{SL}(2, \mathbb{R})$*-invariant measure.*

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