BOUNDED COHOMOLOGY AND CROSS RATIOS

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Abstract. We use cross ratios to describe second real continuous bounded cohomology for locally compact second countable topological groups. We also survey some recent rigidity results for representations which make essential use of bounded cohomology.

1. Introduction

Let G be any locally compact second countable topological group. For every $i \geq 1$, the group G naturally acts on the vector space $C_b(G^i, \mathbb{R})$ of continuous bounded maps $G^i \to \mathbb{R}$. If we denote by $C_b(G^i, \mathbb{R})^G \subset C_b(G^i, \mathbb{R})$ the linear subspace of all G-invariant such maps, then the *second real continuous bounded cohomology group* $H^2_{cb}(G, \mathbb{R})$ of G is defined as the second cohomology group of the complex

$$0 \to C_b(G, \mathbb{R})^G \xrightarrow{d} C_b(G^2, \mathbb{R})^G \xrightarrow{d} \dots$$

with the usual homogeneous coboundary operator d (see [19]). This coboundary operator maps a G-invariant continuous function $\varphi : G^k \to \mathbb{R}$ $(k \geq 1)$ to the G-invariant continuous function

$$d\varphi(g_0, \dots, g_k) = \sum_{i=0}^{k} (-1)^i \varphi(g_0, \dots, \hat{g}_i, \dots, g_k).$$

A function φ is called a *cocycle* if $d\varphi = 0$. With the usual interpretation of an ordered triple of points in G as a "2-simplex" with vertices in G, a function φ is a cocycle if and only if it vanishes on the boundary of any "3-simplex" with vertices in G.

For $k \geq 1$ let $H^k_c(G, \mathbb{R})$ be the k-th ordinary *continuous* second cohomology group of G with real coefficients. Then there is a natural homomorphism $H^k_{cb}(G, \mathbb{R}) \to H^k_c(G, \mathbb{R})$ which in general is neither injective nor surjective. This homomorphism is obtained by viewing a continuous bounded G-invariant function on G^{k+1} as a continuous G-invariant function without any further assumption. If G is a countable group with the discrete topology then we simply write $H_b(\Gamma, \mathbb{R})$ instead of $H_{cb}(\Gamma, \mathbb{R})$.

Bounded cohomology can be used to distinguish various classes of locally compact second countable topological groups and to impose restrictions on the existence of interesting continuous homomorphisms between such groups. The goal of this

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note is to discuss some geometric aspects of second continuous bounded cohomology and to survey some recent applications to rigidity results for homomorphisms between geometrically defined groups.

In Section 2 we introduce anti-symmetric cross ratios as a way to view the cocycle conditions for G-invariant bounded functions $G^3 \to \mathbb{R}$ as a symmetry condition for G-invariant bounded functions $G^4 \to \mathbb{R}$. In Section 3 we discuss strong boundaries for locally compact second countable topological groups and their relevance to computing second continuous bounded cohomology. We describe explicit examples of such strong boundaries for some classes of groups, and we discuss in some detail the case of surface groups. In Section 4 we investigate the image of the natural homomorphism $H^2_{cb}(G,\mathbb{R}) \to H_c(G,\mathbb{R})$ in a specific geometric context. Section 5 is dedicated to a survey of some recent rigidity results using second bounded cohomology.

2. Cross-ratios and bounded cohomology

Let X be an arbitrary topological space with infinitely many points. We define an *anti-symmetric cross ratio* for X to be a bounded function $[\,]$ on the space of quadruples of pairwise distinct points in X with the following additional properties.

i) $[\xi,\xi',\eta,\eta'] = -[\xi',\xi,\eta,\eta']$.
ii) $[\eta,\eta',\xi,\xi'] = -[\xi,\xi',\eta,\eta']$.
iii) $[\xi,\xi',\eta,\eta'] + [\xi',\xi'',\eta,\eta'] = [\xi,\xi'',\eta,\eta']$.

Note that $[\,]$ can naturally be extended by 0 to the set of quadruples of points in X of the form (ξ,ξ,η,ζ) where ξ,η,ζ are pairwise distinct.

Let G be an arbitrary locally compact second countable topological group which acts continuously on X as a group of homeomorphisms. For example, if G is countable and equipped with the discrete topology, then any action on a space X can be assumed to be continuous by equipping X with the discrete topology. We define an *anti-symmetric G-cross ratio* on X to be an anti-symmetric *continuous* cross ratio on X which is invariant under the action of G. Let $\mathcal{C}(G,X)$ be the vector space of all anti-symmetric G-cross ratios on X, equipped with the supremums-norm $\| \, \|$. Thus we have $\|[\,]\| = \sup\{[\xi,\xi',\eta,\eta'] \mid \xi,\xi',\eta,\eta' \in X \text{ are pairwise distinct}\}$, and $(\mathcal{C}(G,X),\| \, \|)$ is naturally a Banach space.

The second continuous real bounded cohomology group $H^2_{cb}(G,\mathbb{R})$ of G is a vector space of equivalence classes of bounded functions $G\times G\times G \to \mathbb{R}$ which are invariant under the free left action of G. It can be equipped with a natural pseudo-norm which assigns to a cohomology class the infimum of the supremums-norms of any representative of the class. For the second continuous bounded cohomology, this pseudo-norm turns out to be a norm which provides $H^2_{cb}(G,\mathbb{R})$ with the structure of a Banach space [14]. We have.

Proposition 2.1. *There is a continuous linear map* $\Omega : \mathcal{C}(G,X) \to H^2_{cb}(G,\mathbb{R})$.

Proof. Let $[\,] \in \mathcal{C}(G,X)$ and let (ξ,ζ,η) be a triple of pairwise distinct points in X. Choose an arbitrary point $\nu \in X$ which is distinct from any of the points in the triple and define

$$\varphi(\xi,\eta,\zeta) = \frac{1}{2}\big([\xi,\zeta,\eta,\nu] + [\zeta,\eta,\xi,\nu] + [\eta,\xi,\zeta,\nu]\big). \tag{1}$$

We claim first that this does not depend on the choice of ν. Namely, let $\nu' \in X - \{\xi,\eta,\zeta\}$ be arbitrary. We compute

$$\begin{aligned} &[\xi,\zeta,\eta,\nu] + [\zeta,\eta,\xi,\nu] + [\eta,\xi,\zeta,\nu] - [\xi,\zeta,\eta,\nu'] - [\zeta,\eta,\xi,\nu'] - [\eta,\xi,\zeta,\nu'] \\ &= [\xi,\zeta,\nu',\eta] + [\xi,\zeta,\eta,\nu] + [\zeta,\eta,\nu',\xi] + [\zeta,\eta,\xi,\nu] + [\eta,\xi,\nu',\zeta] + [\eta,\xi,\zeta,\nu] \\ &= [\xi,\zeta,\nu',\nu] + [\zeta,\eta,\nu',\nu] + [\eta,\xi,\nu',\nu] = 0 \end{aligned} \tag{2}$$

by the defining properties of an anti-symmetric cross ratio. This shows our claim.

As a consequence, the function φ is well defined, moreover it is continuous and invariant under the action of the group G. Its supremums norm is bounded in absolute value by $3\|[\,]\|/2$. We extend φ (non-continuously) by 0 to the set of all triples of points in X for which at least two of the points coincide. Then φ induces a continuous bounded function μ on G^3 which is invariant under the free left action of G as follows. Choose any point $\xi \in X$ and define $\mu(g_1,g_2,g_3) = \varphi(g_1\xi, g_2\xi, g_3\xi)$.

We next show that this function μ is a cocycle, i.e. that its image under the natural coboundary map $C_b(G^3,\mathbb{R}) \to C_b(G^4,\mathbb{R})$ vanishes. For this it is enough to show the corresponding properties for the function φ.

We begin with showing that φ is *alternating*. Namely, observe that for a triple (a,b,c) of pairwise distinct points in X and any $d \in X - \{a,b,c\}$ we have

$$\begin{aligned} 2(\varphi(a,c,b) + \varphi(a,b,c)) &= [a,b,c,d] + [b,c,a,d] + [c,a,b,d] \\ &+ [a,c,b,d] + [c,b,a,d] + [b,a,c,d] = 0 = 2(\varphi(a,c,b) + \varphi(c,a,b)) \end{aligned} \tag{3}$$

and similarly for the remaining elementary permutations. We use this fact to show that

$$\varphi(b,c,d) - \varphi(a,c,d) + \varphi(a,b,d) - \varphi(a,b,c) = 0 \tag{4}$$

for any quadruple (a,b,c,d) of points in X. By our observation (3), equation (4) is valid for every quadruple (a,b,c,d) of points in X for which at least 2 of the points coincide. On the other hand, for every quadruple (a,b,c,d) of pairwise distinct points in X we have

$$\begin{aligned} 2(\varphi(a,b,c) - \varphi(b,c,d)) &= [a,c,b,d] + [c,b,a,d] + [b,a,c,d] \\ &- [b,d,c,a] - [d,c,b,a] - [c,b,d,a] = 2[c,b,a,d] \end{aligned} \tag{5}$$

by the defining properties of an anti-symmetric cross ratio. Together with equation (3) we deduce with the same calculation as before that $\varphi(a,c,d) - \varphi(a,b,d) = -\varphi(c,a,d) + \varphi(a,d,b) = -[d,a,c,b] = -[c,b,a,d]$ and consequently φ satisfies the cocycle equation (4). Therefore the function μ on G defines a continuous bounded cocycle and hence a second continuous bounded cohomology class $\Omega([\,]) \in H^2_{cb}(G,\mathbb{R})$.

We next observe that the cohomology class $\Omega([\,])$ does not depend on the choice of the point $\xi \in X$ which we used to define the function φ. Namely, let η be a different point in X and define $\mu'(g_1, g_2, g_3) = \varphi(g_1\eta, g_2\eta, g_3\eta)$. For $g \in G$ write $\nu(g) = [g\xi, \eta, \xi, g\eta]$. By equation (3), we have $\nu(g) = \varphi(\xi, \eta, g\xi) + \varphi(g\xi, \eta, g\eta)$ and consequently

$$\mu(e, g_1, g_2) - \mu'(e, g_1, g_2) - \nu(g_1) - \nu(g_1^{-1}g_2) + \nu(g_2) \tag{6}$$

is the value of φ on the boundary of a singular polyhedron of dimension 3 whose vertices consist of the points $\xi, \eta, g_1\xi, g_1\eta, g_2\xi, g_2\eta$ and whose sides contain the simplices with vertices $\xi, g_1\xi, g_2\xi$ and $\eta, g_1\eta, g_2\eta$ as well as three quadrangles with the same set of vertices (see [3], in particular Figure 3, for an illustration of this situation in a similar context). Since φ is a cocycle, the evaluation of φ on the boundary of this polyhedron vanishes.

If we define $\tilde{\nu}(\gamma_1, \gamma_2) = \nu(\gamma_1^{-1}\gamma_2)$ then we have

$$d\tilde{\nu}(e, \gamma_1, \gamma_2) = \nu(\gamma_1^{-1}\gamma_2) - \nu(\gamma_2) + \nu(\gamma_1) \tag{7}$$

and hence the formula (6) together with invariance under the diagonal action of G shows that the function $\mu - \mu'$ is the image of the G-invariant bounded function $\tilde{\nu}$ on G^2 under the coboundary map $C_b(G^2) \to C_b(G^3)$. As a consequence, for every $[\,] \in \mathcal{C}(G, X)$ the bounded cohomology class $\Omega([\,]) \in H^2_{cb}(G, \mathbb{R})$ only depends on the cross ratio $[\,]$ but not on the choice of a point $\xi \in X$. The resulting map $\Omega : \mathcal{C}(G, X) \to H^2_{cb}(G, \mathbb{R})$ is clearly linear and continuous with respect to the supremums norm on $\mathcal{C}(G, X)$ and the Gromov norm on $H^2_{cb}(G, \mathbb{R})$. ☐

Every locally compact second countable group G acts freely on itself by left translations. Thus we can use Proposition 2.1 for $X = G$ to deduce that there is a continuous linear map from the vector space of all anti-symmetric G-cross ratios on G into $H^2_{cb}(G, \mathbb{R})$. Namely, we have.

Corollary 2.2. *Let G be a locally compact second countable group; then $H^2_{cb}(G, \mathbb{R})$ is the quotient of the space of continuous anti-symmetric G-invariant cross ratios on G under the coboundary relation.*

Proof. By Proposition 2.1 and its proof there is a linear map Ω from the space of all continuous anti-symmetric G-cross ratios on G into $H^2_{cb}(G, \mathbb{R})$. We have to show that Ω is surjective. For this note that every second continuous bounded cohomology class for G is the class of a G-invariant continuous *alternating* function φ on G^3 which satisfies the cocycle equation (4) (see [14, 19]). For such a function φ define $[g, g', h, h'] = \varphi(h, g', g) - \varphi(g', g, h')$ as in the proof of Proposition 2.1. Then $[\,]$ is continuous and G-invariant; we claim that it satisfies the requirements of an anti-symmetric G-cross ratio.

Namely, first of all we have

$$[b, c, a, d] = \varphi(a, c, b) - \varphi(c, b, d) = -\varphi(a, b, c) + \varphi(b, c, d) = -[c, b, a, d] \tag{8}$$

since φ is alternating and by the cocycle equality (4) moreover

$$[a, d, c, b] = \varphi(c, d, a) - \varphi(d, a, b) = -\varphi(a, b, d) + \varphi(a, c, d) = -[c, b, a, d]. \tag{9}$$

In the same way we obtain that

$$[a, b, c, d] + [b, b', c, d] = [a, b', c, d] \tag{10}$$

which shows that [] is indeed a G-invariant anti-symmetric cross ratio. By the proof of Proposition 2.1, the cohomology class of φ is just $\Omega([\,])$. Thus the map Ω is surjective. □

We conclude this section with some examples.

Examples:

1) Let G be a lcoally compact second countable group. The group G is called *amenable* if for every continuous action of G on a compact metrizable space X there is a G-invariant Borel probability measure on X. In other words, there is a fixed point for the natural induced action of G on the space $\mathcal{P}(X)$ of probability measures on X. For example, all compact groups and all solvable groups are amenable (see [21]). The bounded cohomology of amenable groups vanishes in degree different from 0 (see [14, 19] for more information and for references).

2) Let G be a simple Lie group of non-compact type with trivial center. Then the natural homomorphism $H^2_{cb}(G,\mathbb{R}) \to H^2_c(G,\mathbb{R})$ is an isomorphism [8]. Namely, Burger and Monod showed that the kernel of this homomorphism vanishes (see also the discussion on p.176 of [19]). On the other hand, a class in $H^2_c(G,\mathbb{R})$ is characteristic, and such classes are automatically bounded by an observation of Gromov. Moreover, the group $H^2_c(G,\mathbb{R})$ does not vanish if and only if the symmetric space associated to G is of Hermitian type.

3) A lattice Γ in a product $G_1 \times \cdots \times G_k$ of locally compact second countable non-compact topological groups is called *irreducible* if the projection of Γ to each factor is dense. If G does not admit any product decomposition of this form then any lattice in G is called irreducible. Let Γ be an irreducible lattice in a semi-simple Lie group G of non-compact type without compact factors. If the real rank of G is at least two then the kernel of the natural homomorphism $H^2_{cb}(\Gamma,\mathbb{R}) \to H^2_c(\Gamma,\mathbb{R})$ vanishes, and $H^2_b(\Gamma,\mathbb{R}) = H^2_{cb}(G,\mathbb{R}) = H^2_c(G,\mathbb{R})$ [8]. If the real rank of G equals one then the kernel of the natural homomorphism $H^2_{cb}(\Gamma,\mathbb{R}) \to H^2_c(\Gamma,\mathbb{R})$ is infinite dimensional [10].

3. Strong boundaries

In this section we explain how one can obtain some control on second continuous bounded cohomology for an arbitrary locally compact second countable topological group G.

Let (B,μ) be a *standard Borel space*, i.e. (B,μ) is a measure space equivalent to the unit interval equipped with the Borel σ-algebra and a Borel probability measure. The group G *acts* on (B,μ) if there is a measurable homomorphism of G into the space of measure class preserving Borel maps $(B,\mu) \to (B,\mu)$. The action of G on (B,μ) is called *amenable* if it satisfies a property extending the definition of an amenable group acting trivially on a single point. We refer to [21] for a precise definition. For our purpose, the crucial property of an amenable action of G on (B,μ) is as follows. If G acts as a group of continuous transformations on a compact metrizable space X then there is a G-equivariant measurable map $B \to \mathcal{P}(X)$ (see [21]).

A *separable G-module* is a separable Banach space E together with a continuous homomorphism of G into the isometry group of E. A *strong boundary* for a locally compact second countable topological group G is a standard measure space (B,μ) together with a measure class preserving action of G such that the following two conditions are satisfied.

(1) Amenability: The G-action on (B,μ) is amenable.
(2) Double ergodicity: For any separable Banach-G-module E, any measurable G-equivariant map $B \times B \to E$ is essentially constant.

Note that the second requirement for a strong boundary is more restrictive than just asking for ergodicity of the diagonal action of G on $B \times B$. The existence of a strong boundary for every locally compact second countable topological group G was established by Kaimanovich [15]. Abstractly, such a strong boundary can be obtained as a Poisson boundary of a suitably chosen random walk on G.

We next list some examples of strong boundaries.

Examples:

1) Let G be a semi-simple Lie group of non-compact type without compact factor. Let $P < G$ be a minimal parabolic subgroup. The *Furstenberg boundary* of G is the quotient G/P. The space G/P is a compact G-space and admits a natural G-invariant Lebesgue measure class. If λ is any probability measure defining this class then the space $(G/P,\lambda)$ is a strong boundary for G.

2) An irreducible lattice $\Gamma < G$ in a semi-simple Lie group G of non-compact type acts doubly ergodic on the Furstenberg boundary $(G/P,\lambda)$. It follows that $(G/P,\lambda)$ is a strong boundary for Γ as well.

A strong boundary (B,μ) for a locally compact second countable topological group G was introduced by Burger and Monod [8] as a tool for the calculation of second continuous bounded cohomology for G with arbitrary coefficients. To illustrate their result for the trivial coefficients $\mathbb{R}$, define a *measurable anti-symmetric cross ratio* on B to be a *measurable* essentially bounded G-invariant function on the space of quadruples of distinct points in B which satisfies almost everywhere the defining equation for a cross ratio. Then we have (see [19]).

Lemma 3.1. *Let (B,μ) be a strong boundary for G; then $H^2_{cb}(G,\mathbb{R})$ is naturally isomorphic to the space of G-invariant μ-measurable anti-symmetric cross ratios on B.*

Proof. Let (B,μ) be a strong boundary for G. For $k \geq 1$ denote by $L^\infty(B^k,\mu^k)$ the Banach space of essentially bounded μ^k-measurable functions on B^k; it contains the closed subspace $L^\infty(B^k,\mu^k)^G$ of all G-invariant such functions. By the results of Burger and Monod [9], $H^2_{cb}(G,\mathbb{R})$ coincides with the second bounded cohomology group of the resolution

$$\mathbb{R} \to L^\infty(B,\mu)^G \to L^\infty(B^2,\mu^2)^G \to \dots \tag{11}$$

with the usual homogeneous coboundary operator. By Proposition 2.1 and its proof, there is a natural bijection between the space of μ^4-measurable essentially bounded anti-symmetric G-cross ratios on B and the space of μ^3-measurable bounded cocycles on B, i.e. the space of alternating functions in $L^\infty(B^3,\mu^3)^G$ which satisfy the cocycle equation. Any two such bounded cocycles are cohomologous if and only if they differ by the image under the coboundary operator of an element of $L^\infty(B^2,\mu^2)^G$. However, by ergodicity of the measure class μ^2 under the diagonal action of G, the vector space $L^\infty(B^2,\mu^2)^G$ only contains essentially constant functions. This means that the natural map which assigns to a μ^4-measurable G-cross ratio its corresponding bounded cohomology class is injective. □

We now look at a particularly simple example. Namely, the symmetric space of the group $PSL(2,\mathbb{R}) = PSU(1,1)$ is the (oriented) hyperbolic plane $\mathbf{H}^2$. The Furstenberg boundary admits a natural idenfication with the *ideal boundary* of $\mathbf{H}^2$ which is just the oriented circle S^1. If we denote by λ the usual Lebesgue measure on S^1 then (S^1,λ) is a strong boundary for $PSL(2,\mathbb{R})$.

A triple (a,b,c) of pairwise distinct points in S^1 will be called *ordered* if the point b is contained in the interior of the oriented subinterval (a,c) of S^1. Similarly we define a quadruple (a,b,c,d) of pairwise distinct points on S^1 to be ordered. Ordered quadruples define a $PSL(2,\mathbb{R})$-invariant subset of the space of quadruples of pairwise distinct points in S^1. There is a unique antisymmetric cross ratio $[\,]_0$ on S^1 with the following properties.

(1) $[a,b,c,d]_0 = 0$ if (a,b,c,d) is ordered.
(2) $[a,b,c,d]_0 = 1$ if (a,c,b,d) is ordered.

Clearly this cross ratio is measurable and invariant under the action of $PSL(2,\mathbb{R})$. We have.

Lemma 3.2. (1) *Up to a constant,* $[\,]_0$ *is the unique non-trivial antisymmetric cross ratio on* S^1 *which is invariant under the full group* $PSL(2,\mathbb{R})$ *of orientation preserving isometries of* $\mathbf{H}^2$.
(2) *If* μ *is any Borel measure on* S^1 *without atoms and if* $[\,]$ *is any* μ*-measurable antisymmetric cross ratio on* S^1 *which satisfies* $[a,b,c,d] = 0$ *for* μ*-almost every ordered quadruple of pairwise distinct points in* S^1 *then* $[\,]$ *is a multiple of* $[\,]_0$.

Proof. If $[\,]$ is any antisymmetric cross ratio on S^1 which is invariant under the full group $PSL(2,\mathbb{R})$ then the same is true for the induced function φ on the set of triples of pairwise distinct points in S^1. Recall that the group $PSL(2,\mathbb{R})$ acts transitively on the space of *ordered* triples of distinct points in S^1 and consequently we have $\varphi(c,b,a) = \varphi(d,c,b)$ for every ordered quadruple (a,b,c,d) of pairwise distinct points in S^1. Hence equation (5) in the proof of Proposition 2.1 implies that $[a,b,c,d] = 0$ whenever the quadruple (a,b,c,d) of pairwise distinct points in

S^1 is ordered. As a consequence, if $[\,] \not\equiv 0$ then after possibly multiplying $[\,]$ with a constant we conclude from the definition of an anti-symmetric cross ratio that $(a,b,c,d) = 1$ whenever (a,c,b,d) is ordered. In other words, up to a constant we have $[\,] = [\,]_0$ which shows the first part of the lemma.

To show the second part of the lemma, let μ be a Borel measure on S^1 without atoms and let $[\,]$ be any μ-measurable antisymmetric cross ratio on S^1 such that $[a,b,c,d] = 0$ for almost every ordered quadruple of pairwise distinct points in S^1. Let (a,b,c,d) be a quadruple of pairwise distinct points in S^1 such that $[a,b,c,d] \neq 0$. By the symmetry relation of an antisymmetric cross ratio we may assume that (a,c,b,d) is ordered. Thus the point b is contained in the oriented open subinterval (c,d) of S^1. Now if $b' \in (c,d)$ is another such point then by the defining properties of an antisymmetric cross ratio we have $[a,b,c,d] + [b,b',c,d] = [a,b',c,d]$. Assume first that $b' \in (b,d)$; then the quadruple (b,b',d,c) is ordered and consequently we have $[b,b',c,d] = 0$ by assumption and $[a,b,c,d] = [a,b',c,d]$. Via exchanging b and b' we conclude that $[a,b,c,d]$ does not depend on $b \in (c,d)$. Similarly we deduce that in fact $[a,b,c,d]$ only depends on the order of the points in the quadruple (a,b,c,d). In other words, $[\,]$ is a multiple of $[\,]_0$. $\square$

Lemma 3.2 shows in particular that $[\,]_0$ corresponds to a generator α of the group $H^2_{cb}(PSL(2,\mathbb{R}),\mathbb{R})$.

Now let $\Gamma < PSL(2,\mathbb{R})$ be any torsion free lattice. Then Γ acts freely on the hyperbolic plane $\mathbf{H}^2$ as a properly discontinuous group of isometries, and $S = \mathbf{H}^2/\Gamma$ is an oriented surface of finite volume. The unit circle $S^1 = \partial\mathbf{H}^2$ with the Γ-invariant measure class of the Lebesgue measure λ is a strong boundary for Γ. If Γ is cocompact then we have $H^2(\Gamma,\mathbb{R}) = H^2(M,\mathbb{R}) = \mathbb{R}$ and there is a natural surjective restriction homomorphism $H^2_b(\Gamma,\mathbb{R}) \to H^2(\Gamma,\mathbb{R})$. The diagram

$$\begin{array}{ccc} H^2_b(\Gamma,\mathbb{R}) & \longrightarrow & H^2(\Gamma,\mathbb{R}) \\ \uparrow & & \uparrow \\ H^2_{cb}(PSL(2,\mathbb{R}),\mathbb{R}) & \longrightarrow & H^2_c(PSL(2,\mathbb{R}),\mathbb{R}) \end{array}$$

commutes and hence in our identification of $H^2_b(\Gamma,\mathbb{R})$ with the space of Γ-invariant antisymmetric cross ratios on the unit circle S^1, the $PSL(2,\mathbb{R})$-invariant cross ratio $[\,]_0$ from Lemma 3.2 maps to a multiple of the Euler class of the tangent bundle of S. An easy calculation shows that the multiplication factor is in fact 1 (compare [6] for a detailed explanation of this fact).

If Γ is not cocompact then S is homeomorphic to the interior of a compact oriented surface $\hat{S}$ whose boundary $\partial\hat{S}$ is a finite union of circles. In particular, the fundamental group of each of the boundary circles is amenable. Now the bounded cohomology of any countable cellular space X is naturally isomorphic to the bounded cohomology of its fundamental group $\pi_1(X)$ [14]. Thus $H_b(\partial\hat{S},\mathbb{R}) = \{0\}$ and hence the relative group $H^2_b(\hat{S},\partial\hat{S},\mathbb{R})$ is isomorphic to $H^2_b(\hat{S},\mathbb{R}) = H^2_b(\Gamma,\mathbb{R})$. We refer to [19] for more on exact sequences for bounded cohomology. The image under the natural map $\iota : H^2_b(\hat{S},\partial\hat{S},\mathbb{R}) \to H^2(\hat{S},\partial\hat{S},\mathbb{R})$ of the pull-back $i^*\alpha$ of the distinguished generator α of $H^2_{cb}(PSL(2,\mathbb{R}),\mathbb{R})$ under the inclusion $i : \Gamma \to PSL(2,\mathbb{R})$ is just the Euler class of the (bordered) surface $\hat{S}$. In other words, the evaluation of

$\iota(i^*\alpha)$ on the fundamental class of the pair $(\hat{S}, \partial\hat{S})$ just equals the Euler characteristic of S (we refer to [6] for a detailed discussion of a slightly different viewpoint on these facts).

To describe the kernel Q of the natural map $H_b^2(\Gamma, \mathbb{R}) \to H^2(\hat{S}, \partial\hat{S}, \mathbb{R})$, let Δ be the diagonal in $S^1 \times S^1$. A finitely additive signed measure on $S^1 \times S^1 - \Delta$ is a function μ which assigns to each oriented rectangle of the form $[a, b) \times [c, d)$ where a, b, c, d are pairwise distinct a number $\mu[a, b) \times [c, d) \in \mathbb{R}$ such that

$$\mu[a, b) \times [c, d) + \mu[b, b') \times [c, d) = \mu[a, b') \times [c, d) \tag{12}$$

whenever $[a, b') = [a, b) \cup [b, b')$ is disjoint from $[c, d)$. This measure is flip-antiinvariant if $\mu[a, b) \times [c, d) = -\mu[c, d) \times [a, b)$.

Let λ be the Lebesgue measure on S^1 which defines the unique $PSL(2, \mathbb{R})$-invariant measure class. The collection of all finitely additive flip-antiinvariant finite signed measures μ on $S^1 \times S^1$ with the additional property that the function $(a, b, c, d) \to \mu[a, b) \times [c, d)$ is λ-measurable is clearly a vector space; we call it the space of λ-*measurable* flip-antiinvariant finite signed measures. We have (compare Section 2 of [12]).

Corollary 3.3. *There is a linear isomorphism* Ψ *of the kernel of the map*

$$H_b^2(\Gamma, \mathbb{R}) \to H^2(\hat{S}, \partial\hat{S}, \mathbb{R})$$

onto the vector space of all λ*-measurable finite flip-antiinvariant* Γ*-invariant finitely additive signed measures on* $S^1 \times S^1 - \Delta$.

Proof. Every Γ-invariant λ-measurable anti-symmetric cross ratio $[\,]$ on S^1 defines a finite finitely additive measurable signed measure $\mu = \Psi([\,])$ by assigning to an ordered quadruple (a, b, c, d) of pairwise distinct points in S^1 the value $\mu[a, b) \times [c, d) = [a, b, c, d]$. The map $\Psi : [\,] \to \Psi([\,])$ is clearly linear, moreover by Lemma 2.4 its kernel is spanned by the cross ratio $[\,]_0$ which defines the Euler class of S viewed as a class in $H^2(\hat{S}, \partial\hat{S}, \mathbb{R})$. As as consequence, the restriction of the map Ψ to the kernel Q of the natural map $H_b^2(\Gamma, \mathbb{R}) \to H^2(\hat{S}, \partial\hat{S}, \mathbb{R})$ is injective.

Now let μ be any finite λ-measurable finitely additive Γ-invariant flip anti-invariant signed measure on $S^1 \times S^1 - \Delta$. Define an antisymmetric cross ratio $[\,]$ as follows. First, if (a, b, c, d) is ordered and if a, b, c, d are typical points for λ such that $\mu[a, b) \times [c, d)$ is defined then we define $[a, b, c, d] = \mu[a, b) \times [c, d)$. This determines $[a, b, c, d]$ whenever (a, b, c, d) or (b, a, c, d) or (a, b, d, c) is ordered. Choose an arbitrary fixed typical ordered quadruple (a, b, c, d) and define $[a, c, b, d] = 0$. As in the proof of Lemma 3.2, if $b' \in (a, b)$ is arbitrary (and typical) then (c, a, b', b) is ordered and we define $[a, c, b', d] = [a, c, b', b] + [a, c, b, d] = -\mu[c, a) \times [b', b)$. Similarly, for a point $b' \in (a, b)$ and every $a' \in (d, a)$ define $[a', c, b', d] = [a', a, b', d] + [a, c, b', d] = \mu[a', a) \times [b', d) + [a, c, b', d]$. Successively we can define in this way an anti-symmetric measurable Γ-cross ratio $[]$ on S^1 with $\Psi([]) = \mu$. As a consequence, the map Ψ is surjective. This shows the corollary. $\square$

It follows from our consideration that there is a direct decomposition $H_b^2(\Gamma, \mathbb{R}) = Q \oplus \ker(\Psi)$ where Q is the kernel of the natural map $H_b^2(\Gamma, \mathbb{R}) \to H^2(\hat{S}, \partial\hat{S}, \mathbb{R})$ and

Ψ is the map which associates to a λ-measurable anti-symmetric Γ-cross ratio [ ] its corresponding finitely additive signed measure on $S^1 \times S^1 - \Delta$.

4. Isometries of hyperbolic spaces

Bounded cohomology has shown to be been particularly useful for the investigation of groups of isometries of *hyperbolic geodesic metric spaces*. Here a geodesic metric space X is called δ*-hyperbolic* for some $\delta > 0$ if it satisfies the δ-thin triangle condition: For every geodesic triangle in X with sides a, b, c the side a is contained in the δ-neighborhood of $b \cup c$.

The *Gromov boundary* ∂X of X is defined as follows. For a fixed point $x \in X$, define the *Gromov product* $(y, z)_x$ based at x of two points $y, z \in X$ by

$$(y,z)_x = \frac{1}{2}\big(d(y,x) + d(z,x) - d(y,z)\big). \tag{13}$$

Call two sequences $(y_i), (z_j) \subset X$ *equivalent* if $(y_i, z_i)_x \to \infty$ $(i \to \infty)$. By hyperbolicity of X, this notion of equivalence defines an equivalence relation for the collection of all sequences $(y_i) \subset X$ with the additional property that $(y_i, y_j)_x \to \infty$ $(i, j \to \infty)$ [5]. The boundary ∂X of X is the set of equivalence classes of this relation.

There is a natural topology on $X \cup \partial X$ which restricts to the given topology on X. With respect to this topology, a sequence $(y_i) \subset X$ converges to $\xi \in \partial X$ if and only if we have $(y_i, y_j)_x \to \infty$ $(i, j \to \infty)$ and the equivalence class of (y_i) defines ξ. Every isometry of X acts naturally on $X \cup \partial X$ as a homeomorphism. Moreover, for every $x \in X$ and every $a \in \partial X$ there is a geodesic ray $\gamma : [0, \infty) \to X$ with $\gamma(0) = x$ and $\lim_{t\to\infty} \gamma(t) = a$. If X is *proper*, i.e if closed balls in X of finite radius are compact, then the space $X \cup \partial X$ is compact and metrizable.

For every proper metric space X the isometry group $\mathrm{Iso}(X)$ of X can be equipped with a natural locally compact second countable metrizable topology, the so-called *compact open topology*. With respect to this topology, a sequence $(g_i) \subset \mathrm{Iso}(X)$ converges to some isometry g if and only if $g_i \to g$ uniformly on compact subsets of X. A closed subset $Y \subset \mathrm{Iso}(X)$ is compact if and only if there is a compact subset K of X such that $gK \cap K \neq \emptyset$ for all $g \in Y$. In particular, the action of $\mathrm{Iso}(X)$ on X is proper. In the sequel we always assume that subgroups of $\mathrm{Iso}(X)$ are equipped with the compact open topology.

The *limit set* Λ of a subgroup G of the isometry group of a proper hyperbolic geodesic metric space X is the set of accumulation points in ∂X of one (and hence every) orbit of the action of G on X. If G is non-compact then its limit set is a compact non-empty G-invariant subset of ∂X. The group G is called *elementary* if its limit set consists of at most two points. In particular, every compact subgroup of $\mathrm{Iso}(X)$ is elementary. If G is non-elementary then its limit set Λ is uncountable without isolated points, and G acts as a group of homeomorphisms on Λ.

We showed in [13].

Theorem 4.1. *Let X be a proper hyperbolic geodesic metric space and let $G <$ $\mathrm{Iso}(X)$ be a non-elementary closed subgroup. Then one of the following two possibilities holds.*

(1) *G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ and the kernel of the natural homomorphism $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ is infinite dimensional.*
(2) *G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ and the natural homomorphism $H_{cb}(G, \mathbb{R}) \to H_c(G, \mathbb{R})$ is injective.*

Note that in the case that G is any closed subgroup of $PSL(2, \mathbb{R})$ this theorem is closely related to the discussion in Section 2.

Examples:

1) The automorphism group G of a regular tree X of finite valence acts transitively on the complement of the diagonal in $\partial X \times \partial X$. Thus by Theorem 4.1, its second bounded cohomology embeds into the second usual cohomology. We refer to [8] for a discussion of the bounded cohomology for groups acting on a product of trees.

2) If Γ is a finitely generated group which acts discretely as a group of isometries on a Gromov hyperbolic geodesic metric space then Theorem 4.1 implies that the kernel of the natural map $H_b(\Gamma, \mathbb{R}) \to H(\Gamma, \mathbb{R})$ is infinite dimensional. This result was earlier shown by Fujiwara [10].

3) A finitely generated group Γ is called *hyperbolic* if Γ viewed as a metric space with respect to the word metric defined by a finite symmetric generating set is hyperbolic. If M is a closed Riemannian manifold of bounded negative curvature then the fundamental group Γ of M is hyperbolic and hence by Theorem 4.1 the kernel of the natural map $H^2_b(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is infinite dimensional.

4) Let M be a closed Riemannian manifold of non-positive curvature. By a recent result of Bestvina and Fujiwara [4], the kernel of the natural homomorphism $H_b(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is nontrivial if and only if M is *not* a locally symmetric space of rank at least two. Then this kernel is in fact infinite dimensional.

From now on we assume that X is a proper hyperbolic geodesic metric space of *bounded growth*. This means that there is a number $b > 1$ such that for every $R > 1$, every metric ball of radius R contains at most be^{bR} disjoint metric balls of radius 1. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ. Let (B, μ_0) be a strong boundary for G. Since Λ is a compact metrizable G-space there is an equivariant Furstenberg map $(B, \mu_0) \to \mathcal{P}(\Lambda)$. Now G is non-elementary by assumption and therefore this map induces a G-equivariant map $\varphi : B \to \Lambda$ (see e.g. Lemma 2.2 of [13]). The image μ of μ_0 under the map φ is a measure on Λ whose measure class is invariant under the action of G and such that the diagonal action of G on $\Lambda \times \Lambda$ is ergodic with respect to the class of $\mu \times \mu$. Since X is of bounded growth by assumption, the stabilizer in $\mathrm{Iso}(X)$ of each point in ∂X is amenable and therefore the measure space (Λ, μ) is a strong boundary for G (compare the discussion in [1, 2, 15]). In particular, Lemma 3.1 implies that $H^2_{cb}(G,\mathbb{R})$ is isomorphic to the space of μ-measurable anti-symmetric G-cross ratios on Λ.

Assume now that G acts transitively on the complement of the diagonal in $\Lambda\times\Lambda$. The stabilizer $G_{a,b}$ of (a,b) acts on $\Lambda-\{a,b\}$ as a group of homeomorphisms. Moreover, there is some $g\in G$ which maps (a,b) to (b,a). Define the group G to be *directed* if there is a $G_{a,b}$-invariant subset $U(a,b)$ of $\Lambda-\{a,b\}$ of positive μ-mass such that $gU(a,b)\cap U(a,b)=\emptyset$. We have.

Lemma 4.2. *Let $G<\mathrm{Iso}(X)$ be a closed subgroup with limit set Λ. Assume that G acts transitively on the complement A of the diagonal in $\Lambda\times\Lambda$; if $H^2_{cb}(G,\mathbb{R})\neq\{0\}$ then G is directed.*

Proof. Let $G<\mathrm{Iso}(X)$ be a closed subgroup with limit set Λ which acts transitively on the complement A of the diagonal in $\Lambda\times\Lambda$. If $H^2_{cb}(G,\mathbb{R})\neq 0$ then there is some G-invariant measurable nontrivial alternating function φ on the space of triples of pairwise distinct points in Λ which satisfies the cocycle identity (3).

Let φ be any G-invariant μ-measurable alternating function on the space of triples of pairwise distinct points in Λ. Let $(a,b)\in A$; since the action of G on A is transitive, φ is uniquely determined by its restriction to the set of triples of the form (a,b,u) for $u\in\Lambda-\{a,b\}$. Let $g\in G$ be such that $g(a,b)=(b,a)$. Then $g^{-1}G_{a,b}g=G_{a,b}$ and if G is not directed then for every $G_{a,b}$-invariant measurable subset U of $\Lambda-\{a,b\}$ with $\mu(U)>0$ we have $\mu(U\cap gU)>0$.

Now for every $c\in\mathbb{R}$ and every $\epsilon>0$ the set $U_{c,\epsilon}=\{u\in\Lambda-\{a,b\}\mid\varphi(a,b,u)\in(c-\epsilon,c+\epsilon)\}$ is measurable and $G_{a,b}$-invariant. By assumption, if $\mu(U_{c,\epsilon})>0$ then we have $\mu(gU_{c,\epsilon}\cap U_{c,\epsilon})>0$ as well and therefore there is some $u\in U_{c,\epsilon}$ with $|\varphi(a,b,gu)-c|<\epsilon$. Since φ is alternating we obtain that $\varphi(a,b,gu)=-\varphi(b,a,gu)=-\varphi(ga,gb,gu)=-\varphi(a,b,u)\in(-c-\epsilon,-c+\epsilon)$ and consequently $|c|<2\epsilon$. On the other hand, for μ-almost every $u\in\Lambda$ and every $\epsilon>0$ we have $\mu(U_{\varphi(a,b,u),\epsilon})>0$. From this we deduce that φ vanishes almost everywhere and hence $H^2_{cb}(G,\mathbb{R})=\{0\}$. □

In the case that G is a simple Lie group of non-compact type and rank 1 we have $H_{cb}(G,\mathbb{R})\neq\{0\}$ if and only if $G=SU(n,1)$ for some $n\geq 1$. That this condition is necessary is immediate from Lemma 4.2 and the following observation.

Lemma 4.3. *A simple Lie group G of non-compact type and rank one is directed only if $G=SU(n,1)$ for some $n\geq 1$.*

Proof. Let G be a simple Lie group of non-compact type and rank one. Then G is the isometry group of a symmetric space X of non-compact type and negative curvature with *ideal boundary* ∂X. The action of G on ∂X preserves the measure class of the Lebesgue measure λ and $(\partial X,\lambda)$ is a strong boundary for G. The group G acts transitively on the space A of pairs of distinct points in ∂X. If $G=SO(n,1)$ for some $n\geq 3$ then G acts transitively on the space of triples of pairwise distinct points in ∂X and hence by the above, G is not directed.

Now let $G = Sp(n,1)$ for some $n \geq 1$. Then for every pair $(a,b) \in A$ there is a unique totally geodesic embedded quaternionic line $L \subset X$ of constant curvature -4 whose boundary $\partial L = S^3 \subset \partial X$ contains a and b. The stabilizer $G_{a,b}$ of (a,b) is contained in the stabilizer G_L of L in G which is conjugate to the quotient of the group $Sp(1,1) \times Sp(n-1) < Sp(n,1)$ by its center. Moreover, G_L acts transitively on the space of triples of pairwise distinct points in ∂L. In particular, for every $u \in \partial L$ and every $g \in G$ with $g(a,b) = (b,a)$ the $G_{a,b}$-orbit of u coincides with the $G_{a,b}$-orbit of gu. More generally, this is true for *every* point $u \in \partial X - \{a,b\}$. Namely, let $P : \partial X \to L$ be the shortest distance projection. The subgroup $\tilde{G}_{Pu}$ of G_L which stabilizes Pu is the quotient of the group $Sp(1) \times Sp(n-1)$ by its center. The factor subgroup $Sp(n-1)$ acts transitively on $P^{-1}(Pu)$ while the orbit of Pu under the group $G_{a,b}$ consists of the set of all points in L whose distance to the geodesic connecting b to a coincides with the distance of Pu. As above we conclude that for every $u \in \partial X$ and every $g \in G$ with $g(a,b) = (b,a)$ the $G_{a,b}$-orbit of u coincides with the $G_{a,b}$-orbit of gu. In other words, G is not directed. In the same way we obtain that the exceptional Lie group F_{20}^4 is not directed as well. ☐

Note that for every $n \geq 1$ the group $SU(n,1)$ admits a non-trivial continuous second bounded cohomology class $\alpha \in H^2_{cb}(SU(n,1),\mathbb{R})$ induced by the *Kähler form* ω of the complex hyperbolic space $X = SU(n,1)/S(U(n)U(1))$, and this class generates $H^2_{cb}(SU(n,1),\mathbb{R})$. The anti-symmetric cross ratio $[\,]$ on ∂X defining the class α is a continuous function on ∂X with values in $[-\pi,\pi]$. We have $|[a,b,c,d]| = \pi$ if and only if (a,c,b,d) or (c,a,d,b) is an ordered quadruple of points in the boundary of a complex line in X.

5. Applications and questions

In this section we discuss some applications of second bounded cohomology to the understanding of homomorphisms into the isometry group of a hyperbolic geodesic metric space X of bounded growth.

Let Γ be any ireducible lattice in a semi-simple Lie group G of non-compact type without compact factors and let $\rho : \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. Recall that $(G/P,\lambda)$ is a strong boundary for Γ where $P < G$ is a minimal parabolic subgroup and λ is the Lebesgue measure. Denote by Λ the limit set of $\rho(\Gamma)$. This limit set is empty if and only if the closure H of $\rho(\Gamma)$ is compact. Moreover, since X is of bounded growth, elementary subgroups of $\mathrm{Iso}(X)$ are amenable. Thus as in Section 4, if the closure H of the group $\rho(G) < \mathrm{Iso}(X)$ is not amenable, then there is a measurable ρ-equivariant map $\varphi : G/P \to \Lambda$.

We use φ to pull back *continuous* antisymmetric H-cross ratios to nontrivial measurable cocycles for the action of Γ on G/P. This then implies the following (see [13]).

Corollary 5.1. *Let Γ be an irreducible lattice in a semi-simple Lie group of non-compact type and rank at least 2. Let X be a hyperbolic geodesic metric space of bounded growth and let $\rho : \Gamma \to \mathrm{Iso}(X)$ be a homomorphisms. Then either the closure H of $\rho(\Gamma)$ is amenable or H is a compact extension of a simple Lie group L*

of rank one and there is a continuous surjective homomorphism $G \to L$ factoring through ρ.

Proof. Let Λ be the limit set of the closure H of $\rho(\Gamma)$. If the group H is non-amenable and if it does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then by a refined version of Theorem 4.1, there is an infinite dimensional space of continuous antisymmetric H-cross ratios on Λ. The preimage of such a cross ratio under the map φ is a λ-measurable G-invariant cocycle for the action of Γ on $(G/P, \lambda)$ which defines a nontrivial second bounded cohomology class. This contradicts the results of Burger and Monod.

In the case that H acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$, this argument can not be applied. Instead one has to use bounded cohomology with non-trivial coefficients to obtain the conclusion of the second part of the corollary (see [13] for details). □

We conclude this note with a short discussion of representations of surface groups into $PSU(n, 1)$. We observed in Section 4 that the Kaehler form ω on the corresponding symmetric space $X = PSU(n, 1)/K$ defines a second bounded cohomology class α for $PSU(n, 1)$. Up to normalization, this class is just the first Chern class of the canonical bundle L of X. In particular, if Γ is the fundamental group of a surface S of finite type and if $\rho : \Gamma \to PSU(n, 1)$ is any representation, then $\rho^*\alpha$ is up to normalization the Euler class of the pull-back bundle F^*L under a ρ-equivariant map $F : \mathbf{H}^2 \to X$. We simply write $\chi(\rho^*L)$ do denote this Euler class. If the surface S has cusps then as in the discussion in Section 3, the Euler class is still well defined. We next describe how to calculate this class.

The Kähler form defines a continuous cross ratio $[\,]$ on the boundary ∂X of the negatively curved manifold X which is invariant under the action of $PSU(n, 1)$. This cross ratio is given by

$$[c, b, a, d] = \omega(a, b, c) - \omega(b, c, d)$$

where $\omega(a, b, c)$ denotes the integral of ω over a geodesic triangle in X with ordered ideal endpoints a, b, c for a suitable choice of normalization of ω. Now if the closure H of $\rho(\Gamma)$ is elementary and hence amenable then the second continuous bounded cohomology of H vanishes and we have $\rho^*\alpha = 0$ since ρ factors through the inclusion $H \to G$.

If H is non-elementary then its limit set Λ is uncountable and $[\,]$ restricts to a H-invariant cross ratio on Λ. This cross ratio takes values in $[-1, 1]$, and $[a, b, c, d] = 1$ if and only if the four points a, c, b, d form an orderd quadruple in the boundary of a complex line in X. The pull-back $f^*[\,]$ of the cross-ratio $[\,]$ under a Furstenberg map $f : S^1 \to \Lambda$ is a measurable cross ratio on S^1. By the discussion in Section 3, this cross-ratio has a unique decomposition $f^*[\,] = \nu + c[\,]_0$ where $[\,]_0$ is the unique $PSL(2, \mathbb{R})$-invariant cross ratio on S^1 and where ν is an anti-symmetric bounded signed measure. The Euler class $\chi(\rho^*L)$ is then c times the fundamental class of our surface. This implies in particular the following well-known *Milnor-Wood inequality* (see [7] for a further discussion and references).

Corollary 5.2. *Let S be any oriented surface of finite volume with fundamental group Γ and let $\rho : \Gamma \to PSU(n,1)$ be any representation. Then the Euler class $\chi(\rho^* L)$ of the pull-back of the canonical bundle L satisfies $|\chi(\rho^* L)| \le |\chi(S)|$ where $\chi(S)$ is the Euler characteristic of S. Moreover, equality holds only if $\rho(\Gamma)$ stabilizes a complex line.*

Proof. By our above discussion, we may assume that the closure of $\rho(\Gamma)$ is non-amenable. Then $f^*[\,] = c[\,]+\nu$ where ν is an anti-symmetric signed measure. Now if ν is non-trivial then ν assumes both negative and positive values on a set of positive Lebesgue measure. Since $-1 \le [\,] \le 1$ and since $[\,]_0$ only assumes the values ± 1, this implies that $c < 1$.

If $c = 1$ then almost all quadruples of pairwise distinct points in Λ are contained in the boundary of a complex line. By equivariance, measurability and the defining equations of a cross ratio, these complex lines coincide almost everywhere. Then the limit set Λ of $\rho(\Gamma)$ is contained in this complex line from which the corollary follows (again we refer to [7] for an extended discussion from a slightly different viewpoint). □

As another corollary, we obtain the following stability result.

Corollary 5.3. *Let $\rho : \Gamma \to PSU(1,1) < PSU(n,1)$ be a standard embedding. Then the connected component of ρ in the variety of representations $\Gamma \to PSU(n,1)$ coincides with conjugates of representations into $PSU(1,1)$.*

Proof. The corollary follows from Corollary 5.2 together with the fact that the Euler class $\chi(\rho^* L)$ depends continuously on the representation ρ, on the other hand it only assumes integer values. □

Call a representation $\rho : \Gamma \to PSU(n,1)$ *maximal* if its *Toledo invariant* $\rho^*\alpha$ is maximal. Maximal representations have been investigated by Burger, Iozzi and Wienhard. We look at one particularly easy result (see [6]).

Theorem 5.4. *If $\rho : \Gamma \to PSU(1,1)$ is a maximal representation then ρ is discrete and maps hyperbolic elements to hyperbolic elements.*

Proof. Let $\rho : \Gamma \to PSU(1,2)$ be a maximal representation. Let Λ be the limit set of $\rho(\Gamma)$. We observed above that Λ is uncountable and that there is an equivariant Furstenberg map $f : S^1 \to \Lambda$ which is order preserving. We claim that the kernel of ρ is necessarily trivial.

To see this note that Γ is torsion free and hence if the kernel N of ρ is nontrivial then it is infinite. Since Γ is discrete, the limit set of N is nontrivial and Γ-invariant by normality of N and hence it coincides with the limit set S^1 of Γ.

We follow Burger and Iozzi [6] and choose an open interval $I \subset S^1$ such that $f^{-1}(I)$ contains λ-almost all of an open subset O of S^1. Then the N-translates of O cover S^1, on the other hand these translates are mapped by f into I by equivariance. Since I can be chosen in such a way that $S^1 - I$ contains an open subset of the limit set of $\rho(\Gamma)$, this is impossible.

Similarly we deduce that ρ is discrete by observing that otherwise there is an infinite sequence $g_i \to \infty$ and a triple (x_1, x_2, x_3) of pairwise distinct points in Λ such that $g_i(x_1, x_2, x_3)$ leaves every compact subset of Λ and that $\rho(g_i)(x_1, x_2, x_3)$ remains in a compact subset of the set of triples in the limit set of $\rho(\Gamma)$. Again this is impossible.

To see that ρ maps a hyperbolic element to a hyperbolic element assume otherwise. Let γ be the axis of a hperbolic element $g \in \Gamma$ with $\rho(g) = 0$. Since Γ is a lattice, the endpoints of γ separate the limit set S^1 of Γ into two nontrivial g-invariant connected components. Since the Furstenberg map f is equivariant and order preserving, the limit set Λ of $\rho(\Gamma)$ contains two $f(g)$-invariant ordered subsets. This is impossible if $\rho(g)$ is parabolic, □

We refer to Koziarz and Maubon [16] and Burger and Iozzi [7] for higher dimensional analogs of this ridigidy result.

We conclude our discussion with two open questions.

Question 1: Let F be the free group with $m \geq 2$ generators. The outer automorphism group $\mathrm{Out}(F_m)$ is finitely generated. Is the kernel of the natural map $H_b^2(\mathrm{Out}(F), \mathbb{R}) \to H^2(\mathrm{Out}(F), \mathbb{R})$ infinite dimensional?

The discussion in this note indicates that for a countable group Γ, the kernel of the natural homomorphism $H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is infinite dimensional if there is some hyperbolicity for pairs of points in Γ. If Γ acts as a discrete group of isometries on a simply connected manifold of non-positive curvature then this hyperbolicity can be given the interpretation as a contraction property for the action of Γ on geodesics. Now if G is a simple Lie group of higher rank, then such a contraction property for geodesics does not hold. However, there is a good contraction for the action of Γ on the space of Weyl-chambers or, equivalently, on the space of flat cones of maximal dimension of the symmetric space G/K associated to G. This motivates the following

Question 2: Let Γ be a lattice in a simple Lie group of non-compact type and rank 2. Is the kernel of the natural map $H_b^3(\Gamma, \mathbb{R}) \to H(\Gamma, \mathbb{R})$ infinite dimensional?

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