GEOMETRY OF THE MAPPING CLASS GROUPS II: A BIAUTOMATIC STRUCTURE

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ABSTRACT. We show that the mapping class group $\mathcal{MCG}(S)$ of an oriented surface S of genus g with m punctures and $3g-3+m \geq 2$ admits a biautomatic structure. We also show that various subgroups of $\mathcal{MCG}(S)$ are undistorted.

CONTENTS

1. INTRODUCTION

Let Γ be a finitely generated group and let $\mathcal{G}$ be a finite symmetric set of generators for Γ. Then $\mathcal{G}$ defines a *word norm* $|\,|$ on Γ by assigning to an element $g \in \Gamma$ the smallest length $|g|$ of a word in the generating set $\mathcal{G}$ which represents g. Any two such word norms $|\,|, |\,|_0$ are *equivalent*, i.e. there is a constant $L > 1$ (depending on the generating sets defining the norms) such that $|g|/L \leq |g|_0 \leq L|g|$ for all $g \in \Gamma$. This means that the *word metrics* d, d_0 on Γ defined by $d(g,h) = |g^{-1}h|$ and $d_0(g,h) = |g^{-1}h|_0$ are bilipschitz equivalent. The action of Γ on itself by left translation preserves a word metric.

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Let Γ be a finitely generated group equipped with a word metric d. An *automatic structure* for Γ consists of a finite *alphabet* A, a (not necessarily injective) map $\pi : A \to \Gamma$ and a *regular language* L in A with the following properties (Theorem 2.3.5 of [E92]).

(1) The set $\pi(A)$ generates Γ as a semi-group.
(2) Via concatenation, every word w in the alphabet A is mapped to a word $\pi(w)$ in the generators $\pi(A)$ of Γ. The restriction of the map π to the set of all words from the language L maps L onto Γ.
(3) There is a number $\kappa > 0$ with the following property. For all $x \in A$ and each word $w \in L$ of length $k \geq 0$, the word wx defines a path $s_{wx} : [0, k+1] \to \Gamma$ connecting the unit element to $\pi(wx)$. Since $\pi(L) = \Gamma$, there is a word $w' \in L$ of length $\ell > 0$ with $\pi(w') = \pi(wx)$. Let $s_{w'} : [0, \ell] \to \Gamma$ be the corresponding path in Γ. Then $d(s_{wx}(i), s_{w'}(i)) \leq \kappa$ for every $i \leq \min\{k+1, \ell\}$.

A *biautomatic structure* for the group Γ is an automatic structure (A, L) with the following additional property. The alphabet A admits an inversion ι with $\pi(\iota a) = \pi(a)^{-1}$ for all a, and

$$d(\pi(x)s_w(i), s_{w'}(i)) \leq \kappa$$

for all $w \in L$ all $x \in A$, for any $w' \in L$ with $\pi(w') = \pi(xw)$ and all i (Lemma 2.5.5 of [E92]).

Examples of groups which admit a biautomatic structure are word hyperbolic groups [E92] and groups which admit a proper cocompact action on a CAT(0) cubical complex (Corollary 8.1 of [S06]). The existence of a biautomatic structure for a group Γ has strong implications for the structure of Γ. For example, the word problem is solvable in linear time, and solvable subgroups are virtually abelian [E92, BH99].

Now let S be an oriented surface of finite type, i.e. S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g - 3 + m \geq 2$, i.e. that S is not a sphere with at most 4 punctures or a torus with at most 1 puncture. We then call the surface S *non-exceptional.* The *mapping class group* $\mathcal{MCG}(S)$ of all isotopy classes of orientation preserving self-homeomorphisms of S is finitely generated. We refer to the survey of Ivanov [I02] for the basic properties of the mapping class group and for references. Mosher [M95] showed that the mapping class group $\mathcal{MCG}(S)$ admits an automatic structure. The main goal of this paper is to strengthen this result and to show

Theorem 1. *The mapping class group of a non-exceptional surface of finite type admits a biautomatic structure.*

Most of the known consequences of the existence of a biautomatic structure are well known for mapping class groups. Perhaps the only improvement of known results is a strengthening of a result of Hemion [He79] who showed that the conjugacy problem in $\mathcal{MCG}(S)$ is solvable. From a biautomatic structure we obtain a uniform exponential bound on the length of a conjugating element [BH99].

Corollary. *Let $\mathcal{G}$ be a finite symmetric set of generators of $\mathcal{MCG}(S)$ and let $\mathcal{F}(\mathcal{G})$ be the free group generated by $\mathcal{G}$. There is a constant $\mu > 0$ such that words $u, v \in \mathcal{F}(\mathcal{G})$ represent conjugate elements of $\mathcal{MCG}(S)$ if and only if there is a word $w \in \mathcal{F}(\mathcal{G})$ of length at most $\mu^{\max\{|u|,|v|\}}$ with $w^{-1}uw = v$ in $\mathcal{MCG}(S)$.*

For pseudo-Anosov elements in $\mathcal{MCG}(S)$, the conjugacy problem can be solved in linear time [MM00], and Mosher [M86] gave an explicit algorithm for the solution in this case. For arbitrary elements of the mapping class group, the statement of the corollary was recently also established by Jing Tao [JT09].

A finite symmetric set $\mathcal{G}'$ of generators of a subgroup Γ' of a finitely generated group Γ can be extended to a finite symmetric set of generators of Γ. Thus for any two word norms $|\,|$ on Γ and $|\,|'$ of Γ' there is a number $L > 1$ such that $|g| \leq L|g|'$ for every $g \in \Gamma'$. However, in general the word norm in Γ of an element $g \in \Gamma'$ can not be estimated from below by $L'|g|'$ for a universal constant $L' > 0$. Define the finitely generated subgroup Γ' of Γ to be *undistorted* in Γ if there is a constant $c > 1$ such that $|g|' \leq c|g|$ for all $g \in \Gamma'$. Thus $\Gamma' < \Gamma$ is undistorted if and only if the inclusion $\Gamma' \to \Gamma$ is a quasi-isometric embedding.

There are subgroups of $\mathcal{MCG}(S)$ with particularly nice geometric descriptions. To begin with, an *essential subsurface* of S is a bordered surface S_0 which is embedded in S as a closed subset and such that the following two additional requirements are satisfied.

(1) The homomorphism $\pi_1(S_0) \to \pi_1(S)$ induced by the inclusion is injective.
(2) Each boundary component of S_0 is an *essential* simple closed curve in S, i.e. a simple closed curve which is neither contractible nor freely homotopic into a puncture.

A finite index subgroup $\mathcal{MCG}_0(S_0)$ of the mapping class group $\mathcal{MCG}(S_0)$ of S_0 can be identified with the subgroup of $\mathcal{MCG}(S)$ of all elements which can be represented by a homeomorphism of S fixing $S - S_0$ pointwise. We give an alternative proof of the following result of Masur and Minsky (the result is implicitly but not explicitly contained in Theorem 6.12 of [MM00]).

Theorem 2. *If $S_0 \subset S$ is an essential subsurface of a non-exceptional surface S of finite type then $\mathcal{MCG}_0(S_0) < \mathcal{MCG}(S)$ is undistorted.*

In the case that the subsurface S_0 of S is a disjoint union of essential annuli, the group $\mathcal{MCG}_0(S_0)$ equals the free abelian group generated by the Dehn twists about the center core curves of the annuli. In this case the above theorem was shown by Farb, Lubotzky and Minsky [FLM01] (see also [H09] for an alternative proof).

Now let S be a closed surface of genus $g \geq 2$ and let $\Gamma < \mathcal{MCG}(S)$ be any *finite* subgroup. By the solution of the Nielsen realization problem [Ke83], Γ can be realized as a subgroup of the automorphism group of a marked complex structure h on S. Then the quotient $(S, h)/\Gamma$ is a compact Riemann surface, and the projection $S \to S/\Gamma$ is a branched covering ramified over a finite set Σ of points. Let $S_0 =$

$S/\Gamma - \Sigma$ and let $N(\Gamma)$ be the normalizer of Γ in $\mathcal{MCG}(S)$. Then there is an exact sequence

$$0 \to \Gamma \to N(\Gamma) \to \mathcal{MCG}_0(S_0) \to 0$$

where $\mathcal{MCCG}_0(S_0)$ is the subgroup of the mapping class group of S_0 of all elements which can be represented by a homeomorphism which lifts to a homeomorphism of S [BH73]. We use this to observe (compare [RS07] for a similar statement)

Theorem 3. *Let S be a closed surface of genus $g \geq 2$ and let $\Gamma < \mathcal{MCG}(S)$ be a finite subgroup. Then the normalizer of Γ is undistorted in $\mathcal{MCG}(S)$.*

There are other relations between mapping class groups which can be described by exact sequences of groups. An example of such a relation is as follows. Let S_0 be any non-exceptional surface of finite type and let S be the surface obtained from S_0 by deleting a single point p. Then there is an exact sequence [B74]

$$0 \to \pi_1(S_0) \to \mathcal{MCG}(S) \xrightarrow{\Pi} \mathcal{MCG}(S_0) \to 0.$$

The image in $\mathcal{MCG}(S)$ of an element $\alpha \in \pi_1(S_0)$ is the mapping class obtained by dragging the puncture p of S along a simple closed curve in the homotopy class α. The projection $\Pi : \mathcal{MCG}(S) \to \mathcal{MCG}(S_0)$ is induced by the map $S \to S_0$ defined by closing the puncture p.

Define a *coarse section* for the projection Π to be a map $\Psi : \mathcal{MCG}(S_0) \to \mathcal{MCG}(S)$ with the property that there exists a number $\kappa > 0$ such that

$$d(\Pi\Psi(g), g) \leq \kappa$$

for all $g \in \mathcal{MCG}(S_0)$.

Theorem 4. *The projection $\mathcal{MCG}(S) \to \mathcal{MCG}(S_0)$ admits a coarse section which is a quasi-isometric embedding.*

Theorem 4 contrasts a result of Braddeus, Farb and Putman [BFP07] who showed that the normal subgroup $\pi_1(S_0)$ of $\mathcal{MCG}(S)$ is exponentially distorted.

The proof of Theorem 1 builds on the results of [H09]. In that paper we constructed a locally finite connected directed graph $\mathcal{TT}$ whose vertex set $\mathcal{V}(\mathcal{TT})$ is the set of all isotopy classes of complete train tracks on S. The mapping class group $\mathcal{MCG}(S)$ acts property and cocompactly on $\mathcal{TT}$ as a group of simplicial isometries.

A *discrete path* in a metric space (X, d) is a map $\rho : [0, k_\rho] \cap \mathbb{N} \to X$. For a number $L > 1$, such a path is called an *L-quasi-geodesic* if

$$|j - i|/L - L \leq d(\rho(i), \rho(j)) + L|j - i| + L$$

for all $i, j \in [0, k_\rho] \cap \mathbb{N}$. For convenience, we often consider ρ as an eventually constant map $\mathbb{N} \to X$ by setting $\rho(j) = \rho(k_\rho)$ for all $j \geq k_\rho$. When referring to these eventually constant maps as L-quasi-geodesics, we mean that their restrictions to $[0, k_\rho] \cap \mathbb{N}$ are L-quasi-geodesics.

A *(discrete) bicombing* of a metric space X assigns to any pair of points $x, y \in X$ a discrete path $\rho_{x,y} : [0, k_\rho] \cap \mathbb{N} \to X$ so that $\rho_{x,y}(0) = x, \rho_{x,y}(k_\rho) = y$. The bicombing is called *quasi-geodesic* if there is a number $L > 1$ such that for all $x, y \in X$ the path $\rho_{x,y}$ is a discrete L-quasi-geodesic connecting x to y. The bicombing is called *bounded* if there is a number $L > 1$ such that

$$d(\rho_{x,y}(i), \rho_{z,u}(i)) \le L(d(x,z) + d(y,u)) + L$$

for all $x, y, z, u \in X$ and for all i (here the combing paths $\rho_{x,y}, \rho_{z,u}$ are viewed as eventually constant maps $\mathbb{N} \to X$).

We use directed edge-paths in $\mathcal{TT}$ to construct an $\mathcal{MCG}(S)$-equivariant bounded bicombing of $\mathcal{TT}$. We proceed in three steps.

(1) Show that directed edge-paths connect a coarsely dense set of pairs of points in $\mathcal{TT}$.
(2) If x can be connected to y by a directed edge-path, single out a specific such path so that the resulting path system is invariant under the action of the mapping class group.
(3) Show that these specific directed edge-paths can be equipped with parametrizations in such a way that the resulting path system defines an $\mathcal{MCG}(S)$-equivariant bounded bicombing of $\mathcal{TT}$.

Section 3 of this paper is devoted to the first step above. We show that while in general for two vertices $x, y \in \mathcal{V}(\mathcal{TT})$ there is no directed edge path connecting x to y, for any pair of points $x, y \in \mathcal{TT}$ there is a pair of vertices $x', y' \in \mathcal{V}(\mathcal{TT})$ within a uniformly bounded distance of x, y so that x' can be connected to y' by a directed edge path.

The second and the third step of the proof are much more involved, and they are carried out in Section 5 and Section 6. First we have a closer look at the geometry of the graph $\mathcal{TT}$. Using the fact that directed edge paths are uniform quasi-geodesics, we give in Section 5 a fairly explicit description of the geometry of $\mathcal{TT}$. Then we single out for every pair of vertices $(x, y) \in \mathcal{V}(\mathcal{TT}) \times \mathcal{V}(\mathcal{TT})$ with the property that x can be connected to y by a directed edge path a specific such path. In Section 6 we show that these paths equipped with suitably chosen parametrizations define a $\mathcal{MCG}(S)$-equivariant bounded bicombing of $\mathcal{TT}$.

In Section 7 we observe that this bicombing of $\mathcal{TT}$ is a regular path system in the sense of [S06]. Theorem 1 then follows from the results of [S06].

Theorems 2-4 are derived in Section 4 from coarse density of pairs of vertices in $\mathcal{TT}$ which can be connected by a directed edge path. Namely, let Γ be any finitely generated group equipped with some word metric and let $\varphi : \Gamma \to \mathcal{MCG}(S)$ be any homomorphism with the property that $d(\varphi g, \varphi h) \le \kappa d(g, h)$ for all $g, h \in \Gamma$ and some $\kappa > 1$. Assume moreover that for any $R > 0$ there is some $r > 0$ such that $d(\varphi(g), \varphi(h)) \ge R$ whenever $d(g, h) \ge r$. Then φ is a quasi-isometric embedding if for some choice τ of a basepoint in $\mathcal{TT}$ and for any two points $g, h \in \Gamma$ there is a directed edge path in $\mathcal{TT}$ connecting $\varphi(g)\tau$ to a point in a uniform neighborhood of $\varphi(h)\tau$ and which is entirely contained in a uniformly bounded neighborhood of $\varphi(\Gamma)\tau$. In the situations described in Theorems 2-4, a map φ which has these properties can fairly easily be constructed.

In Section 2 we summarize the properties of the train track complex $\mathcal{TT}$ of S which are needed for our purpose. The appendix contains a technical result about train tracks which is used in Section 5 but which can be established independently of the rest of the paper.

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2. The complex of train tracks

In this section we summarize some results from [H09] which will be used throughout the paper.

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and where $3g - 3 + m \geq 2$. A *train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. In particular, the half-branches which are incident on a fixed switch are divided into two classes according to the orientation of an inward pointing tangent at the switch. Each closed curve component of τ has a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with $0, 1$ or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. A train track is called *maximal* if each of its complementary components either is a trigon, i.e. a topological disc with three cusps at the boundary, or a once punctured monogon, i.e. a once punctured disc with one cusp at the boundary. We always identify train tracks which are isotopic. The book [PH92] contains a comprehensive treatment of train tracks which we refer to throughout the paper.

A train track is called *generic* if all switches are at most trivalent. The train track τ is called *transversely recurrent* if every branch b of τ is intersected by an embedded simple closed curve $c = c(b) \subset S$ of class C^1 which intersects τ transversely and is such that $S - \tau - c$ does not contain an embedded *bigon*, i.e. a disc with two corners at the boundary.

A *trainpath* on a train track τ is a C^1-immersion $\rho : [m, n] \to \tau \subset S$ which maps each interval $[k, k + 1]$ $(m \leq k \leq n - 1)$ onto a branch of τ. The integer $n - m$ is called the *length* of ρ. We sometimes identify a trainpath with its image in τ. Each complementary region of τ is bounded by a finite number of (not necessarily embedded) trainpaths which either are closed curves or terminate at the cusps of the region. A *subtrack* of a train track τ is a subset σ of τ which itself is a train track. Thus every switch of σ is also a switch of τ, and every branch of σ is a trainpath on τ. We write $\sigma < \tau$ if σ is a subtrack of τ.

A *transverse measure* on a train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: for every switch s of τ, the half-branches incident on s are divided into two classes, and the sums of the weights over all half-branches in each of the two classes coincide. The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. We call such a transverse measure μ *positive*, and we write $\mu > 0$. If μ is any transverse measure on a train track τ then the subset of τ consisting of all branches with positive μ-weight is a recurrent subtrack of τ. A train track τ is called *birecurrent* if τ is recurrent and transversely recurrent. We call τ *complete* if τ is generic, maximal and birecurrent.

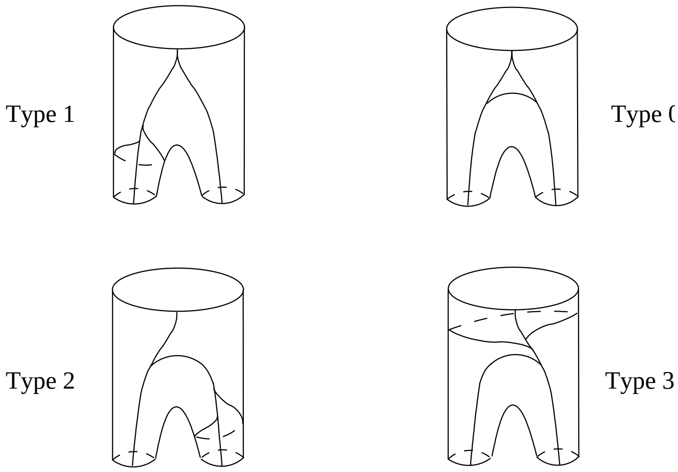
Remark: As in [H09], we require every train track to be generic. Unfortunately this leads to a slight inconsistency of our terminology with the terminology found in the literature.

There is a special collection of complete train tracks on S which were introduced by Penner and Harer [PH92]. Namely, a *pants decomposition* P for S is a collection of $3g-3+m$ simple closed curves which decompose S into $2g-2+m$ pairs of pants. Here a pair of pants is a planar orientable bordered surface of Euler characteristic -1 which may be non-compact. Define a *marking* of S (or complete clean marking in the terminology of [MM00]) to consist of a pants decomposition P for S and a system of *spanning curves* for P. For each pants curve $\gamma \in P$ there is a unique simple closed spanning curve which is contained in the connected component S_0 of $S-(P-\gamma)$ containing γ, which is not freely homotopic into the boundary or a puncture of this component and which intersects γ in the minimal number of points (one point if S_0 is a one-holed torus and two points if S_0 if a four-holed sphere). Note that any two choices of such a spanning curve differ by a *Dehn twist* about γ.

For each marking F of S we can construct a collection of finitely many maximal transversely recurrent train tracks as follows. Let P be the pants decomposition of the marking. Choose an open neighborhood A of P in S whose closure in S is homeomorphic to the disjoint union of $3g-3+m$ closed annuli. Then $S-A$ is the disjoint union of $2g-2+m$ pairs of pants. We require that each train track τ from our collection intersects a component of $S-A$ which does not contain a puncture of S in a train track with stops which is isotopic to one of the four *standard models* shown in Figure A (see Figure 2.6.2 of [PH92]).

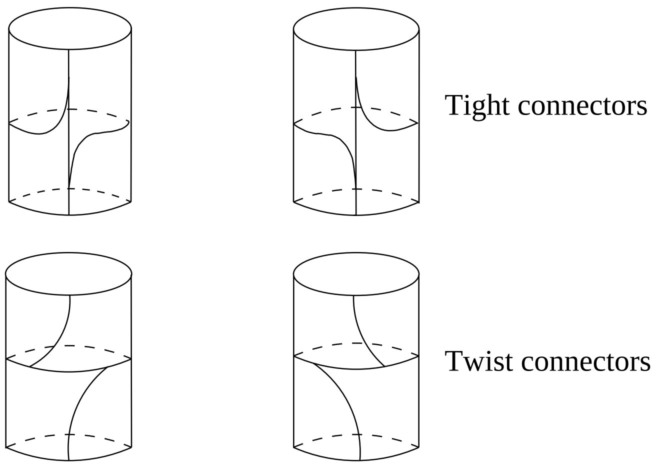
If S_0 is a component of $S-A$ which contains precisely one puncture, then we require that τ intersects S_0 in a train track with stops which we obtain up to diffeomorphism of S_0 from the standard model of type 2 or of type 3 by replacing the top boundary curve by a puncture and by deleting the branch which is incident on the stop of this boundary component. If S_0 is a component of $S-A$ which contains two punctures, then we require that τ intersects S_0 in a train track with stops which we obtain up to diffeomorphism of S_0 from the standard model of type 1 by replacing the two lower boundary components by a puncture and by deleting the two branches which are incident on the stops of these boundary components.

Figure A



The intersection of τ with a component of the collection A of $3g-3+m$ annuli is one of the following four *standard connectors* which are shown in Figure B (see Figure 2.6.1 of [PH92]).

Figure B



From the above standard pieces we can build a train track τ on S by choosing for each component of $S-A$ one of the standard models as described above and choosing for each component of A one of the four standard connectors. These train tracks with stops are then glued at their stops to a connected train track on S. Any two train tracks constructed in this way from the same pants decomposition P, the same choices of standard models for the components of $S-A$ and the same choices of connectors for the components of A differ by Dehn twists about the pants curves of P. The spanning curves of the marking F determine a specific choice of such a glueing [PH92]. We call each of the resulting train tracks *in standard form for F*

provided that it is complete (see p.147 of [PH92] for examples of train tracks built in this way which are not recurrent and hence not complete).

A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ is *minimal* if each of its half-leaves is dense in λ. A geodesic lamination is *maximal* if its complementary regions are all ideal triangles or once punctured monogons (note that a minimal geodesic lamination can also be maximal). The space of geodesic laminations on S equipped with the *Hausdorff topology* is a compact metrizable space.

A geodesic lamination λ is called *complete* if λ is maximal and can be approximated in the Hausdorff topology by simple closed geodesics. The space $\mathcal{CL}$ of all complete geodesic laminations equipped with the Hausdorff topology is compact. The mapping class group $\mathcal{MCG}(S)$ naturally acts on $\mathcal{CL}$ as a group of homeomorphisms. Every geodesic lamination λ which is a disjoint union of finitely many minimal components is a *sublamination* of a complete geodesic lamination, i.e. there is a complete geodesic lamination which contains λ as a closed subset (Lemma 2.2 of [H09]).

A train track or a geodesic lamination σ is *carried* by a transversely recurrent train track τ if there is a map $\varphi : S \to S$ of class C^1 which is homotopic to the identity and maps σ into τ in such a way that the restriction of the differential of φ to the tangent space of σ vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of φ to σ a *carrying map* for σ. Write $\sigma \prec \tau$ if the train track σ is carried by the train track τ. Then every geodesic lamination λ which is carried by σ is also carried by τ. A train track τ is complete if and only if it is generic and transversely recurrent and if it carries a complete geodesic lamination. The space of complete geodesic laminations carried by a complete train track τ is open and closed in $\mathcal{CL}$ (Lemma 2.3 of [H09]). In particular, the space $\mathcal{CL}$ is totally disconnected.

For every pants decomposition P of S there is a finite set of complete geodesic laminations on S which contain (the geodesic representatives of) the components of P as their minimal components. We call such a geodesic lamination *in standard form* for P. If λ is a geodesic lamination in standard form for P then for each component S_0 of $S - P$ which does not contain a puncture of S there are precisely three leaves of λ contained in S_0 which spiral about the three different boundary components of S_0. The leaves of λ spiraling from two different sides about a component γ of P define opposite orientations near γ (as shown in Figure A of [H09]). If S_0 contains exactly one puncture of S there there are two leaves of λ contained in S_0 which spiral about the two boundary components of S_0.

For every marking F of S with pants decomposition P and every train track τ in standard form for F with only twist connectors there is a unique complete geodesic lamination in standard form for P which is carried by τ. This implies that for every marking F of S with pants decomposition P there is a bijection between the complete train tracks in standard form for F with only twist connectors and the complete geodesic laminations in standard form for P. The set of all complete geodesic laminations in standard form for some pants decomposition P is invariant under the action of the mapping class group, moreover there are only finitely many $\mathcal{MCG}(S)$-orbits of such complete geodesic laminations.

Define the *straightening* of a train track τ on S with respect to some complete finite volume hyperbolic structure g on S to be the edgewise immersed graph in S whose vertices are the switches of τ and whose edges are the unique geodesic arcs which are homotopic with fixed endpoints to the branches of τ. For a number $\epsilon > 0$ we say that the train track τ *ϵ-follows* a geodesic lamination λ if the tangent lines of the straightening of τ are contained in the ϵ-neighborhood of the projectivized tangent bundle $PT\lambda$ of λ (with respect to the distance function induced by the metric g) and if moreover the straightening of every trainpath on τ is a piecewise geodesic whose exterior angles at the breakpoints are not bigger than ϵ.

Lemma 2.1. *Let $\lambda \in \mathcal{CL}$ be a complete geodesic lamination on S in standard form for a pants decomposition P of S and let $\epsilon > 0$. Then there is a complete train track τ on S in standard form for a marking F of S with pants decomposition P which carries λ and ϵ-follows λ.*

Proof. The train track τ can be obtained from λ by collapsing a sufficiently small tubular neighborhood of λ. We refer to Theorem 1.6.5 of [PH92] and to Lemma 3.2 of [H09] and its proof for more details of this construction. □

Note that in Lemma 2.1, the marking F of S depends on the number ϵ as well as on choices made in the construction.

A *measured geodesic lamination* is a geodesic lamination equipped with a transverse translation invariant measure of full support. The space $\mathcal{ML}$ of measured geodesic laminations on S equipped with the weak*-topology is homeomorphic to the product of a sphere of dimension $6g - 7 + 2m$ with the real line. A measured geodesic lamination μ is carried by a train track τ if its support is carried by τ. Then μ defines a transverse measure on τ, and every transverse measure on τ arises in this way [PH92].

We use measured geodesic laminations to establish another relation between train tracks in standard form for a marking of S and complete geodesic laminations which is a variant of a result of Penner and Harer (Theorem 2.8.4 of [PH92]).

Lemma 2.2. *For any marking F of S, every complete geodesic lamination on S is carried by a unique train track in standard form for F.*

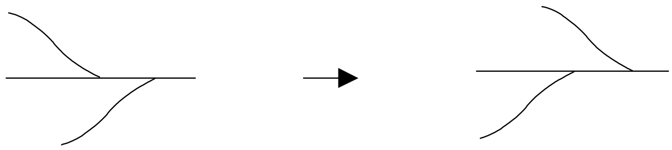
Proof. A complete geodesic lamination λ can be approximated in the Hausdorff topology by a sequence $\{c_i\}$ of simple closed geodesics. For a fixed marking F of S, each such geodesic is carried by a train track in standard form for F (this is contained in Theorem 2.8.4 of [PH92]). Since there are only finitely many train tracks in standard form for F, there is a fixed train track τ in standard form for F which carries infinitely many of the curves c_i. By Lemma 3.2 of [H09], the set of geodesic laminations carried by a fixed train track τ is closed in the Hausdorff topology and hence the geodesic lamination λ is carried by τ.

Now assume that there is a second train track η in standard form for F which carries λ. By Lemma 3.2 and Lemma 3.3 of [H09], there is a complete train track σ which is carried by both τ and η. The train track σ carries a measured geodesic lamination μ whose support is both minimal and maximal (see the top of p.556 of [H09] for a detailed discussion of this fact). Thus μ is carried by two distinct train tracks in standard form for F which violates Theorem 2.8.4 of [PH92]. ☐

A half-branch $\hat{b}$ in a generic train track τ incident on a switch v of τ is called *large* if every trainpath containing v in its interior passes through $\hat{b}$. A half-branch which is not large is called *small*. A branch b in a generic train track τ is called *large* if each of its two half-branches is large; in this case b is necessarily incident on two distinct switches. A branch is called *small* if each of its two half-branches is small. A branch is called *mixed* if one of its half-branches is large and the other half-branch is small (see p.118 of [PH92]).

There are two simple ways to modify a complete train track τ to another complete train track. First, we can *shift* τ along a mixed branch b to a train track τ' as shown in Figure C. If τ is complete then the same is true for τ'. Moreover, a train track

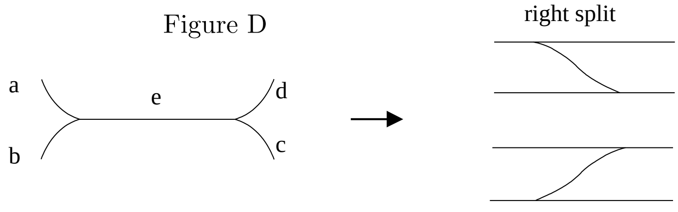
Figure C



or a geodesic lamination is carried by τ if and only if it is carried by τ' (see [PH92] p.119). In particular, the shift τ' of τ is carried by τ. There is a natural bijection $\varphi(\tau, \tau')$ of the set of branches of τ onto the set of branches of τ' which is induced by the identity of the complement of a small neighborhood of b in S. The bijection $\varphi(\tau, \tau')$ also induces a bijection of the set of half-branches of τ onto the set of half-branches of τ' which we denote again by $\varphi(\tau, \tau')$.

Second, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure D. Note that a right split at e is uniquely determined by

Figure D



the orientation of S and does not depend on the orientation of e. Using the labels in the figure, in the case of a right split we call the branches a and c *winners* of the split, and the branches b, d are *losers* of the split. If we perform a left split, then the branches b, d are winners of the split, and the branches a, c are losers of the split. The split τ' of a train track τ is carried by τ, and there is a natural choice of a carrying map which maps the switches of τ' to the switches of τ. The image of a branch of τ' is then a trainpath on τ whose length either equals one or two. There is a natural bijection $\varphi(\tau, \tau')$ of the set of branches of τ onto the set of branches of τ' which maps the branch e to a small branch e' which we call the *diagonal* of the split. This bijection is induced by the identity on the complement of a small neighborhood of e in S. The map $\varphi(\tau, \tau')$ also induces a bijection of the set of half-branches of τ onto the set of half-branches of τ' again denoted by $\varphi(\tau, \tau')$.

Occasionally we also have to consider the *collision* of a train track η at a large branch e. This collision is obtained from η by a split at e and removal of the diagonal in the split track. Such a collision is shown in Figure 2.1.2 of [PH92].

A split of a maximal transversely recurrent generic train track is maximal, transversely recurrent and generic. If τ is a complete train track and if $\lambda \in \mathcal{CL}$ is carried by τ, then for every large branch e of τ there is a unique choice of a right or left split of τ at a large branch e of τ with the property that the split track τ' carries λ (see p. 557 of [H09] for a more complete discussion). We call such a split a λ-*split*. The train track τ' is complete. In particular, a complete train track τ can always be split at any large branch e to a complete train track τ'; however there may be a choice of a right or left split at e such that the resulting train track is not recurrent any more (compare p.120 in [PH92]).

For a number $L \geq 1$, an L-*quasi-isometric embedding* of a metric space (X, d) into a metric space (Y, d) is a map $\varphi : X \to Y$ such that

$$d(x, y)/L - L \leq d(\varphi(x), \varphi(y)) \leq Ld(x, y) + L$$

for all $x, y \in X$. The map φ is called an L-*quasi-isometry* if moreover the L-neighborhood of φX in Y is all of Y. An L-*quasi-geodesic* in a metric space (X, d) is an L-quasi-isometric embedding of a closed connected subset of $\mathbb{R}$ or of the intersection of such a closed connected subset of $\mathbb{R}$ with $\mathbb{Z}$.

Denote by $\mathcal{TT}$ the directed metric graph whose set $\mathcal{V}(\mathcal{TT})$ of vertices is the set of isotopy classes of complete train tracks on S and whose edges are determined as follows. The train track $\tau \in \mathcal{V}(\mathcal{TT})$ is connected to the train track τ' by a directed edge of length one if and only if τ' can be obtained from τ by a single split. The graph $\mathcal{TT}$ is connected (Corollary 2.7 of [H09]). The mapping class group $\mathcal{MCG}(S)$ of S acts properly and cocompactly on $\mathcal{TT}$ as a group of simplicial isometries. In particular, $\mathcal{TT}$ is $\mathcal{MCG}(S)$-equivariantly quasi-isometric to $\mathcal{MCG}(S)$ equipped with any word metric (Corollary 4.4 of [H09]).

Define a *splitting sequence* in $\mathcal{TT}$ to be a sequence $\{\alpha(i)\}_{0 \leq i \leq m} \subset \mathcal{V}(\mathcal{TT})$ with the property that for every $i \geq 0$ the train track $\alpha(i+1)$ can be obtained from $\alpha(i)$ by a single split. Thus splitting sequences in $\mathcal{V}(\mathcal{TT})$ correspond precisely to directed edge-paths in $\mathcal{TT}$. If τ can be connected to η by a splitting sequence then we say that τ is *splittable* to η. If $\{\alpha(i)\}_{0 \leq i \leq m}$ is a splitting sequence then the composition

$$\varphi(\alpha(0), \alpha(m)) = \varphi(\alpha(m-1), \alpha(m)) \circ \cdots \circ \varphi(\alpha(0), \alpha(1))$$

is a bijection of the branches (or half-branches) of $\alpha(0)$ onto the branches (or half-branches) of $\alpha(m)$ which does not depend on the choice of the splitting sequence connecting $\alpha(0)$ to $\alpha(m)$ (Lemma 5.1 of [H09]).

For a complete train track τ and a complete geodesic lamination λ carried by τ define the *cubical Euclidean cone* $E(\tau, \lambda)$ to be the full subgraph of $\mathcal{TT}$ whose vertices consist of the complete train tracks which can be obtained from τ by a splitting sequence and which carry λ. Then $E(\tau, \lambda)$ is a connected subgraph of $\mathcal{TT}$ and hence it can be equipped with an intrinsic path-metric d_E. We showed (Theorem 2 of [H09])

Theorem 2.3. *There is a number $L_0 > 1$ such that for every $\tau \in \mathcal{V}(\mathcal{TT})$ and every complete geodesic lamination λ carried by τ the inclusion $(E(\tau, \lambda), d_E) \to \mathcal{TT}$ is an L_0-quasi-isometric embedding.*

By Corollary 5.2 of [H09], directed edge-paths in $(E(\tau, \lambda), d_E)$ are geodesics and hence we obtain as an immediate consequence that directed edge-paths in $\mathcal{TT}$ are L_0-quasi-geodesics.

3. Density of splitting sequences

The goal of this section is to show the following proposition which is the first step in the proof of Theorem 1 from the introduction and which is the main technical tool for the proof of Theorem 2-4.

Proposition 3.1. *There is a number $d_0 > 0$ with the following property. For any train tracks $\tau, \sigma \in \mathcal{V}(\mathcal{TT})$ there is a train track τ' which is contained in the d_0-neighborhood of τ and which is splittable to a train track σ' contained in the d_0-neighborhood of σ.*

To simplify the argument we reduce Proposition 3.1 to the following

Proposition 3.2. *There is a number $d_1 > 0$ with the following property. Let F be any marking of S. Then for any train track $\sigma \in \mathcal{V}(\mathcal{TT})$ there is a train track $\tau \in \mathcal{V}(\mathcal{TT})$ in standard form for F which carries a train track σ' contained in the d_1-neighborhood of σ. If σ is in standard form for a marking G with pants decomposition Q then σ' can be chosen to contain the pants decomposition Q as an embedded subtrack.*

We begin with explaining how Proposition 3.1 follows from Proposition 3.2. The mapping class group acts properly and cocompactly on $\mathcal{TT}$ preserving the set of train tracks in standard form for some marking of S. Thus every complete train track is contained in a uniformly bounded neighborhood of a train track in standard form for some marking F of S. The mapping class group acts on the set of all markings of S, with finitely many orbits. Therefore the diameter in $\mathcal{TT}$ of any set of train tracks in standard form for a fixed marking is uniformly bounded. This implies that there is a number $d_2 > 0$ and for every complete train track τ on S there is a marking F of S such that $d(\tau, \eta) \leq d_2$ for every train track η in standard form for F.

By Lemma 6.6 of [H09], there is a number $p > 0$ and for two complete train tracks $\sigma \prec \tau$ there is a train track ζ which can be obtained from τ by a splitting sequence and such that $d(\sigma, \zeta) \leq p$. As a consequence, Proposition 3.1 follows from Proposition 3.2.

The idea of proof for Proposition 3.2 is as follows. Define a *splitting and shifting sequence* to be a sequence $\{\alpha(i)\}_{0 \leq i \leq m}$ with the property that for every $i \geq 0$ the train track $\alpha(i+1)$ can be obtained from $\alpha(i)$ by a sequence of shifts followed by a single split. Theorem 2.4.1 of [PH92] relates splitting and shifting to carrying.

Proposition 3.3. *If $\sigma \in \mathcal{V}(\mathcal{TT})$ is carried by $\tau \in \mathcal{V}(\mathcal{TT})$ then τ can be connected to σ by a splitting and shifting sequence.*

Now let F, G be any two markings of S. We attempt to construct a splitting and shifting sequence connecting some train track in standard form for F to some train track in standard form for G.

A train track in standard form for G with only twist connectors carries a complete geodesic lamination λ in standard form for the pants decomposition Q of the marking G of S. By Lemma 2.2, every complete geodesic lamination λ on S is carried by a unique train track τ in standard form for F. We modify τ with a sequence of splits as efficiently as possible to a train track ξ which carries λ and contains the pants decomposition Q as a subtrack. This train track ξ carries a train track η in standard form for some marking of S with pants decomposition Q whose distance to ξ is uniformly bounded. There is a *multi-twist* φ (i.e. a concatenation of mutually commuting Dehn twists) about the pants curves of Q which maps η to a train track $\varphi\eta$ in standard form for G. In general, $\varphi\eta$ is not carried by η, but we obtain enough control that we can find a perhaps different train track in standard form for F which can be connected with a splitting and shifting sequence to a train track in a uniformly bounded neighborhood of $\varphi\eta$ which is in standard form for a marking with pants decomposition Q.

To carry out this strategy we use the pants decomposition Q for the construction of splitting sequences. However, Q may not *fill* τ, i.e. a carrying map $Q \to \tau$ may not be surjective. Therefore we are lead to investigate splitting sequences of complete train tracks which are determined by modifications of subtracks. This will occupy the major part of this section and will also be of crucial importance in Section 5 and Section 6.

Fix a complete Riemannian metric on S of finite volume. With respect to this metric, a complementary region C of a train track σ on S is a hyperbolic surface whose metric completion $\overline{C}$ is a bordered surface with boundary ∂C. This boundary consists of a finite number of arcs of class C^1, called *sides* of C or of $\overline{C}$. Each side of C either is a closed curve of class C^1 (i.e. the boundary component containing the side does not contain any cusp) or an arc with endpoints at two not necessarily distinct cusps of the component. We call a side of C which does not contain cusps a *smooth side* of C. The closure of C in S can be obtained from $\overline{C}$ by some identifications of subarcs of sides (the inclusion $C \to S$ extends to an immersion of each side of C, but the image arc may have tangential self-intersections or may meet another side tangentially). For simplicity we call the image in σ of a side of

C a side of C as well (i.e. most of the time we view a side of C as an immersed arc of class C^1 in σ). Using this abuse of notation, a side of C is just an immersed arc or an immersed closed curve of class C^1 in σ with only tangential self-intersections. However, we reserve the notation $\overline{C}$ for the metric completion of C.

If $T \subset \partial C$ is a smooth side of a complementary region C of σ then we mark a point on T. We view this point as a point on the boundary of the completion $\overline{C}$ of C, i.e. even in the case that the point corresponds to a point of tangential self-intersection of the image of ∂C in σ, passing once through T means crossing the point precisely once, and not passing through T means not crossing through the point.

If C is a complementary region of σ whose boundary contains precisely $k \geq 0$ cusps, then the *Euler characteristic* $\chi(C)$ is defined by $\chi(C) = \chi_0(C) - k/2$ where $\chi_0(C)$ is the usual Euler characteristic of the compact topological surface with boundary $\overline{C}$. Note that the sum of the Euler characteristics of the complementary regions of σ is just the Euler characteristic of S (see the discussion in Chapter 1.1 of [PH92]).

A *complete extension* of a train track σ is a complete train track τ containing σ as a subtrack and whose switches are distinct from the images in σ of the marked points on smooth boundary components of complementary regions of σ. Such a complete extension τ intersects each complementary region C of σ in an embedded graph with smooth edges. The closure of $\tau \cap C$ in the completion $\overline{C}$ of C is a graph whose univalent vertices are contained in the complement of the cusps and marked points of the boundary ∂C of $\overline{C}$. At a univalent vertex, the graph is tangential to ∂C. We call two such graphs $\tau \cap C, \tau' \cap C$ *equivalent* if there is a smooth isotopy of $\overline{C}$ which fixes the cusps and the marked points in ∂C and which maps $\tau \cap C$ onto $\tau' \cap C$. The complete extensions τ, τ' of σ are called *σ-equivalent* if for each complementary region C of σ the graphs $\tau \cap C$ and $\tau' \cap C$ are equivalent in this sense. The purpose of marking a point on a smooth boundary component T of a complementary region of σ is to control the amount of relative twisting about T of two complete extensions τ, τ' of σ.

For two complete extensions τ, τ' of σ define the *intersection number* $i_\sigma(\tau, \tau')$ to be the minimal number of intersection points contained in $S - \sigma$ between any two complete extensions η, η' of σ which are σ-equivalent to τ, τ' and with the following additional properties.

a) A switch v of η (or η') is also a switch of η' (or η) if and only if v is a switch of σ.
b) A switch of η (or η') contained in the interior of a complementary region C of σ is not contained in η' (or η), i.e. an intersection point of η with η' contained in C is an interior point of a branch of η and of a branch of η'.

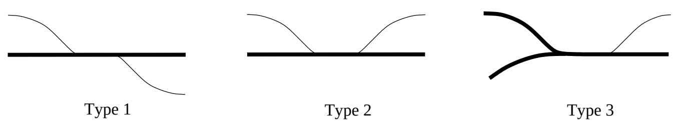
Since the number of switches of a complete train track on S only depends on the topological type of S, for any complete extension τ of σ the intersection number $i_\sigma(\tau, \tau)$ is bounded from above by a universal constant neither depending on σ nor τ. Moreover, for every number $m > 0$ there is a number $q(m) > 0$ not depending on

σ so that for every complete extension τ of σ the number of σ-equivalence classes of complete extensions τ' of σ with $i_\sigma(\tau,\tau') \leq m$ is bounded from above by $q(m)$.

To simplify the notation we do not distinguish between σ as a subgraph of τ (and hence containing switches of τ which are bivalent in σ) and σ viewed as a subtrack of τ, i.e. the graph from which the bivalent switches not contained in simple closed curve components have been removed. A branch e of σ defines an embedded trainpath $\rho : [0,m] \to \tau$, unique up to orientation, whose image is precisely e. We call τ *tight* at e if e is a branch in τ, i.e. if the length m of ρ equals one. If e is a large branch of σ then ρ begins and ends with a large half-branch and hence $\rho[0,m]$ contains a large branch of τ (see Lemma 2.7.2 of [PH92]).

A *proper subbranch* of a branch e of σ is a branch b of τ which is a proper subset of e. Then b is incident on at least one switch v of τ which is not a switch of σ. There is a half-branch c of τ which is incident on v and not contained in σ. We call c a *neighbor* of σ at v. We distinguish three different types of *large* proper subbranches b of a branch e of σ. These types are shown in Figure E. Note that a large branch of any train track on S is embedded in S.

Figure E



Type 1: b is contained in the interior of e and the two neighbors of σ at the two endpoints of b lie on different sides of e in a small tubular neighborhood of b in S.

Type 2: b is contained in the interior of e and both neighbors of σ at the two endpoints of b lie on the same side of e in a small tubular neighborhood of b in S.

Type 3: One endpoint of b is incident on a switch of σ.

A split of τ at a large proper subbranch b of σ (i.e. of a large branch of τ which is a proper subbranch of a branch of σ) is called a *σ-split* if the split track contains σ as a subtrack. Note that such a split always exists. If b is a large proper subbranch of σ of type 2 then any split of τ at b is a σ-split.

Let q be the number of branches of a complete train track on S. The number of switches of a complete train track on S then equals $2q/3 < q$. For a subtrack σ of a complete train track τ let $\beta(\tau,\sigma)$ be the number of neighbors of σ in τ, i.e. the number of half-branches of $\tau - \sigma$ which are incident on a switch contained in σ. If τ_1 is obtained from τ by a split at a large proper subbranch b of σ of type 2 then $\beta(\tau_1,\sigma) = \beta(\tau,\sigma) - 1$.

Now let σ be a recurrent train track on S. Then there is a measured geodesic lamination ν on S which is carried by σ and which defines a positive transverse measure on σ. We call such a measured geodesic lamination *filling* for σ. For every complete extension τ of σ there is a complete geodesic lamination λ which is carried by τ and contains the support of ν as a sublamination. Namely, the positive transverse measure on σ defined by ν can be approximated by positive transverse measures μ_i on τ which define a measured geodesic lamination whose support is a minimal and maximal geodesic lamination carried by τ (see p.556 of [H09] for a detailed proof of this fact). Since the space $\mathcal{CL}$ of all complete geodesic laminations on S is compact, as $\mu_i \to \nu$ in the space of transverse measures on τ, up to passing to a subsequence the supports of μ_i converge in the Hausdorff topology to a complete geodesic lamination λ which contains the support of ν as a sublamination. By Lemma 2.3 of [H09], λ is carried by τ. We call λ a *complete τ-extension* of ν.

The following observation will be used throughout the paper.

Lemma 3.4. *Let σ be a recurrent subtrack of a complete train track τ and let λ be a complete τ-extension of a σ-filling measured geodesic lamination. Then for every large branch e of σ there is a unique train track τ' with the following properties.*

1. *τ' can be obtained from τ by at most q^2 σ-splits at large proper subbranches of e. In particular, τ' contains σ as a subtrack.*
2. *τ' carries λ.*
3. *τ' is tight at e.*

Moreover, given any complete extension η of σ, if no marked point on a smooth side of a complementary region C of σ is mapped to the branch e then

$$i_\sigma(\tau', \eta) \le i_\sigma(\tau, \eta) + q(\beta(\tau, \sigma) - \beta(\tau', \sigma)).$$

Otherwise we have $i_\sigma(\tau', \eta) \le i_\sigma(\tau, \eta) + q^3$.

Proof. If τ is tight at the large branch e of σ then $\tau = \tau'$ satisfies the requirements in the lemma.

Otherwise let a be a neighbor of σ at a switch of τ contained in e. There is a unique maximal trainpath $\rho : [-1, m] \to \tau$ with $\rho[-1/2, 0] = a$ and such that $\rho[0, m] \subset e$. Then $\rho(m)$ is a switch of σ on which e is incident. Let $c(a, e) = m \le q$ be the length of the intersection of the trainpath ρ with e and let

$$c(\tau, e) = \sum_a c(a, e)$$

where the sum is taken over all neighbors of e in τ. Then $c(\tau, e) \le q^2$, and $c(\tau, e) = 0$ if and only if τ is tight at e.

Let b be a large proper subbranch of e. Let λ be a complete τ-extension of a σ-filling measured geodesic lamination ν and let τ_1 be the train track obtained from τ by a λ-split at b. We distinguish two cases according to the type of b.

If b is of type 1 or of type 3 then there is a unique choice of a right or left split of τ at b such that the split track τ_1' contains σ as a subtrack. Since ν is a σ-filling measured geodesic lamination, τ_1' is also the unique train track obtained from τ by a split at b which carries ν. Now λ is a complete extension of ν and therefore $\tau_1' = \tau_1$. The natural bijection $\varphi(\tau, \tau_1)$ of the half-branches of τ onto the half-branches of τ_1 maps any neighbor a of σ in τ to a neighbor $\varphi(\tau, \tau_1)(a)$ of σ

in τ_1. Moreover, we have $c(\varphi(\tau,\tau_1)(a),e) \leq c(a,e)$. If a is a neighbor of σ at an endpoint of b then $c(\varphi(\tau,\tau_1)(a),e) = c(a,e)-1$ (see Figure D), To summarize, we have $c(\tau_1,e) \leq c(\tau,e)-1$.

If b is of type 2 then once again, the train track τ_1 contains σ as a subtrack. Moreover, we have $\beta(\tau_1,\sigma) = \beta(\tau,\sigma)-1$ and $c(\tau_1,e) < c(\tau,e)$. As a consequence, a splitting sequence of length at most q^2 at large proper subbranches of e transforms τ to a train track τ' which contains σ as a subtrack, is tight at e and carries λ. By uniqueness of sequences of λ-splits up to order (Lemma 5.1 of [H09]), the train track τ' is uniquely determined by τ,σ,λ,e.

To estimate intersection numbers between τ,τ' and an arbitrary complete extension η of σ, let again b be a large proper subbranch of e of type 1 or type 3 and let τ_1 be the train track obtained from τ by a λ-split at b. If e does not contain the image of any marked point on a smooth boundary component of a complementary region of σ, then τ and τ_1 are σ-equivalent.

Otherwise there are one or two (not necessarily distinct) complementary regions C_1,C_2 of σ and smooth sides T_i of C_i whose images in σ contain e. Up to isotopy, a split of τ at b can be realized by moving one of the neighbors of σ incident on an endpoint of b, say the neighbor a, across b while leaving the second neighbor (or the branches of σ incident on an endpoint of e in case the branch b is of type 3) fixed. Assume that the half-branch a is contained in the complementary region C_1 of σ and that a terminates at a point in the smooth boundary component T_1 of C_1. There are at most q half-branches of η contained in $\eta \cap C_1$ which terminate at a point in T_1. Up to isotopy of $\overline{C_1} \cup \overline{C_2}$ preserving the cusps and the marked points, moving the half-branch a of τ across the marked point in T_1 increases the number of intersection points between τ and η by at most q. Namely, up to isotopy such a move creates at most one additional intersection point with any half-branch of η with endpoint T_1. As a consequence, we have

$$i_\sigma(\tau_1,\eta) \leq i_\sigma(\tau,\eta)+q. \tag{1}$$

If τ_1 is obtained from τ by a λ-split at a large proper subbranch of e of type 2 then the split which modifies τ to τ_1 can be realized by moving one of the neighbors of σ incident on an endpoint of b across b to a half-branch which is incident on a point in the interior of the neighbor of σ at the second endpoint of b. As before, this implies that the inequality (1) holds true for every complete extension η of σ (independent of whether or not e contains the image of a marked point).

To summarize, if there is no marked point on the boundary of a complementary region of σ which is mapped into e and if η is any complete extension of σ then the above discussion shows that only splits at large proper subbranches of e of type 2 change the intersection number between τ and η. A successive application of the estimate (1) yields that

$$i_\sigma(\tau',\eta) \leq i_\sigma(\tau,\eta)+q(\beta(\tau,\sigma)-\beta(\tau',\sigma)).$$

The second estimate of intersection numbers stated in the lemma follows in the same way from the inequality (1). □

For a recurrent subtrack σ of a complete train track τ, for a large branch e of σ and a complete τ-extension λ of a σ-filling measured geodesic lamination, we call the complete train track τ' constructed in Lemma 3.4 the (e,λ)-*modification* of τ.

Remark: 1) Lemma 3.4 and its proof remain valid if the large branch e of a recurrent subtrack σ of τ is replaced by any embedded trainpath $\rho : [0,m] \to \tau$ which begins and ends with a large half-branch. A large branch e of a non-recurrent subtrack of τ is an example. In this case the complete geodesic lamination λ has to be replaced by a measured geodesic lamination whose support is minimal and complete and is carried by τ and which defines a transverse measure on τ giving positive weight to the set of arcs which are mapped homeomorphically *onto* $\rho[0,m]$ by a carrying map. We call such a complete geodesic lamination ρ-*filling*, and we call the train track obtained from τ, ρ, λ with the procedure from the proof of Lemma 3.4 the (ρ,λ)-*modification* of τ. Note that a ρ-filling complete geodesic lamination may not exist always.

2) Let e_1, e_2 be distinct large branches of a train track σ on S, let τ be a complete extension of σ and let λ be a complete τ-extension of a σ-filling measured geodesic lamination Denote by τ_1, τ_2 the complete train tracks constructed from σ and λ as in Lemma 3.4 which are tight at the large branch e_1, e_2. Then up to isotopy, for every neighborhood U_1, U_2 of e_1, e_2 in S the intersection $\tau_i \cap (S - U_i)$ coincides with the intersection $\tau \cap (S - U_i)$ $(i = 1, 2)$. As a consequence, the train track $\tau_{1,2}$ obtained from τ_1, σ, λ by the construction in Lemma 3.4 which is tight at the large branch e_2 coincides with the train track $\tau_{2,1}$ obtained from τ_2, σ, λ by the construction in Lemma 3.4 which is tight at e_1.

If σ is any train track on S and if τ, η are two complete extension of σ, then we defined an intersection number $i_\sigma(\tau, \eta)$ which depends on the choice of marked points, one on each smooth boundary component of a complementary region of σ. A different choice of a marked point only changes the intersection number up to a uniformly bounded amount (compare the proof of Lemma 3.4 for a more detailed explanation and recall that the choice of the marked point is needed to control twisting of η relative to τ along the smooth boundary components of σ). The last statement of the following proposition then means that there are choices of marked points on σ, σ_ℓ so that the stated inequality holds true for these choices.

For a precise formulation, for a train track τ which is splittable to a train track η (i.e. such that τ can be connected to η by a splitting sequence) denote by $E(\tau, \eta)$ the graph whose vertex set consists of all train tracks which can be obtained from τ by a splitting sequence and which are splittable to η and where such a vertex ξ is connected to a vertex ζ by a directed edge of length one if ζ can be obtained from ξ by a single split.

Call a splitting sequence $\{\sigma_i\}$ of train tracks on S *recurrent* if each of the train tracks σ_i is recurrent.

Proposition 3.5. *Given a recurrent splitting sequence $\{\sigma_i\}_{0\leq i\leq \ell}$ of train tracks on S, there is an algorithm which associates to a complete extension τ of σ_0 and a complete τ-extension λ of a σ_ℓ-filling measured geodesic lamination ν a sequence $\{\tau_i\}_{0\leq i\leq 2\ell} \subset \mathcal{V}(\mathcal{TT})$ with the following properties.*

(1) *$\tau_0=\tau$, and for each $i\leq \ell$ the train tracks τ_{2i},τ_{2i+1} contain σ_i as a subtrack and carry λ.*
(2) *If σ_{i+1} is obtained from σ_i by a right (or left) split at a large branch e_i then τ_{2i+1} is the (e_i,λ)-modification of τ_{2i}, and τ_{2i+2} is obtained from τ_{2i+1} by a right (or left) split at e_i.*
(3) *The train track $\tau_{2\ell}$ only depends on $\tau,\sigma,\sigma_\ell,\lambda$ but not on the choice of a splitting sequence connecting σ to σ_ℓ.*
(4) *Every complete train track $\tau'\in E(\tau,\tau_{2\ell})$ contains a subtrack $\sigma'\in E(\sigma_0,\sigma_\ell)$.*
(5) *If $\{\eta_i\}_{0\leq i\leq 2\ell}$ is another such sequence beginning with a complete extension $\eta=\eta_0$ of σ then*
$$i_{\sigma_\ell}(\tau_{2\ell},\eta_{2\ell})\leq i_\sigma(\tau,\eta)+4q^5.$$

Proof. Let σ' be a train track which can be obtained from a train track σ by a single split at a large branch e. Let U be any neighborhood of e in S. Then up to modifying σ' with an isotopy we may assume that $\sigma'\cap(S-U)=\sigma\cap(S-U)$ and that there is a map $F:S\to S$ of class C^1 which equals the identity on $S-U$ and which restricts to a carrying map $\sigma'\to\sigma$. In particular, there is a natural bijection ψ between the complementary regions of σ and the complementary regions of σ' which preserves the topological type of the regions and which maps a complementary region C of σ to the complementary region $\psi(C)$ of σ' containing $C-U$ (here we assume that U is sufficiently small that $C-U\neq\emptyset$ for every complementary region C of σ).

If T is a smooth side of C then there is a smooth side T' of $\psi(C)$ whose image in σ' is mapped by the carrying map F onto the image of T in σ. Let $\rho:[0,n]\to\sigma$ be a trainpath which parametrizes the image of T in σ. Then ρ passes through any branch of σ at most twice, in opposite direction. In particular, the length n of ρ is at most $2q$ where as before, q is the number of branches of a complete train track on S (which is the maximal number of branches of any train track on S). If $\rho[0,n]$ contains the branch e, then the image in σ' of the side T' of $\psi(C)$ does not pass through the diagonal branch of the split. As a consequence, the length of a trainpath ρ' on σ' parametrizing the image of T' is strictly smaller than the length n of the trainpath on ρ (see Figure D).

The number of distinct smooth boundary components of complementary regions of σ is bounded from above by $3g-3+m<q/2$. If $\{\sigma_i\}_{0\leq i\leq\ell}$ is any splitting sequence, then the discussion in the previous paragraph shows that there are at most q^2 numbers $i\in\{1,\dots,\ell\}$ such that σ_{i+1} is obtained from σ_i by a split at a large branch which is contained in the image of a smooth boundary component of a complementary region of σ_i.

Now let τ,η be complete extensions of a recurrent train track $\sigma=\sigma_0$. As in the beginning of this section, mark a point on each smooth boundary component of a complementary region of σ in such a way that no marked point of σ is a switch of either τ or η (this can always be achieved with a small isotopy of τ,η preserving σ as a set). Let $\{\sigma_i\}_{0\leq i\leq\ell}$ be a recurrent splitting sequence issuing from $\sigma=\sigma_0$ and let λ,μ be complete τ,η-extensions of a σ_ℓ-filling measured geodesic lamination ν. We construct sequences $\{\tau_i\}_{0\leq i\leq 2\ell},\{\eta_i\}_{0\leq i\leq 2\ell}\subset\mathcal{V}(\mathcal{TT})$ with the properties stated in the proposition inductively as follows.

Let $\tau_0=\tau,\eta_0=\eta$ and assume that the train tracks τ_{2i},η_{2i} have already been constructed for some $i\geq 0$. Assume that σ_{i+1} is obtained from σ_i by a right (or left) split at the large branch e_i. Define τ_{2i+1},η_{2i+1} to be the (e_i,λ)-modification (or the (e_i,μ)-modification, respectively) of τ_{2i},η_{2i}. By construction, these train tracks carry the geodesic laminations λ,μ, and they are tight at e_i.

Since ν is σ_ℓ-filling, the right (or left) split of σ_i at e_i is the unique split so that the split track carries ν. Namely, otherwise ν is carried by the train track obtained from σ_i by splitting at e_i and removing the diagonal of the split. But this then means that a carrying map $\nu\to\sigma_{i+1}$ is not surjective which violates the assumption that ν fills $\sigma_\ell\prec\sigma_{i+1}$. Define τ_{2i+2},η_{2i+2} to be the train track obtained from τ_{2i+1},η_{2i+1} by a right (or left) split at the large branch e_i. Then τ_{2i+2},η_{2i+2} contains σ_{i+1} as a subtrack, and by the above reasoning, it is the unique train track obtained from τ_{2i+1},η_{2i+1} by a split at e_i which carries ν. On the other hand, there is a unique choice of a split of τ_{2i+1},η_{2i+1} at e_i so that the split track carries λ,μ and hence ν. But ν is a sublamination of λ,μ and therefore the train tracks τ_{2i+2},η_{2i+2} carry λ,μ. In particular, these train tracks are complete.

As a consequence, the inductively defined sequences $\{\tau_i\}_{0\leq i\leq 2\ell},\{\eta_i\}_{0\leq i\leq 2\ell}$ have properties 1)-2) stated in the proposition. The third property follows from the fact that a splitting sequence connecting σ to σ_ℓ is unique up to order (Lemma 5.1 of [H09] is also valid for splitting sequences of train tracks which are not complete since the assumption of completeness is nowhere used in the proof) and from the second remark after Lemma 3.4. Namely, by this remark, the train track obtained from a complete extension τ of σ and a complete τ-extension of a σ-filling measured geodesic lamination λ by two consecutive applications of Lemma 3.4 at distinct large branches e_1,e_2 of σ only depends on $\tau,\sigma,\lambda,e_1,e_2$ but not on the order in which these two applications of Lemma 3.4 are carried out.

Property 4) follows in the same way by induction on the length of a splitting sequence connecting τ to $\tau_{2\ell}$. If this length vanishes then there is nothing to show, so assume that the claim holds true whenever the length of such a sequence does not exceed $n-1$ for some $n\geq 1$. Under the hypotheses used throughout this proof, assume that the length of a splitting sequence connecting τ to $\tau_{2\ell}$ equals n.

Let $\tau'\in E(\tau,\tau_{2\ell})$. If $\tau'=\tau$ then τ' contains σ as a subtrack and there is nothing to show. Otherwise there is a train track $\tilde\tau\in E(\tau,\tau')\subset E(\tau,\tau_{2\ell})$ which can be obtained from τ by a single split at a large branch b. By uniqueness of splitting sequences (Lemma 5.1 of [H09]), we have $b\subset\sigma$.

If b is a large branch of σ (i.e. if τ is tight at b) then it follows once again by uniqueness of splitting sequences that $\tilde\tau$ contains a subtrack $\tilde\sigma\in E(\sigma,\sigma_\ell)$ which can be obtained from σ by a single split at b. Property 4) now follow s from property 3) and the induction hypothesis, applied to $\tilde\tau,\tau_{2\ell},\tilde\sigma,\sigma_\ell,\tau'$. Otherwise b is a large proper subbranch of σ. If b is of type 2 then any split of τ at b contains σ as a subtrack. If b is of type 1 or type 3 then there is a unique split of τ at b so that the split track contains σ as a subtrack, and by the previous discussion, the split track coincides with $\tilde\tau$. Once again, we can apply the induction hypothesis to $\tilde\tau,\tau_{2\ell},\sigma,\sigma_\ell,\tau'$ to complete the induction step and hence the proof of property 4).

We are left with the verification of property 5). For this we control the increase of intersection numbers between the train tracks τ_{2i}, η_{2i} and τ_{2i+2}, η_{2i+2}. This is done by distinguishing two cases.

Case 1: No marked point of a smooth side of a complementary component of σ_i is mapped into the large branch e_i of σ_i.

By two applications of Lemma 3.4, in this case we have

$$\begin{aligned} i_{\sigma_i}(\tau_{2i+1}, \eta_{2i+1}) &\leq i_{\sigma_i}(\tau_{2i+1}, \eta_{2i}) + q(\beta(\eta_{2i}, \sigma_i) - \beta(\eta_{2i+1}, \sigma_i)) \\ &\leq i_{\sigma_i}(\tau_{2i}, \eta_{2i}) + q(\beta(\tau_{2i}, \sigma_i) - \beta(\tau_{2i+1}, \sigma_i)) + q(\beta(\eta_{2i}, \sigma_i) - \beta(\eta_{2i+1}, \sigma_i)). \end{aligned}$$

Let ψ be the natural bijection between the complementary regions of σ_i and the complementary regions of σ_{i+1} as introduced in the first paragraph of this proof. Up to isotopy, for an arbitrary given neighborhood U of e_i in S and for any complementary region C of σ_i, there is a diffeomorphism F of the completion $\overline{C}$ of C onto the completion $\overline{\psi(C)}$ of $\psi(C)$ respecting cusps and marked points and which equals the identity outside of U. The intersection $\tau_{2i+2} \cap \psi(C)$ is equivalent to $F(\tau_{2i+1} \cap C)$, and $\eta_{2i+2} \cap \psi(C)$ is equivalent to $F(\eta_{2i+1} \cap C)$. This shows that

$$\begin{aligned} i_{\sigma_{i+1}}(\tau_{2i+2}, \eta_{2i+2}) &= i_{\sigma_i}(\tau_{2i+1}, \eta_{2i+1}) \\ &\leq i_{\sigma_i}(\tau_{2i}, \eta_{2i}) + q(\beta(\tau_{2i}, \sigma_i) - \beta(\tau_{2i+1}, \sigma_i)) + q(\beta(\eta_{2i}, \sigma_i) - \beta(\eta_{2i+1}, \sigma_i)). \end{aligned}$$

Case 2: There is a marked point on a smooth boundary component of a complementary region of σ_i which is mapped into the branch e_i.

In this case, e_i is contained in the image of one or two smooth boundary components T_1, T_2 of complementary regions of σ_i. By two applications of Lemma 3.4, we have

$$i_{\sigma_i}(\tau_{2i+1}, \eta_{2i+1}) \leq i_{\sigma_i}(\tau_{2i}, \eta_{2i+1}) + q^3 \leq i_{\sigma_i}(\tau_{2i}, \eta_{2i}) + 2q^3.$$

The marked points on T_1, T_2 determine marked points on smooth sides T_1', T_2' of complementary regions of σ_{i+1} so that we have

$$i_{\sigma_{i+1}}(\tau_{2i+2}, \eta_{2i+2}) = i_{\sigma_i}(\tau_{2i+1}, \eta_{2i+1}) \leq i_{\sigma_i}(\tau_{2i}, \eta_{2i}) + 2q^3.$$

Now by the consideration in the beginning of this proof, Case 2 can occur at most q^2 times. Moreover, the number of neighbors of σ in τ, η is bounded from above by the upper bound q for the number of switches of τ, η and hence this number can not be decreased by more than q in this process. Together we conclude that there are at most $q^2 + 2q \leq 2q^2$ among the numbers $0, \dots, \ell - 1$ such that $i_{\sigma_{i+1}}(\tau_{2i+2}, \eta_{2i+2}) \neq i_{\sigma_i}(\tau_{2i}, \eta_{2i})$. Since

$$|i_{\sigma_{i+1}}(\tau_{2i+2}, \eta_{2i+2}) - i_{\sigma_i}(\tau_{2i}, \eta_{2i})| \leq 2q^3$$

for all i, this completes the proof of the proposition. $\square$

We call the sequence $\{\tau_j\}_{0\leq j\leq 2\ell}$ constructed in Proposition 3.5 from a recurrent splitting sequence $\{\sigma_i\}_{0\leq i\leq \ell}$, a complete extension τ of σ_0 and a complete τ-extension of a σ_ℓ-filling measured geodesic lamination a sequence *induced* by $\{\sigma_i\}$.

If σ_0 is an arbitrary (not necessarily recurrent) subtrack of a complete train track τ_0 and if $\{\sigma_i\}_{0\leq i\leq \ell}$ is a splitting sequence issuing from σ_0 then the definition of a sequence of complete train tracks $\{\tau_i\}_{0\leq i\leq 2\ell}$ induced by the splitting sequence $\{\sigma_i\}_{0\leq i\leq \ell}$ always makes sense. However, if the splitting sequence $\{\sigma_i\}$ is not recurrent then such an induced sequence of complete train tracks may not exist.

Corollary 3.6. *For every $R>0$ there is a number $p(R)>0$ with the following property. Let $\{\sigma_i\}_{0\leq i\leq \ell}$ be a recurrent splitting sequence of train tracks on S. Let τ,η be complete extensions of σ_0 with $d(\tau,\eta)\leq R$ and let τ',η' be the endpoints of a sequence induced by $\{\sigma_i\}$ and issuing from τ,η. Then*

$$d(\tau',\eta')\leq p(R).$$

Remark: Corollary 3.6 corresponds to the fact that splitting a complete train track τ along a subtrack σ does not change the intersection of τ with the complement of a neighborhood of σ in S. Moreover, it commutes with the action of the pure mapping class group of a bordered subsurface of S which is contained in a complementary component of σ. We chose to introduce intersection numbers to find an easy quantitative description of this fact, the main difficulty being the possibility of twisting about boundary components of complementary regions.

For a simple geodesic multi-curve c and a train track τ which carries c we denote by $\tau(c)\subset\tau$ the subgraph of τ of all branches which are contained in the image of c under a carrying map. Note that $\tau(c)$ is a recurrent subtrack of τ and hence either it is a disjoint union of simple closed curves which define the multi-curve c, or it contains a large branch (see [PH92]).

Now let more specifically Q be a pants decomposition of S. If C is any complementary component of $\tau(Q)$, then a simple closed curve c contained in C is disjoint from Q. Thus if c is neither contractible nor freely homotopic into a puncture of S then c is freely homotopic to a component of the pants decomposition Q. In particular, the Euler characteristic of the completion of a complementary component of $\tau(Q)$ is at least -1. Thus this completion is of one of the following seven types, where in our terminology, a pair of pants can be a twice punctured disc or a once punctured annulus or a planar compact bordered surface of Euler characteristic -1 with three boundary components.

(1) A *triangle*, i.e. a disc with three cusps at the boundary.
(2) A *quadrangle*, i.e. a disc with four cusps at the boundary.
(3) A punctured disc with one cusp at the boundary.
(4) A punctured disc with two cusps at the boundary.
(5) An annulus with one cusp at the boundary.
(6) An annulus with two cusps at the boundary.
(7) A pair of pants with no cusps at the boundary.

If C is a complementary component of $\tau(Q)$ which is an annulus then the core curve of this annulus is freely homotopic to a component of Q. Since $\tau(Q)$ carries Q, this implies that if the boundary of C contains two cusps then these cusps are contained in the same boundary component, i.e. one of the boundary components of C is a smooth circle. In other words, every complementary component of type (6) is of the more restricted following type.

(6′) An annulus with one smooth boundary component and one boundary component containing two cusps.

For the proof of Proposition 3.2 we need some control on $\tau(Q)$ where Q is any pants decomposition of S and where τ is a train track in standard form for a marking F of S which carries Q. The next lemma provides such a control. It follows immediately from the work of Penner and Harer [PH92] and uses an argument due to Thurston (see [FLP91]). We present the lemma in the form needed in Section 5.

Lemma 3.7. *Let F be a marking for S with pants decomposition P. Let ν be a measured geodesic lamination; then there is a train track σ with the following properties:*

1. *σ carries ν.*
2. *ν fills σ.*
3. *Every train track in standard form for F which carries ν contains σ as a subtrack.*

Proof. We begin with showing the lemma in the case that ν is supported in a simple geodesic multi-curve.

Thus let F be a marking of S with pants decomposition P and let c be a simple geodesic multi-curve. Let S_0 be a connected component of $S - P$ with boundary circles $\gamma_i \in P$ (the number of these circles is contained in $\{1, 2, 3\}$). Up to homotopy, the multi-curve c intersects S_0 in a (perhaps empty) collection of disjoint simple arcs with endpoints on the boundary of S_0 which are *essential*, i.e. not homotopic with fixed endpoints into the boundary of S_0.

For each i let $n(\gamma_i)$ be the *intersection number* between c and γ_i. Note that if γ_i is a component of c then $n(\gamma_i) = 0$. Since any two essential simple arcs in S_0 with endpoints on the same boundary components of S_0 are isotopic in S_0 relative to the boundary, there is up to isotopy a unique configuration of mutually disjoint simple arcs in S_0 with endpoints on the boundary of S_0 which realizes the intersection numbers $n(\gamma_i)$ (see [FLP91] for details). For this configuration there is a unique isotopy class of a train track (with stops) in S_0 which carries the configuration with a surjective carrying map and which can be obtained from a standard model as shown in Figure A by removing some (perhaps all) of the branches (Figure 2.6.2 of [PH92] shows in detail how to remove some of the branches of a standard model). These train tracks with stops can be glued to connectors obtained from the standard models shown in Figure B by removing some of the branches (see Figure 2.6.1 of [PH92]) in such a way that the resulting train track σ has the following properties.

1. σ carries c.

(2) c fills σ.
(3) There is a train track in standard form for F which contains σ as a subtrack.

Note that the direction of the winding of a component of c relative to a curve from the marking F which intersects the pants curve γ_i determines the connector about γ_i in σ. It is immediate from the construction that a complete train track in standard form for F carrying c is an extension of σ (compare the discussion in [PH92]).

Now if ν is an arbitrary measured geodesic lamination then the support of ν can be approximated in the Hausdorff topology by a sequence $\{c_i\}$ of simple geodesic multi-curves [CEG87]. There are only finitely many train tracks which are subtracks of a train track in standard form for F. Thus if $\{\sigma_i\}$ is a sequence of train tracks as above for the multi-curves c_i then there is some train track σ so that $\sigma = \sigma_i$ for infinitely many i. Since the set of all geodesic laminations carried by the fixed train track σ is closed in the Hausdorff topology, σ carries ν and satisfies the requirements in the lemma. □

We use this to show

Lemma 3.8. *There is a number $\kappa > 0$ with the following properties. Let F be a marking for S and let Q by any pants decomposition of S. Then there is a set $\mathcal{D}$ of complete train tracks in standard form for some marking with pants decomposition Q (depending on the train track from $\mathcal{D}$) with the following properties.*

(1) *Every geodesic lamination in standard form for Q is carried by some train track in the set $\mathcal{D}$.*
(2) *The diameter of $\mathcal{D}$ in $\mathcal{TT}$ is at most κ.*
(3) *For every $\eta \in \mathcal{D}$ there is a train track in standard form for F which carries η.*

Proof. By the discussion in the beginning of this section, it suffices to show the existence of a number $\chi > 0$ with the following properties. Let F be a marking of S and let Q by any pants decomposition of S. Then for every geodesic lamination λ in standard form for Q there is a train track $\tau(\lambda)$ with the following properties.

(1) $\tau(\lambda)$ carries λ.
(2) $\tau(\lambda)$ is in standard form for a marking with pants decomposition Q.
(3) $\tau(\lambda)$ is carried by a train track $\tau_0(\lambda)$ in standard form for F.
(4) If λ' is any other geodesic lamination in standard form for Q then we have $i_Q(\tau(\lambda), \tau(\lambda')) \leq \chi$.

Thus let F be a marking for S with pants decomposition P, let Q be a second pants decomposition and let λ, λ' be two geodesic laminations in standard form for Q. By Lemma 2.2, there are unique train tracks τ, τ' in standard form for F which carry λ, λ'; in particular, τ, τ' carry Q. Let as before $\tau(Q), \tau'(Q)$ be the subtrack of τ, τ' of all branches of positive Q-weight. By Lemma 3.7, the train tracks $\tau(Q), \tau'(Q)$ are isotopic. This means that τ, τ' are complete extensions of $\tau(Q)$. We equip the smooth boundary components of complementary regions of

$\tau(Q)$ with marked points and use these marked points to define the intersection number between the complete extensions τ,τ' of $\tau(Q)$. Then $i_{\tau(Q)}(\tau,\tau')$ is bounded from above by a universal constant.

Assume first that $\tau(Q)=Q$. By invariance under the action of the mapping class group and cocompactness, Lemma 2.1 together with Lemma 3.3 of [H09] shows that there is a number $\beta>0$ and there is a train track η in standard form for some marking of S with pants decomposition Q which carries λ, which is carried by τ and such that $d(\tau,\eta)\leq\beta$. Thus in this case the lemma is obvious (this can also easily seen with a direct combinatorial argument). Therefore we may assume that $\tau(Q)$ contains a large branch.

Let $\{\sigma_i\}_{0\leq i\leq s}$ be a splitting sequence issuing from $\sigma_0=\tau(Q)$ so that for each $i\leq s$ the pants decomposition Q is carried by σ_i and fills σ_i. Then for each i the pants decomposition Q defines an integral transverse counting measure on σ_i by assigning to a branch b the number of connected components of the preimage of b under a carrying map $Q\to\sigma_i$. For $i<s$ the total Q-weight of σ_{i+1}, i.e. the sum of the weights of this counting measure over all branches of σ_{i+1}, is bounded from above by the total Q-weight of σ_i minus two. Namely, if σ_{i+1} is obtained from σ_i by a split at the large branch e and if e' is the diagonal branch of the split in σ_{i+1}, then the Q-weight of e equals the sum of the Q-weights of e' and the Q-weights of the two losing branches of the split. These weights are all positive and integral. The weights of the branches of σ_i which are distinct from e coincide with the weights of their images in σ_{i+1} under the natural bijection $\varphi(\sigma_i,\sigma_{i+1})$ of the branches of σ_i onto the branches of σ_{i+1}. Therefore the length of the splitting sequence $\{\sigma_i\}$ is bounded from above by the total Q-weight of $\tau(Q)$.

Assume that the sequence $\{\sigma_i\}_{0\leq i\leq s}$ is of maximal length. This means that for every large branch e of σ_s the pants decomposition Q is carried by a collision of σ_s at e (i.e. a split followed by the removal of the diagonal).

By Proposition 3.5, there are complete extensions τ_1,τ_1' of σ_s with $\tau_1(Q)=\tau_1'(Q)=\sigma_s$ so that τ_1 carries λ, τ_1' carries λ' and that moreover

$$i_{\tau_1(Q)}(\tau_1,\tau_1')\leq i_{\tau(Q)}(\tau,\tau')+4q^5. \tag{2}$$

Let e be any large branch of $\tau_1(Q)=\sigma_s$. The pants decomposition Q is carried by the train track ξ obtained from σ_s by the collision at e. Let τ_2,τ_2' be the $(\tau_1(Q),\lambda)$-modification (or the $(\tau_1'(Q),\lambda')$-modification) of τ_1,τ_1' at e. Two applications of Lemma 3.4 show that

$$i_{\tau_2(Q)}(\tau_2,\tau_2')\leq i_{\tau_1(Q)}(\tau_1,\tau_1')+2q^2. \tag{3}$$

The train tracks τ_2,τ_2' are tight at e. Let τ_3,τ_3' be the train tracks obtained from τ_2,τ_2' by a split at e with the property that the split tracks carry λ,λ'. The number of branches of $\tau_3(Q)=\tau_3'(Q)=\xi$ is strictly smaller than the number of branches of $\tau_1(Q)$. The diagonal branch $d=\varphi(\tau_2,\tau_3)(e)$ of the split of τ_2 at e is a small branch of τ_3 which is contained in a complementary region C of $\xi=\tau_3(Q)$ and which is attached at both endpoints to a side of C. Let $d'=\varphi(\tau_2',\tau_3')(e)$ be the diagonal of the split of τ_2' at e.

Let U be a neighborhood of e in S which is sufficiently small that σ_s intersects U in the union of e with the four half-branches of σ_s which are incident on the endpoints of e. Up to isotopy, the intersections of σ_s, ξ with $S-U$ coincide. Thus if there is a complementary region C of ξ containing a smooth boundary component T which newly arises in the process then this boundary component intersects U. Mark a point on T which is mapped into U. Note that any marked point on a smooth boundary component of a complementary region of σ_s induces a marked point on a smooth boundary component of a complementary region of ξ.

Let ζ, ζ' be complete extensions of σ_s which are σ_s-equivalent to τ_2, τ_2' and which have the minimal number of intersection points in $S - \sigma_s$, i.e. which realize the intersection number $i_{\sigma_s}(\tau_2, \tau_2')$. By the definition of equivalence, after perhaps replacing ζ, ζ' by equivalent train tracks and after perhaps a modification with an isotopy we may assume that ζ, ζ' are tight at e.

Using once more the definition of equivalence, the train tracks ζ, ζ' can be modified with a single split at e to train tracks ζ_0, ζ_0' which are complete extensions of ξ and which are equivalent to the complete extensions τ_3, τ_3' of ξ. If τ_3, τ_3' is obtained from τ_2, τ_2' by a right (or left) split at e then ζ_0, ζ_0' is obtained from ζ, ζ' by a right (or left) split at e.

Let d_0, d_0' be the diagonal of the split in ζ_0, ζ_0'. The train track ξ intersects U in two disjoint embedded arcs which are joined by the two branches d_0, d_0' of ζ_0, ζ_0'. We may assume that the branches d_0, d_0' either are disjoint (if the train tracks τ_3, τ_3' are both obtained from τ_2, τ_2' by the same type of split, left or right) or that they intersect transversely in a single point. By the definition of intersection numbers, this shows that

$$i_{\tau_3(Q)}(\tau_3, \tau_3') = i_\xi(\zeta_0, \zeta_0') \le i_{\tau_2(Q)}(\tau_2, \tau_2') + 1. \tag{4}$$

Inequalities (4) and (3) now yield that

$$i_{\tau_3(Q)}(\tau_3, \tau_3') \le i_{\tau_2(Q)}(\tau_2, \tau_2') + 1 \le i_{\tau_1(Q)}(\tau_1, \tau_1') + 2q^2 + 1$$

and hence from the estimate (2) we obtain (since $q \ge 2$)

$$i_{\tau_3(Q)}(\tau_3, \tau_3') \le i_{\tau(Q)}(\tau, \tau') + 6q^5.$$

Repeat this procedure with the train track $\xi = \tau_3(Q)$. After at most $k \le q$ such steps where k is the number of branches of $\tau(Q)$ we arrive at train tracks η, η' which contain Q as a disjoint union of simple closed curves and carry λ, λ'.

To summarize, we obtain in at most q steps two splitting sequences connecting τ, τ' to train tracks η, η' so that $\eta(Q) = \eta'(Q) = Q$ and that η, η' carry λ, λ'. Each of these steps increases intersection numbers by at most $6q^5$. In particular, the intersection number $i_Q(\eta, \eta')$ is uniformly bounded and hence the distance in $\mathcal{TT}$ between η, η' is uniformly bounded as well.

Now λ, λ' is carried by η, η' and is in standard form for Q. Hence by the reasoning in the third paragraph of this proof, there are train tracks β, β' in standard form for some marking with pants decomposition Q which carry λ, λ', which are carried by η, η' and such that $d(\eta, \beta) \le \kappa, d(\eta', \beta') \le \kappa$. As a consequence, the distance between β, β' is uniformly bounded. Since λ, λ' were arbitrarily chosen geodesic laminations in standard form for Q the lemma follows. □

Now we are ready to complete the proof of Proposition 3.2. Let F, G be markings of S with pants decompositions P, Q. Let λ be a complete geodesic lamination in standard form for Q and let $\tau(\lambda) \in \mathcal{V}(\mathcal{TT})$ be as in Lemma 3.8. Then $\tau(\lambda)$ is in standard form for a marking G' with pants decomposition Q. Any two markings with pants decomposition Q differ from each other by a multi-twist about the pants curves of Q. Thus if we write $k = 3g - 3 + m$ for simplicity of notation and if we let $\theta_1, \dots, \theta_k$ be the positive Dehn twists about the components $\gamma_1, \dots, \gamma_k$ of Q then there is an integral vector $(n_1, \dots, n_k) \in \mathbb{Z}^k$ such that

$$G = \theta_1^{n_1} \cdots \theta_k^{n_k} G'.$$

Every pants curve γ_i of Q is the core curve of a twist connector for $\tau(\lambda)$. Splitting a standard twist connector at the large branch, with the small branch of the connector as a winner, results in deforming a train track by a (positive or negative) Dehn twist about the core curve of the connector. The sign of the twist is determined by the type of the twist connector which in turn is determined by the spiraling direction of the geodesic lamination λ in standard form for Q about the pants curve γ_i.

Assume after reordering that for some $p \leq k$ and for every $i \leq p$, either $n_i = 0$ or the sign of n_i coincides with the sign determined by the spiraling direction of λ about γ_i, and that for $i > p$ we have $n_i \neq 0$ and the sign of n_i differs from this direction. Let σ be a train track in standard form for the marking G' which is obtained from $\tau(\lambda)$ by reversing the directions in the twist connectors about the curves $\gamma_{p+1}, \dots, \gamma_k$. By Lemma 2.6.1 of [PH92], σ is complete, and σ is splittable to the train track $\theta_1^{n_1} \cdots \theta_k^{n_k} \sigma$ in standard form for G. The train track σ carries a complete geodesic lamination λ' in standard form for Q. By equivariance under the action of the mapping class group and cocompactness, there is a universal constant $\chi > 0$ such that

$$d(\sigma, \tau(\lambda)) \leq \chi.$$

By Lemma 3.8 there is a train track $\tau(\lambda')$ which is in standard form for a marking with pants decomposition Q, which carries λ' and such that

$$d(\tau(\lambda), \tau(\lambda')) \leq \kappa.$$

Since $\mathcal{MCG}(S)$ acts isometrically on $\mathcal{TT}$, we have

$$d(\theta_1^{n_1} \cdots \theta_k^{n_k} \sigma, \theta_1^{n_1} \cdots \theta_k^{n_k} \tau(\lambda')) \leq \kappa + \chi.$$

But $\theta_1^{n_1} \cdots \theta_k^{n_k} \sigma$ is in standard form for G and therefore the train track $\eta = \theta_1^{n_1} \cdots \theta_k^{n_k} \tau(\lambda')$ is at distance at most $\kappa + \chi$ from a train track in standard form for G. It contains the pants decomposition Q as an embedded subtrack.

On the other hand, there is a train track in standard form for F which carries $\tau(\lambda')$, and $\tau(\lambda')$ carries η by construction. Therefore there is train track in standard form for F which carries η. This completes the proof of Proposition 3.2.

Remark: The results in this section are also valid if the surface S is a once puncture torus or a four punctured sphere.

4. Quasi-isometric embeddings

In this section we use Proposition 3.1 to derive Theorems 2-4 from the introduction.

We begin with an investigation of the mapping class group of an essential subsurface S_0 of S. This means that S_0 is a bordered subsurface of S with the property that the inclusion $S_0 \to S$ induces an injection $\pi_1(S_0) \to \pi_1(S)$ of fundamental groups and that moreover every boundary component of S_0 is an essential simple closed curve in S. Let $\mathcal{PMCG}(S_0)$ be the *pure mapping class group* of S_0, i.e. the subgroup of the mapping class group $\mathcal{MCG}(S_0)$ of S_0 of all mapping classes which fix each of the boundary components and each of the punctures. Then $\mathcal{PMCG}(S_0)$ is a subgroup of $\mathcal{MCG}(S_0)$ of finite index. It can be identified with the subgroup of the mapping class group of S of all elements which can be realized by a homeomorphism preserving $S - S_0$ pointwise and fixing each of the punctures of S.

As in the introduction, call a finitely generated subgroup Γ of $\mathcal{MCG}(S)$ *undistorted* if the inclusion $\Gamma \to \mathcal{MCG}(S)$ is a quasi-isometric embedding. For example, every subgroup of $\mathcal{MCG}(S)$ of finite index is undistorted. The following result is implicitly but not explicitly contained in Theorem 6.12 of [MM00].

Proposition 4.1. *For an essential subsurface $S_0 \subset S$ the subgroup $\mathcal{PMCG}(S_0)$ of $\mathcal{MCG}(S)$ is undistorted.*

Proof. If $S_0 = S_1 \cup S_2$ for two disjoint essential subsurfaces S_1, S_2 of S whose fundamental groups as subgroups of $\pi_1(S)$ have trivial intersection then

$$\mathcal{PMCG}(S_0) = \mathcal{PMCG}(S_1) \times \mathcal{PMCG}(S_2).$$

Now a subgroup of a finitely generated group which is a direct product of two undistorted subgroups is undistorted and hence it suffices to show the proposition for connected essential subsurfaces of S. The case that S_0 is an essential annulus is treated in detail in [FLM01, H09], so we assume that the Euler characteristic of S_0 is negative. If S_0 is a thrice punctured sphere then $\mathcal{PMCG}(S_0)$ equals the free abelian group of Dehn twists about the boundary components of S_0. Thus we also may assume that S_0 is different from a thrice punctured sphere.

Our goal is to show that any two elements of $\mathcal{PMCG}(S_0)$ can be connected by a uniform quasi-geodesic in $\mathcal{MCG}(S)$ which is entirely contained in $\mathcal{PMCG}(S_0)$. For this let $\hat{S}_0$ be the surface which we obtain from S_0 by replacing each boundary component by a puncture. There is an exact sequence

$$0 \to \mathbb{Z}^p \to \mathcal{PMCG}(S_0) \xrightarrow{\Pi} \mathcal{PMCG}(\hat{S}_0) \to 0$$

where $\mathbb{Z}^p$ is identified with the free abelian group of Dehn twists about the boundary components of S_0.

Choose a pants decomposition P for S which contains the boundary of S_0 as a subset. Let τ be a complete train track in standard form for a marking F with pants decomposition P and only twist connectors. Let τ_1 be the subtrack of τ which we obtain from τ by removing all branches contained in the interior of S_0. We can choose τ in such a way that any two points in the same connected component of τ_1 can be connected by a trainpath in τ_1 (however, in general τ_1 is neither connected not recurrent).

Let $c_1, \dots, c_p$ be the boundary circles of S_0. Every complete train track σ on $\hat{S}_0$ is a subtrack of a complete train track η on S which contains τ_1 as a subtrack. Namely, up to isotopy, each boundary component c_i of S_0 is contained in a complementary once punctured monogon region C_i of σ. It cuts C_i into an annulus $A_i \subset S_0$ and a once punctured disc. Add a small branch b_i to $\sigma \cup \tau_1$ which is contained in the closure $\overline{A_i}$ of the annulus A_i and connects the boundary of C_i to the boundary circle c_i of S_0. Since σ is complete and hence non-orientable, if for each $i \leq p$ we connect the branch b_i to the circle c_i in such a way that the resulting train track η intersects an annulus neighborhood of c_i in a twist connector as shown in Figure B, then η is recurrent [PH92]. The train track η is also very easily seen to be transversely recurrent and hence it is complete.

The train track η is not uniquely determined by τ_1 and σ. The choices made are the positions of the additional switches on the boundaries of the complementary regions C_i, the inward pointing tangents of the added branches b_i at these switches and the homotopy class with fixed endpoints of the branch $b_i \subset \overline{A_i}$. By invariance under the action of the group $\mathcal{PMCG}(\hat{S}_0)$ and cocompactness, for any two such choices η, η' there is a multi-twist φ about the multi-curve $c = \cup_i c_i$ such that the distance in $\mathcal{TT}$ between $\eta, \varphi\eta'$ is uniformly bounded. The set $\mathcal{E}$ of all such extensions of all complete train tracks on $\hat{S}_0$ is invariant under the action of the group $\mathcal{PMCG}(S_0)$, with finitely many orbits and finite point stabilizers.

Let F be a marking for $\hat{S}_0$. By Theorem 2.3 and by Proposition 3.2 and the following remark, any complete train track η on $\hat{S}_0$ can be obtained from a train track σ in standard form for F by a uniform quasi-geodesic in the train track complex $\mathcal{TT}(\hat{S}_0)$ of $\hat{S}_0$ which is a concatenation of a splitting sequence with an edge-path of uniformly bounded length.

For every train track σ on $\hat{S}_0$ in standard form for F choose an extension $\Psi(\sigma) \in \mathcal{V}(\mathcal{TT})$ as above. By Proposition 3.5, there is a universal number $p > 0$ (depending on the topological type of S) and for every splitting sequence $\{\sigma_i\}_{0 \leq i \leq m}$ of complete train tracks on $\hat{S}_0$ issuing from a train track $\sigma_0 = \sigma$ in standard form for F and for every complete $\Psi(\sigma)$-extension of a σ_m-filling measured geodesic lamination there is an induced sequence $\{\tau_j\}_{0 \leq j \leq 2m} \subset \mathcal{TT}$ connecting $\tau_0 = \Psi(\sigma)$ to a train track τ_{2m} which contains σ_m as well as τ_1 as a subtrack. There is a universal number $s > 0$ such that for every $i < m$ the train track track τ_{2i+2} can be obtained from τ_{2i} by a splitting sequence whose length is contained $[1, s]$.

Since splitting sequences are uniform quasi-geodesics in both $\mathcal{TT}$ and the train track complex $\mathcal{TT}(\hat{S}_0)$ of $\hat{S}_0$ [H09], this shows that there is a number $c > 0$ with the following property. For every train track $\xi \in \mathcal{V}(\mathcal{TT}(\hat{S}_0))$ there is a train track $\Psi(\xi) \in \mathcal{E}$ which contains both ξ and τ_1 as a subtrack and is such that the distance in $\mathcal{TT}(\hat{S}_0)$ between ξ and a train track σ in standard form for F is not bigger than $cd(\Psi(\sigma), \Psi(\xi)) + c$.

The resulting map $\Psi : \mathcal{V}(\mathcal{TT}(\hat{S}_0)) \to \mathcal{E}$ is used to define a map $\rho : \mathcal{PMCG}(\hat{S}_0) \to \mathcal{PMCG}(S_0)$ as follows. Let σ be a fixed train track in standard form for F. For $g \in \mathcal{PMCG}(\hat{S}_0)$ define $\rho(g) \in \mathcal{PMCG}(S_0)$ in such a way that the distance between $\Psi(g\sigma)$ and $\rho(g)(\Psi(\sigma))$ is uniformly bounded. Since $\mathcal{PMCG}(S_0)$ acts on $\mathcal{E}$ with finitely many orbits and finite point stabilizers and since $\mathcal{TT}(\hat{S}_0)$ is equivariantly quasi-isometric to $\mathcal{PMCG}(\hat{S}_0)$, the map ρ is a coarse section of the projection Π. By this we mean that there is a universal constant $\kappa > 0$ such that $d(\Pi\rho(g), g) \leq \kappa$ for all g.

The image of $\mathcal{V}(\mathcal{TT}(\hat{S}_0))$ under the map Ψ consists of train tracks which contain each boundary component c_i of S_0 as the core curve of a twist connector. Splitting such a train track τ at the large branch in this twist connector, with the small branch as the winner, results in replacing τ by $\theta_c(\tau)$ where θ_c is a Dehn twist about c whose direction (positive or negative) depends on the twist connector. Thus if Γ denotes the semi-group of Dehn twists about the boundary components of S_0 determined by the train track τ_1 then for every $g \in \rho(\mathcal{PMCG}(\hat{S}_0))$ and every $\varphi \in \Gamma$ there is a uniform quasi-geodesic in $\mathcal{MCG}(S)$ connecting the identity to $\varphi\rho(g)$ and which is entirely contained in $\mathcal{PMCG}(S_0)$. However, the choice of the twist connector in the train track τ_1 was arbitrary and consequently the unit element in $\mathcal{MCG}(S)$ can be connected to any mapping class $g \in \mathcal{PMCG}(S_0)$ by a uniform quasi-geodesic in $\mathcal{MCG}(S)$ which is entirely contained in $\mathcal{PMCG}(S_0)$. By invariance of the word metrics under left translation, this just means that $\mathcal{PMCG}(S_0) < \mathcal{MCG}(S)$ is undistorted. □

Now let S_0 be any non-exceptional surface of genus $g \geq 0$ with $m \geq 0$ punctures and let S be the surface S_0 punctured at one additional point p. There is an exact sequence [B74]

$$0 \to \pi_1(S_0) \to \mathcal{MCG}(S) \xrightarrow{\Pi} \mathcal{MCG}(S_0) \to 0.$$

The projection Π is induced by the map $S \to S_0$ which consists in closing the puncture p. An element α of the fundamental group $\pi_1(S_0)$ of S_0 is mapped to the element of $\mathcal{MCG}(S)$ obtained by dragging the point p along a loop in S_0 in the homotopy class α. Braddeus, Farb and Putman [BFP07] showed that $\pi_1(S_0)$ is an exponentially distorted subgroup of $\mathcal{MCG}(S)$. We next observe that in contrast, the projection $\Pi : \mathcal{MCG}(S) \to \mathcal{MCG}(S_0)$ has a coarse section which is a quasi-isometric embedding. Here by a coarse section we mean a map $\Psi : \mathcal{MCG}(S_0) \to \mathcal{MCG}(S)$ such that

$$d(\Pi\Psi(g), g) \leq \kappa$$

for all $g \in \mathcal{MCG}(S)$ where $\kappa \geq 0$ is a universal constant.

Proposition 4.2. *Let S_0 be a non-exceptional surface of genus $g \geq 0$ with $m \geq 0$ punctures and let S be the surface of genus g with $m + 1$ punctures. Then there is a coarse section for the projection $\Pi : \mathcal{MCG}(S) \to \mathcal{MCG}(S_0)$ which is a quasi-isometric embedding.*

Proof. Let $\mathcal{TT}(S)$ and $\mathcal{TT}(S_0)$ be the train track complex of S and of S_0. We first define a map $\Psi : \mathcal{V}(\mathcal{TT}(S_0)) \to \mathcal{V}(\mathcal{TT}(S))$ as follows.

For a complete train track τ on S_0 choose any complementary trigon C of τ. Mark a point p in the interior of C and add two switches v_1, v_2 and two branches $b_1, b_2 \subset C - \{p\}$ to τ in the following way. The switch v_1 is an interior point of a branch of τ contained in a side of C, $v_2 \neq p$ is a point in the interior of C, b_1 connects v_1 to v_2 and b_2 is a small branch contained in the interior of C whose endpoints are both incident on v_2 and which is the boundary of a subdisc of C containing p in its interior. Since τ is complete, Proposition 1.3.7 of [PH92] shows that the resulting train track η_0 on $S = S_0 - \{p\}$ is recurrent. It is also easily seen to be transversely recurrent. The train track η_0 decomposes S into trigons, once punctured monogons and one fourgon. The fourgon can be subdivided into two trigons by adding a single small branch. The resulting train track η on S is complete, and it contains τ as a subtrack. This construction defines a map $\Psi : \mathcal{V}(\mathcal{TT}(S_0)) \to \mathcal{V}(\mathcal{TT}(S))$.

The map Ψ depends on some choices among a finite set of possibilities: The choice of the complementary trigon C, the choice of the position of the switch v_1 on a side of C, the orientation of the inward pointing tangent of the branch b_1 at the switch v_1 and the choice of the small branch subdividing the fourgon. Any train track constructed in this way contains τ as a subtrack. Moreover, for $g \in \mathcal{MCG}(S_0)$ there is some $h \in \mathcal{MCG}(S)$ such that $h(\Psi(\tau))$ is one of the possibilities for $\Psi(g\tau)$. Since there are only finitely many orbits of complete train tracks on S_0 under the action of the mapping class group, by coarse equivariance of the construction we conclude that there is a universal number $\kappa_0 > 0$ such that for any other choice Ψ' of such a map we have $d(\Psi(\tau), \Psi'(\tau)) \leq \kappa_0$ for all $\tau \in \mathcal{V}(\mathcal{TT}(S_0))$.

We use the map Ψ to define a map $\Phi : \mathcal{MCG}(S_0) \to \mathcal{MCG}(S)$ as follows. The mapping class groups of S_0, S act properly and cocompactly on $\mathcal{TT}(S_0)$, $\mathcal{TT}(S)$. Choose $\tau \in \mathcal{V}(\mathcal{TT}(S_0))$ and a fundamental domain D for the action of $\mathcal{MCG}(S)$ on $\mathcal{TT}(S)$ containing $\Psi(\tau)$. For $g \in \mathcal{MCG}(S_0)$ choose $\Phi(g) \in \mathcal{MCG}(S)$ in such a way that $\Psi(g\tau) \in \Phi(g)D$. If Φ' is any other such map then $d(\Phi(g), \Phi'(g)) \leq \kappa_1$ where $\kappa_1 > 0$ is a universal constant (and d is any distance on $\mathcal{MCG}(S)$ defined by a word norm of a finite symmetric generating set).

By construction, the map $\Phi : \mathcal{MCG}(S_0) \to \mathcal{MCG}(S)$ is a coarse section for the projection $\mathcal{MCG}(S) \to \mathcal{MCG}(S_0)$. Thus we are left with showing that Φ is a quasi-isometric embedding, and this holds true if this is the case for the map Ψ. To this end, note that τ is a subtrack of $\Psi(\tau)$. By Proposition 3.5, a splitting sequence $\{\tau_i\}_{0 \leq i \leq \ell} \subset \mathcal{TT}(S_0)$ issuing from $\tau_0 = \tau$ induces a splitting sequence in $\mathcal{TT}(S)$ issuing from $\Psi(\tau)$. The length of this sequence is not smaller than the length ℓ of the splitting sequence of τ, and it is not bigger than $q\ell$ for a universal constant $q > 0$. On the other hand, a point on the induced sequence which contains τ_i as a subtrack is a possible choice for $\Psi(\tau_i)$ and hence it is at uniformly bounded distance to $\Psi(\tau_i)$.

By Theorem 2.3 and the remark thereafter, splitting sequences in $\mathcal{TT}(S)$ are uniform quasi-geodesics, and the same holds true for splitting sequences in $\mathcal{TT}(S_0)$.

As a consequence, there is a number $c > 1$ such that

$$d(\tau_0, \tau_\ell)/c - c \leq d(\Psi(\tau_0), \Psi(\tau_\ell)) \leq cd(\tau_0, \tau_\ell) + c$$

whenever $\tau_0 \in \mathcal{V}(\mathcal{TT}(S_0))$ is splittable to $\tau_\ell \in \mathcal{V}(\mathcal{TT}(S_0))$. By Proposition 3.1, splitting sequences connect a coarsely dense set of pairs of points in the train track complex $\mathcal{TT}(S_0)$. This implies that the map $\Psi : \mathcal{V}(\mathcal{TT}(S_0)) \to \mathcal{V}(\mathcal{TT}(S))$ defines a quasi-isometric embedding and hence the same holds true for $\Phi : \mathcal{MCG}(S_0) \to \mathcal{MCG}(S)$. □

Finally, for a closed surface of genus $g \geq 2$ we investigate the normalizer of a finite subgroup Γ of $\mathcal{MCG}(S)$ (see [RS07] for an earlier proof of this result, stated a bit differently).

Proposition 4.3. *For a closed surface S of genus $g \geq 2$, the normalizer in $\mathcal{MCG}(S)$ of a finite subgroup is undistorted.*

Proof. Let $\mathcal{T}(S)$ be the Teichmüller space of S. By the Nielsen realization problem, a finite subgroup Γ of $\mathcal{MCG}(S)$ fixes a point $x \in \mathcal{T}(S)$ [Ke83]. This means that Γ can be realized as a finite group of biholomorphisms of (S, x). The quotient $(S, x)/\Gamma$ is a Riemann surface, and the projection $\pi : (S, x) \to (S, x)/\Gamma$ is a branched covering ramified over a finite number of points $p_1, \dots, p_\ell \in (S, x)/\Gamma$. The marked complex structure x on S projects to a marked complex structure on $(S, x)/\Gamma$.

Let $(S_1, x_1), (S_0, x_0)$ be the punctured Riemann surfaces which are obtained from $(S, x), (S, x)/\Gamma$ by removing the branch points of the covering $(S, x) \to (S, x)/\Gamma$. The projection π restricts to an unbranched covering $S_1 \to S_0$. The Teichmüller spaces $\mathcal{T}(S_0)$ of S_0, $\mathcal{T}(S_1)$ of S_1 are contractible. For every point $y \in \mathcal{T}(S_0)$ which is sufficiently close to x_0 there is a covering $\Psi(y) \in \mathcal{T}(S_1)$ of the Riemann surface (S_0, y) which is of the same topological type as the covering $(S_1, x_1) \to (S_0, x_0)$. The marking of $\Psi(y)$ is determined in such a way that the map Ψ is continuous near x_0. This construction defines a developing map $\Psi : \mathcal{T}(S_0) \to \mathcal{T}(S_1)$. Since $\mathcal{T}(S_0)$ is simply connected, the developing map is in fact single-valued. Moreover, it is clearly injective and hence an embedding. (In fact, its is not hard to see that this construction defines an isometric embedding of $\mathcal{T}(S_0)$ into $\mathcal{T}(S_1)$ for the Teichmüller metrics). There is a natural projection $\Pi : \mathcal{T}(S_1) \to \mathcal{T}(S)$ defined by filling in the punctures.

Let $\mathcal{MCG}_0(S_0)$ be the subgroup of the mapping class group $\mathcal{MCG}(S_0)$ of S_0 of all mapping classes realizable by a homeomorphism of S_0 which lifts to a homeomorphism of S. Let $N(\Gamma)$ be the normalizer of Γ in $\mathcal{MCG}(S)$. Then there is an exact sequence [BH73]

$$0 \to \Gamma \to N(\Gamma) \to \mathcal{MCG}_0(S_0) \to 0.$$

(Theorem 3 in [BH73] states this only in the case that the group Γ is cyclic. However, as pointed out explicitly in [BH73], the result for all finite groups is immediate from the argument given there and the Nielsen realization problem).

Since the group Γ is finite, the groups $N(\Gamma)$ and $\mathcal{MCG}_0(S_0)$ are quasi-isometric. Thus to show the proposition it is enough to show that there is quasi-isometric embedding of $\mathcal{MCG}_0(S_0)$ into $\mathcal{MCG}(S)$ whose image is contained in a uniformly boundedd neighborhood of $N(\Gamma)$. Following Proposition 4.2, it suffices in fact to show that there is a quasi-isometric embedding of $\mathcal{MCG}(S_0)$ into $\mathcal{MCG}(S_1)$ whose image is contained in a uniformly bounded neighborhood of the image of $N(\Gamma)$ under a a coarse section for the projection $\mathcal{MCG}(S_1) \to \mathcal{MCG}(S)$.

For this note that the preimage of a complete train track τ on S_0 under the covering $S_1 \to S_0$ is a Γ-invariant graph ξ in S which decomposes S into polygons and once punctured polygons. The preimage of each trigon component of τ is a union of n trigon components of ξ where $n = |\Gamma|$ is the number of sheets of the covering. Each once punctured monogon in τ encloses one of the points p_i and lifts to a punctured m_i-gon in $S_1 - \xi$ where $2 \leq m_i \leq n$ is the ramification index of p_i.

The branch points of the covering define a set of marked points contained in complementary regions of ξ. Each complementary region of ξ contains at most one such point. Thus ξ defines a (non-complete) train track on the punctured surface S_1, again denoted by ξ. Now a positive transverse measure on τ lifts to a positive transverse measure on ξ and therefore ξ is recurrent. The same argument also shows that ξ is transversely recurrent. Then ξ is a subtrack of a complete train track on S_1 obtained by subdividing some of the complementary regions as in the proof of Proposition 4.2. As before, the resulting complete train track η depends on choices among a uniformly bounded number of possibilities (compare the proof of Proposition 4.2).

Now if the complete train track τ_1 on S_0 is obtained from the complete train track τ by a single split at a large branch e then the preimage ξ_1 of τ_1 can be obtained from the preimage ξ of τ by a splitting sequence of length n. Namely, the preimage of any large branch of τ is the union of n large branches of ξ. Such a splitting sequence then induces a splitting sequence of length at most qn of the complete train track η on S_1 constructed in the previous paragraph where $q > 0$ is a universal constant.

By Proposition 3.1, splitting sequences in the train track complex $\mathcal{TT}(S_0)$ of S_0 connnect a coarsely dense set of pairs of points. By Theorem 2.3 and the remark thereafter, each such splitting sequence defines a uniform quasi-geodesic in the subgroup $\mathcal{MCG}_0(S_0)$ of the mapping class group of S_0. This quasi-geodesic lifts to a uniform quasi-geodesic in $\mathcal{MCG}(S_1)$ contained in a uniformly bounded neighborhood of the image of $N(\Gamma)$ under the coarse section for the projection $\mathcal{MCG}(S_1) \to \mathcal{MCG}(S)$ constructed in Proposition 4.2. As a consequence, the normalizer $N(\Gamma)$ of Γ is undistorted. ☐

Remark: 1) Since splitting sequences define quasi-geodesics in the *curve graph* of a surface of finite type [H06], the above argument immediately implies the following. Let S be a closed surface and let Γ be a finite subgroup of $\mathcal{MCG}(S)$. Then there is a quasi-isometric embedding of the curve graph of S/Γ into the curve graph of S. This was shown in [RS07].

2) In [ALS08], Aramayona, Leininger and Souto constructed for infinitely many $g_i > 0$ injective homomorphisms of the mapping class group of a closed surface of genus g_i into the mapping class group of a closed surface of strictly bigger genus

using unbranched coverings. The reasoning in the proof of Proposition 4.3 can be used to show that these homomorphisms are quasi-isometric embeddings.

5. Distances in the train track complex

In this section we use the results from Section 3 and from [H09] to obtain a control on distances in the train track complex $\mathcal{TT}$. The main goal is the proof of Proposition 5.10 which is the key ingredient for the proof of Theorem 1 from the introduction.

Our strategy is to investigate the geometry of triangles in the train track complex whose sides are bounded extensions of directed edge-paths. There may be many such triangles with given vertices, but we show that we can always find such a triangle which is R-thin for a universal constant $R > 0$. This means that a side of this triangle is contained in the R-neighborhood of the union of the other two sides. This in turn is related to a result of Behrstock, Drutu and Sapir [BDS08] who showed that the asymptotic cone of the mapping class group admits the structure of a median space.

The first step in this direction is the following lemma whose proof uses the results from the appendix. For its proof and for later use, we say that a train track or a geodesic lamination λ *hits a train track* τ *efficiently* if up to isotopy there is no embedded bigon in S whose frontier is composed of two C^1-segments, one from the train track τ and the second one from the train track or geodesic lamination λ. If the train track σ hits the train track τ efficiently then every train track or geodesic lamination which is carried by σ hits τ efficiently as well.

Lemma 5.1. *For every $R > 0$ there is a number $\beta_0 = \beta_0(R) > 0$ with the following property. Let $\tau, \eta \in \mathcal{V}(\mathcal{TT})$ with $d(\tau, \eta) \le R$ and let τ', η' be complete train tracks which can be obtained from τ, η by any splitting sequence. If τ, η do not carry any common geodesic lamination then there is a train track $\tau'' \in \mathcal{V}(\mathcal{TT})$ in the $\beta_0(R)$-neighborhood of τ', a train track $\eta'' \in \mathcal{V}(\mathcal{TT})$ in the $\beta_0(R)$-neighborhood of η' and a splitting sequence connecting τ'' to η'' which passes through the $\beta_0(R)$-neighborhood of η. Moreover, for any complete geodesic lamination ν which is carried by η', we can assume that ν is carried by η''.*

Proof. Fix a complete hyperbolic metric g on S of finite volume. For every $R > 0$ there are only finitely many orbits under the action of the mapping class group of pairs $(\tau, \eta) \in \mathcal{V}(\mathcal{TT}) \times \mathcal{V}(\mathcal{TT})$ where $d(\tau, \eta) \le R$ and such that τ, η do not carry any common geodesic lamination. Thus by invariance under the mapping class group it is enough to show the lemma for two fixed train tracks $\tau, \eta \in \mathcal{V}(\mathcal{TT})$ which do not carry any common geodesic lamination and with a constant $\beta_0 > 0$ depending on τ, η.

For a complete train track ξ denote by $\mathcal{CL}(\xi)$ the set of all complete geodesic laminations which are carried by ξ. By Lemma 2.3 of [H09], the set $\mathcal{CL}(\xi)$ is open and closed in the space $\mathcal{CL}$ of all complete geodesic laminations on S equipped with the Hausdorff topology. We first show that there are finitely many complete train tracks $\tau_1, \dots, \tau_\ell$ and $\eta_1, \dots, \eta_m$ with the following properties.

(1) For each $i \leq \ell$ the train track τ_i is carried by τ and $\cup_i \mathcal{CL}(\tau_i) = \mathcal{CL}(\tau)$.
(2) For each $j \leq m$ the train track η_j is carried by η and $\cup_j \mathcal{CL}(\eta_j) = \mathcal{CL}(\eta)$.
(3) For all $i \leq \ell, j \leq m$ the train tracks τ_i, η_j hit efficiently.

Since τ, η do not carry any common geodesic lamination, every complete geodesic lamination $\lambda \in \mathcal{CL}(\tau)$ intersects every complete geodesic lamination $\mu \in \mathcal{CL}(\eta)$ transversely. Namely, a complete geodesic lamination decomposes the surface S into ideal triangles and once punctured monogons. Thus if ℓ is any simple geodesic on S whose closure in S is compact and if ℓ does *not* intersect the complete geodesic lamination μ transversely then ℓ is contained in μ and hence the closure of ℓ is a sublamination of μ. Now if ℓ is a leaf of the complete geodesic lamination λ then the closure of ℓ is a sublamination of λ as well. Since λ is carried by τ and μ is carried by η, this violates the assumption that τ, η do not carry a common geodesic lamination.

Recall from Section 2 the definition of the straightening of a train track τ with respect to the complete hyperbolic metric g. For a number $a > 0$ call a train track τ *a-long* if the length of each edge from its straightening ist at least a. Recall also from Section 2 the definition of a train track which ϵ-follows a complete geodesic lamination λ for some $\epsilon > 0$. For a fixed number $a > 0$, a trainpath on an a-long train track τ_ϵ which ϵ-follows λ is a uniform quasi-geodesic. As $\epsilon \to 0$, these quasi-geodesics converge locally uniformly to leaves of λ.

For a fixed geodesic lamination $\lambda \in \mathcal{CL}(\tau)$, the compact space $\mathcal{CL}(\eta)$ consists of geodesic laminations which intersect $\lambda \in \mathcal{CL}(\tau)$ transversely. Thus the tangent lines of the leaves of λ and the union of all tangent lines of all leaves of all geodesic laminations in $\mathcal{CL}(\eta)$ are disjoint compact subsets of the projectivized tangent bundle of S. As a consequence, for any fixed number $a > 0$ there is a number $\epsilon > 0$ depending on λ such that every a-long train track ξ which ϵ-follows λ hits every geodesic lamination $\nu \in \mathcal{CL}(\eta)$ efficiently. By Lemma 2.2 and Lemma 2.3 of [H09], this implies that for every complete geodesic lamination $\lambda \in \mathcal{CL}(\tau)$ there is a train track $\tau(\lambda) \in \mathcal{V}(\mathcal{TT})$ which carries λ, which is carried by τ and which hits every geodesic lamination $\nu \in \mathcal{CL}(\eta)$ efficiently. By Lemma 2.3 of [H09], $\mathcal{CL}(\tau(\lambda))$ is an open subset of $\mathcal{CL}(\tau)$. Since $\mathcal{CL}(\tau)$ is compact we conclude that there are finitely many geodesic laminations $\lambda_1, \dots, \lambda_\ell \in \mathcal{CL}(\tau)$ such that $\mathcal{CL}(\tau) = \cup_i \mathcal{CL}(\tau(\lambda_i))$. Write $\tau_i = \tau(\lambda_i)$.

Every geodesic lamination $\nu \in \mathcal{CL}(\eta)$ hits each of the train tracks τ_i efficiently. As a consequence, for any fixed number $a > 0$ there is a number $\epsilon > 0$ depending on ν such that every a-long train track which ϵ-follows ν hits each of the train tracks τ_i $(i \leq \ell)$ efficiently. As before, this implies that we can find a finite family $\eta_1, \dots, \eta_m \in \mathcal{V}(\mathcal{TT})$ of train tracks which are carried by η, which hit each of the train tracks τ_i $(i \leq \ell)$ efficiently and such that $\cup_j \mathcal{CL}(\eta_j) = \mathcal{CL}(\eta)$. This shows the above claim.

Let

$$k = \max\{d(\tau, \tau_i), d(\eta, \eta_j) \mid i \leq \ell, j \leq m\}. \tag{5}$$

Let τ', η' be obtained from τ, η by a splitting sequence and let $\lambda \in \mathcal{CL}(\tau')$ be a complete geodesic lamination carried by τ'. Then $\lambda \in \mathcal{CL}(\tau)$ and hence there is some $i \leq \ell$ such that $\lambda \in \mathcal{CL}(\tau_i)$. By Lemma 6.7 of [H09], there is a number $p_3(k) > 0$ only depending on k and there is a complete train track ξ which carries λ, is carried by both τ' and τ_i and such that

$$d(\tau', \xi) \leq p_3(k). \tag{6}$$

Similarly, for a complete train track η' which can be obtained from η by a splitting sequence and for any $\nu \in \mathcal{CL}(\eta')$ there is some $j \leq m$ and a complete train track ζ which carries ν, is carried by both η' and η_j and such that

$$d(\eta', \zeta) \leq p_3(k). \tag{7}$$

Since τ_i, η_j hit efficiently and τ_i carries ξ, the train track ξ hits the train track η_j efficiently. In particular, the geodesic lamination ν (which is carried by η_j) hits ξ efficiently. Therefore by Proposition A.4 from the appendix, there is a number $p > 0$ not depending on ξ, η_j and there is a ν-collapse ξ^* of the dual bigon track ξ_b^* of ξ with the following property. ξ^* carries a train track β which carries ν and which can be obtained from η_j by a splitting and shifting sequence of length at most p. In particular, we have

$$d(\eta_j, \beta) \leq \kappa \tag{8}$$

where $\kappa > 0$ is a universal constant.

Since η_j carries ζ and ζ carries ν, we conclude from Lemma 6.7 of [H09] that β carries a train track σ which carries ν and such that $d(\sigma, \zeta) \leq p_3(\kappa)$. Together with the estimate (7) above, this implies that

$$d(\sigma, \eta') \leq p_3(k) + p_3(\kappa). \tag{9}$$

Now $d(\tau', \xi) \leq p_3(k)$ by inequality (6) and therefore the distance between ξ^* and τ' is uniformly bounded (compare Proposition A.4 and the following remark). On the other hand, since ξ^* carries β and β carries ν, Proposition A.6 of [H09] shows that ξ^* is splittable to a train track β' which carries ν and is contained in a uniformly bounded neighborhood of β. Since $d(\beta, \eta) \leq k + \kappa$ by inequalities (5,8), we conclude that $d(\beta', \eta)$ is uniformly bounded. Now β carries σ and both β' and σ carry ν and hence another application of Lemma 6.7 of [H09] shows that β' is splittable to a train track σ' which carries ν and is contained in a uniformly bounded neighborhood of σ. Since by inequality (9) the distance between σ and η' is uniformly bounded, this implies the lemma with $\tau'' = \xi^*$ and $\eta'' = \sigma'$. □

Our next goal is to establish an extension of Lemma 5.1 to train tracks $\tau, \eta \in \mathcal{V}(\mathcal{TT})$ containing a common subtrack β which is a union of simple closed curves and such that every minimal geodesic lamination carried by both τ, η is a component of β. The main idea is as follows. If c is an embedded simple closed curve of class C^1 in a complete train track τ then a sequence of c-splits of τ results in modifying τ by a sequence of Dehn twists about c up to a uniformly bounded error. The direction of the twist (positive or negative) is determined by a complete geodesic lamination which is carried by τ and contains c as a minimal component. If η is another complete train track containing c as an embedded simple closed curve and if c is the only geodesic lamination which is carried by both τ, η then we find a version of Lemma 5.1 for the images of τ, η under a suitably chosen multi-twist about c.

We first need a precise control of the distance between a train track obtained from a train track τ by a sequence of c-splits and the image of τ under a suitably chosen multi-twist about c. For this consider for the moment an arbitrary train track τ and let $\rho : [0,k] \to \tau$ be any trainpath which either is embedded in τ or is such that $\rho[0,k-1)$ is embedded, and $\rho(k-1) = \rho(0), \rho(k) = \rho(1)$. For $1 \leq i \leq k-1$ call the switch $\rho(i)$ of τ *incoming* if the half-branch $\rho[i-1/2,i]$ is small at $\rho(i)$, and call the switch $\rho(i)$ *outgoing* otherwise. Note that this depends on the orientation of ρ. We call a neighbor of $\rho[0,k]$ in τ (i.e. a half-branch of τ which is incident on a switch in $\rho[1,k-1]$ but which is not contained in $\rho[0,k]$) incoming if it is incident on an incoming switch, and we call the neighbor outgoing otherwise. Call the switch $\rho(i)$ a *right* (or *left*) switch if with respect to the orientation of S and the orientation of ρ, the neighbor of ρ incident on $\rho(i)$ lies to the right (or left) of ρ in a small neighborhood of $\rho[0,k]$ in S. A neighbor incident on a right (or left) switch is called a *right* (or *left*) neighbor of $\rho[0,k]$. Again this depends on the orientation of ρ.

In the sequel we mean by an embedded simple closed curve c in a train track τ a subtrack of τ which is an embedded simple closed curve freely homotopic to c. Call an embedded simple closed curve c in a train track τ *reduced* if for one (and hence every) trainpath $\rho : [0,k] \to \tau$ with image c and $\rho[0,1] = \rho[k-1,k]$, there are both left and right neighbors of c in τ and either all left neighbors of $c = \rho[0,k]$ in τ are incoming and all right neighbors are outgoing, or all left neighbors are outgoing and all right neighbors are incoming. Note that if τ is recurrent then for any embedded simple closed curve c in τ which is not isolated (i.e. not a component of τ) the following holds true. If with respect to an orientation of c and the orientation of S all left neighbors of c in τ are incoming (or outgoing), then at least one of the right neighbors of c in τ is outgoing (or incoming).

We call a reduced simple closed curve c in τ *positive* if the following holds true. Let S_c be the bordered surface with two boundary circles obtained by cutting S open along c. The orientation of S induces a boundary orientation for each of the two boundary components of S_c. Then c is positive if with respect to this boundary orientation, neighbors which lie to the left of c are incoming. Note that this holds true for both boundary components if it holds true for one. A reduced simple closed curve which is not positive is called *negative*.

A Dehn twist about an essential simple closed curve c in S is defined to be *positive* if the direction of the twist coincides with the direction given by the boundary orientation of c in the surface obtained by cutting S open along c. The following observation gives a quantitative version of the idea that for an embedded reduced simple closed curve c in a train track τ, a sequence of c-splits of τ results in twisting of τ about c.

Lemma 5.2. *Let c be a reduced positive (or negative) embedded simple closed curve in a train track τ and let θ_c be the positive Dehn twist about c.*

(1) *There is a sequence of c-splits of uniformly bounded length which transforms τ to $\theta_c\tau$ (or $\theta_c^{-1}\tau$).*
(2) *There is a number $a_1 > 0$ not depending on τ and c and for every train track η obtained from τ by a sequence of c-splits there is some $i \geq 0$ such that $d(\eta, \theta_c^i\tau) \leq a_1$ (or $d(\eta, \theta_c^{-i}\tau) \leq a_1$).*

Proof. Let c be an embedded reduced simple closed curve in τ. Assume for the purpose of exposition that c is positive (the case of a negative curve is completely analogous). Let $\rho : [0, \ell] \to \tau$ be a parametrization of c as a trainpath with $\rho(\ell) = \rho(0)$. Let $0 \leq i_1 < \cdots < i_k$ be such that the switches $s_j = \rho(i_j)$ are outgoing and that the switches $\rho(j)$ for $j \notin \{i_1, \ldots, i_k\}$ are incoming. Let moreover $b_1, \ldots, b_p$ $(p = \ell - k)$ be the incoming neighbors of c, ordered consecutively along ρ. By assumption, these are precisely the left neighbors of c in τ with respect to the orientation defined by ρ.

Since there are both incoming and outgoing switches along c, there is a large branch contained in c. If $\rho[j, j+1]$ is a large branch in c then the neighbor b of c at $\rho(j)$ is incoming and the neighbor of c at $\rho(j+1)$ is outgoing. Up to isotopy, the c-split of τ at $\rho[j, j+1]$ moves the neighbor b across $\rho(j+1)$ preserving the orders of the right neighbors and the left neighbors of c, respectively, and preserving the types of the neighbors. This means that in the split track η, for every $i \leq p$ the half-branch $\varphi(\tau, \eta)(b_i)$ is an incoming left neighbor of c, and the neighbors $\varphi(\tau, \eta)(b_i)$ are aligned along c in consecutive order as $1 \leq i \leq p$. Only their relative positions to the switches s_j have changed. The image $\theta_c(\tau)$ of τ under the positive Dehn twist θ_c about c is the train track obtained from τ by moving each of the left neighbors of c one full turn about c while leaving the right neighbors fixed. Moreover, $\theta_c(\tau)$ is carried by τ. There is a carrying map $\theta_c(\tau) \to \tau$ which is the identity away from a small annular neighborhood of c in S and which maps every left neighbor of c in $\theta_c(\tau)$ onto the union of a left neighbor of c in τ with the simple closed curve c, traveled through precisely once.

Now let η be a train track which carries $\theta_c(\tau)$ and is obtained from τ by a sequence of c-splits of maximal length with this property. This means that none of the neighbors b_i of τ is moved more that one full turn about c, measured with respect to the position of this neighbor relative to the switches s_j $(1 \leq j \leq k)$ (which we consider fixed). We claim that $\eta = \theta_c(\tau)$. Namely, let e be a large branch of η contained in c and let b be an incoming neighbor of c in τ such that $\varphi(\tau, \eta)(b)$ is the incoming neighbor of c at an endpoint of e. Let moreover s_j be the outgoing switch on which e is incident. If the switch of τ on which b is incident is distinct from $\rho(i_j - 1)$ then b moved less than a full turn about c in the splitting process. Therefore the train track obtained from η by a c-split at e carries $\theta_c(\tau)$ which violates the assumption hat no c-split of η carries $\theta_c(\tau)$.

As a consequence, with respect to the parametrization ρ' of c in η with $\rho'(i_1) = s_1$ which defines the same orientation of c as ρ, the branch $\rho'[j, j+1]$ is large if and only if this holds true for the branch $\rho[j, j+1]$. Moreover, the neighbor of ρ' at $\rho'(j), \rho'(j+1)$ is just $\varphi(\tau, \eta)(b)$ where b is the neighbor of c at $\rho(j), \rho(j+1)$. Since the orders of the incoming and outgoing neighbors along c are preserved in the splitting process, we necessarily have $\eta = \theta_c(\tau)$ as claimed.

To show the second part of the lemma, let η be a train track obtained from τ by a sequence of c-splits and let b be an incoming neighbor of c in τ which crossed the maximal number ℓ of outgoing switches in a splitting sequence connecting τ to η. If as before k is the number of outgoing switches along c and if $\ell \in [ki+1, k(i+1)]$ then b made at least i full turns about c and at most $i+1$ full turns. Since the orders of the incoming neighbors of c is preserved in the process, each of the remaining left neighbors made at least $i-1$ full turns about c and at most $i+1$ full turns. By the first part of this proof, this means that η can be transformed with a uniformly bounded number of splits to the train track $\theta_c^{i+1}\tau$. Then $d(\eta, \theta_c^{i+1}\tau)$ is uniformly bounded. The second part of the lemma follows. □

For an embedded simple closed curve c in a complete train track τ which is not reduced we have the following

Lemma 5.3. *There is a number $a_2 > 0$ with the following property. Let c be an embedded simple closed curve in a complete train track τ. Let λ be a complete geodesic lamination which is carried by τ and which contains c as a minimal component. Then there is a train track τ' with the following properties.*

(1) *τ' can be obtained from τ by a sequence of c-splits of length at most a_2.*
(2) *τ' carries λ.*
(3) *The curve c in τ' is reduced.*

Proof. Let q be the number of branches of a complete train track τ. We show by induction on the number n of switches of τ contained in the simple closed curve c that the lemma holds true for a modification of τ with at most $n(q^2+1)$ c-splits.

If $n = 2$ then c is reduced since τ is complete, so assume that the lemma is known whenever the number of switches of τ contained in c does not exceed $n-1$ for some $n \geq 3$.

Let c be a simple closed curve which is a subtrack of a complete train track τ and such that the number of switches of τ contained in c equals n. Assume that c is *not* reduced. Then with respect to an orientation of c, there are two neighbors b_1, b_2 of c in τ which lie on the same side of c in an annular neighborhood of c in S and such that with respect to some orientation of c, b_1 is incoming and b_2 is outgoing. Let $\rho : [0, \ell] \to \tau$ be a trainpath which parametrizes the oriented subarc of c connecting the switch $\rho(0)$ on which b_1 is incident to the switch $\rho(\ell)$ on which b_2 is incident. Then $\rho[0, \ell]$ is embedded and begins and ends with a large half-branch.

Let λ be a complete geodesic lamination carried by τ which contains c as a minimal component. By Lemma 3.4 and the following remark, there is a sequence of ρ-splits of length at most q^2 which modifies τ to a complete train track τ_1 with the following properties. τ_1 contains c as an embedded simple closed curve, it carries λ, and there is a large branch e in τ_1 contained in c which is of type 2 (here the type of a proper subbranch of c is defined as in Section 3). The λ-split of τ_1 at e is a complete train track τ_2 which contains c as an embedded simple closed curve and such that the number of switches of τ_2 contained in c does not exceed $n-1$. It is obtained from τ by a sequence of c-splits of length at most q^2+1. Thus we can now use the induction hypothesis to complete the proof of the lemma. □

Recall from Section 2 the definition of the cubical Euclidean cone $E(\tau,\lambda)$ for a complete train track τ and a complete geodesic lamination λ carried by τ. From Lemma 5.2 and Lemma 5.3 we obtain

Corollary 5.4. *There is a number $a_3 > 0$ with the following property. Let $\tau \in \mathcal{V}(\mathcal{TT})$, let $c < \tau$ be an embedded simple closed curve and let θ_c be the positive Dehn twist about c. Let $\lambda \in \mathcal{CL}$ be a complete geodesic lamination which is carried by τ and contains c as a minimal component and let $\{\tau_i\}_{0\leq i\leq k} \subset E(\tau,\lambda)$ be a splitting sequence issuing from $\tau_0 = \tau$ which consists of c-splits at large proper subbranches of c. Then there is some $i \in \mathbb{Z}$ such that*

$$d(\tau_k, \theta_c^i\tau) \leq a_3.$$

To use Corollary 5.4 for an extension of Lemma 5.1 we have to overcome a technical difficulty arising from the fact that for a given complete train track τ and a large branch e of τ, there may be a choice of a right or left split of τ at e so that the split track is not recurrent. We call such a large branch *rigid*. Note that e is a rigid large branch of τ if and only if the (unique) complete train track obtained from τ by a split at e carries *every* complete geodesic lamination which is carried by τ. It follows from Lemma 2.1.3 of [PH92] that a branch e in τ is rigid if and only if the train track ζ obtained from τ by a collision at e, i.e. a split at e followed by the removal of the diagonal of the split, is *not* recurrent.

Lemma 5.5. *There is a number $a_4 > 0$ with the following property. For every $\eta \in \mathcal{V}(\mathcal{TT})$ there is a splitting sequence $\{\eta(i)\}_{0\leq i\leq s} \subset \mathcal{V}(\mathcal{TT})$ issuing from $\eta = \eta(0)$ of length $s \leq a_4$ such that for every i, $\eta(i+1)$ is obtained from $\eta(i)$ by a split at a rigid large branch and such that $\eta(s)$ does not contain any rigid large branches.*

Proof. We show the existence of a number $a_4 > 0$ as in the lemma with an argument by contradiction. Assume to the contrary that the lemma does not hold true. Then there is a sequence of pairs $(\beta_i, \xi_i) \in \mathcal{V}(\mathcal{TT}) \times \mathcal{V}(\mathcal{TT})$ such that ξ_i can be obtained from β_i by a splitting sequence of length at least i consisting of splits at rigid large branches. Every complete geodesic lamination which is carried by β_i is also carried by ξ_i.

By invariance under the action of the mapping class group and the fact that there are only finitely many orbits for the action of $\mathcal{MCG}(S)$ on $\mathcal{V}(\mathcal{TT})$, by passing to a subsequence we may assume that there is some $\eta \in \mathcal{V}(\mathcal{TT})$ such that $\beta_i = \eta$ for all i. Since η has only finitely many large branches, with a standard diagonal procedure we can construct from (η, ξ_i) an *infinite* splitting sequence $\{\eta(i)\}_{i\geq 0} \subset \mathcal{V}(\mathcal{TT})$ issuing from $\eta = \eta(0)$ such that for every i the train track $\eta(i+1)$ is obtained from $\eta(i)$ by a split at a rigid large branch. Then for every i, every complete geodesic lamination which is carried by η is also carried by $\eta(i)$. Now for every *projective measured geodesic lamination* whose support ν is carried by η there is a complete geodesic lamination λ carried by η which contains ν as a sublamination (see the discussion in the proof of Lemma 2.3 of [H09] and the discussion in Section 3 of this paper). Thus for each i, the space $\mathcal{PM}(i)$ of projective measured geodesic laminations carried by $\eta(i)$ coincides with the space $\mathcal{PM}(0)$ of projective measured geodesic laminations carried by η. Since η is complete. the set $\mathcal{PM}(0)$ contains an open subset of the space of all projective measured geodesic laminations (Lemma 2.1.1 of [PH92]). Therefore $\cap_i \mathcal{PM}(i) = \mathcal{PM}(0)$ contains an open subset of projective measured geodesic lamination space which contradicts Theorem 8.5.1 of [M03]. This shows the lemma. □

A multi-twist θ about a simple geodesic multi-curve $c = \cup_i c_i$ is a mapping class which can be represented in the form $\theta = \theta_{c_1}^{m_1} \circ \cdots \circ \theta_{c_k}^{m_k}$ for some $m_i \in \mathbb{Z}$ and where as before, θ_{c_i} is the positive Dehn twist about c_i. The next lemma is a version of Lemma 5.1 for train tracks which carry a common multi-curve. As in Section 3, for a train track $\tau \in \mathcal{V}(\mathcal{TT})$ which is splittable to a train track $\tau' \in \mathcal{V}(\mathcal{TT})$ let $E(\tau, \tau')$ be the maximal subgraph of $\mathcal{TT}$ whose set of vertices consists of all train tracks which can be obtained from τ by a splitting sequence and which are splittable to τ'.

Lemma 5.6. *For every $R > 0$ there is a number $\beta_1(R) > 0$ with the following property. Let $\tau, \eta \in \mathcal{V}(\mathcal{TT})$ with $d(\tau, \eta) \leq R$. Assume that τ, η contain a common subtrack which is a multi-curve c and that the components of c are precisely the minimal geodesic laminations which are carried by both τ and η. Let τ', η' be complete train tracks which can be obtained from τ, η by a splitting sequence. Then there is a multi-twist θ about c with the following property. There is a train track τ'' in the $\beta_1(R)$-neighborhood of τ', a train track η'' in the $\beta_1(R)$-neighborhood of η' and a splitting sequence connecting τ'' to η'' which passes through the $\beta_1(R)$-neighborhood of $\theta(\eta)$. For any complete geodesic lamination ν which is carried by η', we can assume that ν is carried by η''. Moreover, $d(\theta(\tau), E(\tau, \tau')) \leq a_5, d(\theta(\eta), E(\eta, \eta')) \leq a_5$ for a universal constant $a_5 > 0$.*

Proof. As in the proof of Lemma 5.1, by invariance under the mapping class group it suffices to show the lemma for some fixed train tracks $\tau, \eta \in \mathcal{V}(\mathcal{TT})$ which satisfy the assumptions in the lemma with a number $\beta_1 > 0$ depending on τ, η.

Thus let τ, η be train tracks containing a common subtrack c which is a multi-curve. Assume that every minimal geodesic lamination which is carried by both τ and η is a component of c. By Lemma 5.1, we may assume that c is not empty.

Assume that τ is splittable to a train track τ' and that η is splittable to a train track η'. The strategy is now as follows. By Corollary 5.4, splitting at large proper subbranches of c results in twisting about the components of c. We first determine the direction of the twist and the amount of twisting about each of the components of c for train tracks $\tilde{\tau} \in E(\tau, \tau')$ and $\tilde{\eta} \in E(\eta, \eta')$ which are obtained from τ, η by a sequence of c-splits of maximal length. Then we compare these data and determine a maximal multi-twist θ about the components of c with the property that there are train tracks $\hat{\tau} \in E(\tau, \tau'), \hat{\eta} \in E(\eta, \eta')$ which are contained in a uniformly bounded neighborhood of $\theta(\tau), \theta(\eta)$. Here by a maximal multi-twist we mean an element of maximal distance to the identity in the free abelian group generated by the Dehn twists about the components of c with respect to the distance function induced by the standard word metric. Finally we show that up to replacing $\hat{\tau}, \hat{\eta}$ and τ', η' by their images under suitably chosen splitting sequences of uniformly bounded length, we may assume that $\hat{\tau}, \tau'$ and $\hat{\eta}, \eta'$ satisfy the assumptions in Lemma 5.1.

To carry out this strategy, note first that by Lemma 5.5, via replacing τ',η' by their images under a splitting sequence of uniformly bounded length we may assume that τ',η' do not contain any rigid large branch.

Let $c_1,\dots,c_k$ be the components of c. To count the amount of twisting about a component of c along a splitting sequence connecting τ to τ' we define inductively a sequence $\{\tilde\tau(i)\}_{0\leq i\leq k}\subset E(\tau,\tau')$ consisting of train tracks $\tilde\tau(i)$ which contain c as a subtrack as follows. Put $\tilde\tau(0)=\tau$ and assume that $\tilde\tau(i-1)$ has been defined for some $i\in\{1,\dots,k\}$. Let $\tilde\tau(i)\in E(\tau,\tau')$ be the train track which can be obtained from $\tilde\tau(i-1)$ by a sequence of c_i-splits of maximal length at large subbranches of c_i. Since the components c_i,c_j of c are disjoint, splits at large subbranches of c_i,c_j for $i\neq j$ commute. Therefore the train track $\tilde\tau(k)$ only depends on τ,τ',c but not on the ordering of the components c_i of c.

By Corollary 5.4, there are numbers $b_i\in\mathbb{Z}$ such that for $\theta_{\tau,\tau'}=\theta_{c_1}^{b_1}\circ\dots\circ\theta_{c_k}^{b_k}\in\mathcal{MCG}(S)$ we have

$$d(\tilde\tau(k),\theta_{\tau,\tau'}(\tau))\leq ka_3. \tag{10}$$

Similarly, there are numbers $p_i\in\mathbb{Z}$ such that for $\theta_{\eta,\eta'}=\theta_{c_1}^{p_1}\circ\dots\circ\theta_{c_k}^{p_k}\in\mathcal{MCG}(S)$ and the train track $\tilde\eta(k)$ obtained from η and η' by the above procedure we have

$$d(\tilde\eta(k),\theta_{\eta,\eta'}(\eta))\leq ka_3. \tag{11}$$

By Lemma 5.3, we may choose $b_i=0$ (or $p_i=0$) if there is no train track which can be obtained from τ (or η) by a sequence of c_i-splits and which contains c_i as a reduced simple closed curve.

For $i\leq k$ define $m(i)=0$ if either the signs of b_i,p_i are distinct or if $b_i=0$ or $p_i=0$. If the signs of b_i,p_i coincide then define $m(i)=\operatorname{sgn}(b_i)\min\{|b_i|,|p_i|\}$. Write $\theta=\theta_{c_1}^{m(1)}\circ\dots\circ\theta_{c_k}^{m(k)}$. Our goal is to show that the lemma holds true for this multi-twist θ. Note that by Corollary 5.4, we have

$$d(\theta(\tau),E(\tau,\tau'))\leq ka_3,\ d(\theta(\eta),E(\eta,\eta'))\leq ka_3.$$

The remainder of the proof is divided into two steps. For convenience of notation, if $b_i=0$ or if $p_i=0$ then we write $\operatorname{sgn}(b_i)=\operatorname{sgn}(p_i)$.

Step 1: Reduction to the case that $\operatorname{sgn}(b_i)\neq\operatorname{sgn}(p_i)$ for all i.

After reordering we may assume that there is some $s\leq k$ such that $\operatorname{sgn}(b_i)\neq\operatorname{sgn}(p_i)$ for $i\leq s$ and that $\operatorname{sgn}(b_i)=\operatorname{sgn}(p_i)$ for $i\geq s+1$. Choose train tracks $\tau_1\in E(\tau,\tau'),\eta_1\in E(\eta,\eta')$ contained in the ka_3-neighborhood of $\theta(\tau),\theta(\eta)$, which contain the multi-curve c as a subtrack and such that for each $i\geq s+1$ the following holds true. If $|b_i|\leq|p_i|$ then the cubical euclidean cone $E(\tau,\tau')$ does not contain any train track which can be obtained from τ_1 by a c_i-split at any large subbranch of c_i in τ_1, and similarly for η_1 in the case that $|p_i|\leq|b_i|$.

Every minimal geodesic lamination which is carried by both τ_1,η_1 is a component of c. Moreover,

$$d(\tau_1,\theta(\tau))\leq ka_3,\ d(\eta_1,\theta(\eta))\leq ka_3 \tag{12}$$

and therefore by invariance of the distance function on $\mathcal{TT}$ under the action of the mapping class group we have

$$d(\tau_1,\eta_1)\leq d(\tau,\eta)+2ka_3. \tag{13}$$

Let $i\geq s+1$ and assume without loss of generality that $|b_i|\leq |p_i|$. Then for each large proper subbranch e of c_i in τ_1, the train track obtained from τ_1 by a c_i-split at e is *not* contained in the cubical euclidean cone $E(\tau,\tau')$.

Assume first that τ_1 contains a large proper subbranch e of c_i of type 1 as defined in Section 3. Note that this is for example the case if c_i is reduced in τ_1. There is a unique choice of a split of τ_1 at e such that the split track $\hat\tau_1$ does *not* carry c_i. We claim that the train track $\hat\tau_1$ is complete. Namely, either we have $\hat\tau_1\in E(\tau,\tau')$, in particular $\hat\tau_1$ is complete, or no train track which can be obtained from τ_1 by a split at e is splittable to τ'. In the second case, $\varphi(\tau_1,\tau')(e)$ is a large branch of τ' by uniqueness of splitting sequences (Lemma 5.1 of [H09]). Since τ' does not contain any rigid large branch by assumption, the train tracks which are obtained from τ' by a single right or left split at $\varphi(\tau_1,\tau')(e)$ are both complete. As a consequence of uniqueness of splitting sequences, the train track $\hat\tau_1$ is splittable to a complete train track which is obtained from τ' by a single right or left split at $\varphi(\tau_1,\tau')(e)$. Thus the train track $\hat\tau_1$ is complete and does not carry the simple closed curve c_i. Moreover, $\hat\tau_1$ is splittable to a train track obtained from τ' by at most one split.

If every large proper subbranch of c_i in τ_1 is of type 2 then no train track obtained from τ_1 by a split at a large branch in c_i is splittable to τ'. Then $\varphi(\tau_1,\tau')(c_i)$ is just the embedded simple closed curve c_i in τ', and $\varphi(\tau_1,\tau')(c_i)$ does not contain any proper large subbranch of type 1. By Lemma 5.3 and Lemma 5.5, the image ξ of τ' under a (suitably chosen) splitting sequence of uniformly bounded length is complete, it contains c_i as an embedded reduced curve, and it does not contain any rigid large branch. Then we can use the consideration in the previous paragraph with the pair $\tau'\prec\tau$ replaced by $\xi\prec\tau$.

To summarize, up to replacing τ' by its image under a splitting sequence of uniformly bounded length we may assume that there is a train track $\hat\tau_1\in E(\tau,\tau')$ in a uniformly bounded neighborhood of $\theta(\tau)$ which does not carry the simple closed curve c_i. In particular, a minimal geodesic lamination which is carried by both τ_1 and η is a component c_j of c for some $j\neq i$.

Since splits at large proper subbranches of the distinct components of c commute, we construct in this way successively in $k-s$ steps from the train tracks τ_1,η_1 complete train tracks τ_2,η_2 with the following properties. τ_2,η_2 can be obtained from τ_1,η_1 by a splitting sequence of uniformly bounded length. In particular, by the estimate (13) and (12) there is a universal constant $a>0$ such that

$$\begin{aligned} &d(\tau_2,\eta_2)\leq d(\tau,\eta)+a \text{ and}\\ &d(\tau_2,\theta(\tau))\leq a, d(\eta_2,\theta(\eta))\leq a.\end{aligned} \tag{14}$$

The train tracks τ_2,η_2 contain the simple closed curves $c_1,\dots,c_s$ as embedded subtracks, and they are splittable to train tracks τ_2',η_2' which can be obtained from τ',η' by splitting sequences of uniformly bounded length. A minimal geodesic lamination which is carried by both τ_2,η_2 coincides with one of the curves c_i for

$i \leq s$. This shows that via replacing τ, η by τ_2, η_2 and replacing τ', η' by τ_2', η_2' we may assume that $\text{sgn}(b_i) \neq \text{sgn}(p_i)$ for all i.

Step 2: The case $\text{sgn}(b_i) \neq \text{sgn}(p_i)$ for all i.

By the choice of b_i, p_i and by Lemma 5.3, there are train tracks $\tau_0 \in E(\tau, \tau'), \eta_0 \in E(\eta, \eta')$ which are obtained from τ, η by a splitting sequence of length at most ka_2 and such that τ_0, η_0 contain each of the components c_i of c $(i \leq k)$ as a reduced simple closed curve. By Lemma 5.2, for each i the train track τ_0 is splittable to $\theta_{c_i}^{b_i}\tau_0$. Since $\text{sgn}(b_i) \neq \text{sgn}(p_i)$, for each i the train track $\theta_{c_i}^{b_i}\eta_0$ is splittable to η_0.

Let τ_1 be the train track which can be obtained from τ_0 by a sequence of $\cup_{i=1}^k c_i$-splits of maximal length and which is splittable to τ'. By the definition of the multiplicities b_i, inquality (10) above shows that for $\psi = \theta_{c_1}^{b_1} \circ \cdots \circ \theta_{c_k}^{b_k}$ we have

$$d(\psi(\tau_0), \tau_1) \leq ka_3.$$

The train track $\psi(\eta_0)$ is splittable to η_0. By invariance under the action of the mapping class group, a minimal geodesic lamination carried by both τ_1 and $\psi(\eta_0)$ is a component of c.

As in Step 1 above, there is a train track τ_2 which is the image of τ_1 under a splitting sequence of uniformly bounded length, which is splittable to the image τ_2' of τ' under a splitting sequence of uniformly bounded length and such that τ_2 does not carry any of the curves c_i. Since ψ acts on $\mathcal{TT}$ as an isometry, we have

$$d(\tau_2, \psi(\eta_0)) \leq d(\tau, \eta) + a + 2ka_2$$

where $a > 0$ is a universal constant as in the estimate (14) in Step 1 of this proof. The train tracks $\psi(\eta_0)$ and τ_2 do not carry any common geodesic lamination. Moreover, $\psi(\eta_0)$ is splittable to η' with a splitting sequence which passes through η_0.

Let ν be a complete geodesic lamination which is carried by η'. By Lemma 5.1, applied to the train tracks $\tau_2, \psi(\eta_0)$ which are splittable to the train tracks τ_2', η', there is a number $\beta > 0$ only depending on $d(\tau_2, \psi(\eta_0))$ and hence on $d(\tau, \eta)$ and there is a splitting sequence $\{\alpha(i)\}_{0 \leq i \leq n}$ which connects a train track $\alpha(0)$ with

$$d(\alpha(0), \tau') \leq \beta$$

to a train track $\alpha(n)$ with

$$d(\alpha(n), \eta') \leq \beta$$

and which carries ν. Moreover, this splitting sequence passes through the β-neighborhood of $\psi(\eta_0)$ and hence it passes through a uniformly bounded neighborhood of $\psi(\eta)$.

Let $j \in \{0, \dots, n\}$ be such that

$$d(\alpha(j), \psi(\eta_0)) \leq \beta.$$

By Lemma 6.7 of [H09], applied to the train tracks $\alpha(j), \psi(\eta_0), \eta_0$ which carry the common complete geodesic lamination ν, there is a number $p_3(\beta) > 0$ and there is a train track ζ which carries ν, which is carried by both $\alpha(j)$ and η_0 and such that

$$d(\zeta, \eta_0) \leq p_3(\beta).$$

Proposition A.6 of [H09] then shows that $\alpha(j)$ is splittable to a train track ξ which is contained in a uniformly bounded neighborhood of ζ and hence η and which carries ν.

An application of Lema 6.7 of [H09] to the train tracks ξ, η_0, η' which carry ν produces a train track ζ' in a uniformly bounded neighborhood of η' which carries ν and is carried by ξ. Proposition A.6 of [H09] then shows that ξ is splittable to a train track ξ' which carries ν and is contained in a uniformly bounded neighborhood of η'. This shows that $\alpha(0)$ can be connected to ξ' by a splitting sequence which passes through a uniformly bounded neighborhood of η and completes the proof of the lemma. □

Finally we investigate splitting sequences issuing from complete train tracks τ, η which contain a common proper recurrent subtrack σ without closed curve components carrying every geodesic lamination which is carried by both τ, η. Our goal is to establish an analog of Lemma 5.6 in this case. To apply the above strategy we recall from Section 3 the definition of a sequence issuing from τ which is induced by a splitting sequence $\{\sigma_i\}$ of σ.

Lemma 5.7. *For every complete train track $\tau \in \mathcal{V}(\mathcal{TT})$ which is splittable to a complete train track τ' and for every subtrack σ of τ there is a unique train track $\xi \in E(\tau, \tau')$ with the following properties.*

1. *There is a splitting sequence $\{\sigma(i)\}_{0 \leq i \leq p}$ issuing from $\sigma(0) = \sigma$ such that ξ can be obtained from τ by a sequence induced by $\{\sigma(i)\}$.*
2. *If $\tilde{\tau} \in E(\tau, \tau')$ can be obtained from τ by a sequence induced by a splitting sequence of σ then $\tilde{\tau}$ is splittable to ξ.*

Proof. Let $\tau \in \mathcal{V}(\mathcal{TT})$ be a complete train track which is splittable to a train track $\tau' \in \mathcal{V}(\mathcal{TT})$ and let σ be a subtrack of τ. Recall that a *σ-split* of τ is a split of τ at a large proper subbranch of σ with the property that the split track contains σ as a subtrack. We proceed similarly to the procedure in the proof of Lemma 3.4.

The large branches $e_1, \dots, e_k$ of σ define pairwise disjoint embedded trainpaths in τ. For each i let $\tilde{\tau}_i$ be the train track obtained from τ by a sequence of σ-splits of maximal length at proper subbranches of e_i and with the additional property that $\tilde{\tau}_i \in E(\tau, \tau')$. By Lemma 3.4, $\tilde{\tau}_i$ is found in at most q^2 steps where $q > 0$ is the number of branches of a complete train track on S.

If $\tilde{\tau}_i$ is *not* tight at e_i then put a mark on the branch e_i in σ. If $\tilde{\tau}_i$ is tight at e_i then e_i is a large branch in $\tilde{\tau}_i$. In this case we put a mark on e_i if there is no train track contained in $E(\tau, \tau')$ which can be obtained from $\tilde{\tau}_i$ by a single split at e_i.

After reordering we may assume that there is some $s \leq k$ such that the branches $e_1, \dots, e_s$ are unmarked and that the branches $e_{s+1}, \dots, e_k$ are marked. If $s = 0$ then there is no train track which can be obtained from a sequence induced by any splitting sequence of σ and which is splittable to τ' and we define $\xi = \tau$. Otherwise let τ_1 be the train track obtained from τ by a sequence of σ-splits of maximal length at proper large subbranches of the branches $e_1, \dots, e_s$ and such that $\tau_1 \in E(\tau, \tau')$. Then τ_1 is tight at each of the branches $e_1, \dots, e_s$ of σ. Moreover, there is a train track $\tau_2 \in E(\tau,\tau')$ which can be obtained from τ_1 by a single split at each of the large branches e_i $(1 \leq i \leq s)$. The train track τ_2 contains a subtrack σ_1 which is obtained from σ by a single split at each of the branches $e_1,\dots,e_s$.

Repeat this procedure with the train track τ_2 and its subtrack σ_2. After finitely many steps we obtain a train track $\xi \in E(\tau,\tau')$ which clearly satisfies the requirements in the lemma. □

For a convenient formulation of the following lemma, we say that a train track η is splittable to a complete geodesic lamination λ if η carries λ.

Lemma 5.8. *Let $\tau \in \mathcal{V}(\mathcal{TT})$ and let $\mathcal{S}(\tau) \subset \mathcal{V}(\mathcal{TT})$ be the set of all complete train tracks which can be obtained from τ by a splitting sequence. Let $E(\tau,\eta)$ be a cubical euclidean cone where either $\eta \in \mathcal{S}(\tau)$ or η is a complete geodesic lamination carried by τ. Then there is a projection*

$$\Pi^1_{E(\tau,\eta)} : \mathcal{S}(\tau) \to E(\tau,\eta)$$

with the following property. For every $\zeta \in \mathcal{S}(\tau)$, $\Pi^1_{E(\tau,\eta)}(\zeta)$ is a train track which is splittable to both ζ,η, and there is no train track $\chi \in E(\Pi^1_{E(\tau,\eta)}(\zeta),\eta) - \Pi^1_{E(\tau,\eta)}(\zeta)$ with this property. Moreover, $\Pi^1_{E(\tau,\eta)}(\zeta) = \Pi^1_{E(\tau,\zeta)}(\eta)$ for all $\eta,\zeta \in \mathcal{S}(\tau)$.

Proof. Let $\tau \in \mathcal{V}(\mathcal{TT})$ and let $E(\tau,\eta)$ be a cubical euclidean cone where either $\eta \in \mathcal{V}(\mathcal{TT})$ is such that τ is splittable to η or where η is a complete geodesic lamination carried by τ. For $\zeta \in \mathcal{S}(\tau)$ we construct by induction on the length n of a splitting sequence connecting τ to ζ a projection point $\Pi^1_{E(\tau,\eta)}(\zeta) = \Pi^1_\eta(\zeta) \in E(\tau,\eta)$ with the required properties.

If $n = 0$, i.e. if $\tau = \zeta$, then we define $\Pi^1_\eta(\zeta) = \tau$. By induction, assume that for some $n \geq 1$ we determined for all $\tau \in \mathcal{V}(\mathcal{TT})$ which are splittable to some $\eta \in \mathcal{V}(\mathcal{TT}) \cup \mathcal{CL}$ such a projection into $E(\tau,\eta)$ of the subset of $\mathcal{S}(\tau)$ of all complete train tracks which can be obtained from τ by a splitting sequence of length at most $n-1$. Let $\{\alpha(i)\}_{0\leq i\leq n} \subset \mathcal{V}(\mathcal{TT})$ be a splitting sequence of length n connecting the train track $\tau = \alpha(0)$ to $\zeta = \alpha(n)$. Let $\{e_1,\dots,e_\ell\}$ be the collection of all large branches of τ with the property that the splitting sequence $\{\alpha(i)\}_{0\leq i\leq n}$ includes a split at e_i. Note that $\ell \geq 1$ since $n \geq 1$. For each i, the choice of a right or left split at e_i is determined by the requirement that the split track carries ζ.

Assume first that there is a large branch $e \in \{e_1,\dots,e_\ell\}$ such that the train track $\tilde{\alpha}(1)$ obtained from τ by a split at e and which is splittable to ζ is also splittable to η. There is a splitting sequence $\{\tilde{\alpha}(i)\}_{1\leq i\leq n}$ of length $n-1$ connecting $\tilde{\alpha}(1)$ to $\tilde{\alpha}(n) = \alpha(n) = \zeta$ (Lemma 5.1 of [H09]). The cubical euclidean cone $E(\tilde{\alpha}(1),\eta)$ is contained in the cubical euclidean cone $E(\tau,\eta)$, and we have $\zeta \in \mathcal{S}(\tilde{\alpha}(1))$. By induction hypothesis, there is a unique train track $\Pi^1_{E(\tilde{\alpha}(1),\eta)}(\zeta) \in E(\tilde{\alpha}(1),\eta) \subset E(\tau,\eta)$ with the property that $\Pi^1_{E(\tilde{\alpha}(1),\eta)}(\zeta)$ is splittable to ζ but that this is not the case for any train track contained in $E(\Pi^1_{E(\tilde{\alpha}(1),\eta)}(\zeta),\eta) - \Pi^1_{E(\tilde{\alpha}(1),\eta)}(\zeta)$.

Define $\Pi^1_\eta(\zeta) = \Pi^1_{E(\tilde{\alpha}(1),\eta)}(\zeta)$. Then $\Pi^1_\eta(\zeta)$ is splittable to ζ and this is not the case for any train track in $E(\Pi^1_\eta(\zeta),\eta) - \Pi^1_\eta(\zeta)$. On the other hand, a splitting sequence connecting τ to ζ is unique up to order (see Lemma 5.1 of [H09] for a detailed discussion of this fact). Therefore if $\xi \in E(\tau,\eta)$ is such that ξ is splittable to ζ and such that a splitting sequence connecting τ to ξ does not include a split at e, then ξ contains $\varphi(\tau,\xi)(e)$ as a large branch, and there is a train track $\xi' \in E(\tau,\eta)$ which can be obtained from ξ by a split at $\varphi(\tau,\xi)(e)$ and which is splittable to ζ. In particular, ξ does not satisfy the requirement in the lemma. As a consequence, the point $\Pi^1_\eta(\zeta)$ does not depend on the above choice of the large branch e.

If none of the train tracks $\xi \in E(\tau,\zeta)$ obtained from τ by a split at one of the branches $e_1,\dots,e_\ell$ is splittable to η, then no train track $\beta \in E(\tau,\eta)-\tau$ is splittable to ζ and we define $\Pi^1_\eta(\zeta)=\tau$. This completes the inductive construction of the map $\Pi^1_\eta : \mathcal{S}(\tau)\to E(\tau,\eta)$. Note that we have $\Pi^1_\eta(\zeta)=\Pi^1_\zeta(\eta)$ for all $\zeta,\eta\in\mathcal{S}(\tau)$. Namely, $\Pi^1_\eta(\zeta)$ is splittable to both ζ,η, but this is not the case for any train track which can be obtained from $\Pi^1_\eta(\zeta)$ by a split. This shows the lemma. □

Remark: Lemma 5.8 is valid without modification for any train track σ which is splittable to train tracks η,ζ. Namely, the proof of Lemma 5.8 nowhere uses the assumption of completeness of the train tracks considered. We used complete train tracks in the formulation of the lemma to include projections to a cubical euclidean cone $E(\tau,\lambda)$ where λ is a complete geodesic lamination carried by τ.

We use Lemma 5.8 to establish a technical statement which enables us to control distances in the train track complex.

Lemma 5.9. *For every $R\geq 0$ there is a number $\beta_2(R)>0$ with the following property. Let $\tau,\eta\in\mathcal{V}(\mathcal{TT})$ with $d(\tau,\eta)\leq R$ and assume that τ,η have a common subtrack σ which carries every measured geodesic lamination carried by both τ,η. Let τ,η be splittable to complete train tracks τ',η'. Then there are train tracks $\tau_1,\eta_1,\tau_1',\eta_1'\in\mathcal{V}(\mathcal{TT})$ with the following properties.*

1. *τ_1',η_1' can be obtained from τ',η' by a splitting sequence of uniformly bounded length (the bound does not depend on R).*
2. *$\tau_1\in E(\tau,\tau_1'),\eta_1\in E(\eta,\eta_1')$.*
3. *$d(\tau_1,\eta_1)\leq\beta_2(R)$.*
4. *τ_1,η_1 contain a (perhaps trivial) multi-curve c as a common subtrack, and every minimal geodesic lamination which is carried by both τ_1,η_1 is a component of c.*
5. *If $\tau=\eta$ and if $\tau',\eta'\in E(\tau,\lambda)$ for a complete geodesic lamination λ carried by τ then $d(\tau_1,\Pi^1_{E(\tau,\tau')}(\eta'))\leq\beta_2(0)$.*

Proof. Let $R>0$ and let $\tau,\eta\in\mathcal{V}(\mathcal{TT})$ be such that $d(\tau,\eta)\leq R$. Assume that τ,η are splittable to $\tau',\eta'\in\mathcal{V}(\mathcal{TT})$. By Lemma 5.5, via possibly replacing τ',η' by their images under a splitting sequence of uniformly bounded length we may assume that τ',η' do not contain any rigid large branch.

Assume that τ,η contain a common subtrack σ which carries every measured geodesic lamination carried by both τ,η. Since a union of recurrent subtracks of σ is recurrent, all measured geodesic laminations which are carried by both τ and η are carried by the largest recurrent subtrack of σ and hence we may assume that

σ is recurrent. Let σ_0 be the complement in σ of the closed curve components of σ. We show the lemma by induction on the number n of branches of σ_0 with a constant $\beta_2(R,n) > 0$ depending on n and R. Since n does not exceed the number q of branches of a complete train track on S, this suffices for the purpose of the lemma.

In the case $n = 0$ we can choose $\tau_1 = \tau, \eta_1 = \eta$ and there is nothing to show. So assume that the lemma holds true whenver the number of branches of σ_0 does not exceed $n-1$ for some $n \geq 1$. Let $\tau, \eta \in \mathcal{V}(\mathcal{TT})$ be such that the number of branches of σ_0 is bounded from above by n.

By Lemma 5.7, there is a splitting sequence $\{\sigma_i\}_{0\leq i\leq p}$ of maximal length issuing from σ_0 which induces a sequence $\{\tilde{\tau}_i\}_{0\leq i\leq 2p} \subset E(\tau,\tau')$ of maximal length issuing from $\tilde{\tau}_0 = \tau$. The train track σ_p is a subtrack of $\tilde{\tau}_{2p}$, and σ_p and $\tilde{\tau}_{2p}$ only depend on τ, τ', σ_0 but not on any choices made in the construction. Similarly, there is a splitting sequence $\{\tilde{\sigma}_i\}_{0\leq i\leq u}$ issuing from $\tilde{\sigma}_0 = \sigma_0$ which induces a sequence $\{\tilde{\eta}_j\}_{0\leq j\leq 2u} \subset E(\eta,\eta')$ of maximal length issuing from η.

The pairs of train tracks (σ_0,σ_p) and $(\sigma_0,\tilde{\sigma}_u)$ determine cubical euclidean cones $E(\sigma_0,\sigma_p), E(\sigma_0,\tilde{\sigma}_u)$ whose vertex sets consist all train tracks which can be obtained from σ_0 by a splitting sequence and which are splittable to $\sigma_p, \tilde{\sigma}_u$. By the remark following Lemma 5.8, we can apply Lemma 5.8 to $E(\sigma_0,\sigma_p)$ and $E(\sigma_0,\tilde{\sigma}_u)$. We obtain a train track $\xi_0 = \Pi^1_{E(\sigma_0,\sigma_p)}\tilde{\sigma}_u = \Pi^1_{E(\sigma_0,\tilde{\sigma}_u)}\sigma_p \in E(\sigma_0,\sigma_p) \cap E(\sigma_0,\tilde{\sigma}_u)$ with the property that ξ_0 is splittable to both $\sigma_p, \tilde{\sigma}_u$ but that this is not the case for any train track which can be obtained from ξ_0 by a split. Define $\xi = \xi_0$ if ξ_0 is recurrent. Otherwise let $\xi \in E(\sigma_0,\xi_0)$ be a recurrent train track with the following property. For every large branch e of ξ, either a splitting sequence connecting ξ to ξ_0 does not include a split at e or the train track obtained from ξ by a split at e and which is splittable to ξ_0 is not recurrent.

By Lemma 5.7, a splitting sequence connecting σ_0 to ξ induces sequences $\{\tilde{\tau}_i\} \subset E(\tau,\tau'), \{\tilde{\eta}_j\} \subset E(\eta,\eta')$ issuing from $\tau_0 = \tau, \eta_0 = \eta$ and connecting τ, η to train tracks τ_1, η_1 which contain ξ as a subtrack and which are splittable to τ', η'. Since $d(\tau,\eta) \leq R$ by assumption, Corollary 3.6 shows the existence of a number $\chi_1(R) > 0$ only depending on R such that

$$d(\tau_1,\eta_1) \leq \chi_1(R). \tag{15}$$

A measured geodesic lamination which is carried by both τ_1, η_1 is carried by the union of ξ with the closed curve components of σ. Note that by the definition of an induced splitting sequence, a closed curve component of σ is a subtrack of both τ_1 and η_1.

Let $\mathcal{E}_\xi(\tau_1), \mathcal{E}_\xi(\eta_1)$ be the set of all large branches e of ξ with the property that there is a train track obtained from ξ by a split at e which is splittable to $\sigma_p, \tilde{\sigma}_u$. We distinguish two cases.

Case 1: $\mathcal{E}_\xi(\tau_1) \cap \mathcal{E}_\xi(\eta_1) \neq \emptyset$.

Let $e \in \mathcal{E}_\xi(\alpha_1) \cap \mathcal{E}_\xi(\beta_1)$. Then there are train tracks $\tau_2 \in E(\tau_1,\tau')$ and $\eta_2 \in E(\eta_1,\eta')$ obtained from τ_1, η_1 by a sequence of splits at large proper subbranches of e (of uniformly bounded length) which are tight at e. There is a train track τ_3, η_3

obtained from τ_2,η_2 by a single split at e which is splittable to τ',η'. The estimate (15) implies that

$$d(\tau_3,\eta_3)\leq \chi_2(R)$$

for a number $\chi_2(R)>0$ only depending on R.

By the definition of the train track ξ, there are now possibilities. In the first case, both τ_3,η_3 are obtained from τ_2,η_2 by a right (or left) split at e. The train track ξ' obtained from ξ by the right (or left) split at e is a common subtrack of τ_3,η_3, but it is not recurrent. Let ζ be the largest recurrent subtrack of ξ'. The number of branches of ζ does not exceed $n-1$. Moreover, ζ carries every measured geodesic lamination which is carried by both τ_3,η_3.

In the second case, the train track τ_3 is obtained from τ_2 by a right (or left) split at e, and the train track η_3 is obtained from η_2 by a left (or right) split at e. Let ζ be the train track obtained from ξ by a collision at the large branch $e\in\mathcal{E}_\xi(\tau_1)\cap\mathcal{E}_\xi(\eta_1)$ of ξ. Then ζ carries every measured geodesic lamination which is carried by both τ_3,η_3. The number of branches of ζ equals $n-1$.

We can now apply the induction hypothesis to τ_3,η_3 which are splittable to τ',η' and to the common subtrack of τ_3,η_3 which is the union of ζ with the simple closed curve components of σ. The statement of the lemma follows.

Case 2: $\mathcal{E}_\xi(\tau_1)\cap\mathcal{E}_\xi(\eta_1)=\emptyset$.

Let e be an arbitrary large branch of ξ. Up to exchanging τ and η we may assume that $e\not\in\mathcal{E}_\xi(\tau_1)$.

Let α,β be the train tracks obtained from τ_1,η_1 by a sequence of ξ-splits of maximal length at proper large subbranches of e and which are splittable to τ',η'. The length of a splitting sequence connecting τ_1,η_1 to α,β is uniformly bounded and hence we have

$$d(\alpha,\beta)\leq\chi_3(R)$$

for a number $\chi_3(R)>0$ only depending on R. The train tracks α,β contain ξ as a common subtrack, and every geodesic lamination which is carried by both α,β is carried by the union of ξ with the closed curve components of σ.

We distinguish three subcases.

Subcase 2.1: Both α and β are tight at e.

Since $e\not\in\mathcal{E}_\xi(\tau_1)$, no train track which can be obtained from α by a single split at e is splittable to τ'. By uniqueness of splitting sequences, $\varphi(\alpha,\tau')(e)$ is large branch of τ'. By assumption, the branch $\varphi(\alpha,\tau')(e)$ of τ' is not rigid. Thus the train tracks obtained from τ' by the right and the left split at $\varphi(\alpha,\tau')(e)$, respectively, are both recurrent and hence complete. Then the train tracks obtained from α by the right split and the left split at e, respectively, are complete as well since these train tracks are splittable to a complete train track obtained from τ' by a split at $\varphi(\alpha,\tau')(e)$.

If there is a train track β' which is obtained from β by a split at e and which is splittable to η', say if β' is obtained from β by the right split, then let α' be the train track obtained from α by the left split at e and let τ'' be the train track obtained from τ' by the left split at $\varphi(\alpha,\tau')(e)$. Then $\alpha' \in E(\alpha,\tau'')$, moreover a measured geodesic lamination which is carried by both α' and β' is carried by the union of the simple closed curve components of σ with the subtrack ζ of α' which is obtained from a collision of ξ at e, i.e. a split followed by the removal of the diagonal of the split. As before, the number of branches of ζ is strictly smaller than the number of branches of ξ. Moreover, we have

$$d(\alpha',\beta') \le \chi_4(R)$$

for a number $\chi_4(R) > 0$ only depending on R. As a consequence, we can now apply the induction hypothesis to $\alpha',\beta',\tau'',\eta'$ and the common subtrack of α',β' which is the union of ζ with the simple closed curve components of σ to deduced the statement of the lemma.

If no train track obtained from a split of β at e is splittable to η' then $\varphi(\beta,\eta')(e)$ is a large branch in η'. Since $\varphi(\beta,\eta')(e)$ is not rigid, the train tracks obtained from β by a right and a left split, respectively, are both complete (see the above discussion). Replace α,β by their images α',β' under a right and left split at e, respectively, and replace τ',η' by their images τ'',η'' under a right split and a left split at $\varphi(\alpha,\tau')(e),\varphi(\beta,\eta')(e)$, respectively. As before, a measured geodesic lamination which is carried by both α',β' is carried by the union of the simple closed curve components of σ with the train track ζ obtained from ξ by a collision at e. The statement of the lemma now follows as above from the induction hypothesis, applied to $\alpha',\beta',\tau'',\eta''$ and the union of ζ with the closed curve components of σ.

Subcase 2.2: Up to exchanging α and β, the train track α contains a large proper subbranch b of e of type 1 or type 3 as introduced in Section 3.

Then precisely one choice of a (right or left) split of α at b contains ξ as a subtrack. If this is say the right split, then the train track α' obtained from α by the left split at e is recurrent. Namely, either $\alpha' \in E(\tau,\tau')$ or α' is splittable to a train track obtained from τ' by a single split at the large branch $\varphi(\alpha,\tau')(b)$ which is not rigid by assumption (see the discussion under subcase 2.1). Moreover, α' does not contain ξ as a subtrack. Note that either $\alpha' \in E(\tau,\tau')$ or $\alpha' \in E(\tau,\tau'')$ where τ'' can be obtained from τ' by a single split at $\varphi(\alpha,\tau')(b)$. A measured geodesic lamination which is carried by both α',β is carried by the union of the simple closed curve components of σ with the largest subtrack ζ of ξ which does not contain the branch e. In particular, the number of branches of ζ is strictly smaller than the number of branches of ξ. As before, the statement of the lemma now follows from the induction hypothesis.

Subcase 2.3: Up to exchanging α and β, the train track α contains a proper large subbranch b of e, and every proper large subbranch b of e in α is of type 2.

Since in this case every split at a large proper subbranch b of e is a ξ-split, no train track obtained from α by a split at a large proper subbranch b of e is splittable to τ'. For such a large proper subbranch b of e, the branch $\varphi(\alpha,\tau')(b)$ is a large branch in τ'.

Let τ'' be a complete train track obtained from τ' by a split at $\varphi(\alpha,\tau')(b)$ and let α' be the train track obtained from α by a split at b and which is splittable to

τ''. Then the number of half-branches of $\alpha' - \xi$ which are incident on a switch in ξ is strictly smaller than the number of half-branches of $\alpha - \xi$ which are incident on a switch in ξ. Via replacing τ'' by its image under a splitting sequence of uniformly bounded length we may assume that τ'' does not contain any rigid large branch.

Carry out the above construction with the train tracks $\alpha', \beta, \tau'', \eta'$. We find a common subtrack ξ' of train tracks $\alpha'' \in E(\alpha', \tau''), \beta'' \in E(\beta, \eta')$ such that ξ' can be obtained from ξ by a splitting sequence and such that ξ' carries every geodesic lamination which is carried by both α'', β''. Moreover, the distance between α'', β'' is bounded from above by a universal constant only depending on R. The number of half-branches of $\alpha'' - \xi'$ which are incident on a switch in ξ' is strictly smaller than the number of half-branches of α which are incident on a switch in ξ. Since the number of neighbors of ξ in α is uniformly bounded, after a uniformly bounded number of such steps, Case 1 or Subcase 2.1 or Subcase 2.2 must occur. Use the reasoning for these cases as before to reduce the statement of the lemma to the induction hypothesis.

Together this completes the proof of the lemma. ☐

Let F be a marking for S and let $X \subset \mathcal{V}(\mathcal{TT})$ be the set of all train tracks which can be obtained from a train track in standard form for F by a splitting sequence. By Proposition 3.2 and the following discussion, there is a number $p > 0$ such that the p-neighborhood of X in $\mathcal{TT}$ is all of $\mathcal{TT}$. Thus if we equip X with the restriction of the metric on $\mathcal{TT}$ then the inclusion $X \to \mathcal{TT}$ is a quasi-isometry.

If λ is any complete geodesic lamination then by Lemma 2.2, λ is carried by a unique train track τ in standard form for F. As a consequence, for $\eta \in X$ there is a *unique* train track τ in standard form for F which is splittable to η. Write $E(F, \eta) = E(\tau, \eta)$.

The next result is the key to an understanding of the geometry of the train track complex.

Proposition 5.10. *There is a number $\kappa > 0$ with the following property. Let F be a marking for S and let $X \subset \mathcal{V}(\mathcal{TT})$ be the set of all complete train tracks which can be obtained from a train track in standard form for F by a splitting sequence. Then for every $\eta \in X$ there is a map $\Pi_{E(F,\eta)} : X \to E(F, \eta)$ such that for every $\zeta \in X$ the following is satisfied.*

(1) *There is a splitting sequence connecting a train track τ' in standard form for F to ζ which passes through the κ-neighborhood of $\Pi_{E(F,\eta)}(\zeta)$.*
(2) *There is a splitting sequence connecting a point in the κ-neighborhood of ζ to a point in the κ-neighborhood of η which passes through the κ-neighborhood of $\Pi_{E(F,\eta)}(\zeta)$.*
(3) *$d(\Pi_{E(F,\eta)}(\zeta), \Pi_{E(F,\zeta)}(\eta)) \le \kappa$ for all $\eta, \zeta \in X$.*
(4) *If $\eta, \zeta \in E(\tau, \lambda)$ for a train track τ in standard form for F which carries the complete geodesic lamination $\lambda \in \mathcal{CL}$ then $\Pi_{E(F,\eta)}(\zeta) = \Pi^1_{E(\tau,\eta)}(\zeta) = \Pi^1_{E(\tau,\zeta)}(\eta)$.*

Proof. Let F be any marking of S, let X be the set of all complete train tracks which can be obtained from a train track in standard form for F by a splitting sequence and let $\tau', \eta' \in X$ be arbitrary. Then there are unique train tracks τ, η in standard form for F so that τ is splittable to τ' and η is splittable to η'. Let $\mathcal{V}(\tau), \mathcal{V}(\eta)$ be the set of all measured geodesic laminations carried by τ, η. Since every geodesic lamination λ contains a minimal component and hence supports a transverse measure (whose support may be a proper sublamination of λ), if $\mathcal{V}(\tau) \cap \mathcal{V}(\eta) = \{0\}$ then by Lemma 5.1 we can define $\Pi_{E(F,\tau')}(\eta') = \tau$ and $\Pi_{E(F,\eta')}(\tau') = \eta$.

On the other hand, if $\mathcal{V}(\tau) \cap \mathcal{V}(\eta) \neq \{0\}$ and if the measured geodesic lamination λ with minimal support is carried by both τ and η then the set ξ, ζ of all branches of τ, η whose λ-weight is positive is a recurrent subtrack of τ, η. By Lemma 3.7, the train tracks ξ, ζ are isotopic. Now the union of two subtracks of τ is again a subtrack and therefore τ, η contain a common recurrent subtrack σ which carries every measured geodesic lamination in $\mathcal{V}(\tau) \cap \mathcal{V}(\eta)$.

Since the diameter in $\mathcal{TT}$ of the set of all train tracks in standard form for F is uniformly bounded, Lemma 5.9, applied to the train tracks τ, η which are splittable to τ', η' and their common subtrack σ, yields the existence of a universal constant $\beta_2 > 0$ and of train tracks τ_1', η_1' and train tracks $\tau_1 \in E(\tau, \tau_1'), \eta_1 \in E(\eta, \eta_1')$ with the following properties.

(1) $d(\tau_1, \eta_1) \leq \beta_2$.
(2) The train track τ_1', η_1' can be obtained from τ', η' by a splitting sequence of length at most β_2.
(3) τ_1, η_1 contain a common embedded multi-curve c such that every minimal geodesic lamination carried by both τ_1, η_1 is a component of c.
(4) If $\tau', \eta' \in E(\tau, \lambda)$ for some complete geodesic lamination λ then $d(\tau_1, \Pi^1_{E(\tau,\tau')}\eta') \leq \beta_2$.

Apply Lemma 5.6 to the train tracks τ_1, η_1 which are splittable to τ_1', η_1'. We find a multi-twist θ about the multi-curve c, train tracks τ'', η'' in a uniformly bounded neighborhood of τ_1', η_1' and hence of τ', η' and a splitting sequence which connects τ'' to η'' and which passes through a uniformly bounded neighborhood of $\theta(\tau_1), \theta(\eta_1)$. Moreover, $d(\theta(\tau_1), E(\tau, \tau')), d(\theta(\eta_1), E(\eta, \eta'))$ is uniformly bounded. Note that $d(\theta(\tau_1), \theta(\eta_1)) \leq \beta_2$.

Define $\Pi_{E(\tau,\tau')}\eta'$ to be a point in $E(\tau, \tau')$ of smallest distance to $\theta(\tau_1)$, and define $\Pi_{E(\eta,\eta')}\tau'$ to be a point in $E(\eta, \eta')$ of smallest distance to $\theta(\eta_1)$. By construction, the train tracks $\Pi_{E(\tau,\tau')}\eta', \Pi_{E(\eta,\eta')}\tau'$ satisfy properties 1),2),3) stated in the proposition. Moreover, we may assume that property 4) holds true as well. □

6. A bounded bicombing of the train track complex

A *discrete bicombing* of a connected metric space (X, d) assigns to every ordered pair of points $(x, y) \in X \times X$ a discrete path $\rho_{x,y} : [0, k_\rho] \cap \mathbb{N} \to X$ connecting $x = \rho_{x,y}(0)$ to $y = \rho_{x,y}(k_\rho)$. The path $\rho_{x,y}$ is called the *combing line* connecting x to y. We view each such path as an eventually constant map defined on $\mathbb{N}$. The bicombing is *reflexive* if $\rho_{x,x}(i) = x$ for all $x \in X$ and all i, and *L-quasi-geodesic* for some $L \geq 1$ if for all $x, y \in X$ the path $i \to \rho_{x,y}(i)$ ($i \in [0, k_\rho] \cap \mathbb{N}$) is an L-quasi-geodesic. Call the bicombing *L-bounded* for some $L > 0$ if for all $x, y, x', y' \in X$ and all $i \geq 0$ we have

$$d(\rho_{x,y}(i), \rho_{x',y'}(i)) \leq L(d(x, x') + d(y, y')) + L.$$

As an example, if X is a Cat(0)-space then any two points in X can be connected by a unique geodesic parametrized by arc length, and these geodesics define a reflexive 1-quasi-geodesic 1-bounded bicombing of X.

The purpose of this section is to construct for some $L \geq 1$ a reflexive L-quasi-geodesic L-bounded $\mathcal{MCG}(S)$-equivariant bicombing of the train track complex $\mathcal{TT}$. We begin with defining for a train track $\tau \in \mathcal{V}(\mathcal{TT})$ which is splittable to a train track $\eta \in \mathcal{V}(\mathcal{TT})$ a combing line connecting τ to η. This combing line is entirely contained in the cubical euclidean cone $E(\tau, \eta)$, and it is constructed from a particular splitting sequence connecting τ to η. Here as before, $E(\tau, \eta)$ is the full subgraph of $\mathcal{TT}$ whose vertex set consists of all train tracks which can be obtained from τ by a splitting sequence and which are splittable to η.

Note first that any bicombing constructed directly from splitting sequences is not bounded. The difficulty arises already in the case of a simple closed curve c embedded in a train track τ as the waist curve of a twist connector as shown in Figure B. The curve consists of a large branch and a small branch, connected at two common switches. A single split at the large branch, with the small branch as the winner, results in the train track $\theta_c(\tau)$ where θ_c is a (positive or negative) Dehn twist about c.

If the twist connector is attached to a standard train track with stops of type 0 as shown in Figure A, then τ can be modified with a single shift to a train track τ' with the property that c is an embedded simple closed curve in τ' which consists of three branches, one large branch, one small branch and one mixed branch. Now a splitting sequence of length 2 results in $\theta_c\tau'$. By invariance, the distance in $\mathcal{TT}$ between $\theta_c^k\tau$ and $\theta_c^k\tau'$ does not depend on $k \in \mathbb{Z}$. On the other hand, there are $2k$ splits needed to transform τ' to $\theta_c^k\tau'$, but k splits suffice to transform τ to $\theta_c^k\tau$. Therefore splitting sequences do not give rise to a bounded bicombing of $\mathcal{TT}$: Their parametrizations are not synchronized.

The main observation for the construction of a bounded bicombing of $\mathcal{TT}$ is that we can "accelerate" splitting sequences in a suitable way to yield indeed a quasi-geodesic bounded bicombing of $\mathcal{TT}$ as required.

We begin with singling out subsets of a recurrent train track which are used for "acceleration" of splitting sequences. Namely, let $\rho : [0, m] \to \tau$ be an embedded trainpath which begins and ends with a large half-branch $\rho[0, 1/2], \rho[m - 1/2, m]$. We call such a trainpath *two-sided large*. By the results of [PH92] (see Lemma 2.3.2 of [PH92] and the discussion thereafter), such a trainpath contains at least one large branch.

As in Section 5, with respect to the orientation of S and the orientation of ρ, we can distinguish right and left neighbors of ρ, namely half-branches of τ incident on switches in the interior of ρ which lie to the right or to the left of ρ, respectively, in a small neighborhood of $\rho[0,m]$ in S. A switch on which a right (or left) neighbor is incident will be called a right (or left) switch. Call a switch $\rho(i)$ for some $i \in \{1,\dots,m-1\}$ incoming if the half-branch $\rho[i-1/2,i]$ is small, and call the switch outgoing otherwise. A neighbor of ρ incident on an incoming (or outgoing) switch is called incoming (or outgoing).

Define a two-sided large embedded trainpath ρ in τ to be *reduced* if either all left neighbors of ρ are incoming and all right neighbors are outgoing, or if all left neighbors of ρ are outgoing and all right neighbors are incoming. If ρ is reduced and if all left neighbors of ρ are incoming then the same holds true for the trainpath ρ with the orientation reversed. In this case we call ρ a *positive reduced trainpath*, and otherwise ρ is called a *negative reduced trainpath.* Note that a subpath ρ' of a reduced trainpath ρ is reduced if and only if it is two-sided large. An embedded trainpath which consists of a single large branch will also be called reduced.

If $m \geq 2$ and if $\rho : [0,m] \to \tau$ is an embedded reduced trainpath then for every large branch $e = \rho[i,i+1]$ of τ contained in $\rho[0,m]$ there is a unique split of τ at e so that the branches $\rho[i-1,i], \rho[i+1,i+2]$ are winners of the split. We called such a split a *ρ-split.* For convenience of terminology, if ρ is a reduced trainpath of length one then we call every split of τ at this branch a ρ-split.

For every train track τ which is splittable to a train track σ there is a natural bijection $\varphi(\tau,\sigma)$ from the branches of τ to the branches of σ (compare Lemma 5.1 of [H09] and its proof). This bijection also induces a natural bijection of the half-branches of τ onto the half-branches of σ. We denote this bijection again by $\varphi(\tau,\sigma)$.

For every embedded reduced trainpath $\rho : [0,m] \to \tau$ and every large branch e in $\rho[0,m]$, the train track τ_1 obtained from τ by a ρ-split at e contains $\varphi(\tau,\tau_1)(\rho[0,m])$ as an embedded trainpath. Namely, the split just consists in moving one of the neighbors of ρ incident on an endpoint of e along ρ across the second endpoint of e (see Figure D). If $e = \rho[0,1]$ or $e = \rho[m-1,m]$ then this trainpath is not two-sided large any more. However, if we denote this trainpath on τ_1 again by ρ then a ρ-split of τ_1 at a large branch in ρ is defined. Thus we can talk about a sequence of splits at large branches contained in $\rho[0,m]$. A successive application of this observation shows that whenever τ' can be obtained from τ by a sequence of ρ-splits then $\varphi(\tau,\tau')(\rho)$ is an embedded trainpath in τ'. Moreover, every two-sided large subpath of this trainpath is reduced.

The following lemma is a fairly immediate consequence of Lemma 3.4 and the remark thereafter. For its formulation, denote as before by q the number of branches of a complete train track on S.

Lemma 6.1. *Let τ be a train track and let $\rho : [0,m] \to \tau$ be any embedded reduced trainpath. Then there is a unique train track τ' with the following properties.*

1. *τ' is obtained from τ by a sequence of at most q^2 ρ-splits at large branches contained in $\rho[0,m]$.*
2. *$\varphi(\tau,\tau')(\rho[0,m])$ is an embedded trainpath in τ' which does not contain a large branch.*

The splitting sequence connecting τ to τ' includes a split at each branch in $\rho[0,m]$.

Proof. Our goal is to show that after a transformation of τ with at most $m(m+1)/2$ ρ-splits which includes at least one split at each branch in $\rho[0,m]$ we arrive at a train track τ' with the property that $\varphi(\tau,\tau')(\rho[0,m])$ does not contain any large branch.

To see that this is the case we proceed by induction on the length m of the trainpath ρ. If this length equals one then the train track τ' obtained from τ by a single split at $\rho[0,1]$ contains $\varphi(\tau,\tau')(\rho[0,1])$ as a small branch and there is nothing to show. Thus assume that the claim holds true whenever the length of ρ does not exceed $m-1$ for some $m \geq 2$.

Let $\rho : [0,m] \to \tau$ be an embedded reduced trainpath of length m and let $i \geq 0$ be the smallest number such that the branch $\rho[i,i+1]$ is large. Since the half-branch $\rho[0,1/2]$ is large, the trainpath $\rho[0,i+1]$ is *one-way* in the terminology used on p.127 of [PH92]. In particular, the branches $\rho[j,j+1]$ are all mixed for $j < i$ (Lemma 2.3.2 of [PH92]). Let τ_1 be the train track obtained from τ by a ρ-split at $\rho[i,i+1]$.

Assume first that $i = 0$. Then the branch $\varphi(\tau,\tau_1)(\rho[0,1])$ is small. The trainpath $\varphi(\tau,\tau_1)(\rho[1,m])$ is two-sided large and hence reduced, and its length equals $m-1$. By induction hypothesis, τ_1 can be transformed with at most $(m-1)m/2$ $\varphi(\tau,\tau_1)(\rho[1,m])$-splits including a split at each branch in $\varphi(\tau,\tau_1)(\rho[1,m])$ to a train track τ' so that $\varphi(\tau,\tau')(\rho[1,m])$ does not contain any large branch. Now the half-branch $\varphi(\tau,\tau')(\rho[0,1/2])$ is small and hence $\varphi(\tau,\tau')(\rho)$ does not contain any large branch as claimed. The number of splits needed to transform τ to τ' is at most $(m-1)m/2+1$.

By possibly reversing the orientation of ρ we are left with the case that both branches $\rho[0,1]$ and $\rho[m-1,m]$ are mixed. Then we have $1 \leq i \leq m-2$, and $\varphi(\tau,\tau_1)(\rho[0,m])$ is a reduced trainpath in τ_1 which contains $\varphi(\tau,\tau_1)(\rho[i-1,i])$ as a large branch. The branches $\varphi(\tau,\tau_1)(\rho[j,j+1])$ are mixed for $j \leq i-2$. Successively we can modify τ in this way with $i \leq m-2$ ρ-splits at the branches in the subarc $\rho[1,i]$ of ρ to a train track $\tilde{\tau}$ which contains $\tilde{\rho} = \varphi(\tau,\tilde{\tau})(\rho[0,m])$ as a reduced trainpath and such that the branch $\varphi(\tau,\tilde{\tau})(\rho[0,1])$ is large. Then we can apply the consideration in the previous paragraph to $\tilde{\tau}$ and the reduced trainpath $\tilde{\rho}$ to deduce the lemma. □

We say that the train track τ' constructed in Lemma 6.1 from τ and an embedded reduced trainpath $\rho : [0,m] \to \tau$ is obtained from τ by the *full ρ-multi-split*.

Lemma 6.1 and its proof also show the following. If $\rho : [0,m] \to \tau$ is an embedded reduced trainpath and if $0 \leq i < j \leq m$ are such that the embedded trainpath $\rho[i,j]$ is two-sided large then the train track obtained from τ by the full $\rho[i,j]$-multi-split is splittable to the train track obtained from τ by the full ρ-multi-split.

Recall from Section 5 the definition of a reduced circle (i.e. a reduced embedded simple closed curve of class C^1) in a train track τ. Using the notations from Section

5, if c is a reduced circle in τ and if b is any small branch contained in c then $c - b$ is an embedded arc in τ which can be parametrized as a reduced trainpath in τ.

As before, let θ_c be the positive Dehn twist about c. By Lemma 5.2, there is a splitting sequence consisting of c-splits which transforms τ to $\theta_c\tau$ (in case c is positive) or to $\theta_c^{-1}\tau$ (in case c is negative). To simplify the notation, we write $\theta_c^{\pm}\tau$ to denote this train track. We have

Lemma 6.2. *Let c be a reduced circle in a train track τ which consists of at least three branches. Let b be a small branch of τ contained in c and let ρ be a trainpath parametrizing $c - b$. Then the train track obtained from τ by the full ρ-multi-split is splittable to $\theta_c^{\pm}\tau$.*

Proof. Let c be a reduced circle in τ and let b be a small branch in c. Then $c - b$ can be parametrized as a reduced trainpath $\rho : [0, m] \to c \subset \tau$. We assume that the length m of ρ is at least 2. For simplicity of notation, assume also that the neighbor b of c at $\rho(m)$ is right outgoing with respect to the orientation of c induced by the orientation of ρ (note that this neighbor is outgoing since the branch b is small). Then the train track obtained from τ by the full ρ-multi-split can be described as the train track obtained by sliding all left (and hence incoming) neighbors of ρ successively along ρ past the endpoint of the half-branch b (compare the discussion in the proof of Lemma 5.2 and of Lemma 6.1). On the other hand, $\theta_c^{\pm}\tau$ is obtained from τ by sliding all left neighbors of c successively along ρ precisely once across each endpoint of each right neighbor of c. This implies the lemma. ☐

Let τ be any (not necessarily recurrent or maximal) train track which is splittable to a train track η. Define a *splittable η-path* in τ to be an embedded reduced trainpath $\rho : [0, m] \to \tau$ with the following property. There is a train track τ' which is splittable to η and which is obtained from τ by a sequence of ρ-splits including at least one split at each of the branches in $\rho[0, m]$. This means that for every $i \leq m - 1$ there is a train track $\tau_i \in E(\tau, \eta)$ which can be obtained from τ by a sequence of ρ-splits and with the following additional properties. The branch $\varphi(\tau, \tau_i)(\rho[i, i + 1])$ is large, and the train track obtained from τ_i by the ρ-split at this large branch is splittable to η. We call a sequence of ρ-splits of maximal length with the property that the split track is splittable to η the *$\rho - \eta$-multi-split* of τ.

The splittable η-path ρ is called *maximal* if for every splittable η-path ρ' which intersects ρ in at least one branch we have $\rho' \subset \rho$.

A single large branch e in τ such that no train track obtained from τ by a split at e is splittable to η is called a *maximal non-splittable η-path.* A maximal splittable or non-splittable η-path is simply called a *maximal η-path.*

A *splittable η-circle* is a reduced simple closed curve c embedded in τ such that there is a train track σ which is splittable to η and which is obtained from τ by a sequence of c-splits including a split at every branch of c. Note that if c is a reduced circle in τ of length two, i.e. if c consists of a single large branch and a single small branch, then c is a splittable η-circle only if $\theta_c^{\pm}\tau$ is splittable to η.

Lemma 6.3. *Let τ be a train track which is splittable to a train track η. Then every large branch e of τ is contained either in a unique maximal η-path or in a unique splittable η-circle.*

Proof. By the definition of a maximal η-path in a train track τ and by uniqueness of splitting sequences, a large branch e of τ such that no train track which can be obtained from τ by a split at e is splittable to η is a maximal non-splittable η-path, and it is the unique maximal η-path containing e.

If e is any large branch of τ such that a splitting sequence connecting τ to η includes a split at e then e is contained in a splittable η-path or in a splittable η-circle.

Let for the moment $\rho : [0, m] \to \tau$ be any two-sided large trainpath with the property that with respect to the given orientation of ρ and the orientation of S, either all left switches are incoming and all right switches are outgoing or all left switches are outgoing and all right switches are incoming (this is a local property). We claim that either ρ is embedded in τ or ρ is a circle of class C^1.

We argue by contradiction and we assume that ρ is not embedded and not a circle of class C^1. Since ρ is two-sided large, the self-intersection of ρ can not consist of a single switch. Thus up to reversing the orientation of ρ there is some $i \in \{1, \dots, m-1\}$ such that the half-branch $\rho[i, i+1/2]$ is large and that both half-branches which are incident and small at $\rho(i)$ are contained in the image of ρ. Since ρ is two-sided large, there is some $j \neq i$ such that $\rho[j-1, j+1]$ is a subarc of ρ with $\rho(j) = \rho(i)$ and such that $\rho[i-1, i+1] \cup \rho[j-1, j+1]$ contains all three half-branches which are incident on $\rho(i)$. Now if $\rho(j+1) = \rho(i+1)$ then the half-branch $\rho[i-1/2, i]$ is an incoming neighbor of $\rho[j-1, j+1]$, and $\rho[j-1/2, j]$ is an incoming neighbor of $\rho[i-1, i+1]$. Moreover, one of these neighbors is a left neighbor along ρ, and the other neighbor is a right neighbor. This violates the assumption on ρ. Similarly, if $\rho(j-1) = \rho(i+1)$ and if $\rho(i)$ is a right (or left) incoming switch for $\rho[i-1, i+1]$ then $\rho(i)$ is a right (or left) outgoing switch for $\rho[j-1, j+1]$. Again this violates the assumption on ρ and shows the claim.

As a consequence, the union of two intersecting two-sided large embedded positive (or negative) reduced trainpaths is necessarily either embedded in τ or an embedded circle of class C^1 and hence it is a two-sided large positive (or negative) reduced trainpath or a reduced circle. Moreover, two such paths either are disjoint or they intersect in at least one branch.

Now let $\rho : [0, m] \to \tau$ be any splittable η-path or splittable η-circle of length $m \geq 2$. Then there is some $i < m$ such that the branch $\rho[i, i+1]$ is large. Since ρ is an η-path, the η-split of τ at the branch $\rho[i, i+1]$ is just the unique ρ-split of τ at $\rho[i, i+1]$. This implies that if $\rho' : [0, n] \to \tau$ is another splittable η-path whose image in τ contains $\rho[i, i+1]$ then either ρ' consists of the single large branch $\rho[i, i+1]$ and hence $\rho' \subset \rho$, or the type of ρ' (positive or negative) coincides with the type of ρ (positive or negative). Thus by the discussion in the previous paragraph, the union of two splittable η-paths which intersect in at least one branch is a splittable η-path or a splittable η-circle.

Let e be any large branch of τ so that a splitting sequence connecting τ to η includes a split at e. Let ρ be the union of all splittable η-paths in τ which pass through e. By the discussion in the previous paragraph, ρ is a splittable η-path or a splittable η-circle, and it is the unique maximal η-path or splittable η-circle containing e. Moreover, any two such maximal η-paths or splittable η-circles either coincide or are disjoint. This shows the lemma. ☐

Let again τ be a (not necessarily complete) train track on S which is splittable to a train track η. A *splittable η-configuration* in τ is defined to be a maximal splittable η-path or a splittable η-circle. If ρ is a maximal splittable η-path then there is a unique train track $\tilde{\tau}$ which can be obtained from τ by a sequence of ρ-splits of maximal length and which is splittable to η. In other words, we require that no train track which can be obtained from $\tilde{\tau}$ by a $\varphi(\tau,\tilde{\tau})(\rho)$-split at a large branch in $\varphi(\tau,\tilde{\tau})(\rho)$ is splittable to η. If c is a splittable η-circle then there is a unique train track $\tilde{\tau}$ which can be obtained from τ by a splitting sequence of maximal length and which is splittable to both $\theta_c^{\pm}(\tau)$ and η. In both cases we say that $\tilde{\tau}$ is obtained from τ by the *$\rho-\eta$-multi-split.* A *non-splittable η-configuration* is a single large branch e in τ such that no train track which can be obtained from τ by a split at e is splittable to η.

We define the train track obtained from τ by an *η-move* to be the unique train track τ' with the following property. Let $\rho_1,\dots,\rho_k$ be the splittable η-configurations of τ. By Lemma 6.3, these are uniquely determined pairwise disjoint embedded reduced trainpaths or reduced circles in τ. In particular, splits at branches contained in ρ_i,ρ_j for $i\neq j$ commute, and we define τ' to be the train track obtained from τ by a successive modification with a $\rho_i-\eta$-multi-split where $i=1,\dots,k$. The train track τ' is uniquely determined by τ and η. Moreover, the length of a splitting sequence connecting τ to τ' is bounded from above by a number $p>0$ only depending on the topological type of S.

Extending slightly the notations used earlier on, for an arbitrary train track τ on S which is splittable to a train track η let $E(\tau,\eta)$ be the connected directed graph whose set of vertices is the set of all train tracks which can be obtained from τ by a splitting sequence and which are splittable to η. Connect $\sigma\in E(\tau,\eta)$ to $\sigma'\in E(\tau,\eta)$ by a directed edge if σ' can be obtained from σ by a single split.

For a recurrent train track τ which is splittable to a recurrent train track η define now inductively a sequence $\{\gamma(\tau,\eta)(i)\}_{0\leq i\leq k}\subset E(\tau,\eta)$ beginning at $\tau=\gamma(\tau,\eta)(0)$ and ending at η by requiring that for each $i<k$ the train track $\gamma(\tau,\eta)(i+1)$ is obtained from $\gamma(\tau,\eta)(i)$ by an η-move. We call the sequence the *balanced splitting path* connecting τ to η, and we denote it by $\gamma(\tau,\eta)$. By construction, if τ is splittable to η and if $g\in\mathcal{MCG}(S)$ is arbitrary then $g\gamma(\tau,\eta)$ is the balanced splitting path connecting $g\tau$ to $g\eta$.

For convenience of notation, we extend a discrete path in a metric space (X,d) defined on a subset $[0,n]\cap\mathbb{N}$ of the natural numbers to a path defined on $\mathbb{N}$ which is constant on $[n,\infty)\cap\mathbb{N}$. Call two eventually constant paths $c_1:\mathbb{N}\to X, c_2:\mathbb{N}\to X$ *weight-L fellow travellers* if

$$d(c_1(i),c_2(i))\leq L(d(c_1(0),c_2(0))+d(c_1(\infty),c_2(\infty)))\text{ for all } i$$

where $c_i(\infty)$ is defined by requiring that $c_i(j) = c_i(\infty)$ for all sufficiently large j. If c_1, c_2 are weight-L fellow travellers then the Hausdorff distance in X between the images $c_1(\mathbb{N})$ and $c_2(\mathbb{N})$ is bounded from above by $L(d(c_1(0), c_2(0)) + d(c_1(\infty), c_2(\infty)))$.

Now we are ready to formulate the main result of this section.

Theorem 6.4. *There is a number $L > 0$ with the following property. Let $\tau, \tau' \in \mathcal{V}(\mathcal{TT})$ be splittable to $\eta, \eta' \in \mathcal{V}(\mathcal{TT})$. Then the balanced splitting paths*

$$\gamma(\tau, \eta), \gamma(\tau', \eta')$$

are weight-L fellow travellers in $\mathcal{TT}$.

The remainder of this section is devoted to the proof of Theorem 6.4. We begin the proof with collecting some first basic properties of balanced splitting paths.

Lemma 6.5. *Let τ be a train track which is splittable to a train track η and let $\sigma \in E(\tau, \eta)$. Then the following holds true.*

(1) $\gamma(\tau, \sigma)(1) \in E(\tau, \gamma(\tau, \eta)(1))$.
(2) *If $\xi \in E(\tau, \gamma(\tau, \eta)(1))$ then $\gamma(\tau, \xi)(1) = \xi$ and $\gamma(\tau, \eta)(1) \in E(\tau, \gamma(\xi, \eta)(1))$.*

Proof. The first part of the lemma is fairly immediate from the definitions. Namely, assume that the train track τ is splittable to the train track η and let $\sigma \in E(\tau, \eta)$. Let ρ be a σ-configuration in τ. If ρ is a splittable σ-circle then by the definitions and uniqueness of splitting sequences (Lemma 5.1 of [H09] which is also valid for train tracks which are not complete), ρ is an η-configuration in τ and the $\rho - \sigma$-multi-split of τ is splittable to the $\rho - \eta$-multi-split of τ.

On the other hand, if ρ is a splittable σ-path then by Lemma 6.3 and the definitions, ρ is a subpath of a maximal splittable η-path ρ' or of a splittable η-circle. Once again, the $\rho - \sigma$-multi-split of τ is splittable to the $\rho' - \eta$-multi-split of τ by construction, by uniqueness of splitting sequences and by Lemma 6.2. Note that the η-configuration ρ' may contain several distinct σ-configurations. The first part of the lemma follows.

The same argument also shows the second part of the lemma. Namely, let $\rho : [0, n] \to \tau$ be any embedded reduced trainpath and assume that ν is obtained from τ by a sequence of ρ-splits. Then $\rho' = \varphi(\tau, \nu)(\rho)$ is an embedded trainpath in ν, and any two-sided large trainpath in ν which is contained in ρ' is reduced. Now if $i \geq 0$ is the smallest number such that the half-branch $\rho'[i, i + 1/2]$ is large then no sequence of ρ'-splits can include a split at any of the branches $\rho'[u, u + 1]$ for $0 \leq u \leq i - 1$. Thus by symmetry, if $j \geq i$ is the largest number such that $\rho'[j - 1/2, j]$ is a large half-branch then $\rho'[i, j]$ is embedded and reduced, and the train track obtained from ν by the full $\rho'[i, j]$-multi-split equals the train track obtained from τ by a full ρ-multi-split. This implies that $\gamma(\nu, \zeta)(1) = \zeta = \gamma(\tau, \zeta)(1)$ for any train track ζ which can be obtained from τ by a sequence of ρ-splits and such that $\nu \in E(\tau, \zeta)$. In particular, we have $\gamma(\nu, \gamma(\tau, \eta)(1))(1) = \gamma(\tau, \eta)(1)$. The same argument is also valid in the case that ρ is a reduced circle c in τ and that ν, ζ are obtained from τ by a sequence of ρ-splits and are splittable to $\theta_c^{\pm}\tau$.

Now let τ be splittable to η and let $\xi \in E(\tau, \gamma(\tau, \eta)(1))$. Let $\rho_1, \dots, \rho_k$ be the η-configurations in τ. Then ξ can be obtained from τ by successively splitting τ with a sequence of ρ_i-splits for $i = 1, \dots, k$. Since any ρ_i-split commutes with any ρ_j-split for $i \neq j$, the discussion in the previous paragraph shows that $\xi = \gamma(\tau, \xi)(1)$ and that $\gamma(\tau, \eta)(1)$ is splittable to $\gamma(\xi, \eta)(1)$. This shows the second part of the lemma. □

As a first step towards a proof of Theorem 6.4 we investigate balanced splitting paths whose images are contained in a fixed cubical euclidean cone $E(\tau, \lambda)$. As in Section 3, denote by d_E the intrinsic path metric on the connected graph $E(\tau, \lambda)$.

Lemma 6.6. *There is a number $L_1 > 0$ with the following property. Let λ be a complete geodesic lamination carried by a complete train track τ and let $\sigma, \eta \in E(\tau, \lambda)$. Then the balanced splitting paths $\gamma(\tau, \sigma)$ and $\gamma(\tau, \eta)$ are weight-L_1 fellow travellers in $(E(\tau, \lambda), d_E)$.*

Proof. By Lemma 5.4 of [H09], for $\sigma, \eta \in E(\tau, \lambda)$ there is a train track $\Theta_-(\sigma, \eta) \in E(\tau, \lambda)$ so that $\Theta_-(\sigma, \eta)$ is splittable to both σ, η and that there is a geodesic in $(E(\tau, \lambda), d_E)$ connecting σ to η which passes through $\Theta_-(\sigma, \eta)$. In particular, we have

$$d_E(\sigma, \eta) = d_E(\sigma, \Theta_-(\sigma, \eta)) + d_E(\Theta_-(\sigma, \eta), \eta). \tag{16}$$

This implies that the balanced splitting paths $\gamma(\tau, \sigma), \gamma(\tau, \eta)$ are weight-L fellow travellers in $(E(\tau, \lambda), d_E)$ if the balanced splitting paths $\gamma(\tau, \sigma), \gamma(\tau, \Theta_-(\sigma, \eta))$ and $\gamma(\tau, \eta)$, $\gamma(\tau, \Theta_-(\sigma, \eta))$ are weight-L fellow travellers in $(E(\tau, \lambda), d_E)$. As a consequence, it suffices to show the lemma in the particular case that σ is splittable to η.

By Corollary 5.2 of [H09], splitting paths in $(E(\tau, \lambda), d_E)$ are geodesics. Thus it is enough to show the lemma for balanced splitting paths $\gamma(\tau, \sigma), \gamma(\tau, \eta)$ connecting τ to train tracks $\sigma, \eta \in E(\tau, \lambda)$ with the additional property that η can be obtained from σ by a single split at a large branch e.

Assume that this is the case. Let $\{\sigma(i)\}_{0 \leq i \leq k}$ be the balanced splitting path connecting $\tau = \sigma(0)$ to $\sigma = \sigma(k)$ and let $\{\eta(i)\}_{0 \leq i \leq \ell}$ be the balanced splitting path connecting $\tau = \eta(0)$ to $\eta = \eta(\ell)$. Then there is a largest number $i \leq k$ such that $\eta(i) = \sigma(i)$. If $i = k$ then we have $\sigma(k) = \sigma = \eta(k)$. Since η can be obtained from σ by a single split, by definition of an η-move we obtain $\ell = k + 1$, and the balanced splitting paths $\gamma(\tau, \sigma), \gamma(\tau, \eta)$ connecting τ to σ, η are weight-1 fellow travellers for the distance d_E.

If $i < k$ then also $i < \ell$ and $\sigma(i+1) \neq \eta(i+1)$. By uniqueness of splitting sequences (Lemma 5.1 of [H09]), there is a train track ξ which can be obtained from $\eta(i) = \sigma(i)$ by a splitting sequence, which is splittable to σ and which is splittable with a single split at a large branch e' to $\eta(i+1)$. Moreover, we have $\varphi(\xi, \sigma)(e') = e$.

By the first part of Lemma 6.5, applied to $\sigma(i) = \eta(i)$ and to $\sigma \in E(\sigma(i), \eta)$, the train track $\sigma(i+1)$ is splittable to $\eta(i+1)$ and therefore $\sigma(i+1)$ is splittable to ξ. The second part of Lemma 6.5, applied to $\xi \in E(\eta(i), \eta(i+1))$, then implies that

$\sigma(i+1) = \xi$. As a consequence, the train track $\eta(i+1)$ can be obtained from $\sigma(i+1)$ by a single split at e'. Inductively we deduce that for every $j \in \{i+1, \dots, k\}$ the train track $\eta(j)$ can be obtained from $\sigma(j)$ by a single split at $\varphi(\sigma(i+1), \sigma(j))(e')$. In other words, we have $k = \ell$ and the balanced splitting paths $\gamma(\tau, \sigma), \gamma(\tau, \eta)$ are weight-1 fellow travellers for the distance d_E. This completes the proof of the lemma. □

If a train track τ is splittable to a train track η then the intrinsic path metric d_E on the cone $E(\tau, \eta)$ is defined.

Lemma 6.7. *There is a number $L_2 > 0$ with the following property. Let τ be a train track which is splittable to a train track η and let $\sigma \in E(\tau, \eta)$. Then the balanced splitting paths $\gamma(\tau, \eta), \gamma(\sigma, \eta)$ are weight-L_2 fellow travellers in $(E(\tau, \eta), d_E)$.*

Proof. Let the train track τ be splittable to the train track η. By definition of a balanced splitting path, if e is a large branch in τ such that a splitting sequence connecting τ to η includes a split at e then a splitting sequence connecting τ to $\gamma(\tau, \eta)(1)$ includes a split at e as well. Since directed edge-paths in $(E(\tau, \eta), d_E)$ are geodesics (Corollary 5.2 of [H09] is also valid if the train tracks τ, η are not complete), it therefore suffices to show the lemma in the particular case that $\sigma \in E(\tau, \gamma(\tau, \eta)(1))$.

For this we claim that if τ is splittable to η and if $\sigma \in E(\tau, \gamma(\tau, \eta)(1))$ then for each u the train track $\gamma(\tau, \eta)(u)$ is splittable to $\gamma(\sigma, \eta)(u)$, and $\gamma(\sigma, \eta)(u)$ is splittable to $\gamma(\tau, \eta)(u+1)$.

We proceed by induction on the length of a splitting sequence connecting τ to η. If this length vanishes then $\tau = \eta = \sigma$ and there is nothing to show, so assume that the claim is known whenever the length of a splitting sequence connecting τ to η is at most $n-1$ for some $n \geq 1$.

Let τ be splittable to η and assume that the length of a splitting sequence connecting τ to η equals n. If $\sigma = \tau$ then once again there is nothing to show, so assume that σ can be obtained from τ by a non-trivial splitting sequence and is splittable to $\gamma(\tau, \eta)(1)$. By the second part of Lemma 6.5, $\gamma(\tau, \eta)(1)$ is splittable to $\gamma(\sigma, \eta)(1)$. Now the length of a splitting sequence connecting σ to η is at most $n-1$ and hence we can apply the induction hypothesis to $\sigma, \gamma(\tau, \eta)(1), \eta$ and to the balanced splitting paths $\gamma(\sigma, \eta)$ and $\gamma(\gamma(\tau, \eta)(1), \eta)$. This yields the above claim.

As a consequence, we have

$$d_E(\gamma(\tau, \eta)(u), \gamma(\sigma, \eta)(u)) \leq p$$

for all u where $p > 0$ is an upper bound for the length of any splitting sequence connecting a train track to its image under a move. Thus the balanced splitting paths $\gamma(\tau, \eta), \gamma(\sigma, \eta)$ are weight-p fellow travellers in $E(\tau, \eta), d_E)$. This shows the lemma. □

As an immediate corollary we conclude that balanced splitting paths connecting points in the same cubical euclidean cone are uniform fellow travellers.

Corollary 6.8. *There is a number $L_3 > 0$ with the following property. Let $E(\tau, \lambda) \subset \mathcal{TT}$ be any cubical euclidean cone and let $\sigma_1, \sigma_2 \in E(\tau, \lambda)$ be train tracks which are splittable to train tracks $\eta_1, \eta_2 \in E(\tau, \lambda)$. Then the balanced splitting paths $\gamma(\sigma_1, \eta_1), \gamma(\sigma_2, \eta_2)$ are weight-L_3 fellow travellers.*

Proof. Let σ_i, η_i be as in the corollary. Let d_E be the intrinsic path metric on $E(\tau, \lambda)$. By Lemma 5.4 of [H09], there is a unique train track $\Theta_-(\sigma_1, \sigma_2) \in E(\tau, \lambda)$ such that $\sigma_1, \sigma_2 \in E(\Theta_-(\sigma_1, \sigma_2), \lambda)$ and that there is a geodesic in $(E(\tau, \lambda), d_E)$ connecting σ_1 to σ_2 which passes through $\Theta_-(\sigma_1, \sigma_2)$. In particular, we have $d_E(\sigma_1, \sigma_2) = d_E(\sigma_1, \Theta_-(\sigma_1, \sigma_2)) + d_E(\Theta_-(\sigma_1, \sigma_2), \sigma_2)$.

By Lemma 6.6, there is a number $L_1 > 0$ such that the balanced splitting paths $\gamma(\Theta_-(\sigma_1, \sigma_2), \eta_1)$ and $\gamma(\Theta_-(\sigma_1, \sigma_2), \eta_2)$ with the same starting point $\Theta_-(\sigma_1, \sigma_2)$ are weight-L_1 fellow travellers in $(E(\Theta(\sigma_1, \sigma_2), \lambda), d_E) \subset (E(\tau, \lambda), d_E)$. Moreover, by Lemma 6.7 and its proof, there is a number $L_2 > 0$ such that the balanced splitting paths $\gamma(\Theta_-(\sigma_1, \sigma_2), \eta_i)$ and $\gamma(\sigma_i, \eta_i)$ $(i = 1, 2)$ with the same endpoints are weight-L_2 fellow travellers in $(E(\Theta_-(\sigma_1, \sigma_2), \lambda), d_E) \subset (E(\tau, \lambda), d_E)$. Since $d_E(\sigma_1, \sigma_2) = d_E(\sigma_1, \Theta_-(\sigma_1, \sigma_2)) + d_E(\Theta_-(\sigma_1, \sigma_2), \sigma_2)$, the balanced splitting paths $\gamma(\sigma_1, \eta_1), \gamma(\sigma_2, \eta_2)$ are weight-$(L_1 + L_2)$-fellow travellers in $(E(\tau, \lambda), d_E)$. Together with Theorem 2.3 this yields the corollary. □

Next we extend Corollary 6.8 to train tracks related by carrying rather than splitting. We begin with the most basic case.

Lemma 6.9. *There is a number $L_4 > 0$ with the following property. Let τ, σ be complete train tracks which are splittable to the same complete train track η. If σ can be obtained from τ by a single shift then*

$$d(\gamma(\tau, \eta)(i), \gamma(\sigma, \eta)(i)) \leq L_4 \text{ for all } i.$$

Proof. Let τ, σ be splittable to η and assume that σ is obtained from τ by a single shift at a mixed branch b. Let v, w be the two switches of τ on which b is incident. Note that since τ is complete and hence recurrent, the switches v, w are distinct: otherwise b defines an embedded circle of class C^1 in τ which contains a single switch. There is a trainpath $\zeta : [0, 3] \to \tau$ where $b = \zeta[1, 2]$ and such that the half-branches $\zeta[1, 3/2], \zeta[2, 5/2]$ are large (see Figure C). Up to isotopy, the shift then moves the neighbor of ζ at $\zeta(1) = v$ across the switch $\zeta(2) = w$. This means that shifting exchanges the two switches v, w along ζ. The natural bijection $\varphi(\tau, \sigma)$ of the branches of τ onto the branches of σ preserves the type of the branches (i.e. large, mixed, small). In particular, the branch $\varphi(\tau, \sigma)(b)$ in σ is mixed, and τ can be obtained from σ by a single shift at $\varphi(\tau, \sigma)(b)$.

Throughout we use the following immediate consequence of uniqueness of splitting sequences (Lemma 5.1 of [H09]). If $\sigma \in \mathcal{V}(\mathcal{TT})$ is splittable to $\eta \in \mathcal{V}(\mathcal{TT})$ and if ξ is obtained from σ by a splitting sequence and carries η then ξ is splittable to η. We subdivide the argument into seven steps.

Step 1:

Let e be a large branch in τ and assume that a splitting sequence connecting τ to η includes a split at e. Then the minimal cardinality of the preimage of a point x contained in the interior of e under *any* carrying map $F : \eta \to \tau$ is at least two (see the proof of Lemma 5.1 of [H09]). Up to isotopy, for any neighborhood U of the branch b in S there is a map $F : S \to S$ of class C^1 which is homotopic to the identity, whose restriction to σ is a carrying map $\sigma \to \tau$ and which is the identity on $S - U$. Therefore the minimal cardinality of the preimage of any point y in the interior of $\varphi(\tau, \sigma)(e)$ under any carrying map $\eta \to \sigma$ is at least two as well. This implies that a splitting sequence connecting σ to η includes a split at $\varphi(\tau, \sigma)(e)$ (see the proof of Lemma 5.1 of [H09]). In other words, a large branch e of τ is a non-splittable η-configuration in τ if and only if $\varphi(\tau, \sigma)(e)$ is a non-splittable η-configuration in σ.

Step 2:

Let e be a large branch in τ which is not incident on an endpoint of b. Then e is contained in the complement of a small neighborhood U of b in S. Up to isotopy, the intersection of τ with $S - U$ coincides with the intersection of σ with $S - U$. The intersection with $S - U$ of the train track τ_1 obtained from τ by a right (or left) split at e is isotopic to the intersection with $S - U$ of the train track σ_1 obtained from σ by a right (or left) split at e. Thus σ_1 carries τ_1. Therefore if τ_1 is splittable to η then σ_1 carries η and hence σ_1 is splittable to η.

Step 3:

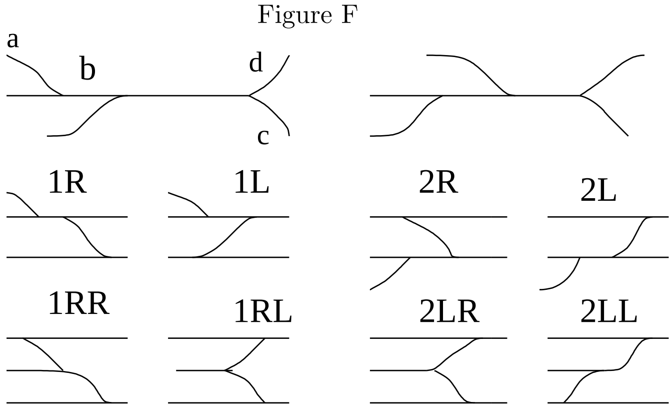
Let e be a large branch in τ such that one of the endpoints of e is a switch v on which the branch b is incident. This means that there is a trainpath $\rho : [0, 2] \to \tau$ of length two such that $\rho[0, 1] = b$ and $\rho[1, 2] = e$. Assume that the train track τ_1 obtained from τ by a right (or left) split at e is splittable to η and that the same holds true for the train track σ_1 obtained from σ by a right (or left) split at $\varphi(\tau, \sigma)(e)$. Figure F (which is just Figure 2.3.3 of [PH92], reproduced here for convenience) shows that if b is a winner of the split connecting τ to τ_1 then $\varphi(\tau, \sigma)(b)$ is a loser of the split connecting σ to σ_1. In Figure F, this is for example the case if τ is as in the left hand side of the figure and if τ_1 is obtained from τ by a right split.

Assume without loss of generality that b is a winner of the right (or left) split connecting τ to τ_1. Then there is a train track τ_2 which is obtained from τ_1 by a single right (or left) split at $\varphi(\tau, \tau_1)(b)$ and such that τ_2 can be obtained from σ_1 by a single shift. Since σ_1 is splittable to η, the train track τ_2 carries η and hence τ_2 is splittable to η. Therefore, in this case $\rho[0, 2]$ is contained in an η-configuration of τ. There is a train track $\xi_1 \in E(\tau, \gamma(\tau, \eta)(1))$ which can be obtained from a train track $\xi_2 \in E(\sigma, \gamma(\sigma, \eta)(1))$ by a single shift.

Step 4:

Let e be a large branch in τ such that one of the endpoints of e is a switch v on which the branch b is incident, i.e. that there is a trainpath $\rho : [0, 2] \to \tau$ of length two with $\rho[0, 1] = b$ and $\rho[1, 2] = e$. Assume that the train track τ_1 obtained from τ by a right (or left) split at e is splittable to η and that the same holds true for

Figure F



the train track σ_1 obtained from σ by a left (or right) split at $\varphi(\tau,\sigma)(e)$. We claim that in this case both τ and σ can be modified with a splitting sequence of length 2 to a train track $\xi \in E(\tau,\eta)\cap E(\sigma,\eta)$.

By the discussion in Step 3 above and uniqueness of splitting sequences, the branch b is a winner of the split connecting τ to τ_1, and $\varphi(\tau,\sigma)(b)$ is a winner of the split connecting σ to σ_1. The branches $b_1 = \varphi(\tau,\tau_1)(b)$ and $c_1 = \varphi(\sigma,\sigma_1)(\varphi(\tau,\sigma)(b))$ are large. In Figure F, this corresponds for example to the case that τ is as in the left had side of the figure, that τ_1 is obtained from τ by a right split and that σ_1 is obtained from σ by a left split.

If the train track τ_1' obtained from τ_1 by a right (or left) split at b_1 is splittable to η then τ_1' can be modified with a single shift to the train track σ_1' obtained from σ by a right (or left) split at $\varphi(\tau,\sigma)(e)$. Then σ_1' carries η and hence is splittable to η. This violates uniqueness of splitting sequences connecting σ to η (Lemma 5.1 of [H09]). Thus if a train track obtained from τ_1 by a split at b_1 is splittable to η, then this is the train track τ_2 obtained from τ_1 by a left (or right) split at b_1. Figure F shows that τ_2 is isotopic to a train track obtained from σ_1 by a right (or left) split at c_1 and the claim holds true.

If no train track obtained from τ_1 by a split at b_1 is splittable to η then $\varphi(\tau_1,\eta)(b_1)$ is a large branch in η. The train track τ' obtained from τ_1 by a right (or left) split at b_1 is splittable to the train track η' obtained from η by a right (or left) split at $\varphi(\tau_1,\eta)(b_1)$. Since σ_1 is splittable to η, it is also splittable to η'. On the other hand, the train track σ' is obtained from σ by a right (or left) split at $\varphi(\tau,\sigma)(e)$ is a modification of τ' with a single shift and hence it carries η'. This contradicts uniqueness of splitting sequences connecting σ to η' (which also holds true in the case that η' is not recurrent and hence not complete). The above claim is proven.

Step 5:

Let $\rho : [0,n] \to \tau$ be an η-configuration of τ which contains the mixed branch b.

Figure F shows that $\rho' = \varphi(\tau,\sigma)(\rho - b)$ is a reduced trainpath in σ or a reduced circle. Namely, via perhaps reversing the orientation of ρ we may assume that

$b = \rho[i, i+1]$ for some $i \in \{0, \dots, n-2\}$ and that the neighbor of ρ at $\rho(i)$ is incoming. (This means that in Figure C, ρ passes from the left to the right.) The neighbor of ρ at $\rho(i+1)$ is also incoming. Since ρ is reduced, the neighbor of ρ at $\rho(i+1)$ is contained in the same side of ρ as the neighbor at $\rho(i)$ in a small neighborhood of $\rho[0, n]$ in S. (Thus in Figure C, if a neighborhood of b in τ is as in the left part of the picture then $\rho[i-1, i]$ is the branch beginning at the top left corner of the picture, and if a neighborhood of b in τ is as in the right part of the picture then $\rho[i-1, i]$ is the branch beginning at the bottom left corner.)

The trainpath $\rho[i, n]$ is two-sided large and reduced. Since $\rho[i+1, i+3/2]$ is a large half-branch, by definition of an η-configuration there is a train track τ_1 which can be obtained from τ by a sequence of ρ-splits at branches contained in $\rho[i+2, n]$ such that $\varphi(\tau, \tau_1)(\rho[i+1, i+2])$ is a large branch. There also is a train track σ_1 which can be obtained from σ by a sequence of ρ'-splits and which can be obtained from τ_1 by a single shift at $\varphi(\tau, \tau_1)(b)$. By the definition of an η-configuration, the train track τ_2 obtained from τ_1 by a ρ-split at the branch $\varphi(\tau, \tau_1)(\rho[i+1, i+2])$ (with the mixed branch $\varphi(\tau, \tau_1)(b)$ as a winner) followed by a ρ-split at the branch $\varphi(\tau, \tau_1)(\rho[i, i+1])$ is splittable to η. By Step 3 above, the train track τ_2 can be modified with a single shift at the mixed branch $\varphi(\tau, \tau_2)(b)$ to a train track σ_2 obtained from σ by a sequence of ρ'-splits.

Reapply this consideration to the maximal two-sided large trainpath contained in $\varphi(\tau, \tau_2)(\rho)$. Since the winner of any split in a splitting sequence which modifies τ to its image under the $\rho - \eta$-multi-split is contained in ρ, together with Steps 1-3 above we conclude that there is a train track $\tilde{\sigma}$ which can be obtained from $\gamma(\tau, \eta)(1)$ by a single shift and which is splittable to $\gamma(\sigma, \eta)(1)$.

Step 6:

Let $\rho : [0, n] \to \tau$ be an η-configuration such that the mixed branch b is incident on a switch contained in $\rho[0, n]$ but is not contained in $\rho[0, n]$. There are now two possibilities.

In the first case, b is incident on a switch $\rho(i)$ for some $i \in \{1, \dots, n-1\}$. Assume that b is an incoming neighbor of ρ, i.e. that the half-branch $\rho[i, i+1/2]$ is large. By the discussion in Step 5 above, there is a reduced trainpath ρ' in σ which contains $\varphi(\tau, \sigma)(\rho \cup b)$. Moreover, if τ_1 is obtained from τ by a sequence of ρ-splits at branches in $\rho[i+1, n]$ and if $\varphi(\tau, \tau_1)(\rho[i, i+1]$ is a large branch, then $\varphi(\tau, \tau_1)(b)$ is a loser of the ρ-split of τ_1 at $\varphi(\tau, \tau_1)(\rho[i, i+1])$. As a consequence, this case corresponds to an exchange of the roles of τ and σ in Step 5 above.

Together with Step 1 and Step 5 above, we conclude the following. If either the mixed branch b is contained in an η-configuration of τ or if b is incident on a switch contained in the interior of an η-configuration of τ then $\gamma(\sigma, \eta)(1)$ can be obtained from $\gamma(\tau, \eta)(1)$ by a single shift.

If b is incident on the endpoint $\rho(0)$ of the η-configuration ρ of τ then $\varphi(\tau, \sigma)(\rho)$ is a reduced trainpath in σ. After modifying τ, σ with a sequence of ρ-splits (or $\varphi(\tau, \sigma)(\rho)$-splits) and perhaps reversing the orientation of ρ we may assume that

$\rho[0,1]$ is a large branch. By Step 3 above, since b is not contained in the η-configuration ρ, the branch b can not be a winner of the η-split at $\rho[0,1]$. This means that either $\varphi(\tau,\sigma)(b) \cup \varphi(\tau,\sigma)(\rho)$ is contained in an η-configuration of σ and $\gamma(\sigma,\eta)(1)$ can be obtained from $\gamma(\tau,\eta)(1)$ by a single shift, or $\varphi(\tau,\sigma)(b)$ is a loser of an η-split of σ at $\varphi(\tau,\sigma)(\rho[0,1])$ and by Step 4 above, both τ,σ are splittable to the same train track $\xi \in E(\tau,\eta) \cap E(\sigma,\eta)$ with a splitting sequence of uniformly bounded length.

Step 7:

Steps 1-6 above and Corollary 6.8 now imply the following. Either $\gamma(\sigma,\eta)(1)$ can be obtained from $\gamma(\tau,\eta)(1)$ by a single shift, or

$$d(\gamma(\tau,\eta)(i), \gamma(\sigma,\eta)(i)) \leq 2L_3 D \text{ for all } i$$

where $L_3 > 0$ is as in Corollary 6.8 and where $D > 0$ is a universal constant.

If $\gamma(\sigma,\eta)(1)$ can be obtained from $\gamma(\tau,\eta)(1)$ by a single shift then use the above consideration for to $\gamma(\tau,\eta)(1)$ and $\gamma(\sigma,\eta)(1)$. Since by invariance under the action of the mapping class group and cocompactness, the distance in $\mathcal{TT}$ between any two shift equivalent train tracks is uniformly bounded, we conclude that $\gamma(\tau,\eta)$ and $\gamma(\sigma,\eta)$ are weight-L_4 fellow travellers for a universal constant $L_4 > 1$ as claimed. This completes the proof of the lemma. □

As a preparation for a general control of balanced splitting paths for train tracks related by shifting we need

Lemma 6.10. *For every $k > 0$ there is a number $n(k) > 0$ with the following property. Let τ,η be complete train tracks with $d(\tau,\eta) \leq k$ which carry a common complete geodesic lamination λ. Then τ,η are splittable to the same complete train track σ with a splitting sequence of length at most $n(k)$ each.*

Proof. Let τ,η be any two train tracks which carry a common complete geodesic lamination λ. Then the set of all complete geodesic laminations carried by both τ,η is open and closed in $\mathcal{CL}$ (Lemma 2.3 of [H09]), and it is non-empty. Therefore there is a minimal complete geodesic lamination μ carried by both τ,η (this is explained in detail in the proof of Lemma 3.3 and Lemma 3.4 of [H09]). By Corollary 2.4.3 of [PH92], the train tracks τ,η are splittable to a common train track ζ which carries μ. Since μ is complete, the train track ζ is complete as well.

Now there are only finitely many orbits under the action of the mapping class group of pairs of complete train tracks which carry a common complete geodesic lamination and whose distance is at most k. By invariance under the action of the mapping class group, the lemma follows. □

The next lemma is the main technical step towards the proof of Theorem 6.4.

Lemma 6.11. *There is a number $L_5 > 0$ with the following property. Let $\tau,\sigma \in \mathcal{V}(\mathcal{TT})$ be splittable to $\eta \in \mathcal{V}(\mathcal{TT})$ and assume that σ is carried by τ. Then the balanced splitting paths $\gamma(\tau,\eta), \gamma(\sigma,\eta)$ are weight-L_5 fellow travellers.*

Proof. For convenience of terminology, call two complete train tracks ζ_1, ζ_2 *shift equivalent* if ζ_1 can be obtained from ζ_2 by a sequence of shifts. This clearly defines an equivalence relation on the set of all complete train tracks on S.

We divide the proof of the lemma into two steps.

Step 1:

By Theorem 2.4.1 of [PH92], if $\sigma \in \mathcal{V}(\mathcal{TT})$ is carried by $\tau \in \mathcal{V}(\mathcal{TT})$ then σ can be obtained from τ by a splitting and shifting sequence. In other words, τ can be connected to σ by a sequence $\{\eta(i)\}_{0\leq i\leq k}$ of minimal length such that for all i, $\eta(i+1)$ can be obtained from $\eta(i)$ either by a single split or a single shift. We call such a sequence a *stretched-out splitting and shifting sequence.*

We claim that there is a number $L > 1$ with the following property. Assume that σ can be obtained from τ by a stretched-out splitting and shifting sequence of length at most n. If τ, σ are splittable to a common complete train track η then the balanced splitting paths $\gamma(\tau,\eta)$ and $\gamma(\sigma,\eta)$ satisfy

$$d(\gamma(\tau,\eta)(i), \gamma(\sigma,\eta)(i)) \leq nL \text{ for all } i. \tag{17}$$

To determine a number $L > 1$ with this property we need the following preparation. Let $\alpha, \beta \in \mathcal{V}(\mathcal{TT})$. Assume that β is obtained from α by a single shift and that α is splittable to a train track ζ. Then ζ is carried by β. Let λ be a complete geodesic lamination carried by ζ. By Proposition A.6 of [H09], there is a number $k > 0$ such that β is splittable to a train track ξ which carries λ and is such that $d(\zeta, \xi) \leq k$. By Lemma 6.10, there is a train track ζ' which can be obtained from both ζ, ξ by a splitting sequence of length at most $n(k)$. Thus $\alpha, \beta, \zeta, \xi$ are all splittable to ζ'.

By Corollary 6.8, the balanced splitting paths $\gamma(\alpha,\zeta), \gamma(\alpha,\zeta')$ are weight-L_3 fellow travellers. Thus Lemma 6.9, applied to α, β, ζ', implies that there is a number $L > 2L_3 n(k)$ so that the balanced splitting paths $\gamma(\alpha,\zeta)$ and $\gamma(\beta,\zeta')$ satisfy

$$d(\gamma(\alpha,\zeta)(i), \gamma(\beta,\zeta')(i)) \leq L/2 \text{ for all } i. \tag{18}$$

We show by induction on $n \geq 0$ that the inequality (17) holds true for this number L. Note first that if the minimal length of a stretched out splitting and shifting sequence connecting τ to σ vanishes then $\tau = \sigma$ and there is nothing to show. So assume that the claim holds true whenever there is a stretched out splitting and shifting sequence connecting τ to σ whose length does not exceed $n-1$ for some $n \geq 1$.

Let σ be obtained from τ by a stretched-out splitting and shifting sequence of length n. If this sequence can be arranged to begin with a split at a large branch e of τ then this split is an η-split since σ is splittable to η by assumption. Let $\tilde{\tau}$ be the split track. Then $\tilde{\tau}$ can be connected to σ by a stretched-out splitting and shifting sequence of length $n-1$. Therefore by the induction hypothesis, the balanced splitting paths $\gamma(\tilde{\tau},\eta), \gamma(\sigma,\eta)$ satisfy the estimate (17) for $n-1$. By Corollary 6.8, the balanced splitting paths $\gamma(\tau,\eta), \gamma(\tilde{\tau},\eta)$ are weight-L_3 fellow travellers and hence they satisfy the estimate (17) with $n = 1$. Together we conclude that the estimate (17) holds true for the balanced splitting paths $\gamma(\tau, \eta), \gamma(\sigma, \eta)$.

If there is no splitting and shifting sequence of length n connecting τ to σ which begins with a single split then let $\tilde{\sigma}$ be a train track obtained from τ by a single shift which can be connected to σ by a stretched-out splitting and shifting sequence of length $n-1$. Let ζ be a train track with the property that both $\eta, \tilde{\sigma}$ are splittable to ζ and that the length of a splitting sequence connecting η to ζ is at most $n(k)$. Such a train track exists by the above discussion. Apply the induction hypothesis to the balanced splitting paths $\gamma(\tilde{\sigma}, \zeta), \gamma(\sigma, \zeta)$. We conclude that

$$d(\gamma(\tilde{\sigma}, \zeta)(i), \gamma(\sigma, \zeta)(i)) \leq (n-1)L \text{ for all } i. \tag{19}$$

On the other hand, the estimate (18) yields that

$$d(\gamma(\tau, \eta)(i), \gamma(\tilde{\sigma}, \zeta)(i)) \leq L/2 \text{ for all } i. \tag{20}$$

Moreover, since $L > 2L_3 n(k)$ by assumption, Corollary 6.8 applied to σ, η, ζ implies that

$$d(\gamma(\sigma, \zeta)(i), \gamma(\sigma, \eta)(i)) \leq L/2 \text{ for all } i. \tag{21}$$

The estimates (20,19,21) together yield the inequality (17) which completes the induction step. The estimate (17) is proven.

Step 2:

Using Step 1 above, we are now able to complete the proof of the lemma. Namely, assume that $\sigma \prec \tau$ are both splittable to a complete train track η. Let λ be a complete geodesic lamination carried by η. By Proposition A.6 of [H09], τ is splittable to a complete train track τ' which carries λ and is such that

$$d(\tau', \sigma) \leq \kappa_0$$

where $\kappa_0 > 0$ is a universal constant.

By Lemma 6.7 of [H09], there is a number $\kappa_1 > 0$ only depending on κ_0 and there is a train track σ' which carries λ and which can be obtained from both τ', σ by a stretched-out splitting and shifting sequence of length at most κ_1. Another application of Lemma 6.7 of [H09] yields a universal number $\kappa_2 > 0$ and a train track η' with

$$d(\eta, \eta') \leq \kappa_2$$

which is carried by both σ' and η and hence by τ', σ. Proposition A.6 of [H09] and Lemma 6.10 then show the existence of a train track η'' with

$$d(\eta, \eta'') \leq \kappa_3$$

for a universal constant $\kappa_3 > 0$ such that both σ' and η are splittable to η''.

By Corollary 6.8, applied to $\tau, \eta, \tau', \eta''$, the balanced splitting paths

$$\gamma(\tau, \eta), \gamma(\tau', \eta'')$$

are weight-L_3-fellow travellers. Since τ' can be connected to σ' by a stretched-out splitting and shifting sequence of length at most κ_1 and since σ', τ' are both splittable to η'', Step 1 above shows that the balanced splitting paths

$$\gamma(\tau', \eta''), \gamma(\sigma', \eta'')$$

are uniform fellow travellers. Another application of Step 1 yields that the balanced splitting paths

$$\gamma(\sigma',\eta''),\gamma(\sigma,\eta'')$$

are uniform fellow travellers as well. Finally Corollary 6.8 shows that the balanced splitting paths

$$\gamma(\sigma,\eta''),\gamma(\sigma,\eta)$$

are uniform fellow travellers. Together the lemma follows. □

We use Lemma 6.11 to show

Corollary 6.12. *For every $R>0$ there is a number $L_6=L_6(R)>0$ with the following property. Let $\tau,\eta\in\mathcal{V}(\mathcal{TT})$ be splittable to τ',η'. Assume that τ',η' carry a common complete geodesic lamination λ and that $d(\tau,\eta)\leq R, d(\tau',\eta')\leq R$. Then*

$$d(\gamma(\tau,\tau')(i),\gamma(\eta,\eta')(i))\leq L_6 \text{ for all } i.$$

Proof. Let τ,τ',η,η' be as in the lemma. By Lemma 6.10, τ',η' are splittable to the same complete train track ζ with a splitting sequence of length at most $n(R)$. Let μ be a complete geodesic lamination carried by ζ.

By Lemma 6.7 of [H09], there is a complete train track β_0 which is carried by both τ,η, which carries μ and such that

$$d(\tau,\beta_0)\leq p_0(R), d(\eta,\beta_0)\leq p_0(R)$$

where $p_0(R)>0$ only depends on R. Another application of Lemma 6.7 of [H09] and of Lemma 6.10 shows that β_0 is splittable to a train track ζ' which can be obtained from ζ by a splitting sequence of length bounded from above by a number $p_1(R)>0$ only depending on R (compare the discussion in Step 2 of the proof of Lemma 6.11). Then Lemma 6.11 yields that

$$d(\gamma(\tau,\zeta')(i),\gamma(\beta_0,\zeta')(i))\leq p_2(R), d(\gamma(\eta,\zeta')(i),\gamma(\beta_0,\zeta')(i))\leq p_2(R)$$

for all i where $p_2(R)>0$ only depends on R.

On the other hand, since the length of a splitting sequence connecting τ',η' to ζ' is bounded form above by a number only depending on R, Lemma 6.6 shows that there is a number $p_3(R)>0$ such that

$$d(\gamma(\tau,\tau')(i),\gamma(\tau,\zeta')(i))\leq p_3(R)$$

and that

$$d(\gamma(\eta,\eta')(i),\gamma(\eta,\zeta')(i))\leq p_3(R)$$

for all i. Together this shows the corollary. □

For any recurrent train track σ which is splittable to a recurrent train track σ' there is a unique balanced splitting path $\gamma_0(\sigma,\sigma')$ connecting σ to σ'. Let τ be a complete extension of σ and assume that the complete train track τ' is the endpoint of a sequence issuing from τ which is induced by a splitting sequence connecting σ to σ' as in Proposition 3.5. For each i there is a train track $\tau(i)\in E(\tau,\tau')$ which contains the train track $\gamma_0(\sigma,\sigma')(i)$ as a subtrack. An example is the endpoint of a sequence issuing from τ which is induced by a splitting sequence connecting σ to

$\gamma_0(\sigma,\sigma')(i)$. Such a train track is not unique, but by Lemma 3.4, the diameter in $\mathcal{TT}$ of the set of all train tracks with this property is uniformly bounded (by q^3).

Our next goal is to estimate the distance between the train tracks $\tau(i)$ and $\gamma(\tau,\tau')(i)$.

Lemma 6.13. *There is a number $L_7 > 0$ with the following property. Let σ be a recurrent train track which is splittable to the recurrent train track σ'. Let τ be a complete extension of σ and let τ' be the endpoint of a sequence issuing from τ which is induced by a splitting sequence connecting σ to σ'. For each i let $\tau(i) \in E(\tau,\tau')$ be a train track which contains $\gamma_0(\sigma,\sigma')(i)$ as a subtrack. Then*

$$d(\gamma(\tau,\tau')(i),\tau(i)) \leq L_7 \text{ for all } i.$$

Proof. We divide the proof of the lemma into four steps.

Step 1:

Let σ be a recurrent train track which is splittable to a recurrent train track σ'. Then σ is connected to σ' by a balanced splitting path $\gamma_0(\sigma,\sigma')$. Let τ be a complete extension of σ and assume that there is a complete train track τ' which can be obtained from τ by a sequence induced by a splitting sequence connecting σ to σ'. Let $\gamma(\tau,\tau')$ be the balanced splitting path connecting τ to τ'. By the forth property in Proposition 3.5, for each i the train track $\gamma(\tau,\tau')(i)$ contains a subtrack $\sigma_i \in E(\sigma,\sigma')$. We claim that σ_i is splittable to $\gamma_0(\sigma,\sigma')(i)$.

For this we proceed by induction on the length of the balanced splitting path $\gamma(\tau,\tau')$. If this length vanishes then there is nothing to show, so assume the claim holds true whenever the length of this path is at most $k-1$ for some $k \geq 1$.

Let $\sigma,\sigma',\tau,\tau'$ be such that the length of the balanced splitting path $\gamma(\tau,\tau')$ equals k. The splittable τ'-configurations in τ are pairwise disjoint trainpaths or circles with images in σ. Let $\rho : [0,m] \to \tau$ be such a splittable τ'-configuration and let ξ be the train track obtained from τ by a $\rho - \tau'$-multi-split. By construction, ξ contains a subtrack ζ which can be obtained from σ by a splitting sequence at branches contained in $\rho[0,m]$. Since splits at large branches of σ contained in the distinct τ'-configurations of τ commute, it now suffices to show that ζ is splittable to $\gamma_0(\sigma,\sigma')(1)$.

Consider first the case that ρ is a reduced trainpath in τ. Then every two-sided large trainpath $\rho_0 : [0,n] \to \sigma$ with $\rho_0[0,n] \subset \rho[0,m]$ is reduced. There is a maximal two-sided large trainpath ρ_0 in σ which contains every two-sided large trainpath in σ with image in $\rho[0,m]$. If $\{\xi_j\}$ is a sequence of ρ-splits transforming τ to ξ, then for each j there is a subtrack $\zeta_j \in E(\sigma,\sigma')$ of ξ_j so that either $\zeta_j = \zeta_{j-1}$ or that ζ_j can be obtained from ζ_{j-1} by a ρ_0-split. As a consequence, ζ is indeed splittable to $\gamma_0(\sigma,\sigma')(1)$.

Now let $\rho[0,m] = c$ be a splittable reduced circle in τ. Since c is splittable, by the definition of an induced sequence the image of c in σ contains a large branch in σ and hence c is a reduced circle in σ. Let as before θ_c be the positive Dehn twist about c. Then $\theta_c^{\pm}\tau$ contains $\theta_c^{\pm}\sigma$ as a subtrack. By definition of a balanced sequence, any train track which can be obtained from σ by a sequence of c-splits and which is splittable to both σ' and $\theta_c^{\pm}(\sigma)$ is also splittable to $\gamma_0(\sigma,\sigma')(1)$. On the other hand, by the definition of a $\rho-\tau'$-multi-split, the train track ξ is splittable to $\theta_c^{\pm}(\tau)$. Together this just means once again that ζ is splittable to $\gamma_0(\sigma,\sigma')(1)$.

As a consequence, the subtrack σ_1 of $\gamma(\tau,\tau')(1)$ is splittable to $\gamma_0(\sigma,\sigma')(1)$. An inductive application of Lemma 6.5 shows that for each $i\geq 1$ the train track $\gamma_0(\sigma_1,\sigma')(i-1)$ is splittable to $\gamma_0(\sigma,\sigma')(i)$. The induction hypothesis, applied to $\gamma(\tau,\tau')(1),\tau',\sigma_1,\sigma'$, then implies that for each i the train track σ_i is splittable to $\gamma_0(\sigma,\sigma')(i)$. This completes the proof of the above claim.

Step 2:

Let $\rho_0:[0,n]\to\sigma$ be an embedded reduced trainpath. Then ρ_0 defines an embedded two-sided large trainpath $\rho:[0,m]\to\tau$ with $\rho[0,m]=\rho_0[0,n]$ as sets. The trainpath ρ may not be reduced. Similarly, a reduced circle in σ defines an embedded circle in τ which may not be reduced.

Assume however for the moment that each σ'-configuration in σ determines in this way a reduced trainpath or a reduced circle in τ. We claim that then the train track $\gamma(\tau,\tau')(1)$ contains the train track $\gamma_0(\sigma,\sigma')(1)$ as a subtrack.

To see that this is indeed the case let $\{\sigma_i\}_{0\leq i\leq j}$ be a splitting sequence connecting $\sigma_0=\sigma$ to $\sigma_j=\gamma_0(\sigma,\sigma')(1)$. Let τ_1 be the endpoint of a sequence issuing from τ which is induced by the sequence $\{\sigma_i\}_{0\leq i\leq j}$ as in Proposition 3.5. If ρ_i $(i=1,\dots,k)$ are the trainpaths or circles in τ whose images are the σ'-configurations of σ then it follows from the definition of an induced sequence that τ_1 can be obtained from τ by a successive transformation with ρ_i-splits. The splitting sequence connecting τ to τ_1 includes a split at each branch of τ contained in ρ_i. Since the trainpaths or circles ρ_i are reduced by assumption, we conclude that for each i, ρ_i is contained in a τ'-configuration of τ and τ_1 is splittable to $\gamma(\tau,\tau')(1)$.

By the fourth part of Proposition 3.5, the train track $\gamma(\tau,\tau')(1)$ contains a subtrack $\hat\sigma$ which can be obtained from σ by a splitting sequence and which is splittable to σ'. By Step 1 above, $\hat\sigma$ is splittable to $\gamma_0(\sigma,\sigma')(1)$. On the other hand, the discussion in the previous paragraph shows that $\gamma_0(\sigma,\sigma')(1)$ is splittable to $\hat\sigma$. This shows that $\gamma(\tau,\tau')(1)$ contains $\gamma_0(\sigma,\sigma')(1)$ as a subtrack as claimed.

Step 3:

The discussion in Step 2 above shows the following. If for every σ'-configuration ρ_0 in σ the trainpath (or circle) defined by ρ_0 in τ is reduced, then $\gamma(\tau,\tau')(1)$ contains $\gamma_0(\sigma,\sigma')(1)$ as a subtrack. As a consequence, by induction the lemma holds true for the balanced splitting path $\gamma(\tau,\tau')$ (with a universal constant $L=L_7>0$) if for every $i\geq 0$ the following is satisfied. Let σ_i be the subtrack of the train track $\gamma(\tau,\tau')(i)$ which can be obtained from σ by a splitting sequence and which is splittable to σ'. Then every σ'-configuration in σ_i defines a reduced trainpath or a reduced circle in $\gamma(\tau,\tau')(i)$.

The goal of this step is to determine a set of complete extensions of σ with this property. For this define a branch a of σ to be σ'-*split* if a splitting sequence connecting σ to σ' includes a split at a (under the natural identification of the branches of train tracks in a splitting sequence, see Lemma 5.1 of [H09]). Call a complete extension τ of σ *simple for* σ' if the following holds true.

(1) Let b be a half-branch of τ which is contained in a complementary region C of σ and which is incident on a switch v contained in a non-smooth side ξ of C. There is a unique trainpath $\zeta : [0,\ell] \to \tau$ such that $\zeta[1/2,1] = b$, $\zeta[1,\ell] \subset \xi$ and that $\zeta(\ell)$ is a cusp of C. If the branch a of σ containing the switch v in its interior is σ'-split then we require that a is incident on the switch $\zeta(\ell)$ of σ, i.e. $\zeta[1,\ell]$ is contained in a single branch of σ. Moreover we require that $\zeta[1,\ell]$ consists of mixed branches. The neighbors of $\zeta[1,\ell]$ are all contained in C.
(2) Let $\xi : [0,n] \to \sigma$ be a smooth side of a complementary region C of σ. We require that there is no half-branch of τ contained in C which is incident on a switch contained in the interior of any σ'-split branch in $\xi[0,n]$.

In the sequel we call a complete extension of σ which is simple for σ' a simple complete extension of σ whenever no confusion is possible. We claim that a simple complete extension τ of σ has the property described in the first paragraph of this step.

For this we first show that every σ'-configuration $\rho_0 : [0,n] \to \sigma$ of σ defines a reduced trainpath or a reduced circle in a simple complete extension τ of σ. Namely, by definition of a σ'-configuration, every branch of σ contained in $\rho_0[0,n]$ is σ'-split. Let b be a half-branch of τ contained in a complementary region C of σ which is incident on a switch $v \in \rho_0[0,n]$. By the second part in the definition of a simple complete extension of σ, the switch v of τ is contained in a non-smooth side ξ of C. Let $\zeta : [0,\ell] \to \tau$ be the trainpath with $\zeta[1/2,1] = b$ and $\zeta[1,\ell] \subset \xi$ which terminates at a cusp $\zeta(\ell) = v_0$ of C. By the first part in the definition of a simple complete extension of σ, $\zeta[1,\ell]$ is contained in a single branch a of σ. The branch a is small at v_0. Since ρ_0 is two-sided large and contains a, the switch v_0 is contained in the interior of $\rho_0[0,n]$. Now by assumption, the subarc of a connecting v to v_0 consists of mixed branches of τ, and the neighbor of ρ_0 at v_0 is a half-branch of σ contained in the boundary of C which is of the same type (incoming or outgoing and left or right) as b along ρ_0.

As a consequence, if $b \subset \tau - \sigma$ is any neighbor of the embedded arc $\rho_0[0,n]$ in τ then there is a neighbor of $\rho_0[0,n]$ in σ of the same type as b. Since ρ_0 is reduced by assumption, this implies that the trainpath ρ in τ which coincides with ρ_0 as a set is reduced as well. In other words, if τ is a simple complete extension of σ then every σ'-configuration in σ defines a reduced trainpath or a reduced circle in τ.

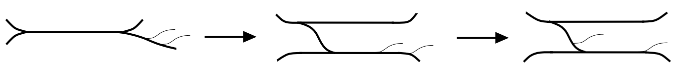
Step 2 implies that $\tau_1 = \gamma(\tau,\tau')(1)$ contains the train track $\sigma_1 = \gamma_0(\sigma,\sigma')(1)$ as an embedded subtrack. We claim that τ_1 is a simple complete extension of σ_1.

To see that this is the case, let again ρ_0 be a σ'-configuration in σ and let e be a large branch of σ contained in ρ_0. Then e is σ'-split and hence since τ is a simple complete extension of σ, the train track τ is tight at e. Let $\tilde\sigma$ be the train track obtained from σ by a σ'-split at e and let $\tilde\tau$ be the complete extension of $\tilde\sigma$ obtained from τ by a single split at e. Then up to isotopy, for any small neighborhood U in

S of the branch e the intersection of σ, τ with $S-U$ coincides with the intersection of $\tilde\sigma, \tilde\tau$ with $S-U$. Therefore if there is a neighbor $\tilde b$ of $\tilde\sigma$ in $\tilde\tau$ which does not meet the requirements in the definition of an extension of $\tilde\sigma$ which is simple for σ' then $\tilde b = \varphi(\tau, \tilde\tau)(b)$ where b is incident on a switch contained in the interior of a σ'-split branch of σ which is a winner of the split connecting σ to $\tilde\sigma$.

Now let the branch a of σ be a winner of the split connecting σ to $\tilde\sigma$. Assume that a is σ'-split. Let C be the complementary region of σ which has a cusp at an endpoint v_0 of e and which contains a in one of its sides, say in the side ξ. The complementary region $\tilde C$ of the train track $\tilde\sigma$ which contains up to isotopy the complement in C of a small neighborhood of the branch e has a side containing $\varphi(\sigma, \tilde\sigma)(e \cup \xi)$ (see Figure G).

Figure G



Let $\zeta : [0, \ell] \to a$ be the parametrization of the branch a as a trainpath on τ with $\zeta(0) = v_0$. For $0 < i < \ell$ denote by b_i the neighbor of a in τ incident on the switch $\zeta(i)$. Since a is a winner of the split transforming σ to $\tilde\sigma$, the branch $\varphi(\tau, \tilde\tau)(\zeta[0, 1]) \subset \tilde\tau$ is large (and of type 3). Since a is σ'-split by assumption, the train track α_1 obtained from $\tilde\tau$ by the (unique) $\tilde\sigma$-split at the large branch $\varphi(\tau, \tilde\tau)(\zeta[0, 1])$ is splittable to τ'. The train track α_1 contains $\tilde\sigma$ as a subtrack, and $\varphi(\tau, \alpha_1)(b_1)$ is a neighbor of $\tilde\sigma$ in α_1 which satisfes the first requirement in the definition of a simple complete extension of $\tilde\sigma$ (see Figure G for an illustration of this fact). Repeat this construction with the large branch $\varphi(\tau, \alpha_1)(\zeta[1, 2])$ and the neighbor $\varphi(\tau, \alpha_1)(b_2)$ of $\tilde\sigma$. After a uniformly bounded number of such steps we obtain a train track α_2 which is splittable to τ' and such that for every neighbor b of a in τ, the half-branch $\varphi(\tau, \alpha_2)(b)$ is incident on the branch $\varphi(\sigma, \tilde\sigma)(e)$.

A second application of this procedure to the second winner of the split connecting σ to $\tilde\sigma$ shows that there is a train track $\alpha_3 \in E(\tau, \gamma(\tau, \tau')(1))$ which is a simple complete extension of $\tilde\sigma$. By induction on the length of a splitting sequence connecting σ to $\gamma_0(\sigma, \sigma')(1)$ we conclude that there is a train track $\alpha_4 \in E(\tau, \gamma(\tau, \tau')(1))$ which is a simple complete extension of $\gamma_0(\sigma, \sigma')(1)$. This means in particular that every large branch of α_4 contained in a σ'-split branch of $\gamma_0(\sigma, \sigma')(1)$ is a large branch of α_4. Then necessarily $\alpha_4 = \gamma(\tau, \tau')(1)$ (see Step 1 above).

A successive application of this argument shows the following. If τ is a simple complete extension of σ then for every i the train track $\gamma(\tau, \tau')(i)$ contains $\gamma_0(\sigma, \sigma')(i)$ as a subtrack. In particular, the balanced splitting path $\gamma(\tau, \tau')$ satisfies

$$d(\gamma(\tau, \tau')(i), \tau(i)) \le L \tag{22}$$

with a universal constant $L > 0$.

Step 4:

In this final step we use steps 1)-3) to complete the proof of the lemma.

Let τ be any complete extension of a recurrent train track σ. Let $\{\sigma_i\}_{0\leq i\leq u}$ be a recurrent splitting sequence issuing from $\sigma=\sigma_0$ and let τ' be a train track obtained from τ by a sequence induced by the splitting sequence $\{\sigma_i\}_{0\leq i\leq u}$. Then τ' contains σ_u as a subtrack. Let λ be the complete τ'-extension of a σ_u-filling measured geodesic lamination. The goal of this step is to show that τ can be modified to a complete train track η with the following properties.

1. η carries λ.
2. η contains a train track $\tilde{\sigma}\in E(\sigma,\sigma')$ as a subtrack which can be obtained from σ by a splitting sequence of uniformly bounded length.
3. η can be obtained from τ by a splitting and shifting sequence of uniformly bounded length.
4. η is a complete extension of $\tilde{\sigma}$ which is simple for σ'.

Before we show how to construct such a train track η, we assume that such a train track η exists and use it to complete the proof of the lemma. Namely, assume a train track η with properties 1)-4) above is given which contains $\tilde{\sigma}\in E(\sigma,\sigma')$ as a subtrack. Let $\tilde{\tau}\in E(\tau,\tau')$ be obtained from τ by a sequence induced by a splitting sequence connecting σ to $\tilde{\sigma}$. Since the length of a splitting sequence connecting σ to $\tilde{\sigma}$ is uniformly bounded, the distance between τ and $\tilde{\tau}$ is uniformly bounded and hence by the third property above, the distance between $\tilde{\tau}$ and η is uniformly bounded as well.

Since η carries the complete extension λ of a σ'-filling measured geodesic lamination, Proposition 3.5 shows that there is a sequence issuing from η which is induced by a splitting sequence connecting $\tilde{\sigma}$ to σ' and which consists of train tracks carrying λ. Let η' be the endpoint of this sequence. Corollary 3.6 shows that the distance between τ' and η' is uniformly bounded. If for $i\geq 0$ we denote by $\eta(i)\in E(\eta,\eta')$ a train track which contains $\gamma_0(\tilde{\sigma},\sigma')(i)$ as a subtrack then by the fourth property above, inequality (22) can be applied and shows that

$$d(\gamma(\eta,\eta')(i),\eta(i))\leq L.$$

Apply Lemma 6.7 to the balanced splitting paths $\gamma_0(\sigma,\sigma'),\gamma_0(\tilde{\sigma},\sigma')$ connecting $\sigma,\tilde{\sigma}$ to σ'. We conclude that for each i, the train track $\gamma_0(\sigma,\sigma')(i)$ is splittable to $\gamma_0(\tilde{\sigma},\sigma')(i)$ with a splitting sequence of uniformly bounded length. Corollary 3.6 then shows that the distance between $\eta(i)$ and a train track $\tau(i)\in E(\tau,\tau')$ which contains $\gamma_0(\sigma,\sigma')(i)$ as a subtrack is bounded from above by a universal constant. Since by Corollary 6.12 the balanced splitting paths $\gamma(\tau,\tau'),\gamma(\eta,\eta')$ are uniform fellow travellers, the distance between $\gamma(\tau,\tau')(i)$ and $\tau(i)$ is uniformly bounded as claimed in the lemma.

To construct a modification of τ to a complete train track with the properties described in the second paragraph of this step, observe first that by the discussion in the proof of Proposition 3.5, there is a train track $\tilde{\sigma}\in E(\sigma,\sigma')$ which can be obtained from σ by a splitting sequence of uniformly bounded length and such that a splitting sequence connecting $\tilde{\sigma}$ to σ' does not include a split at any large branch which is contained in a smooth boundary component of a complementary region of

$\tilde{\sigma}$. By definition of an induced sequence, there is a train track $\tilde{\tau} \in E(\tau, \tau')$ which can be obtained from τ by a splitting sequence of uniformly bounded length and which contains $\tilde{\sigma}$ as a subtrack.

Let ξ be a non-smooth side of a complementary region C of $\tilde{\sigma}$ and let b be a half-branch of $\tilde{\tau}$ which is contained in C and is incident on a switch $v \in \xi$. Let $\zeta : [0, \ell] \to \xi$ be the unique trainpath with $\zeta[1/2, 1] = b$, $\zeta[1, \ell] \subset \xi$ and such that $\zeta(\ell)$ is a cusp of C. Note that $\zeta[1, 3/2]$ is large half-branch. In particular, $\zeta[1, 2]$ is the initial branch of a one-way trainpath as defined on p.127 of [PH92]. Thus if $\zeta[1, \ell]$ does not contain any large branch then all switches of τ along ζ are incoming, and all branches $\zeta[i, i+1]$ $(1 \leq i \leq \ell - 1)$ are mixed. Assume that b lies to the left (or right) of the branch a of $\xi \subset \sigma$ containing v in its interior with respect to the orientation of a induced by the oriented trainpath ζ. Then all left (or right) neighbors of $\zeta[1, \ell]$ can be shifted forward along ξ past every right (or left) neighbor. In the resulting train track τ_1, these shifted neighbors satisfy the requirements for a simple extension of $\tilde{\sigma}$. The train track τ_1 is shift equivalent to $\tilde{\tau}$ and hence it carries λ.

If $\zeta[0, \ell]$ contains a large branch then there is a sequence of shifts and σ-splits of uniformly bounded length which transforms τ to a train track τ_2 which carries λ, which contains σ as a subtrack and which is of the required form (compare the argument in the proof of Lemma 3.4). Together this completes the proof of the lemma. ☐

As an immediate corollary we obtain

Corollary 6.14. *For every $R > 0$ there is a number $L_8 = L_8(R) > 0$ with the following property. Let σ_0 be a recurrent train track and let $\{\sigma_i\}$ be a recurrent splitting sequence issuing from σ_0. Let τ, η be complete extensions of σ_0 with $d(\tau, \eta) \leq R$ and let τ', η' be the endpoints of a sequence induced by the splitting sequence $\{\sigma_i\}$ and issuing from τ, η. Then the balanced splitting paths $\gamma(\tau, \tau'), \gamma(\eta, \eta')$ are weight-L_8 fellow travellers.*

Proof. Let σ be a recurrent train track which is splittable to a recurrent train track σ' and let $\gamma_0(\sigma, \sigma')$ be the balanced splitting path connecting σ to σ'.

Let $R > 0$ and let τ, η be complete extensions of σ with $d(\tau, \eta) \leq R$. Let τ', η' be a train track obtained from τ, η by a sequence induced by a splitting sequence connecting σ to σ' as in Proposition 3.5. Then for every $i \geq 0$ there are train tracks $\beta(\tau, \tau')(i) \in E(\tau, \tau'), \beta(\eta, \eta')(i) \in E(\eta, \eta')$ which are obtained from τ, η by a sequence induced by a splitting sequence connecting σ to $\gamma_0(\sigma, \sigma')(i)$. The train tracks $\beta(\tau, \tau')(i), \beta(\eta, \eta')(i)$ contain $\gamma_0(\sigma, \sigma')(i)$ as a subtrack. By Corollary 3.6, there is a number $p = p(R) > 0$ such that

$$d(\beta(\tau, \tau')(i), \beta(\eta, \eta')(i)) \leq p(R) \text{ for all } i.$$

On the other hand, by Lemma 6.13 there is a number $L_7 > 0$ such that

$$d(\beta(\tau, \tau')(i), \gamma(\tau, \tau')(i)) \leq L_7, d(\beta(\eta, \eta')(i), \gamma(\tau, \tau')(i)) \leq L_7 \text{ for all } i.$$

Together this shows the corollary. ☐

By the discussion in Section 5, for a complete train track τ which contains an embedded simple closed curve c, splitting τ with a sequence of c-splits results in twisting τ along c. On the other hand, c-splits of τ commute with splits at large branches of τ which are disjoint from c. Thus if τ is splittable to a complete train track τ' which contains c as an embedded simple closed curve, then there is a splitting sequence $\{\tau_i\}$ connecting τ to τ' and there is a number $\ell \geq 0$ with the following property. The train track τ_ℓ can be obtained from τ by a splitting sequence at large branches disjoint from c, and τ' can be obtained from τ_ℓ by a sequence of c-splits. We call τ_ℓ the $(S-c)$-*contraction* of the pair (τ, τ'). Let θ_c be the positive Dehn twist about c. By Corollary 5.4, there is a number $p \in \mathbb{Z}$ such that

$$d(\tau', \theta_c^p(\tau_\ell)) \leq a_3.$$

We call p a *c-twist parameter* of the pair (τ, τ').

The following observation is immediate from the definitions, from Lemma 5.2 and Corollary 5.4 and from Corollary 6.8.

Lemma 6.15. *There is a number $L_9 > 0$ with the following property. Let $\tau \in \mathcal{V}(\mathcal{TT})$ be splittable to $\tau' \in \mathcal{V}(\mathcal{TT})$ and let c be a simple closed curve which is embedded in both τ, τ'. Let $\eta \in \mathcal{V}(\mathcal{TT})$ be the $(S-c)$-contraction of the pair (τ, τ') and let $p \in \mathbb{Z}$ be the c-twist parameter of (τ, τ'). Then for every $i \leq |p|$ we have*

$$d(\gamma(\tau, \tau')(i), \theta_c^{(\mathrm{sgn}\, i)i}\gamma(\tau, \eta)(i)) \leq L_9,$$

and $d(\gamma(\tau, \tau')(i), \theta_c^p\gamma(\tau, \eta)(i)) \leq L_9$ for $i \geq |p|$.

Proof. Let τ, τ', c be as in the lemma. Choose a complete τ'-extension λ of a c-filling measured geodesic lamination. By Lemma 5.3, there is a complete train track $\tau_1 \in E(\tau, \lambda)$ which can be obtained from τ by a sequence of c-splits of length at most a_2 and which contains c as a reduced circle. Let η be the $(S-c)$-contraction of the pair (τ, τ'). Then η is splittable with a sequence of c-splits of length at most a_2 to a train track $\tau_2 \in E(\tau_1, \lambda)$. By Lemma 5.2 and Corollary 5.4 and by the choice of λ, we may assume that $\theta_c^p\tau_2 \in E(\tau_1, \lambda)$ and that τ_2 is splittable to $\theta_c^p\tau_2$. By the definition of a move, for each $i \leq |p|$ we have

$$\gamma(\tau_1, \theta_c^p\tau_2)(i) = \theta_c^{(\mathrm{sgn}\, i)i}\gamma(\tau_1, \tau_2)(i).$$

The lemma is now immediate from this observation and from Corollary 6.8. ☐

We use Corollary 6.8 and Lemma 6.15 to show

Lemma 6.16. *There is a number $L_{10} > 0$ with the following properties. Let F be a marking for S and let τ, η be complete train tracks in standard form for F. Assume that τ, η are splittable to $\tau', \eta' \in \mathcal{V}(\mathcal{TT})$. Then the balanced splitting paths $\gamma(\tau, \tau'), \gamma(\eta, \eta')$ are weight-L_{10} fellow travellers.*

Proof. Let τ, η be train tracks in standard form for a marking F which are splittable to train tracks τ', η'. By Proposition 5.10 and by Theorem 2.3 and the following remark, there is a number $a > 0$ such that

$$d(\tau', \eta') \geq a(d(\tau', \Pi_{E(\tau,\tau')}(\eta')) + d(\Pi_{E(\eta,\eta')}(\tau'), \eta')) - 1/a.$$

Thus by two applications of Corollary 6.8, applied to the train track τ which is splittable to the train tracks $\tau', \Pi_{E(\tau,\tau')}(\eta')$ and to the train track η which is splittable to the train tracks $\eta', \Pi_{E(\eta,\eta')}(\tau')$, it suffices to show that the balanced splitting paths

$$\gamma(\tau, \Pi_{E(\tau,\tau')}(\eta')), \gamma(\eta, \Pi_{E(\eta,\eta')}(\tau'))$$

are uniform fellow travellers.

By the construction of the projections $\Pi_{E(\tau,\tau')}(\eta'), \Pi_{E(\eta,\eta')}(\tau')$ (see in particular the proof of Lemma 5.9) and by Lemma 6.15, for this it suffices to show that for every $R > 0, k > 0$ there is a constant $L(R,k) > 0$ with the following property. Let σ be a recurrent train track on S without closed curve components. Assume that the number of branches of σ is at most k. Let τ, η be complete extensions of σ with $d(\tau,\eta) \leq R$. Assume that every minimal geodesic lamination which is carried by both τ, η and which is not a simple closed curve is carried by σ. Let τ', η' be complete train tracks which can be obtained from τ, η by a splitting sequence and such that $d(\tau', \eta') \leq R$. Then the balanced splitting paths $\gamma(\tau,\tau'), \gamma(\eta,\eta')$ satisfy

$$d(\gamma(\tau,\tau')(i), \gamma(\eta,\eta')(i)) \leq L(R,k) \text{ for all } i. \tag{23}$$

Since k is bounded from above by the number of branches of complete train tracks on S, this is sufficient for the proof of the lemma.

We show this claim by induction on k. Note first that by the results from Section 5 and by Corollary 6.8 we may assume that τ', η' are obtained from τ, η be a splitting sequence consisting of splits at branches contained in a common proper subtrack of τ, η. Thus the case $k = 0$ is immediate from Lemma 6.15.

Now assume that the lemma holds true for some $k - 1 \geq 0$. Let $R > 0$ and let $\tau, \eta \in \mathcal{V}(\mathcal{TT})$ be such that $d(\tau,\eta) \leq R$ and that τ, η contain a common recurrent subtrack σ without closed curve components which carries every minimal geodesic lamination which is carried by both τ, η and which is not a simple closed curve. Let τ, η be splittable to τ', η' and assume that $d(\tau', \eta') \leq R$. By the above remark, for the purpose of the proof of inequality (23) we may assume that a splitting sequence connecting τ, η to τ', η' does not include a split at a large branch disjoint from σ.

As in the proof of Lemma 5.9, we use Lemma 5.7 and Lemma 5.8 to find a train track σ' with the following properties.

(1) σ' can be obtained from σ by a splitting sequence.
(2) There are train tracks $\hat{\tau} \in E(\tau,\tau'), \hat{\eta} \in E(\eta,\eta')$ which are obtained from τ, η by a sequence induced from a splitting sequence connecting σ to σ'.
(3) If $\tilde{\sigma}$ can be obtained from σ by a splitting sequence and if there are train tracks $\tilde{\tau} \in E(\tau,\tau'), \tilde{\eta} \in E(\eta,\eta')$ obtained from τ, η by a sequence induced by a splitting sequence connecting σ to $\tilde{\sigma}$ then $\tilde{\sigma} \in E(\sigma,\sigma')$.

Let $\ell, n > 0$ be the length of the balanced splitting path $\gamma(\tau,\tau'), \gamma(\eta,\eta')$. Let $i_0(\tau) \geq 0$ be the smallest number with the property that $\gamma(\tau,\tau')(i_0 + 1)$ does *not* contain a *recurrent* subtrack contained in $E(\sigma,\sigma')$. If no such number exists then define $i_0(\tau) = \ell$. Define similarly $i_0(\eta) \geq 0$.

Consider first the case that $i_0(\tau) = \ell$. Then the train track σ' is recurrent. Let as above $\hat{\tau} \in E(\tau, \tau')$ be the endpoint of a sequence issuing from τ which is induced by a splitting sequence connecting σ to σ'. Since τ' contains σ' as a subtrack, the distance between τ' and $\hat{\tau}$ is uniformly bounded. Let $\hat{\eta} \in E(\eta, \eta')$ be the train track obtained from η by a sequence induced by a splitting sequence connecting σ to σ'. By Corollary 3.6, the distance between $\hat{\eta}$ and $\hat{\tau}$ is bounded from above by a constant only depending on R. As a consequence, there is a number $\chi_1(R) > 0$ only depending on R such that

$$d(\eta', \hat{\eta}) \leq \chi_1(R).$$

Thus in this case the estimate (23) follows from Corollary 6.8, applied to τ which is splittable to $\hat{\tau}, \tau'$ and to η which is splittable to $\hat{\eta}, \eta'$, and Corollary 6.14.

In the case that $i_0(\tau) < \ell, i_0(\eta) < n$ let $i_0 = \min\{i_0(\tau), i_0(\eta)\}$ and assume that $i_0 = i_0(\tau)$. We claim that the distance between $\gamma(\tau, \tau')(i_0)$ and $\gamma(\eta, \eta')(i_0)$ is bounded from above by a constant only depending on R.

Namely, by the definition of $i_0(\tau)$, the train track $\gamma(\tau, \tau')(i_0)$ contains a recurrent subtrack $\zeta_1 \in E(\sigma, \sigma')$. Since by assumption a splitting sequence connecting τ to τ' does not include a split at a large branch which is disjoint from σ, the distance between $\gamma(\tau, \tau')(i_0)$ and the train track $\hat{\tau}$ obtained from τ by a sequence induced by a splitting sequence connecting σ to ζ_1 is uniformly bounded. Note that $\hat{\tau} \in E(\tau, \gamma(\tau, \tau')(i_0))$ by the definition of an induced sequence. By Corollary 6.8, applied to the train tracks $\tau, \hat{\tau}, \gamma(\tau, \tau')(i_0) \in E(\tau, \tau')$, if $i_1 \geq 0$ denotes the length of the balanced splitting path $\gamma(\tau, \hat{\tau})$ then $|i_1 - i_0|$ is uniformly bounded. Moreover, there is a constant $\chi_2 > 0$ such that

$$d(\gamma(\tau, \tau')(i), \gamma(\tau, \hat{\tau})(i)) \leq \chi_2 \text{ for all } i \leq \min\{i_0, i_1\}. \tag{24}$$

Let $i_2 \geq 0$ be the length of the balanced splitting path connecting σ to ζ_1. The first step in the proof of Lemma 6.13, applied to $\tau, \hat{\tau}, \gamma(\tau, \tau')(i_0)$ and their recurrent subtracks σ, ζ_1, shows that $i_2 \leq \min\{i_0, i_1\}$. Moreover, $i_1 - i_2$ is uniformly bounded. The estimate (24) then implies that there is a universal constant $\chi_3 > 0$ such that

$$d(\gamma(\tau, \tau')(i_0), \gamma(\tau, \hat{\tau})(i_2)) \leq \chi_3, \; d(\gamma(\tau, \hat{\tau})(i_2), \hat{\tau}) \leq \chi_3. \tag{25}$$

Let $\hat{\eta}$ be the endpoint of a sequence issuing from η which is induced by a splitting sequence connecting σ to ζ_1. By Corollary 3.6, the distance between $\hat{\tau}, \hat{\eta}$ is bounded from above by a constant $p(R) > 0$ only depending on R. Corollary 6.14 shows that there is a constant $\chi_4(R) > 0$ only depending on R such that

$$d(\gamma(\tau, \hat{\tau})(i_2), \gamma(\eta, \hat{\eta})(i_2)) \leq \chi_4(R). \tag{26}$$

Since $d(\gamma(\tau, \hat{\tau})(i_2), \hat{\tau}) \leq \chi_3$ by the estimate (25), this implies that

$$d(\gamma(\eta, \hat{\eta})(i_2), \hat{\eta}) \leq \chi_4(R) + \chi_3 + p(R). \tag{27}$$

In particular, if $\nu_2 \in E(\sigma, \zeta_1)$ is a recurrent subtrack of $\gamma(\eta, \hat{\eta})(i_3)$ then the length of a splitting sequence connecting ν_2 to ζ_1 is bounded by a number only depending on R.

On the other hand, since $i_2 \leq i_0$ and since $\hat{\eta}$ is splittable to η', Lemma 6.5 shows that $\gamma(\eta, \hat{\eta})(i_2)$ is splittable to $\gamma(\eta, \eta')(i_0)$. Therefore the recurrent subtrack

$\nu_2 \in E(\sigma,\zeta_1)$ of $\gamma(\eta,\hat\eta)(i_2)$ is splittable to a recurrent subtrack $\zeta_2 \in E(\sigma,\sigma')$ of $\gamma(\eta,\eta')(i_0)$. Reversing the roles of τ,τ' and η,η' (which is possible since so far we only used the fact that both $\gamma(\tau,\tau')(i_0)$ and $\gamma(\eta,\eta')(u_0)$ contain a recurrent subtrack in $E(\sigma,\sigma')$) yields a train track $\nu_1 \in E(\sigma,\sigma')$ which is splittable to ζ_2 with a splitting sequence of uniformly bounded length and which is splittable to ζ_1.

Lemma 5.4 of [H09] (which is valid for train tracks which are not necessarily complete) shows that there is a unique train track $\nu = \Theta_-(\zeta_1,\zeta_2) \in E(\nu_2,\sigma') \cap E(\nu_1,\sigma') \subset E(\sigma,\sigma')$ which is splittable to both ζ_1,ζ_2 and such that there is a geodesic in $E(\sigma,\sigma')$ connecting ζ_1 to ζ_2 which passes through ν. Since ν_1,ν_2 are both splittable to ζ_1,ζ_2, the length of a splitting sequence connecting ν to ζ_1,ζ_2 does not exceed the length of a splitting sequence connecting ν_2 to ζ_1, ν_1 to ζ_2. In particular, this length is bounded from above by a number only depending on R. Moreover, both ν_1,ν_2 are splittable to ν with a splitting sequence of uniformly bounded length.

As a consequence, there is a number $\chi_5(R) > 0$ only depending on R such that

$$d(\hat\eta,\gamma(\eta,\eta')(i_0)) \le \chi_5(R). \tag{28}$$

The estimates (28,27,26,25) then show that

$$d(\gamma(\tau,\tau')(i_0),\gamma(\eta,\eta')(i_0)) \le \chi_5(R) + 2\chi_4(R) + 2\chi_3 + p(R)$$

and hence the distance between $\gamma(\tau,\tau')(i_0+1)$ and $\gamma(\eta,\eta')(i_0+1)$ is bounded from above by a constant only depending on R. Now by the choice of i_0+1, the train track $\gamma(\tau,\tau')(i_0+1)$ does not contain a recurrent subtrack which can be obtained from σ by a splitting sequence. Therefore a minimal geodesic lamination which is carried by both $\gamma(\tau,\tau')(i_0+1)$ and $\gamma(\eta,\eta')(i_0+1)$ and which is not a simple closed curve is carried by a subtrack of $\gamma(\tau,\tau')(i_0+1)$ with fewer than k branches (compare the discussion in the proof of Lemma 5.9). The lemma now follows from the induction hypothesis, applied to $\gamma(\tau,\tau')(i_0),\tau'$ and $\gamma(\eta,\eta')(i_0),\eta'$. □

Finally we are ready to complete the proof of Theorem 6.4. We have to show the existence of a number $L > 1$ such that for any complete train tracks $\tau,\eta \in \mathcal{V}(\mathcal{TT})$ which are splittable to complete train tracks τ',η' the balanced splitting paths $\gamma(\tau,\tau'),\gamma(\eta,\eta')$ are weight-L fellow travellers.

For this observe first that it suffices to consider the particular case that τ,η are in standard form for some marking of S. Namely, let λ be a complete geodesic lamination carried by τ'. By invariance under the action of the mapping class group and cocompactness, there is a number $\chi_0 > 0$ and there is a train track τ_1 in standard form for a marking F of S which carries λ and such that

$$d(\tau,\tau_1) \le \chi_0.$$

By Lemma 6.7 and Proposition A.6 of [H09], τ_1 is splittable to a train track τ_1' in a uniformly bounded neighborhood of τ' which carries λ. Corollary 6.12 then shows that the balanced splitting paths $\gamma(\tau,\tau'),\gamma(\tau_1,\tau_1')$ are uniform fellow travellers. Hence indeed we may assume that τ,η are in standard form for a marking of S.

Thus let τ,η be complete train tracks in standard form for markings F,G of S. Assume that τ is splittable to a complete train track τ'. By Proposition 3.2 and by

Proposition A.6 of [H09], there is a universal constant $\chi_1 > 0$ and there is a train track η_0 with

$$d(\eta, \eta_0) \le \chi_1$$

which can be obtained from a train track in standard form for F by a splitting sequence. Using the notations from Section 5, let $\beta = \Pi_{E(\tau,\tau')}(\eta_0) \in E(\tau, \tau')$ be as in Proposition 5.10. Let λ be any complete geodesic lamination carried by τ'. By Proposition 5.10, there is a universal number $\kappa > 0$ and there are train tracks $\eta_1' \in \mathcal{V}(\mathcal{TT})$ with $d(\tau', \eta_1') \le \kappa$ and $\eta_1 \in \mathcal{V}(\mathcal{TT})$ with $d(\eta_0, \eta_1) \le \kappa$ and hence $d(\eta, \eta_1) \le \chi_1 + \kappa$ with the following properties.

(1) η_1' carries λ.
(2) η_1 can be connected to η_1' by a splitting sequence.
(3) There is a train track $\beta' \in E(\eta_1, \eta_1')$ with $d(\beta', \beta) \le \kappa$.
(4) There is a splitting sequence connecting a point in the κ-neighborhood of τ to a point in the κ-neighborhood of η_1 which passes through the κ-neighborhood of β.

We begin with showing that there is a universal number $L_{11} > 0$ not depending on $\tau, \eta_1, \tau', \eta_1'$ such that the balanced splitting paths $\gamma(\tau, \tau'), \gamma(\eta_1, \eta_1')$ are weight-L_{11} fellow travellers.

Since by Theorem 2.3 and the remark thereafter splitting sequences are uniform quasi-geodesics in $\mathcal{TT}$, by the third and the forth property above there is a universal constant $a > 0$ such that

$$d(\tau, \eta) \ge a(d(\tau, \beta) + d(\beta', \eta_1)) - 1/a.$$

Together with Corollary 6.8, applied to the train tracks τ, β which are splittable to τ' and to the train tracks η_1, β' which are splittable to η_1', it now suffices to show that the balanced splitting paths $\gamma(\beta, \tau'), \gamma(\beta', \eta_1')$ are uniform fellow travellers. However, since τ', η_1' both carry λ and since $d(\beta, \beta') \le \kappa, d(\tau', \eta_1') \le \kappa$ this follows from Corollary 6.12.

We are left with showing that the balanced splitting paths $\gamma(\eta_1, \eta_1')$ and $\gamma(\eta, \eta')$ are weight-L-fellow travellers for a universal constant $L > 0$. For this recall that $d(\eta, \eta_1) \le \kappa + \chi_1$. Choose complete geodesic laminations ν, λ which are carried by η', η_1'. By invariance under the action of the mapping class group and cocompactness, there is a marking F of S such that the distance between η, η_1 and any train track in standard form for F is uniformly bounded. Let ξ_1, ξ_2 be train tracks in standard form for F which carry the complete geodesic laminations ν, λ. By Lemma 6.7 and Proposition A.6 of [H09], ξ_1, ξ_2 are splittable to complete train tracks ξ_1' and ξ_2' which carry ν, λ and which are contained in a uniformly bounded neighborhood of η', η_1'. By Corollary 6.12, the balanced splitting paths $\gamma(\eta, \eta'), \gamma(\xi_1, \xi_1')$ and $\gamma(\eta_1, \eta_1'), \gamma(\xi_2, \xi_2')$ are uniform fellow travellers. Thus it suffices to assume that η, η_1 are in standard form for the same marking of S. However, Lemma 6.16 shows that in this case the balanced splitting paths $\gamma(\eta, \eta')$ and $\gamma(\eta_1, \eta_1')$ are weight-L_{10} fellow travellers.

Together we conclude that the balanced splitting paths

$$\gamma(\tau, \tau') \text{ and } \gamma(\eta, \eta')$$

are weight-L-fellow travellers for a universal constant $L > 0$. This completes the proof of Theorem 6.4.

7. A biautomatic structure

In this section we show Theorem 1 from the introduction. Our strategy is to use balanced splitting paths to construct a regular path system on $\mathcal{TT}$ (or, rather, on an $\mathcal{MCG}(S)$-graph obtained from $\mathcal{TT}$ by adding some edges) in the sense of [S06].

For this we first add three families of edges to $\mathcal{TT}$. The purpose of the edges in the different families is distinct, and to keep track of this purpose we color the edge in each of the families with a fixed color (red, yellow, green). We then can talk about a colored graph.

For every marking F of S, connect any two train tracks in standard form for F by a yellow edge. The resulting graph G_1 is of finite valence and contains $\mathcal{TT}$ as a subgraph. The mapping class group acts properly and cocompactly on G_1. Note that the stabilizer in $\mathcal{MCG}(S)$ of a fixed train track in standard form for F is a subgroup of the stabilizer of the marking F which in turn is a finite subgroup of $\mathcal{MCG}(S)$.

If η can be obtained from τ by a move as described in Section 6 then add a green edge to the graph G_1 which connects τ to η. As before, since η can be obtained from τ by a splitting sequence of uniformly bounded length, the resulting extension G_2 of G_1 is of finite valence, and the mapping class group acts cocompactly on this graph.

By Proposition 3.2 and Proposition A.6 of [H09], there is a number $\chi > 0$ and for every marking F of S and every train track $\eta \in \mathcal{V}(\mathcal{TT})$ there is a train track $\eta' \in \mathcal{V}(\mathcal{TT})$ which can be obtained from a train track in standard form for F by a splitting sequence and such that $d(\eta, \eta') \leq \chi$. For any two vertices $\eta, \eta' \in \mathcal{TT}$ with $d(\eta, \eta') \leq \chi$ add a red edge to G_2 which connects η to η'. The resulting colored graph X is locally finite and admits a properly discontinuous cocompact simplicial action by $\mathcal{MCG}(S)$. In particular, the inclusion $\mathcal{TT} \to X$ defines a $\mathcal{MCG}(S)$-equivariant quasi-isometry between $\mathcal{TT}$ and X.

Following [S06], a *path* in X is a finite sequence $e_1, \dots, e_n$ of oriented edges in X such that for each i the terminal point of e_i equals the initial point of e_{i+1}. The mapping class group naturally acts on paths in X. A $\mathcal{MCG}(S)$-invariant set of paths in X is called a $\mathcal{MCG}(S)$-invariant *path system* $\mathcal{P}$ on X.

Fix a marking F of S and let τ be a train track in standard form for S. Define a $\mathcal{MCG}(S)$-invariant path system $\mathcal{P}$ on X to consist of all paths $\gamma = (e_1, \dots, e_n)$ with the following properties.

(1) e_1 is a yellow edge connecting a train track $\tau' = g\tau$ for some $g \in \mathcal{MCG}(S)$ to a (not necessarily different) train track τ'' in standard form for gF.
(2) Each of the edges $e_2, \dots, e_{n-1}$ is green, and the concatenation of these edges defines a balanced splitting path connecting the terminal point of e_1 (a train track in standard form for F) to the terminal point of e_{n-1}.

(3) The edge e_n is red and connects the terminal point of e_{n-1} to a train track in the orbit $\mathcal{MCG}(S)\tau$ of τ.

By construction, each path in $\mathcal{P}$ starts and terminates at vertices from the orbit $\mathcal{MCG}(S)\tau$, and for any two points in $\mathcal{MCG}(S)\tau$ there is a path in $\mathcal{P}$ connecting these two points. Following [S06], this means that $\mathcal{P}$ is *transitive* on $\mathcal{MCG}(S)\tau$. Moreover, $\mathcal{P}$ is $\mathcal{MCG}(S)$-invariant. By Theorem 6.4, the system $\mathcal{P}$ satisfies moreover the *2-sided fellow traveller property*: There is a constant $c > 0$ such that

$$d(\gamma_1(j), \gamma_2(j)) \le c(d(\gamma_1(0), \gamma_2(0)) + d(\gamma_1(\infty), \gamma_2(\infty))) + c \text{ for all } j$$

where as before, we view a path as an eventually constant map defined on $\mathbb{N}$.

Theorem 1 from the introduction is now an immediate consequence of Theorem 1.1 of [S06] and the following

Proposition 7.1. *The path system $\mathcal{P}$ is regular.*

Proof. Following p.24 of [S06], an *automaton* is a triple (M, S_0, S_Ω) where M is a directed graph and where S_0, S_Ω are subsets of the vertex set V_M of M called the sets of *start* and *accept* vertices, respectively. An *automaton over a graph X* is a quadruple $\mathcal{M} = (M, m, S_0, S_\Omega)$ in which (M, S_0, S_Ω) is an automaton and $m : M \to X$ is a simplicial map. If G is a group of automorphisms of X then an automaton over (X, G) is an automaton $\mathcal{M} = (M, m, S_0, S_\Omega)$ over X equipped with an action of G on M satisfying the following properties.

(1) The sets S_0 and S_Ω are G-invariant.
(2) The map $m : M \to X$ is G-equivariant.

By [S06], a path system in a graph X is called regular if there is a finite-to-one automaton $\mathcal{M}$ over (X, G) such that $\mathcal{P} = \mathcal{P}(\mathcal{M})$.

We construct an automaton $\mathcal{M}$ over the graph X defined in the beginning of this section as follows. Define a *train track with traffic control* to be a complete train track equipped with a labeling of each large branch with one of four labels green, red, right, left. The set V_M of vertices of the graph M is a union $V_M = V_T \cup V_0 \cup V_\Omega$ where V_T is the set of all train tracks with traffic control. The set V_0 is the set of start vertices, and the set V_Ω is the set of accept vertices which are given as follows. Let $\Gamma < \mathcal{MCG}(S)$ be the stabilizer of the fixed train track τ in standard form for the marking F. For each $g \in \mathcal{MCG}(S)/\Gamma$ there is an additional vertex $v_0(g\Gamma) \in V_0$ and a vertex $v_\Omega(g\Gamma) \in V_\Omega$.

The directed edges of the graph M are determined as follows. Every vertex $v_0(g\Gamma) \in V_0$ determines a train track $g\tau \in \mathcal{V}(\mathcal{TT})$ in the $\mathcal{MCG}(S)$-orbit of τ. The train track $g\tau$ is in standard form for the marking gF. For every train track η in standard form for gF connect $v_0(g\Gamma)$ to the train track with traffic control which is just η with all large branches labeled green. Note that the number of directed edges issuing from $v_0(g\Gamma)$ is uniformly bounded.

A train track $[\xi]$ with traffic control is connected to a train track $[\eta]$ with traffic control by a directed edge if the train track η underlying $[\eta]$ can be obtained from the train track ξ underlying $[\xi]$ by a move and if the traffic controls of $[\xi],[\eta]$ satisfy the following compability conditions.

(1) If a large branch e of ξ is labeled red in $[\xi]$ then we require that a splitting sequence connecting ξ to η does not include a split at e. Then $\varphi(\xi,\eta)(e)$ is a large branch of η. We require that the label of this large branch in the train track with traffic control $[\eta]$ is red.
(2) If e is a large branch in ξ labeled green, right or left in $[\xi]$ and if a splitting sequence connecting ξ to η does not include a split at e then $\varphi(\xi,\eta)(e)$ is a large branch of η. We require that the label of this large branch $\varphi(\xi,\eta)(e)$ in $[\eta]$ is red.
(3) Let e be a large branch in ξ whose label in $[\xi]$ is right (or left). If a splitting sequence connecting ξ to η includes a split at e then we require that this split is a right (or left) split.
(4) Assume that η can be obtained from ξ by a move and let $\rho:[0,n]\to\xi$ be a splittable η-configuration which is a (left or right) reduced trainpath. Let $e_1,\dots,e_s$ $(s\geq 0)$ be the large branches of η contained in $\varphi(\xi,\eta)(\rho)$. If for $i\leq s$ the $\varphi(\xi,\eta)(\rho)$-split at e_i is a right (or left) split then we require that the label in $[\eta]$ of the large branch e_i is left (or right).
(5) If ρ is a splittable η-circle c and if the train track ξ' obtained from ξ by a $\rho-\eta$-multi split is distinct from $\theta_c^{\pm}(\xi)$ then there are large branches $e_1,\dots,e_s$ in $\xi'\cap c$ such that for each i the train track obtained from ξ' by a right (or left) split at e_i is splittable to $\theta_c^{\pm}(\xi)$. The branch $\varphi(\xi',\eta)(e_i)$ in η is large in $[\eta]$, and we require that the label of this branch in $[\eta]$ is left (or right).
(6) If η is obtained from τ by a move and if the label of a large branch e of η is not determined from τ by one of the five rules above then we require that the label of e is green.

This list of requirements determines for every train track $[\xi]$ with traffic control a uniformly bounded number of directed edges issuing from $[\xi]$. Finally, if $[\zeta]$ is any train track with traffic control and if $g\in\mathcal{MCG}(S)$ is such that $d(\zeta,g\tau)\leq\chi$ then we require that there is a directed edge connecting $[\zeta]$ to $v_\Omega(g\Gamma)$.

The action of the mapping class group on $\mathcal{V}(\mathcal{TT})$ induces an action on the graph M. Moreover, there is a finite-to-one $\mathcal{MCG}(S)$-equivariant simplicial map $M\to X$ which maps the set of directed path in M connecting a vertex in V_0 to a vertex in V_Ω onto the path system $\mathcal{P}$. This shows that the path system $\mathcal{P}$ is indeed regular and complete the proof of the proposition. □

Remark: The automaton which defines a biautomatic structure on $\mathcal{MCG}(S)$ is completely explicit. Since the number of branches of a complete train track on a surface S of genus $g\geq 0$ with m punctures does not exceed $18g-18+6m$ [PH92] and since for every $p>0$ the number of trivalent graphs with p vertices is uniformly exponential in p, the complexity of the automaton (i.e. the number of its states and the cardinality of its alphabet) is uniformly exponential in the complexity of S.

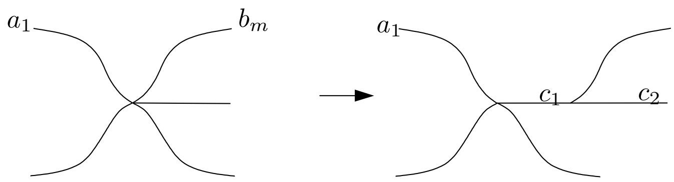
Appendix A. Train tracks hitting efficiently

In this appendix we construct complete train tracks with some specific properties which are used to obtain a geometric control on the train track complex $\mathcal{TT}$. First, define a *bigon track* on S to be an embedded 1-complex on S which satisfies all the requirements of a train track except that we allow the existence of complementary bigons. Such a bigon track is called *maximal* if all complementary components are either bigons or trigons or once punctured monogons. Recurrence, transverse recurrence, birecurrence and carrying for bigon tracks are defined in the same way as they are defined for train tracks. Any complete train track is a maximal birecurrent bigon track in this sense.

A *tangential measure* ν for a maximal bigon track ζ assigns to each branch b of ζ a nonnegative weight $\nu(b) \in [0,\infty)$ with the following properties. Each side of a complementary component of ζ can be parametrized as a trainpath ρ on ζ (here as in the case of train tracks, a trainpath is a C^1-immersion $\rho : [0,m] \to \zeta$ which maps each segment $[i,i+1]$ onto a branch of ζ). Denote by $\nu(\rho)$ the sum of the weights of the branches contained in $\rho[0,m]$ counted with multiplicities. If ρ_1,ρ_2 are the two distinct sides of a complementary bigon then we require that $\nu(\rho_1) = \nu(\rho_2)$, and if ρ_1,ρ_2,ρ_3 are the three distinct sides of a complementary trigon then we require that $\nu(\rho_i) \leq \nu(\rho_{i+1})+\nu(\rho_{i+2})$ (where indices are taken modulo 3). A bigon track is transversely recurrent if and only if it admits a tangential measure which is positive on every branch [PH92].

A bigon track is called *generic* if all switches are at most trivalent. A bigon track τ which is not generic can be *combed* to a generic bigon track by successively modifying τ as shown in Figure H. By Proposition 1.4.1 of [PH92] (whose proof is also valid for bigon tracks), the combing of a recurrent bigon track is recurrent. However, the combing of a transversely recurrent bigon track need not be transversely recurrent (see the discussion on p.41 of [PH92]).

Figure H



The next Lemma gives a criterion for a non-generic maximal transversely recurrent bigon track to be combable to a generic maximal transversely recurrent bigon track. For its formulation, we say that a positive tangential measure ν on a maximal bigon track σ satisfies the *strict triangle inequality for complementary trigons* if for every complementary trigon of σ with sides e_1,e_2,e_3 we have $\nu(e_i) < \nu(e_{i+1}) + \nu(e_{i+2})$. By Theorem 1.4.3 of [PH92], a *generic* maximal train track is transversely recurrent if and only if it admits a positive tangential measure satisfying the strict triangle inequality for complementary trigons.

Lemma A.1. *Let ζ be a maximal bigon track which admits a positive tangential measure satisfying the strict triangle inequality for complementary trigons. Then ζ can be combed to a generic transversely recurrent bigon track.*

Proof. Let σ be an arbitrary maximal bigon track. Then σ does not have any bivalent switches. For a switch s of σ denote the valence of s by $V(s)$ and define the excessive total valence $\mathcal{V}(\sigma)$ of σ to be $\sum_s(V(s)-3)$ where the sum is taken over all switches s of σ; then $\mathcal{V}(\sigma)=0$ if and only if σ is generic. By induction it is enough to show that a maximal non-generic bigon track σ which admits a positive tangential measure ν satisfying the strict triangle inequality for complementary trigons can be combed to a bigon track σ' which admits a positive tangential measure ν' satisfying the strict triangle inequality for complementary trigons and such that $\mathcal{V}(\sigma')<\mathcal{V}(\sigma)$.

For this let σ be such a non-generic maximal bigon track with tangential measure ν satisfying the strict triangle inequality for complementary trigons and let s be a switch of σ of valence at least 4. For a fixed orientation of the tangent line of σ at s assume that there are ℓ incoming and m outgoing branches where $1\leq \ell\leq m$ and $\ell+m\geq 4$. We number the incoming branches in counter-clockwise order $a_1,\dots,a_\ell$ (for the given orientation of S) and do the same for the outgoing branches $b_1,\dots,b_m$. Then the branches b_m and a_1 are contained in the same side of a complementary component of σ, and the branches b_{m-1},b_m are contained in adjacent (not necessarily distinct) sides e_1,e_2 of a complementary component T of σ. Assume first that T is a complementary trigon. Denote by e_3 the third side of T; by assumption, the total weight $\nu(e_3)$ is strictly smaller than $\nu(e_1)+\nu(e_2)$ and therefore there is a number $\epsilon\in(0,\min\{\nu(b_i)\mid 1\leq i\leq m\})$ such that $\nu(e_3)<\nu(e_1)+\nu(e_2)-2\epsilon$. Move the endpoint of the branch b_m to a point in the interior of b_{m-1} as shown in Figure H; we obtain a bigon track σ' with $\mathcal{V}(\sigma')<\mathcal{V}(\sigma)$.

The branch b_{m-1} decomposes in σ' into the union of two branches c_1,c_2 where c_1 is incident on s and on an endpoint of the image b_m' of b_m under our move. Assign the weight ϵ to the branch c_1, the weight $\nu(b_m)-\epsilon$ to the branch b_m' and the weight $\nu(b_{m-1})-\epsilon$ to the branch c_2. The remaining branches of σ' inherit their weight from the tangential measure ν on σ. This defines a positive weight function on the branches of σ'. By the choice of ϵ, this weight function defines a tangential measure on σ' which satisfies the strict triangle inequality for complementary trigons.

Similarly, if the complementary component T containing b_m and b_{m-1} in its boundary is a bigon or a once punctured monogon, then we can shift b_m along b_{m-1} as before and modify our tangential measure to a positive tangential measure on the combed track with the desired properties. □

Our next goal is to transform a maximal recurrent bigon track η which admits a positive tangential measure satisfying the strict triangle inequality for complementary trigons to a complete train track. This will be done with splits, combings and *collapses of embedded bigons*, where by an embedded bigon we mean a bigon B in η whose boundary does not have self-intersections. A collapse of an embedded bigon B consists in identifying the two boundary arcs of B with a map $F:S\to S$ which is homotopic to the identity and equals the identity in the complement of an arbitrarily small neighborhood of B. This procedure is described more explicitly in the following lemma.

Lemma A.2. *Let η be a maximal bigon track which admits a positive tangential measure ν satisfying the strict triangle inequality for complementary trigons. Let $B \subset \eta$ be an embedded bigon. Then there is a maximal bigon track $\tilde{\eta}$ which carries η with a carrying map which equals the identity outside a small neighborhood of B. The tangential measure ν on η induces a positive tangential measure $\tilde{\nu}$ on $\tilde{\eta}$ satisfying the strict triangle inequality for complementary trigons. The number of bigons of $\tilde{\eta}$ is strictly smaller than the number of bigons of η.*

Proof. Let η be a maximal bigon track which admits a positive tangential measure ν satisfying the strict triangle inequality for complementary trigons. Assume that η contains an embedded bigon B.

We construct from η and ν a maximal birecurrent bigon track $\tilde{\eta}$ which carries η. Namely, assume that the side E of the bigon B is the image of an embedded trainpath $\rho_E : [0,\ell] \to \eta$ and that the second side F of B is the image the embedded trainpath $\rho_2 : [0,m] \to \eta$. Assume also that $\rho_1(0) = \rho_2(0)$. We collapse the bigon B to a single arc in S with a homeomorphism $\Psi : E \to F$ which is defined as follows.

If for some $p \geq 1$, $q \geq 1$ we have

$$\sum_{j=1}^{q-1} \nu(\rho_2[j-1,j]) < \sum_{i=1}^{p} \nu(\rho_1[i-1,i]) < \sum_{j=1}^{q} \nu(\rho_2[j-i,j])$$

then Ψ maps the subarc $\rho_1[0,p]$ of E homeomorphically onto a subarc of F which contains $\rho_2[0,q-1]$ and has its endpoint in the interior of the branch $\rho_2[q-1,q]$. If

$$\sum_{i=1}^{p} \nu(\rho_1[i-1,i]) = \sum_{j=1}^{q} \nu(\rho_2[j-1,j])$$

then Ψ maps $\rho_1[0,p]$ onto $\rho_2[0,q]$, i.e. an endpoint of $\rho_1[p-1,p]$ is mapped to an endpoint of $\rho_2[q-1,q]$. The resulting bigon track $\tilde{\eta}$ carries η, and it is maximal.

There is a natural carrying map $\Phi : \eta \to \tilde{\eta}$ which maps each complementary trigon of η to a complementary trigon of $\tilde{\eta}$. The positive tangential measure ν on η induces a positive weight function $\tilde{\nu}$ on the branches of $\tilde{\eta}$. Note that the total weight of $\tilde{\nu}$ is strictly smaller than the total weight of ν and that the ν-weight of a side ρ of a complementary component $T \neq B$ in η coincides with the $\tilde{\nu}$-weight of the side $\Phi(\rho)$ of the complementary component $\Phi(T)$ in $\tilde{\eta}$. In particular, the weight function $\tilde{\nu}$ is a tangential measure on $\tilde{\eta}$ which satisfies the strict triangle inequality for complementary trigons. The number of complementary bigons in $\tilde{\eta}$ is strictly smaller than the number of complementary bigons in η. More precisely, there is a bijection between the complementary bigons of $\tilde{\eta}$ and the complementary bigons of η distinct from B. The image of the bigon B under the map Φ is an embedded arc in $\tilde{\eta}$. The number of branches of $\tilde{\eta}$ does not exceed the number of branches of η. □

We use Lemma A.2 to show

Lemma A.3. *For every $m > 0$ there is a number $h(m) > 0$ with the following property. Let η be a maximal bigon track with at most m bigon which admits a positive tangential measure satisfying the strict triangle inequality for complementary trigons. Let λ be a complete geodesic lamination carried by η. Then there is a complete train track η' which carries λ and which is obtained from η by at most $h(m)$ splits, combings and collapses.*

Proof. We modify a bigon track η as in the lemma in a uniformly bounded number of steps to a complete train track η' with a (non-deterministic) algorithm as follows.

The set of input data for the algorithm is the set $\mathcal{B}$ of quadruples (η, λ, ν, B) which consist of a maximal birecurrent bigon track η, a complete geodesic lamination λ carried by η, a positive tangential measure ν on η which satisfies the strict triangle inequality for complementary trigons and a complementary bigon B of η. If η does not have any complementary bigons, i.e. if η is a train track, then we put $B = \emptyset$. The algorithm modifies the quadruple $(\eta, \lambda, \nu, B) \in \mathcal{B}$ to a quadruple $(\eta', \lambda, \nu', B') \in \mathcal{B}$ with $B' = \emptyset$ as follows.

Step 1:

Let $(\eta, \lambda, \nu, B) \in \mathcal{B}$ be an input quadruple with $B \neq \emptyset$. Check whether the boundary ∂B of B is embedded in η. If this is not the case then go to Step 2.

Otherwise collapse B with the procedure described in Lemma A.2. This yields a bigon track $\tilde{\eta}$ which carries η and hence λ and which admits a positive transverse measure $\tilde{\nu}$ satisfying the strict triangle inequality for complementary trigons. The number of bigons of $\tilde{\eta}$ equals the number of bigons of η minus one.

Choose an input quadruple of the form $(\tilde{\eta}, \lambda, \tilde{\nu}, \tilde{B}) \in \mathcal{B}$ for a complementary bigon $\tilde{B}$ of $\tilde{\eta}$ (or $\tilde{B} = \emptyset$ if $\tilde{\eta}$ is a train track) and repeat Step 1 with this input quadruple.

Step 2:

Let $(\eta, \lambda, \nu, B) \in \mathcal{B}$ be an input quadruple such that $B \neq \emptyset$ and that the boundary ∂B of B is *not* embedded. Then the sides E, F of B are immersed arcs of class C^1 in S which intersect tangentially or have tangential self-intersections. Check whether the two cusp points of B meet at the same switch of η. If this is not the case, continue with Step 3.

Otherwise the two cusps of B meet at a common switch s of η which is necessarily at least 4-valent. By Lemma A.1 and its proof, we can modify η with a sequence of combings near s to a maximal birecurrent bigon track $\tilde{\eta}$ in such a way that the two cusps of the complementary bigon $\tilde{B}$ in $\tilde{\eta}$ corresponding to B under the combing are distinct and such that the tangential measure ν on η induces a tangential measure $\tilde{\nu}$ on $\tilde{\eta}$ which satisfies the strict triangle inequality for complementary trigons. The bigon track $\tilde{\eta}$ carries the complete geodesic lamination λ, moreover the number of its bigons coincides with the number of bigons of η. Continue with Step 1 above with the input quadruple $(\tilde{\eta}, \lambda, \tilde{\nu}, \tilde{B}) \in \mathcal{B}$.

Step 3:

Let $(\eta,\lambda,\nu,B)\in\mathcal{B}$ be an input quadruple where B is a bigon in η whose boundary ∂B is not embedded and whose sides E,F have distinct endpoints. Check whether the boundary ∂B of B contains any isolated self-intersection points. Such a point is a switch s contained in the interior of at least two distinct embedded subarcs ρ_1,ρ_2 of ∂B of class C^1 with the additional property that $(\rho_1-\{s\})\cap(\rho_2-\{s\})=\emptyset$. If ∂B does not contain such an isolated self-intersection point then go to Step 4.

Otherwise any such isolated self-intersection point s is a switch of η which is at least 4-valent. Thus using once more Lemma A.1, we can modify η with a sequence of combings to a complete birecurrent bigon track $\tilde\eta$ with the property that all self-intersection points of the boundary of the bigon $\tilde B$ in $\tilde\eta$ corresponding to B are non-isolated, i.e. they are contained in a self-intersection branch, and that the tangential measure ν on η induces a tangential measure $\tilde\nu$ on $\tilde\eta$ satisfying the strict triangle inequality for complementary trigons. Continue with Step 4 with the input quadruple $(\tilde\eta,\lambda,\tilde\nu,\tilde B)\in\mathcal{B}$.

Step 4:

Let $(\eta,\lambda,\nu,B)\in\mathcal{B}$ be an input quadruple where B is a complementary bigon for η whose boundary ∂B does not contain any isolated self-intersection points, whose sides have distinct endpoints and such that ∂B has tangential self-intersection branches.

Check whether there is a branch e of η contained in the self-intersection locus of ∂B which is not incident on any one of the two cusps s_1,s_2 of B. If there is no such branch continue with Step 5.

Otherwise note that since the interior of the bigon B is an embedded topological disc in S, such a self-intersection branch e is necessarily a large branch. Now η carries λ and therefore there is a bigon track $\tilde\eta$ which is the image of η under a split at e and which carries λ. To each complementary region of $\tilde\eta$ naturally corresponds a complementary region of η of the same topological type. In particular, the number of complementary bigons in $\tilde\eta$ and η coincide, and the bigon B in η corresponds to a bigon $\tilde B$ in $\tilde\eta$. The bigon track $\tilde\eta$ is recurrent, and it admits a positive tangential measure $\tilde\nu$ induced from the measure ν on η which satisfies the strict triangle inequality for complementary trigons. The number of branches contained in the self-intersection locus of the boundary $\partial\tilde B$ of the bigon $\tilde B$ is strictly smaller than the number of branches in the self-intersection locus of ∂B.

After a number of splits of this kind which is bounded from above by the number of branches of the bigon track η and hence by the number of bigons of η we obtain from η a bigon track η_1 which is maximal and birecurrent. There is a natural bijection from the set of complementary bigons of η onto the set of complementary bigons of η_1. If B_1 is the bigon of η_1 corresponding to B then the self-intersection locus of the boundary ∂B_1 of B_1 is a union of branches which are incident on one of the two cusps $s_1\neq s_2$ of B_1. As before, η_1 admits a positive tangential measure ν_1 which satisfies the strict triangle inequality for complementary trigons and is induced from ν. Moreover, η_1 carries the complete geodesic lamination λ.

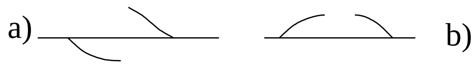
If the boundary ∂B_1 of B_1 is embedded then we proceed with Step 1 above for the input quadruple $(\eta_1, \lambda, \nu_1, B_1)$. Otherwise proceed with Step 5 using again the input quadruple $(\eta_1, \lambda, \nu_1, B_1)$.

Step 5:

Let (η, λ, ν, B) be an input quadruple such that the cusps of B are distinct and that there is no branch of self-intersection of the boundary ∂B of B which is entirely contained in the interior of a side of B. Check whether there is a self-intersection branch b of ∂B which is incident on a cusp s of B. Note that the branch b can *not* be large, so it is either small or mixed.

For a small branch b, there are two possibilities which are shown in Figure I.

Figure I



Assume first that b is as in the left hand side of Figure I. The boundary of a bigon in a bigon track is the boundary of an embedded disc in S and hence it admits a natural orientation induced from the orientation of S. This easily implies that the small branch b is contained in the intersection of the two *distinct* sides of B. Since there is no branch in the interior of a side of ∂B which is a branch of self-intersection of ∂B, and since the tangential measure ν on η is positive by assumption, the construction in Lemma A.2 can be used to collapse the bigon B in η to a single simple closed curve. We obtain a maximal birecurrent bigon track $\tilde{\eta}$ which carries λ and admits a positive tangential measure $\tilde{\nu}$ satisfying the strict triangle inequality for complementary trigons. The number of bigons of $\tilde{\eta}$ is strictly smaller than the number of bigons of η. Choose an arbitrary complementary bigon $\tilde{B}$ in $\tilde{\eta}$ or put $\tilde{B} = \emptyset$ if there is no such bigon and continue with Step 1 above and the input quadruple $(\tilde{\eta}, \lambda, \tilde{\nu}, \tilde{B}) \in \mathcal{B}$.

A small branch b of self-intersection of ∂B as shown on the right hand side of Figure I can not be collapsed. In this case the branch b coincides with a side E of the bigon B, and the second side F of B contains b as a proper subarc. However, this configuration violates the assumption that the tangential measure ν on η is *positive.*

If the branch b is mixed then b and the cusp s of B are contained in the interior of a side E of B. The bigon track η can be modified with a single shift to a maximal birecurrent bigon track η_1 such that the switch s is not contained any more in the interior of a side of the bigon B_1 corresponding to B in η. The tangential measure ν on η naturally induces a positive tangential measure ν_1 on η_1 which satisfies the strict triangle inequality for complementary trigons. We now proceed with Step 1 for the input quadruple $(\eta_1, \lambda, \nu_1, B_1)$. The algorithm stops if there is no bigon left. This completes the description of the algorithm.

The algorithm produces a not necessarily generic maximal recurrent train track which admits a positive tangential measure satisfying the strict triangle inequality for complementary trigons and hence which can be combed to a complete train track η' carrying λ. The train track η' depends on choices made in the above construction. However, the number of possibilities in each step is uniformly bounded and hence η' depends on choices among a uniformly bounded number of possibilities. The lemma follows. □

A train track τ on the surface S *hits efficiently* a train track or a geodesic lamination σ if τ can be isotoped to a train track τ' which intersects σ transversely in such a way that $S - \tau' - \sigma$ does not contain any embedded bigon. A *splitting and shifting sequence* is a sequence $\{\tau_i\} \subset \mathcal{V}(\mathcal{TT})$ such that for every i the train track τ_{i+1} can be obtained from τ_i by a sequence of shifts and a single split. Denote by d the distance on $\mathcal{TT}$. The following technical result relates a train track τ to train tracks which hit τ efficiently. Its proof relies on a construction in Section 3.4 of [PH92] and uses Lemma A.3.

Proposition A.4. *There is a number $p > 0$ and for every $\tau \in \mathcal{V}(\mathcal{TT})$ and every complete geodesic lamination λ which hits τ efficiently there is a complete train track $\tau^* \in \mathcal{V}(\mathcal{TT})$ with the following properties.*

(1) $d(\tau, \tau^*) \leq p$.
(2) τ^* *carries* λ.
(3) *Let $\sigma \in \mathcal{V}(\mathcal{TT})$ be a train track which hits τ efficiently and carries λ; then τ^* carries a train track σ' which carries λ and can be obtained from σ by a splitting and shifting sequence of length at most p.*

Proof. By Lemma 3.4.4 and Proposition 3.4.5 of [PH92], for every complete train track τ there is a maximal birecurrent *dual bigon track* τ_b^* with the following property. A geodesic lamination or a train track σ hits τ efficiently if and only if σ is carried by τ_b^*. We construct the train track τ^* with the properties stated in the proposition from this dual bigon track and a complete geodesic lamination $\lambda \in \mathcal{CL}$ which hits τ efficiently and hence is carried by τ_b^*.

For this we recall from p.194 of [PH92] the precise construction of the dual bigon track τ_b^* of a complete train track τ. Namely, for each branch b of τ choose a compact embedded arc b^* of class C^1 meeting τ transversely in a single point in the interior of b and such that all these arcs are pairwise disjoint. Let $T \subset S - \tau$ be a complementary trigon of τ and let E be a side of T which is composed of the branches $b_1, \dots, b_\ell$. Choose a point $x \in T$ and extend all the arcs $b_1^*, \dots, b_\ell^*$ within T in such a way that they end at x, with the same inward pointing tangents at x. In the case $\ell \geq 2$ we then add an arc of class C^1 which connects x within T to a point $x' \in T$ and whose inward pointing tangent at x equals the outward pointing tangent at x of the arcs $b_1^*, \dots, b_\ell^*$. We do this in such a way that the different configurations from the different sides of T are disjoint. If $y' \in T$ is the point in T arising in this way from a second side, then we connect x' (or x if $\ell = 1$) and y' by an embedded arc of class C^1 whose outward pointing tangent at x', y' coincides with the inward pointing tangents of the arcs constructed before which end at x', y'. In a similar way we construct the intersection of τ_b^* with a complementary once punctured monogon of τ. Note that the resulting graph τ_b^* is in general not generic, but its only vertices which are not trivalent arise from the sides of the complementary components of τ. Figure J shows the intersection of the dual bigon track τ_b^* with a neighborhood in S of a complementary trigon of τ and with a neighborhood in S of a complementary once punctured monogon.

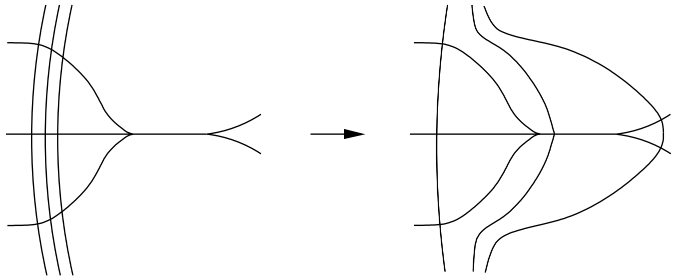
Figure J

Following [PH92], τ_b^* is a maximal birecurrent bigon track, and the number of its branches is bounded from above by a constant only depending on the topological type of S. Each complementary trigon of τ contains exactly one complementary trigon of τ_b^* in its interior, and these are the only complementary trigons. Each complementary once punctured monogon of τ contains exactly one complementary once punctured monogon of τ_b^* in its interior. All other complementary components of τ_b^* are bigons. The number of these bigons is uniformly bounded.

Now let μ be a positive integral transverse measure on τ with the additional property that the μ-weight of every branch of τ is at least 4. This weight then defines a *simple multi-curve* c carried by τ in such a way that μ is just the counting measure for c (see [PH92]). Here a simple multi-curve consists of a disjoint union of essential simple closed curves which can be realized disjointly; we allow that some of the component curves are freely homotopic. For every side ρ of a complementary component of τ there are at least 4 connected subarcs of c which are mapped by the natural carrying map $c \to \tau$ *onto* ρ. Namely, the number of such arcs is just the minimum of the μ-weights of a branch contained in ρ.

Assign to a branch b^* of τ_b^* which is dual to the branch b of τ the weight $\nu(b^*) = \mu(b)$, and to a branch of τ_b^* which is contained in the interior of a complementary region of τ assign the weight 0. The resulting weight function ν is a tangential measure for τ_b^*, but it is not positive (this relation between transverse measures on τ and tangential measures on τ_b^* is discussed in detail in Section 3.4 of [PH92]). However by construction, every branch of vanishing ν-mass is contained in the interior of a complementary trigon or once punctured monogon of τ, and positive mass can be pushed onto these branches by "sneaking up" as described on p.39 and p.200 of [PH92]. Namely, the closed multi-curve c defined by the positive integral transverse measure μ on τ hits the bigon track τ_b^* efficiently. For every branch b of τ the weight $\nu(b^*) = \mu(b)$ equals the number of intersections between b^* and c. For each side of a complementary component T of τ there are at least 4 arcs from c which are mapped by the carrying map onto this side. If the side consists of more than one branch then we pull two of these arcs into T as shown in Figure K. If the side consists of a single branch then we pull a single arc into T in the same way.

Figure K



For a branch e of τ_b^* define $\mu^*(e)$ to be the number of intersections between e and the deformed multi-curve. The resulting weight function μ^* is a positive integral tangential measure for τ_b^*. Note that the weight of each side of a complementary trigon in τ_b^* is exactly 2 by construction, and the weight of a side of a once punctured monogon is 2 as well. In particular, the tangential measure μ^* satisfies the strict triangle inequality for complementary trigons: If T is any complementary trigon with sides e_1, e_2, e_3 then $\mu^*(e_i) < \mu^*(e_{i+1}) + \mu^*(e_{i+2})$ (compare the proof of Lemma A.1).

Apply the algorithm from the proof of Lemma A.3 to the bigon track τ_b^*, the tangential measure μ^* for τ_b^* constructed from an integral transverse measure μ on τ with $\mu(b) \geq 4$ for every branch b and a complete geodesic lamination λ which hits τ efficiently and hence is carried by τ_b^*. Since the number of branches of τ_b^* is bounded from above by a constant only depending on the topological type of S, in at most p steps for a universal number $p > 0$ the algorithm constructs from these data a complete train track τ^* which carries λ. The train track τ^* is not unique, and it depends on λ and μ. However, since the number of steps in the algorithm is uniformly bounded and since each step involves only a uniformly bounded number of choices, the number of such train tracks which can be obtained from τ_b^* by this procedure is uniformly bounded. Moroever, the algorithm is equivariant with respect to the action of the mapping class group and therefore by invariance under the action of $\mathcal{MCG}(S)$, the distance between τ and τ^* is uniformly bounded. In other words, τ^* has properties 1) and 2) stated in the proposition.

To show property 3), let σ be a complete train track on S which hits τ efficiently and carries a complete geodesic lamination $\lambda \in \mathcal{CL}$. Then σ is carried by τ_b^* and hence it is carried by every bigon track which can be obtained from τ_b^* by a sequence of combings, shifts and collapses. On the other hand, if η_i $(0 \leq i \leq k)$ are the successive bigon tracks obtained from an application of the algorithm to $\eta_0 = \tau_b^*$ and if η_i is obtained from η_{i-1} by a split at a large branch e, then this split is a λ-split. By Lemma A.3 and Lemma 6.7 of [H09] (which are local statements and hence they are also valid for bigon tracks) there is a universal number $m > 0$ such that if σ is carried by η_{i-1} and carries λ then there is a train track $\tilde{\sigma}$ which carries λ, which is carried by η_i and which can be obtained from σ by a splitting and shifting sequence of length at most m. Since the number of splits which occur during the modification of τ_b^* to τ^* is uniformly bounded, this means that τ^* also satisfies the third requirement in the proposition. This completes the proof of the proposition. □

Remark: We call the train track τ^* constructed in the proof of Proposition A.4 from a complete train track τ and a complete geodesic lamination λ which hits τ efficiently a λ-*collapse* of τ_b^*. Note that a λ-collapse is not unique, but the number of different train tracks which can be obtained from the construction is bounded by a constant only depending on the topological type of S. In general, a λ-collaps of τ_b^* is neither carried by the dual bigon track of τ nor carries this bigon track. In particular, in general a λ-collapse of τ does not hit τ efficiently.

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