WORD HYPERBOLIC EXTENSIONS OF SURFACE GROUPS

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ABSTRACT. Let S be a closed surface of genus $g \geq 2$. A finitely generated group Γ_S is an extension of the fundamental group $\pi_1(S)$ of S if $\pi_1(S)$ is a normal subgroup of Γ_S. We show that the group Γ_S is hyperbolic if and only if the orbit map for the action of the quotient group $\Gamma = \Gamma_S/\pi_1(S)$ on the complex of curves is a quasi-isometric embedding.

1. INTRODUCTION

Let Γ be a finitely generated group. A finite symmetric set $\mathcal{G}$ of generators induces a *word norm* $\| \, \|$ on Γ by defining $\|\varphi\|$ to be the smallest length of a word in the generating set $\mathcal{G}$ which represents φ. For $\varphi, \psi \in \Gamma$ let $d(\varphi, \psi) = \|\varphi^{-1}\psi\|$; then d is a distance function on Γ which is invariant under the left action of Γ on itself. Any two such distance functions on Γ are bilipschitz equivalent. The group Γ is called *word hyperbolic* if equipped with the distance induced by one (and hence every) word norm, Γ is a hyperbolic metric space.

In this note we are interested in word hyperbolic groups which are *extensions* of the fundamental group $\pi_1(S)$ of a closed orientable surface S of genus $g \geq 2$. By definition, this means that such a group Γ_S contains $\pi_1(S)$ as a normal subgroup. Our main goal is to give a geometric characterization of such groups via the action of the quotient group $\Gamma = \Gamma_S/\pi_1(S)$ on the *complex of curves* for the surface S.

Our approach builds on earlier work of Mosher [Mo96, Mo03] and Farb and Mosher [FM02]. First, recall that by a classical result of Dehn-Nielsen-Baer (see [I02]), the *extended mapping class group* $\mathcal{M}_g^0$ of all isotopy classes of diffeomorphisms of S is just the group of outer automorphisms of the fundamental group $\pi_1(S)$ of S. Since the center of $\pi_1(S)$ is trivial we can identify $\pi_1(S)$ with its group of inner automorphisms and therefore we obtain an exact sequence

$$1 \to \pi_1(S) \to \mathrm{Aut}(\pi_1(S)) \xrightarrow{\Pi} \mathcal{M}_g^0 \to 1.$$

In particular, for every subgroup Γ of $\mathcal{M}_g^0$ the pre-image $\Pi^{-1}(\Gamma)$ of Γ under the projection Π is an extension of $\pi_1(S)$ with quotient group Γ. Vice versa, if Γ_S is any group which contains $\pi_1(S)$ as a normal subgroup then the quotient group $\Gamma = \Gamma_S/\pi_1(S)$ acts as a group of outer automorphisms on $\pi_1(S)$ and therefore there is a natural homomorphism $\rho : \Gamma \to \mathcal{M}_g^0$.

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Now consider any finitely generated extension Γ_S of $\pi_1(S)$ with quotient group $\Gamma = \Gamma_S/\pi_1(S)$. If Γ_S is word hyperbolic, then the kernel K of the natural homomorphism $\rho : \Gamma \to \mathcal{M}_g^0$ is finite. Namely, Γ_S contains the direct product $\pi_1(S) \times K$ as a subgroup and no word hyperbolic group can contain the direct product of two infinite subgroups (see [FM02]). As a consequence, the extension of $\pi_1(S)$ defined by the subgroup $\rho(\Gamma)$ of $\mathcal{M}_g^0$ is the quotient of Γ_S by a finite normal subgroup. Since passing to the quotient by a finite normal subgroup preserves hyperbolicity, we may assume without loss of generality that $\Gamma = \Gamma_S/\pi_1(S)$ is a subgroup of $\mathcal{M}_g^0$.

For every subgroup Γ of $\mathcal{M}_g^0$, the intersection Γ_0 of Γ with the *mapping class group* $\mathcal{M}_g$ of all isotopy classes of orientation preserving diffeomorphisms is a subgroup of Γ of index at most 2 and hence the pre-image $\Pi^{-1}(\Gamma_0)$ of Γ_0 under the projection Π is a subgroup of $\Gamma_S = \Pi^{-1}(\Gamma)$ of index at most 2 as well. Thus $\Pi^{-1}(\Gamma_0)$ is quasi-isometric to Γ_S and word hyperbolic if and only if this is the case for Γ_S. In other words, we may assume that $\Gamma = \Gamma_S/\pi_1(S)$ is a subgroup of $\mathcal{M}_g$.

The vertices of the *complex of curves* $\mathcal{C}(S)$ for S are nontrivial free homotopy classes of simple closed curves on S. The simplices in $\mathcal{C}(S)$ are spanned by collections of such curves which can be realized disjointly. In the sequel we restrict our attention to the one-skeleton of $\mathcal{C}(S)$ which we denote again by $\mathcal{C}(S)$ by abuse of notation. Since $g \geq 2$ by assumption, $\mathcal{C}(S)$ is a nontrivial graph which moreover is connected [Ha81]. However, this graph is locally infinite. Namely, for every simple closed curve α on S the surface $S - \alpha$ which we obtain by cutting S open along α contains at least one connected component of Euler characteristic at most -2, and such a component contains infinitely many pairwise distinct free homotopy classes of simple closed curves which viewed as curves in S are disjoint from α.

Providing each edge in $\mathcal{C}(S)$ with the standard euclidean metric of diameter 1 equips the complex of curves with the structure of a geodesic metric space. Since $\mathcal{C}(S)$ is not locally finite, this metric space $(\mathcal{C}(S), d)$ is not locally compact. Masur and Minsky [MM99] showed that nevertheless its geometry can be understood quite explicitly. Namely, $\mathcal{C}(S)$ is hyperbolic of infinite diameter (see also [B02, H05] for alternative shorter proofs). The extended mapping class group $\mathcal{M}_g^0$ of all isotopy classes of diffeomorphisms of S acts naturally on $\mathcal{C}(S)$ as a group of simplicial isometries. In fact, Ivanov showed that if $g \neq 2$ then $\mathcal{M}_g^0$ is precisely the isometry group of $\mathcal{C}(S)$ (see [I02] for a sketch of a proof and for references).

A map Φ of a finitely generated group Γ into a metric space (Y, d) is called a *quasi-isometric embedding* if for some (and hence every) choice of a word norm $\| \, \|$ for Γ there exists a number $L > 1$ such that

$$d(\Phi\psi, \Phi\eta)/L - L \leq \|\psi^{-1}\eta\| \leq Ld(\Phi\psi, \Phi\eta) + L.$$

Note that a quasi-isometric embedding need not be injective. The following definition extends the well known notion of a convex cocompact group of isometries of a simply connected Riemannian manifold of bounded negative curvature to subgroups of the extended mapping class group, viewed as the isometry group of the complex of curves.

Definition: A finitely generated subgroup Γ of $\mathcal{M}_g^0$ is called *convex cocompact* if for some $\alpha \in \mathcal{C}(S)$ the orbit map $\varphi \in \Gamma \to \varphi\alpha \in \mathcal{C}(S)$ is a quasi-isometric embedding.

Since $\mathcal{C}(S)$ is a hyperbolic geodesic metric space, every convex cocompact subgroup Γ of $\mathcal{M}_g$ is word hyperbolic.

Farb and Mosher [FM02] introduce another notion of a convex cocompact subgroup Γ of $\mathcal{M}_g$ via its action on the *Teichmüller space* $\mathcal{T}_g$ of all marked hyperbolic metrics on S. Namely, they define a group $\Gamma < \mathcal{M}_g$ to be convex cocompact if a Γ-orbit in $\mathcal{T}_g$ is *quasi-convex* with respect to the *Teichmüller metric*; this means that for every fixed $h \in \mathcal{T}_g$ and any two elements $\varphi, \psi \in \Gamma$ the unique Teichmüller geodesic connecting φh to ψh is contained in a uniformly bounded neighborhood of the orbit Γh. Answering among other things a question raised by Farb and Mosher, we show.

Theorem: *For a finitely generated subgroup Γ of $\mathcal{M}_g$, the following are equivalent.*

(1) *Γ is convex cocompact.*
(2) *Some Γ-orbit on $\mathcal{T}_g$ is quasi-convex.*
(3) *The natural extension of $\pi_1(S)$ with quotient group Γ is word hyperbolic.*

The implication 3) $\Longrightarrow$ 2) in our theorem is due to Farb and Mosher [FM02] and was the main motivation for this work. In the particular case that the subgroup Γ of $\mathcal{M}_g$ is *free*, it was observed in [FM02] that the reverse implication 2) $\Longrightarrow$ 3) is an easy consequence of a difficult result of Bestvina and Feighn [BF92].

Examples of convex cocompact subgroups of $\mathcal{M}_g$ are *Schottky groups* which are defined to be *free* convex cocompact subgroups of $\mathcal{M}_g$. There is an abundance of such groups: Since every *pseudo-Anosov* element of $\mathcal{M}_g$ acts with north-south dynamics on the Gromov boundary of the complex of curves, the classical ping-pong lemma shows that for any two non-commuting pseudo-Anosov elements $\varphi, \psi \in \mathcal{M}_g$ there are numbers $k \geq 1, \ell \geq 1$ such that the subgroup of $\mathcal{M}_g$ generated by φ^k, ψ^ℓ is free and convex cocompact. However, to our knowledge there are no known examples of convex cocompact groups which are not virtually free. On the other hand, there are examples of surface-subgroups of $\mathcal{M}_g$ with interesting geometric properties [GDH99, LR05], but these groups contain elements which are not pseudo-Anosov. Since the orbit on $\mathcal{C}(S)$ of an infinite cyclic subgroup of $\mathcal{M}_g$ generated by an element which is not pseudo-Anosov is bounded, these groups are not convex cocompact.

The organization of this paper is as follows. In Section 2, we define a map $\Psi : \mathcal{T}_g \to \mathcal{C}(S)$ which is roughly equivariant with respect to the action of $\mathcal{M}_g$. We characterize quasi-geodesics in Teichmüller space which are mapped by Ψ to quasi-geodesics in the complex of curves and deduce from this as an immediate corollary the equivalence of 1) and 2) in our theorem. Section 3 contains the main technical result of this note. We give a geometric description of hyperbolic fibrations with fibre a tree and base a proper hyperbolic geodesic space. In the case that the fibre and the base are metric trees which admit a cocompact isometry group, our characterization coincides with the one given by Bestvina and Feighn [BF92] (our proof is however completely different). The results in Section 2 and Section 3 are used in Section 4 to show the equivalence of 1) and 3) in our theorem.

2. Quasi-geodesics in Teichmüller space which project to quasi-geodesics in the complex of curves

In this section we consider an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures. We require that S is *non-exceptional*, i.e. that $3g-3+m \geq 2$. The *Teichmüller space* $\mathcal{T}_{g,m}$ of all marked isometry classes of complete hyperbolic metrics on S of finite volume is homeomorphic to $\mathbb{R}^{6g-6+2m}$. The *mapping class group* $\mathcal{M}_{g,m}$ of all isotopy classes of orientation preserving diffeomorphisms of S acts properly discontinuously on $\mathcal{T}_{g,m}$ preserving the *Teichmüller metric*. The Teichmüller metric is a complete Finsler metric on $\mathcal{T}_{g,m}$.

The one-skeleton $\mathcal{C}(S)$ of the *complex of curves* for S is defined to be the metric graph whose vertices are free homotopy classes of simple closed *essential* curves, i.e. curves which are not contractible or homotopic into a puncture, and where two such vertices are connected by an edge of length 1 if and only if they can be realized disjointly. Since S is non-exceptional by assumption, the graph $\mathcal{C}(S)$ is connected. Moreover, as a metric space it is hyperbolic in the sense of Gromov [MM99, B02, H05].

For every marked hyperbolic metric $h \in \mathcal{T}_{g,m}$, every essential free homotopy class α on S can be represented by a closed geodesic which is unique up to parametrization. This geodesic is simple if the free homotopy class admits a simple representative. The *h-length* $\ell_h(\alpha)$ of the class is defined to be the length of its geodesic representative; equivalently, $\ell_h(\alpha)$ equals the minimum of the h-lengths of all closed curves representing the class α.

A *pants decomposition* for S is a collection of $3g-3+m$ pairwise disjoint simple closed essential curves on S which decompose S into $2g-2+m$ *pairs of pants*, i.e. planar surfaces homeomorphic to a three-holed sphere. By a classical result of Bers (see [Bu92]), there is a number $\chi > 0$ only depending on the topological type of S such that for every complete hyperbolic metric h on S of finite volume there is a pants decomposition for S consisting of simple closed curves of h-length at most χ. Define a map $\Psi : \mathcal{T}_{g,m} \to \mathcal{C}(S)$ by associating to a complete hyperbolic metric h on S of finite volume an essential simple closed curve $\Psi(h) \in \mathcal{C}(S)$ whose h-length is at most χ. By the collar theorem for hyperbolic surfaces (see [Bu92]), the number of intersection points between any two simple closed geodesics of length at most χ for any metric $h \in \mathcal{T}_{g,m}$ is bounded from above by a universal constant. On the other hand, the distance between two curves $\alpha, \beta \in \mathcal{C}(S)$ is bounded from above by the minimal number of intersection points between any representatives of α, β plus one [MM99, B02]. Thus the diameter in $\mathcal{C}(S)$ of the set of all simple closed curves of h-length at most χ is bounded from above by a universal constant $R > 0$ not depending on h and the map Ψ is roughly equivariant with respect to the action of $\mathcal{M}_{g,m}$. This means that for every $\varphi \in \mathcal{M}_{g,m}$ and every $h \in \mathcal{T}_{g,m}$ we have $d(\varphi(\Psi h), \Psi(\varphi h)) \leq R$.

Let $J \subset \mathbb{R}$ be a closed connected subset, i.e. either J is a closed interval or a closed ray or the whole line. For some $p > 1$, a map $\gamma : J \to \mathcal{C}(S)$ is called a *p-quasi-geodesic* if for all $s, t \in J$ we have

$$d(\gamma(s), \gamma(t))/p - p \leq |s - t| \leq pd(\gamma(s), \gamma(t)) + p.$$

The map $\gamma : J \to \mathcal{C}(S)$ is called an *unparametrized p-quasi-geodesic* if there is a closed connected set $I \subset \mathbb{R}$ and a homeomorphism $\zeta : I \to J$ such that $\gamma \circ \zeta : I \to \mathcal{C}(S)$ is a p-quasi-geodesic. By a result of Masur and Minsky (Theorem 2.6 and Theorem 2.3 of [MM99]; the precise quantitative version which we will use is Theorem 4.1 of [H05]), there is a number $p > 1$ such that the image under Ψ of *every* Teichmüller geodesic (i.e. every geodesic in $\mathcal{T}_{g,m}$ with respect to the Teichmüller metric) is an unparametrized p-quasi-geodesic. However, in general this curve is not a quasi-geodesic with its proper parametrization (see [MM99]).

For $\epsilon > 0$ let $\mathcal{T}_{g,m}^{\epsilon}$ be the collection of all hyperbolic metrics $h \in \mathcal{T}_{g,m}$ for which the length of the shortest closed h-geodesic is at least ϵ. Informally we think of $\mathcal{T}_{g,m}^{\epsilon}$ as the ϵ-thick part of Teichmüller space. The mapping class group preserves the set $\mathcal{T}_{g,m}^{\epsilon}$ and acts on it cocompactly. Moreover, every $\mathcal{M}_{g,m}$-invariant subset of $\mathcal{T}_{g,m}$ on which $\mathcal{M}_{g,m}$ acts cocompactly is contained in $\mathcal{T}_{g,m}^{\epsilon}$ for some $\epsilon > 0$.

Define for $\epsilon > 0$ a *quasi-convex curve* in $\mathcal{T}_{g,m}^{\epsilon}$ to be a closed subset of $\mathcal{T}_{g,m}$ whose *Hausdorff distance* to the image of a geodesic arc $\zeta : J \to \mathcal{T}_{g,m}^{\epsilon}$ is at most $1/\epsilon$. Recall that the Hausdorff distance between two closed subsets A, B of a metric space is the infimum of all numbers $r > 0$ such that A is contained in the r-neighborhood of B and B is contained in the r-neighborhood of A. The main goal of this section is to show the following result of independent interest.

Theorem 2.1:

(1) *For every $\nu > 1$ there is a constant $\epsilon = \epsilon(\nu) > 0$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set of diameter at least $1/\epsilon$ and let $\gamma : J \to \mathcal{T}_{g,m}$ be a ν-quasi-geodesic. If $\Psi \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{C}(S)$ then $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}_{g,m}^{\epsilon}$.*
(2) *For every $\epsilon > 0$ there is a constant $\nu(\epsilon) > 1$ with the following property. Let $\gamma : J \to \mathcal{T}_{g,m}$ be a $1/\epsilon$-quasi-geodesic in $\mathcal{T}_{g,m}$ whose image $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}_{g,m}^{\epsilon}$; then $\Psi \circ \gamma$ is a $\nu(\epsilon)$-quasi-geodesic in $\mathcal{C}(S)$.*

We begin with establishing the second part of our theorem. For this we need the following simple no-retraction lemma for quasi-geodesics in the hyperbolic geodesic metric space $\mathcal{C}(S)$.

Lemma 2.2: *For $p > 1$ there is a constant $c = c(p) > 0$ with the following property. Let $\gamma : J \to \mathcal{C}(S)$ be any unparametrized p-quasi-geodesic; if $t_1 < t_2 < t_3 \in J$ then $d(\gamma(t_1), \gamma(t_3)) \geq d(\gamma(t_1), \gamma(t_2)) + d(\gamma(t_2), \gamma(t_3)) - c$.*

Proof: Let $p>1$; by the definition of an unparametrized p-quasi-geodesic, it is enough to show the existence of a number $c>0$ such that for every (parametrized) p-quasi-geodesic $\gamma:[0,n]\to\mathcal{C}(S)$ and all $0<t<n$ we have $d(\gamma(0),\gamma(n))\geq d(\gamma(0),\gamma(t))+d(\gamma(t),\gamma(n))-c$.

Since $\mathcal{C}(S)$ is hyperbolic, there is a constant $R>0$ only depending on p such that the Hausdorff distance between every p-quasi-geodesic and every geodesic connecting the same endpoints is at most R. Let $\gamma:[0,n]\to\mathcal{C}(S)$ be any p-quasi-geodesic and let $\zeta:[0,m]\to\mathcal{C}(S)$ be a geodesic connecting $\gamma(0)$ to $\gamma(n)$. Then for every $t\in(0,n)$ there is a point $s\in(0,m)$ such that $d(\gamma(t),\zeta(s))\leq R$. Thus we have $d(\gamma(0),\gamma(t))+d(\gamma(t),\gamma(n))\leq d(\zeta(0),\zeta(s))+d(\zeta(s),\zeta(m))+2R=d(\gamma(0),\gamma(n))+2R$ which shows the lemma. □

A *geodesic lamination* for a hyperbolic metric $h\in\mathcal{T}_{g,m}$ is a *compact* subset of S foliated by simple h-geodesics [CEG87]. A *measured geodesic lamination* μ on S is a geodesic lamination together with a nontrivial transverse invariant measure. An example of a measured geodesic lamination on S is a simple closed curve with the transverse counting measure. The space $\mathcal{ML}$ of measured geodesic laminations on S can be equipped with the weak*-topology, and with this topology, it is homeomorphic to $\mathbb{R}^{6g-6+2m}-\{0\}$. There is a natural continuous action of the multiplicative group $(0,\infty)$ of positive reals on $\mathcal{ML}$ by scaling, and the quotient of $\mathcal{ML}$ under this action is the space $\mathcal{PML}$ of *projective measured laminations* which is homeomorphic to the sphere $S^{6g-7+2m}$. It can naturally be identified with the projectivized cotangent space of $\mathcal{T}_{g,m}$ at h. The space $\mathcal{PML}$ also is the boundary of a compactification of $\mathcal{T}_{g,m}$, called the *Thurston boundary* of Teichmüller space. This is used to show.

Lemma 2.3: *For every $\epsilon>0$ there is a number $\nu_0=\nu_0(\epsilon)>0$ with the following property. Let $\gamma:J\to\mathcal{T}_{g,m}^{\epsilon}$ be a Teichmüller geodesic; then the curve $\Psi\circ\gamma:J\to\mathcal{C}(S)$ is a ν_0-quasi-geodesic.*

Proof: Let $p>1$ be such that the image under Ψ of every Teichmüller geodesic is an unparametrized p-quasi-geodesic in $\mathcal{C}(S)$; such a number exists by the results of Masur and Minsky [MM99, H05]. Let $c=c(p)>0$ be as in Lemma 2.2.

We claim that for every $\epsilon>0$ there is a constant $k_0=k_0(\epsilon)>0$ with the following property. Let $k\geq k_0$ and let $\gamma:[0,k]\to\mathcal{T}_{g,m}^{\epsilon}$ be a geodesic arc of length at least k_0; then $d(\Psi(\gamma(0)),\Psi(\gamma(k)))\geq 2c$.

To see that this is the case, we argue by contradiction and we assume otherwise. With the number c as above, by Lemma 2.2 there is then a number $\epsilon>0$ and for every $k>0$ there is a geodesic arc $\gamma_k:[0,k]\to\mathcal{T}_{g,m}^{\epsilon}$ such that $d(\Psi\gamma_k(0),\Psi\gamma_k(t))\leq 3c$ for every $t\in[0,k]$. Let $R>0$ be an upper bound for the diameter in $\mathcal{C}(S)$ of the set of all simple closed curves whose length with respect to some fixed metric $h\in\mathcal{T}_{g,m}$ is at most χ. Since the action of $\mathcal{M}_{g,m}$ on $\mathcal{T}_{g,m}^{\epsilon}$ is isometric and cocompact, via replacing our constant $3c$ by $3c+2R$ we may assume that the initial points $\gamma_k(0)$ ($k\geq 1$) of the geodesic arcs γ_k are contained in a fixed compact subset of $\mathcal{T}_{g,m}^{\epsilon}$. Thus by passing to a subsequence we may assume that the geodesics γ_k converge locally uniformly as $k\to\infty$ to a geodesic $\gamma:[0,\infty)\to\mathcal{T}_{g,m}^{\epsilon}$. By the definition of the map Ψ and continuity of the length functions on Teichmüller space we then have $d(\Psi\gamma(s),\Psi\gamma(0))\leq 3c+4R$ for all $s\geq 0$.

Let $\lambda\in\mathcal{PML}$ be the projective measured geodesic lamination which defines the direction of γ at $\gamma(0)$, viewed as a point in the projectivized cotangent space of $\mathcal{T}_{g,m}$ at $\gamma(0)$. Since γ is cobounded, i.e. it projects into a compact subset of moduli space $\mathcal{T}_{g,m}/\mathcal{M}_{g,m}$, by a result of Masur [Ma82a] the lamination λ *fills up* S; this means that every simple closed curve on S intersects λ transversely. Moreover, $\gamma(t)$ converges as $t\to\infty$ in the Thurston compactification of $\mathcal{T}_{g,m}$ to λ [Ma82b]. By the definition of the Thurston compactification of $\mathcal{T}_{g,m}$ (see [FLP91]), this implies that the curves $\Psi(\gamma(t))$, viewed as projective measured laminations, converge as $t\to\infty$ in $\mathcal{PML}$ to λ. As a consequence, the curve $\Psi\circ\gamma$ is an unparametrized quasi-geodesic in $\mathcal{C}(S)$ of *infinite* diameter (see [K99], [H04]) which is a contradiction and shows our claim.

Now let $n>0$ and let $\gamma:[0,k_0n]\to\mathcal{T}^{\epsilon}_{g,m}$ be any Teichmüller geodesic. The image under Ψ of every geodesic in $\mathcal{T}_{g,m}$ is an unparametrized p-quasi-geodesic; thus by the choice of c, for all $0\leq s\leq t$ we have $d(\Psi\gamma(t),\Psi\gamma(0))\geq d(\Psi\gamma(t),\Psi\gamma(s))+d(\Psi\gamma(s),\Psi\gamma(0))-c$. On the other hand, from our above consideration and the choice of k_0 we conclude that for every $u<n$ we have $d(\Psi\gamma(uk_0),\Psi\gamma((u+1)k_0))\geq 2c$ and therefore $d(\Psi\gamma((u+1)k_0)),\Psi\gamma(s))\geq d(\Psi\gamma(uk_0),\Psi\gamma(s))+c$ for all $s\leq uk_0$. Inductively we deduce that $d(\Psi\gamma(uk_0),\Psi\gamma(vk_0))\geq c|u-v|$ for all integers $u,v\leq n$. The map $\Psi:\mathcal{T}_{g,m}\to\mathcal{C}(S)$ is *coarsely Lipschitz* by which we mean that there is a constant $a>0$ such that $d(\Psi h,\Psi h')\leq ad(h,h')+a$ for all $h,h'\in\mathcal{T}_{g,m}$ and where $d(h,h')$ denotes the Teichmüller distance between h and h'. Together with above, it follows that $\Psi\gamma$ is a ν_0-quasi-geodesic for a constant $\nu_0>0$ only depending on ϵ (more precisely, we have $c|s-t|/k_0-k_0a-a\leq d(\Psi\gamma(s),\Psi\gamma(t))\leq a|s-t|+a$ for all $s,t\in[0,k_0n]$). This shows the lemma. □

The following corollary shows the second part of Theorem 2.1.

Corollary 2.4: *For every $\epsilon>0$ there is a number $\nu=\nu(\epsilon)>1$ with the following property. Let $\gamma:J\to\mathcal{T}_{g,m}$ be a $1/\epsilon$-quasi-geodesic such that $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}^{\epsilon}_{g,m}$; then $\Psi\circ\gamma:J\to\mathcal{C}(S)$ is a ν-quasi-geodesic.*

Proof: Let $\epsilon>0$ and let $\gamma:J\to\mathcal{T}_{g,m}$ be a $1/\epsilon$-quasi-geodesic such that $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}^{\epsilon}_{g,m}$. Then there is a Teichmüller geodesic $\zeta:I\to\mathcal{T}^{\epsilon}_{g,m}$ and a map $\rho:J\to I$ such that $d(\zeta(\rho(t)),\gamma(t))\leq 1/\epsilon$ for all t. Since by assumption the map γ is a $1/\epsilon$-quasi-geodesic in $\mathcal{T}_{g,m}$ and since ζ realizes the distance between any of its points, the map ρ is necessarily a b-quasi-isometry for a constant $b>1$ only depending on ϵ. On the other hand, the map Ψ is coarsely Lipschitz and therefore the distances $d(\Psi\gamma(t),\Psi(\zeta\circ\rho(t)))$ are bounded from above by a number only depending on ϵ. This implies by Lemma 2.3 that $\Psi\circ\gamma$ is a ν-quasi-geodesic for a constant $\nu>0$ only depending on ϵ. □

To show the first part of Theorem 2.1, we begin again with a simple observation.

Lemma 2.5: *For every $\nu > 1$ there is a number $\epsilon_0 = \epsilon_0(\nu) > 0$ with the following properties. Let $\gamma : [0, n] \to \mathcal{T}_{g,m}$ be a ν-quasi-geodesic whose projection $\Psi\gamma$ to $\mathcal{C}(S)$ is a ν-quasi-geodesic. If $n \geq 1/\epsilon_0$ then $\gamma[0, n] \subset \mathcal{T}_{g,m}^{\epsilon_0}$.*

Proof: Let $n > 0, \nu > 1$ and let $\gamma : [0, n] \to \mathcal{T}_{g,m}$ be a ν-quasi-geodesic such that $\Psi \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{C}(S)$. Then we have $d(\Psi(\gamma(t)), \Psi(\gamma(s))) \geq |s-t|/\nu - \nu$ for all $s, t \in [0, n]$. Let $R > 0$ be an upper bound for the diameter in $\mathcal{C}(S)$ of the collection of all simple closed curves on S whose length with respect to some metric $h \in \mathcal{T}_{g,m}$ is at most χ where as before, $\chi > 0$ is determined by Bers' theorem. Let $[a, b] \subset [0, n]$ be an interval for which there is a simple closed curve $\alpha \in \mathcal{C}(S)$ so that $\ell_{\gamma(t)}(\alpha) \leq \chi$ for all $t \in [a, b]$; then we have $d((\Psi(\gamma(a)), \alpha) \leq R, d(\Psi(\gamma(b)), \alpha) \leq R$ and therefore $|b - a| \leq 2\nu R + \nu^2$.

Now by a result of Wolpert (see [IT99]), for all $\alpha \in \mathcal{C}(S)$ and all $h, h' \in \mathcal{T}_{g,m}$ the distance between h and h' is at least $|\log \ell_h(\alpha) - \log \ell_{h'}(\alpha)|$. Thus if there is a point $t \in [0, n]$ with $\log(\ell_{\gamma(t)}(\alpha)) < \log(\chi) - 2\nu R - 2\nu^2$ then the $\gamma(s)$-length of α is smaller than χ for every $s \in [0, n]$ with $|s - t| \leq 2\nu R + 2\nu^2$ and consequently by our above consideration, $\Psi \circ \gamma$ is *not* a ν-quasi-geodesic provided that $n \geq 4\nu R + 4\nu^2$. □

Every Teichmüller geodesic line $\gamma : \mathbb{R} \to \mathcal{T}_{g,m}$ is defined by a *quadratic differential* on S. More precisely, for each $t \in \mathbb{R}$ there is a holomorphic quadratic differential q_t on the Riemann surface $\gamma(t)$ defining a singular euclidean metric on S in the conformal class of $\gamma(t)$ and of area one. The differential q_t and the corresponding piecewise euclidean metric are determined by the *horizontal* and the *vertical* foliation of q_t. These foliations have a common finite set of singular points and are equipped with a transverse invariant measure. For $s \neq t$, the horizontal foliation for q_s coincides with the horizontal foliation for q_t, but its transverse measure is obtained from the transverse measure for q_t by scaling with the factor e^{t-s}. Similarly, the vertical foliation of q_s coincides with the vertical foliation of q_t, but its transverse measure is obtained from the transverse measure for q_t by scaling with the factor e^{s-t}. We use this description of Teichmüller geodesics together with the arguments in Section 3.9 of [Mo03] to show the first part of Theorem 2.1.

Lemma 2.6: *For every $\nu > 1$ there is a constant $\epsilon = \epsilon(\nu) > 0$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected subset of diameter at least $1/\epsilon$ and let $\gamma : J \to \mathcal{T}_{g,m}$ be a ν-quasi-geodesic such that $\Psi \circ \gamma : J \to \mathcal{C}(S)$ is a ν-quasi-geodesic in $\mathcal{C}(S)$; then $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}_{g,m}^{\epsilon}$.*

Proof: For $\nu > 1$ define a *ν-Lipschitz curve* in $\mathcal{T}_{g,m}$ to be a ν-Lipschitz map $\gamma : J \to \mathcal{T}_{g,m}$ with respect to the standard metric on $\mathbb{R}$ and the Teichmüller metric on $\mathcal{T}_{g,m}$. Since $\mathcal{T}_{g,m}$ is a smooth manifold and the Teichmüller metric is a complete Finsler metric, every ν-quasi-geodesic $\gamma : J \to \mathcal{T}_{g,m}$ can be replaced by a piecewise geodesic $\zeta : J \to \mathcal{T}_{g,m}$ which is a 2ν-Lipschitz curve and which satisfies $d(\gamma(t), \zeta(t)) \leq 2\nu$ for all $t \in J$. Thus it is enough to show the statement of the lemma for ν-Lipschitz curves $\gamma : J \to \mathcal{T}_{g,m}$ which are ν-quasi-geodesics and such that $\Psi \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{C}(S)$. In the sequel we also assume that the diameter $|J|$ of the set J is bigger than $1/\epsilon_0$ where $\epsilon_0 = \epsilon_0(\nu)$ is as in Lemma 2.5; then $\gamma(J) \subset \mathcal{T}_{g,m}^{\epsilon_0}$.

Since $\mathcal{C}(S)$ is hyperbolic and $\Psi \circ \gamma$ is a ν-quasi-geodesic by assumption, there is a geodesic arc in $\mathcal{C}(S)$ whose Hausdorff distance to $\Psi \circ \gamma(J)$ is bounded from above by a universal constant. As a consequence, if J is one-sided infinite, say if $[0,\infty) \subset J$, then the points $\Psi(\gamma(t))$ converge as $t \to \infty$ to a point in the *Gromov boundary* $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ (see [BH] for the definition of the Gromov boundary of a hyperbolic geodesic metric space). The Gromov boundary of $\mathcal{C}(S)$ can naturally be identified with the space of *minimal* geodesic laminations on S which fill up S, equipped with a *coarse Hausdorff topology* (see [H04]). Here a geodesic lamination is minimal if each of its half-leaves is dense, and it fills up S if it intersects every essential simple closed curve on S transversely.

A simple closed curve $\alpha \in \mathcal{C}(S)$ defines a projective measured lamination which we denote by $[\alpha]$. Similarly, for a measured lamination $\lambda \in \mathcal{ML}$ we denote by $[\lambda]$ the projective class of λ. Following Mosher [Mo03], we say that the projective measured lamination $[\alpha]$ defined by a simple closed curve $\alpha \in \mathcal{C}(S)$ is *realized* at some $t \in J$ if the length of α with respect to the metric $\gamma(t) \in \mathcal{T}_{g,m}$ is at most χ. Note that the number of projective measured laminations which are realized at a given point $t \in J$ is uniformly bounded and that $[\Psi(\gamma(t))]$ is realized at $\gamma(t)$. Similarly, we say that the projectivization $[\lambda]$ of a measured geodesic lamination λ is realized at an infinite "endpoint" of J if the support of λ equals the corresponding endpoint of the quasi-geodesic $\Psi\gamma(J)$ in the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$, viewed as a minimal geodesic lamination. The set of projective measured laminations which are realized at an infinite endpoint of J is a nonempty closed subset of $\mathcal{PML}$ (see [K99], [H04]). We call a projective measured lamination which is realized at a (finite or infinite) endpoint of J an *endpoint lamination.*

Now $\Psi\gamma$ is a ν-quasi-geodesic in $\mathcal{C}(S)$ by assumption and the diameter in $\mathcal{C}(S)$ of the set of all curves of length at most χ with respect to some fixed hyperbolic metric $h \in \mathcal{T}_{g,m}$ is bounded from above by a universal constant. Since any two curves $\alpha, \beta \in \mathcal{C}(S)$ with $d(\alpha,\beta) \geq 3$ *jointly fill up* S, i.e. are such that every simple closed essential curve $\zeta \in \mathcal{C}(S)$ intersects either α or β transversely, by possibly increasing the lower bound for the diameter of J we may assume that any two projective measured laminations $[\alpha],[\beta]$ which are realized at the two distinct endpoints of J jointly fill up S.

There is a 1-1-correspondence between measured geodesic laminations and equivalence classes of *measured foliations* on S (see e.g. [Ke92] for a precise statement and references). Via this identification, any pair of distinct points $[\lambda] \neq [\mu] \in \mathcal{PML}$ which jointly fill up the surface S define a unique Teichmüller geodesic line. Thus for every ν-quasi-geodesic $\zeta : J \to \mathcal{T}_{g,m}$ with $|J| \geq 1/\epsilon_0$ such that $\Psi\zeta$ is a ν-quasi-geodesic in $\mathcal{C}(S)$, any pair of projective measured laminations $[\lambda],[\mu]$ realized at the two (possibly infinite) endpoints of ζ defines a unique Teichmüller geodesic $\eta([\lambda],[\mu])$.

Choose a number $R > 2\chi$ and a smooth function $\sigma : [0,\infty) \to [0,1]$ with $\sigma[0,\chi] \equiv 1$ and $\sigma[R,\infty) \equiv 0$. For each $h \in \mathcal{T}_{g,m}$, the number of simple closed geodesics α for h with $\ell_h(\alpha) \leq R$ is bounded from above by a universal constant not depending on h, and the diameter of the subset of $\mathcal{C}(S)$ containing these curves is uniformly bounded as well. Thus we obtain for every $h \in \mathcal{T}_{g,m}$ a finite Borel measure μ_h on $\mathcal{C}(S)$ by defining $\mu_h = \sum_\beta \sigma(\ell_h(\beta))\delta_\beta$ where δ_β denotes the Dirac mass at

β. The total mass of μ_h is bounded from above and below by a universal positive constant, and the diameter of the support of μ_h in $\mathcal{C}(S)$ is uniformly bounded as well. Moreover, the measures μ_h depend continuously on $h \in \mathcal{T}_{g,m}$ in the weak*-topology. This means that for every bounded function $f : \mathcal{C}(S) \to \mathbb{R}$ the function $h \to \int f d\mu_h$ is continuous.

We define now a new "distance" function ρ on $\mathcal{T}_{g,m}$ by

$$\rho(h,h') = \int_{\mathcal{C}(S)\times\mathcal{C}(S)} d(\cdot,\cdot) d\mu_h \times d\mu'_h / \mu_h(\mathcal{C}(S))\mu_{h'}(\mathcal{C}(S)).$$

Clearly the function ρ is positive and continuous on $\mathcal{T}_{g,m} \times \mathcal{T}_{g,m}$ and invariant under the action of $\mathcal{M}_{g,m}$. Moreover, it is immediate that there is a universal constant $a > 0$ such that $\rho(h,h')/a - a \le d(\Psi(h),\Psi(h')) \le a\rho(h,h') + a$. As a consequence, for every $\nu > 1$ there is a constant $p = p(\nu) > 1$ with the following property. If $\gamma : J \to \mathcal{T}_{g,m}$ is such that $\Psi\gamma$ is a ν-quasi-geodesic, then γ is a p-quasi-geodesic with respect to the "distance" function ρ. By this we mean that

$$\rho(\gamma(s),\gamma(t))/p - p \le |s-t| \le p\rho(\gamma(s),\gamma(t)) + p$$

for all $s,t \in J$. Moreover, for every $p > 1$ there is a constant $\nu = \nu(p) > 1$ such that if $\gamma : J \to \mathcal{T}_{g,m}$ is a Lipschitz curve which is a p-quasi-geodesic with respect to ρ, then $\Psi \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{C}(S)$.

Let $h \in \mathcal{T}_{g,m}$ and let $\mu \in \mathcal{ML}$ be a measured geodesic lamination. The product of the transverse measure for μ together with the length element of h defines a measure on the support of μ whose total mass is called the *h-length* of μ; we denote it by $\ell_h(\mu)$. Following Mosher [Mo03], for $p > 1$ define Γ_p to be the set of all triples $(\gamma : J \to \mathcal{T}_{g,m}, \lambda_+, \lambda_-)$ with the following properties.

(1) $0 \in J$ and the diameter of J is at least $1/\epsilon_0$ where $\epsilon_0 = \epsilon_0(\nu(p))$ is as in Lemma 2.5.
(2) $\gamma : J \to \mathcal{T}_{g,m}$ is a p-Lipschitz curve which is a p-quasi-geodesic with respect to the "distance" ρ.
(3) $\lambda_+, \lambda_- \in \mathcal{ML}$ are laminations of $\gamma(0)$-length 1, and the projective measured lamination $[\lambda_+]$ is realized at the right end, the projective measured lamination $[\lambda_-]$ is realized at the left end of γ.

We equip Γ_p with the product topology, using the weak*-topology on $\mathcal{ML}$ for the second and the third component of our triple and the compact-open topology for the arc γ in $\mathcal{T}_{g,m}$. Note that this topology is metrizable.

We follow Mosher (Proposition 3.17 of [Mo03]) and show that the action of $\mathcal{M}_{g,m}$ on Γ_p is cocompact. Namely, recall from Lemma 2.5 that there is a constant $\epsilon_0 > 0$ such that for every $(\gamma,\lambda_+,\lambda_-) \in \Gamma_p$ the image of γ is contained in $\mathcal{T}_{g,m}^{\epsilon_0}$. Since $\mathcal{M}_{g,m}$ acts cocompactly on $\mathcal{T}_{g,m}^{\epsilon_0}$ it is therefore enough to show that the subset of Γ_p consisting of triples with the additional property that $\gamma(0)$ is contained in a fixed compact subset A of $\mathcal{T}_{g,m}^{\epsilon_0}$ is compact. Since our topology is metrizable, this follows if every sequence of points $(\gamma,\lambda_+,\lambda_-)$ with $\gamma(0) \in A$ has a convergent subsequence.

However, by the Arzela-Ascoli theorem, the set of p-Lipschitz maps into $\mathcal{T}_{g,m}^{\epsilon_0}$ issuing from a point in A is compact. Moreover, the function ρ on $\mathcal{T}_{g,m} \times \mathcal{T}_{g,m}$ is continuous and invariant under the action of $\mathcal{M}_{g,m}$ and hence if γ_i converges locally uniformly to γ and if γ_i is a p-quasi-geodesic with respect to ρ for all i then the same is true for γ. Since the function on $\mathcal{T}_{g,m} \times \mathcal{ML}$ which assigns to a metric $h \in \mathcal{T}_{g,m}$ and a measured lamination $\mu \in \mathcal{ML}$ the h-length of μ is continuous and since for every fixed $h \in \mathcal{T}_{g,m}$ the set of measured laminations of h-length 1 is compact and naturally homeomorphic to $\mathcal{PML}$, the action of $\mathcal{M}_{g,m}$ on Γ_p is indeed cocompact provided that the following holds: If $(\gamma_i : J_i \to \mathcal{T}_{g,m}^{\epsilon_0})$ is a sequence of p-Lipschitz curves which converge locally uniformly to $\gamma : J \to \mathcal{T}_{g,m}^{\epsilon_0}$, if the projective measured lamination $[\lambda_i]$ is realized at the right endpoint of J_i and if $[\lambda_i] \to [\lambda]$ in $\mathcal{PML}$ $(i \to \infty)$ then $[\lambda]$ is realized at the right endpoint of J.

To see that this is indeed the case, assume first that $J \cap [0,\infty) = [0,b]$ for some $b \in (0,\infty)$. Then for sufficiently large i we have $J_i \cap [0,\infty) = [0,b_i]$ with $b_i \in (0,\infty)$ and $b_i \to b$. Thus $\gamma_i(b_i) \to \gamma(b)$ $(i \to \infty)$ and therefore for sufficiently large i there is only a *finite* number of curves $\alpha \in \mathcal{C}(S)$ whose length with respect to one of the metrics $\gamma_j(b_j), \gamma(b)$ $(j \geq i)$ is at most χ. By passing to a subsequence we may assume that there is a simple closed curve $\alpha \in \mathcal{C}(S)$ with $[\lambda_j] = [\alpha]$ for all large j. The $\gamma_j(b_j)$-length of α is at most χ for all sufficiently large j and hence the same is true for the $\gamma(b)$-length of α by continuity of the length function. As a consequence, the limit $[\lambda] = [\alpha]$ of the sequence $([\lambda_i])$ is realized at the endpoint $\gamma(b)$ of γ.

In the case that $[0,\infty) \subset J$ we argue as before. Assume first that $b_i < \infty$ for all i and that $b_i \to \infty$. Recall that each of the curves $\Psi\gamma_i$ is a uniform quasi-geodesic in $\mathcal{C}(S)$ and that the map Ψ is coarsely Lipschitz. Let $\alpha_i \in \mathcal{C}(S)$ be the simple closed curve such that $[\alpha_i] = [\lambda_i]$. Then for each i, the curve α_i is contained in a ball about $\Psi(\gamma_i(b_i))$ of radius $R > 0$ independent of i and hence as $i \to \infty$, the curves α_i converge in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ to the endpoint $\mu \in \partial\mathcal{C}(S)$ of $\Psi \circ \gamma$ in the Gromov boundary of $\mathcal{C}(S)$. As a consequence, the curves α_i converge to μ in the *coarse Hausdorff topology* [H04]. This means that every accumulation point of (α_i) in the Hausdorff topology contains μ as a sublamination. Since μ is a minimal geodesic lamination which fills up S, the complement of μ in every lamination ζ containing μ as a sublamination consists of a finite number of isolated leaves and therefore every transverse measure supported in ζ is in fact supported in μ. Thus after passing to a subsequence, the projective measured laminations $[\lambda_i]$ converge as $i \to \infty$ to a projective measured lamination supported in μ. But $[\lambda_i] \to [\lambda]$ in $\mathcal{PML}$ by assumption and hence the lamination $[\lambda]$ is realized at the endpoint of γ. Similarly, if $b_i = \infty$ for infinitely many i then the endpoints $\beta_i \in \partial\mathcal{C}(S)$ of the quasi-geodesics $\Psi\gamma_i$ converge in $\partial\mathcal{C}(S)$ to the endpoint of $\Psi\gamma$. As before, this implies that the limit λ of the projective measured lamination λ_i is realized at the endpoint of γ (see [H04]). This shows our above claim and implies that the action of $\mathcal{M}_{g,m}$ on Γ_p is indeed cocompact.

Now we follow Section 3.10 of [Mo03]. Namely, each point $(\gamma, \lambda_+, \lambda_-) \in \Gamma_p$ determines the geodesic $\eta([\lambda_+],[\lambda_-])$ in $\mathcal{T}_{g,m}$. This geodesic defines a family q_t of quadratic differentials whose horizontal foliation corresponds to the lamination $e^{-t}\lambda_+$ and whose vertical foliation corresponds to $e^t\lambda_-$ (note that the area of these differentials may not be one, however this is of importance for our argument, compare [Mo03]).

For $(\gamma, \lambda_+, \lambda_-)$ define $\sigma(\gamma, \lambda_+, \lambda_-)$ to be the point on the geodesic $\eta([\lambda_+], [\lambda_-])$ which corresponds to the quadratic differential defined by the measured geodesic laminations λ_+, λ_-. The map taking $(\gamma, \lambda_+, \lambda_-)$ to $(\gamma(0), \sigma(\gamma, \lambda_+, \lambda_-)) \in \mathcal{T}_{g,m} \times \mathcal{T}_{g,m}$ is continuous and equivariant with respect to the natural action of $\mathcal{M}_{g,m}$ on Γ_p and on $\mathcal{T}_{g,m} \times \mathcal{T}_{g,m}$. Since the action of $\mathcal{M}_{g,m}$ on Γ_p is cocompact, the same is true for the action of $\mathcal{M}_{g,m}$ on the image of our map (see [Mo03]). Thus the distance between $\gamma(0)$ and $\sigma(\gamma, \lambda_+, \lambda_-)$ is bounded from above by a universal constant $p > 0$.

Let again $(\gamma, \lambda_+, \lambda_-) \in \Gamma_p$. For each $s \in J$ define

$$a_-(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_-)}, \quad a_+(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_+)}$$

where as before, $\ell_{\gamma(s)}(\lambda_\pm)$ is the $\gamma(s)$-length of $\lambda_\pm$. These are continuous functions of $s \in J$. Define for $s \in \mathbb{R}$ the shift $\gamma'(t) = \gamma(t+s)$; then the ordered triple $(\gamma'(0), a_+(s)\lambda_+, a_-(s)\lambda_-)$ lies in the $\mathcal{M}_{g,m}$-cocompact set Γ_p and hence the distance between $\gamma(s)$ and a suitably chosen point on the geodesic $\eta([\lambda_+], [\lambda_-])$ is at most p. As a consequence, the arc γ is contained in the p-neighborhood of the geodesic $\eta([\lambda_+], [\lambda_-])$. Since the curve γ is a p-quasi-geodesic, this implies that the Hausdorff distance between $\gamma(J)$ and a subarc of $\eta([\lambda_+], [\lambda_-])$ is uniformly bounded and shows the lemma. □

Recall that for every finitely generated group Γ, every finite symmetric set of generators induces a word norm on Γ, and any two such word norms are equivalent. As in the introduction, we define a convex cocompact subgroup of the mapping class group $\mathcal{M}_{g,m}$ for S as follows.

Definition 2.7: Let Γ be a finitely generated subgroup of $\mathcal{M}_{g,m}$. The group Γ is called *convex cocompact* if some orbit map $\varphi \in \Gamma \to \varphi\alpha \in \mathcal{C}(S)$ for the action of Γ on $\mathcal{C}(S)$ is a quasi-isometric embedding.

The following observation follows immediately from the fact that the map Ψ is coarsely Lipschitz.

Lemma 2.8: *For every convex cocompact group $\Gamma < \mathcal{M}_{g,m}$ and every $h \in \mathcal{T}_{g,m}$, the orbit map $\varphi \in \Gamma \to \varphi h \in \mathcal{T}_{g,m}$ is a quasi-isometric embedding.*

Proof: Let $\Gamma < \mathcal{M}_{g,m}$ be a convex cocompact group with a finite symmetric set $\mathcal{G}$ of generators and let $h \in \mathcal{T}_{g,m}$. Write $\ell = \max\{d(h, \varphi h) \mid \varphi \in \mathcal{G}\}$; since Γ acts on $\mathcal{T}_{g,m}$ as a group of isometries we have $d(\varphi h, \psi h) \leq \ell \|\varphi^{-1}\psi\|$ for all $\varphi, \psi \in \Gamma$. On the other hand, the map Ψ is coarsely equivariant with respect to the action of $\mathcal{M}_{g,m}$ on $\mathcal{T}_{g,m}$ and $\mathcal{C}(S)$ and coarsely Lipschitz; therefore there is a number $\nu > 0$ such that $d(\varphi h, \psi h) \geq d(\varphi \Psi h, \psi \Psi h)/\nu - \nu$ for all $\varphi, \psi \in \mathcal{M}_{g,m}$. Since Γ is convex cocompact by assumption, we moreover have $d(\varphi \Psi h, \psi \Psi h) \geq \|\varphi^{-1}\psi\|/\nu' - \nu'$ for some $\nu' > 0$ and all $\varphi, \psi \in \Gamma$. This shows the lemma. □

For $h \in \mathcal{T}_{g,m}$, the Γ-orbit Γh of a finite generated subgroup Γ of $\mathcal{M}_{g,m}$ is called *quasi-convex* if for any two $\varphi, \psi \in \Gamma$, the Teichmüller geodesic connecting φ to ψ is contained in a uniformly bounded neighborhood of Γh. The following result shows the first part of our theorem from the introduction in the more general context of non-exceptional surfaces of finite type.

Theorem 2.9: *A finitely generated subgroup Γ of $\mathcal{M}_{g,m}$ is convex cocompact. if and only if some Γ-orbit on $\mathcal{T}_{g,m}$ is quasi-convex.*

Proof: Let Γ be a finitely generated convex cocompact subgroup of $\mathcal{M}_{g,m}$. Let $h \in \mathcal{T}_{g,m}^{\epsilon}$ for some $\epsilon > 0$. By Lemma 2.8, the orbit map $\varphi \in \Gamma \to \varphi h \in \mathcal{T}_{g,m}$ is a quasi-isometric embedding. In particular, for any two $\varphi, \psi \in \Gamma$ the orbit of a geodesic in Γ connecting φh to ψh is a uniform quasi-geodesic in $\mathcal{T}_{g,m}$ which is contained in $\mathcal{T}_{g,m}^{\epsilon}$ and is mapped by Ψ to a uniform quasi-geodesic in $\mathcal{C}(S)$. By Theorem 2.1, this curve is contained in a uniformly bounded neighborhood of the Teichmüller geodesic connecting φh to ψh. In other words, the orbit $\Gamma h \subset \mathcal{T}_{g,m}$ is quasi-convex.

Vice versa, let $\Gamma < \mathcal{M}_{g,m}$ be a finitely generated group and assume that there is some $h \in \mathcal{T}_{g,m}$ such that the orbit Γh of h for the action of Γ on $\mathcal{T}_{g,m}$ is quasi-convex. This means that there is a number $D > 0$ such that each Teichmüller geodesic with both endpoints in Γh is contained in the D-neighborhood of Γh. Write $\alpha = \Psi h \in \mathcal{C}(S)$ and assume to the contrary that the orbit map $\varphi \to \varphi\alpha$ for the action of Γ on the complex of curves is *not* a quasi-isometry. Choose a finite symmetric set $\mathcal{G}$ of generators for Γ which induces the word norm $\| \|$. Since Γ acts on $\mathcal{C}(S)$ by isometries, the orbit map is coarsely Lipschitz with respect to the word norm $\| \|$ on Γ and the metric on $\mathcal{C}(S)$. Thus our assumption implies that for every $L > 0$ there is a word $w = w_1 \dots w_p \in \Gamma$ in the generators $w_i \in \mathcal{G}$ with $\|w\| = p$ and such that $d(w\alpha, \alpha) \le p/L$. Choose a geodesic $\zeta : [0, m] \to \mathcal{T}_{g,m}$ connecting $h = \zeta(0) \in \mathcal{T}_{g,m}^{\epsilon}$ to $wh = \zeta(m)$. Since the orbit $\Gamma h \subset \mathcal{T}_{g,m}$ is quasi-convex by assumption, the geodesic is contained in the D- neighborhood of Γh, in particular it is contained in $\mathcal{T}_{g,m}^{\epsilon}$ for a universal number $\epsilon > 0$. By Theorem 2.1, there is a number $\nu > 1$ not depending on w such that the length m of ζ is at most $\nu p/L - 1$.

On the other hand, the number of elements $\varphi \in \mathcal{M}_{g,m}$ with $d(\varphi h, h) \le 2D + 1$ is bounded from above by a constant $\kappa > 0$ only depending on the topological type of the surface S and on ϵ [Bu92]. Therefore the word-norm of an element $\varphi \in \Gamma$ with $d(\varphi h, h) \le 2D + 1$ is bounded from above by a constant $\ell > 0$. For an integer $k < m$ choose some $\varphi(k) \in \Gamma$ with $d(\varphi(k)h, \zeta(k)) \le D$. Then we have $d(\varphi(k)^{-1}\varphi(k+1)h, h) \le 2D + 1$ and hence the word norm of $\varphi(k)^{-1}\varphi(k+1)$ is at most ℓ. As a consequence, the word norm p of w is at most $\ell(m+1) \le \ell\nu p/L$. For $L > \ell\nu$, this is a contradiction which shows that Γ is indeed convex cocompact. □

3. Hyperbolic tree-bundles over proper hyperbolic spaces

In [FM02], Farb and Mosher introduce *metric fibrations* as a generalization of Riemannian submersions between complete Riemannian manifolds. The following definition is adapted to our needs from their paper.

Definition 3.1: Let (X,d) be a proper geodesic metric space. A *metric fibration* over X with fibre a topological space F is a geodesic metric space $(Y=X\times F,d)$ with the following properties.

(1) For all $x,x'\in X$ and every $y\in F$ we have $d((x,y),(x',y))=d(x,x')=d(\{x\}\times F,\{x'\}\times F)$.
(2) For each $x\in X$, the metric on Y induces a complete geodesic metric on $\{x\}\times F$ which defines the given topology on $\{x\}\times F\sim F$.

Recall that a geodesic metric space (X,d) is called *δ-hyperbolic* for some $\delta\geq 0$ if the *δ-thin triangle condition* holds for X: For every geodesic triangle in X with sides a,b,c, the side a is contained in the δ-neighborhood of $b\cup c$. In this section we consider a metric fibration $Y=X\times T\to X$ over a δ-hyperbolic geodesic metric space (X,d) with fibre a simplicial tree T of bounded valence. Our goal is to give a necessary and sufficient condition for the space Y to be hyperbolic.

We begin with analyzing the case when T is a closed subset of the real line $\mathbb{R}$. We use an idea of Bestvina and Feighn who introduced in [BF92] the following "rectangle flare" condition. Let (X,d) be a geodesic metric space and let $r:X\to(0,\infty)$ be a positive function. Given $\kappa>1,n\in\mathbb{Z}_+$, the *$\kappa,n$-flaring property* for r with threshold $A\geq 0$ says that if $J\subset\mathbb{R}$ is a closed connected subset, if $t-n,t,t+n\in J$ and if $\gamma:J\to X$ is a geodesic so that $r(\gamma(t))\geq A$ then $\max\{r(\gamma(t-n)),r(\gamma(t+n))\}\geq\kappa r(t)$. We say that r satisfies the *bounded* κ,n-flaring property with threshold A if in addition r is continuous and if its growth is uniformly exponentially bounded with exponent n at large scales, i.e. if $r(y)\leq e^{d(x,y)n}r(x)$ for all $x,y\in X$ such that $r(x)\geq A$.

For a constant $c>0$ define a subset B of X to be *c-quasi-convex* if every geodesic in X connecting two points in B is contained in the c-neighborhood of B. We have.

Lemma 3.2: *Let X be a proper geodesic metric space and let $r:X\to(0,\infty)$ be a function which satisfies the bounded κ,n-flaring property with threshold $A>0$. Let $\mu=\inf_{x\in X}r(x)$. There is a constant $D=D(\kappa,n,A)>0$ only depending on κ,n,A with the following properties.*

(1) *If $\mu\geq A$ then the function r assumes a minimum on X. The diameter of the set $\{x\in X\mid r(x)=\mu\}$ is bounded from above by D.*
(2) *If $\mu<A$ then the set $\{x\mid r(x)\leq A\}$ is D-quasi-convex.*

Proof: Let (X,d) be a proper geodesic metric space and let $r:X\to(0,\infty)$ be a function which satisfies the bounded κ,n-flaring property with threshold A. Assume that $\mu=\inf_{x\in X}r(x)\geq A$. We have to show that r assumes a minimum on X. Namely, let x,y be two points in X whose distance χ is at least $2n$ and such that $r(x)<\kappa\mu,r(y)<\kappa\mu$. Let $\gamma:[0,\chi]\to X$ be a geodesic connecting $\gamma(0)=x$ to $\gamma(\chi)=y$ and let $\ell\geq 2$ be such that $\chi\in[\ell n,(\ell+1)n)$. By the flaring property for r and the fact that $r(\gamma(n))\geq A$ we have $r(\gamma(2n))\geq\kappa r(\gamma(n))\geq\kappa\mu$ and inductively we conclude that $r(\gamma(\ell n))\geq\kappa^{\ell-1}\mu$. On the other hand, the growth of r is uniformly exponentially bounded and therefore $\kappa\mu>r(y)\geq e^{-n^2}r(\gamma(\ell n))\geq e^{-n^2}\kappa^{\ell-1}\mu$. This implies that the distance between x and y is bounded from above by a constant $D > 0$ only depending on κ, n, A. Since X is proper and r is continuous, we conclude that the function r assumes a minimum, and the diameter of the set of points at which such a minimum is achieved is at most D.

Now assume that $\mu < A$ and let $E = \{z \mid r(z) \leq A\}$. We have to show that E is D'-quasi-convex for a constant $D' > 0$ only depending on κ, n, A. For this let $x, y \in E$ and let $\gamma : [0, \chi] \to X$ be a geodesic arc connecting x to y. Let $\ell \geq 0$ be such that the length χ of γ is contained in the interval $[\ell n, (\ell + 1)n)$. If $\ell \leq 1$ then there is nothing to show, so assume otherwise. If $\gamma(n) \notin E$ then we have $r(\gamma(n)) > A$ and it follows as above from the bounded flaring property that $r(y) \geq e^{-n^2}\kappa^{\ell-1}A$ (recall that the function r is continuous by assumption and therefore if $r(\gamma(t)) \leq A$ for some $t \in [\ell n, (\ell+1)n)$ then there is some $\tilde{t} \in [\ell n, t]$ with $r(\gamma(\tilde{t})) = A$ to which we can apply our growth condition for r). Hence the distance χ between x and y is bounded from above by a universal constant. Otherwise we have $\gamma(n) \in E$ and we can apply the same consideration to the points $\gamma(n), y$. Inductively we conclude that the set E is D'-quasi-convex for a constant $D' > 0$ only depending on κ, n, A. ☐

In the sequel, we will use the following criterion for hyperbolicity of a geodesic metric space (Proposition 3.5 in [H05]).

Lemma 3.3: *Let (Y, d) be a geodesic metric space. Assume that there is a number $D > 0$ and for every pair of points $x, y \in Y$ there is a path $c(x, y) : [0, 1] \to Y$ connecting $c(x, y)(0) = x$ to $c(x, y)(1) = y$ with the following properties.*

1. *If $d(x, y) \leq 1$ then the diameter of the set $c(x, y)[0, 1]$ is at most D.*
2. *For $x, y \in X$ and $0 \leq s \leq t \leq 1$, the Hausdorff distance between $c(x, y)[s, t]$ and $c(c(x, y)(s), c(x, y)(t))[0, 1]$ is at most D.*
3. *For any triple (x, y, z) of points in X, the arc $c(x, y)[0, 1]$ is contained in the D-neighborhood of $c(x, z)[0, 1] \cup c(z, y)[0, 1]$.*

Then the space (Y, d) is δ-hyperbolic for a constant $\delta > 0$ only depending on D. Moreover, for all $x, y \in Y$ the Hausdorff distance between $c(x, y)$ and a geodesic connecting x to y is at most δ.

Now consider a metric fibration whose fibre J either is the closed interval $[0, 1]$ or the half-line $[0, \infty)$. By assumption, for every compact interval $[s, t] \subset J$ and every $x \in X$ the arc $\{x\} \times [s, t]$ is rectifiable. As a consequence, we can define a function on X by associating to $x \in X$ the length of the arc $\{x\} \times [s, t]$; we call such a function a *vertical distance function.* The next lemma is the main technical result of this section.

Lemma 3.4: *Let (X, d) be a proper δ-hyperbolic geodesic metric space and let $Y = X \times J \to X$ be a bounded metric fibration. Assume that the vertical distance functions satisfy the bounded κ, n-flaring property with flaring threshold A for some $\kappa > 1, n > 0, A > 0$. Assume moreover that the infimum of every vertical distance* *function is not bigger than A. Then Y is δ_0-hyperbolic for a number $\delta_0 > 0$ only depending on δ, κ, n, A.*

Proof: Let (X, d) be a proper δ-hyperbolic geodesic metric space and let $Y = X \times J \to X$ be a metric fibration with fibre $J = [0,1]$ or $J = [0,\infty)$ such that the vertical distance functions satisfy the bounded κ, n-flaring property with threshold $A > 0$ for some $\kappa > 1, n > 0, A > 0$. Denote the distance on Y again by d. For $t \in J$ let $\ell_t : X \to [0,\infty)$ be the function which associates to a point $x \in X$ the length of the vertical path $\{x\} \times [0,t]$. By assumption, the function ℓ_t is continuous. Write $\mu(t) = \inf_{x \in X} \ell_t(x)$; the function $t \to \mu(t)$ is continuous and increasing. We assume that μ is bounded from above by A. Our goal is to construct for any two points $x, y \in Y$ a curve $c(x,y)$ connecting x to y so that the resulting curve system satisfies the properties 1-3 in Lemma 3.3. For this we proceed in four steps.

Step 1:

In a first step, we construct for every $y = (x,t) \in Y$ and every $s \leq t$ a curve $\eta_s(x,t) : [0,1] \to Y$ connecting (x,t) to $X \times \{s\}$. For this let $\overline{X}$ be the union of X with its Gromov boundary ∂X. Since X is proper, the space $\overline{X}$ is compact. For $s \geq 0$ define a set $C_s \subset \overline{X}$ as follows. If $J = [0,1]$ then define $C_s = \{x \in X \mid (\ell_1 - \ell_s)(x) \leq A\}$. By Lemma 3.2, the set C_s is D_1-quasi-convex for a universal constant $D_1 > 0$. If $J = [0,\infty)$ then for $t \geq s$ write $Q_{t,s} = \{\ell_t - \ell_s \leq A\}$. By Lemma 3.2 the sets $Q_{t,s}$ are D_1-quasi-convex for our constant $D_1 > 0$. If we denote by $\overline{Q}_{t,s}$ the closure of $Q_{t,s}$ in $\overline{X}$ then the sets $\overline{Q}_{t,s}$ are compact and non-empty and we have $\overline{Q}_{t,s} \supset \overline{Q}_{u,s}$ for $t \leq u$. Thus $C_s = \cap_{t \geq s} \overline{Q}_{t,s} \neq \emptyset$, moreover C_s is D_1-quasi-convex. This means that either C_s consists of a single point $\zeta \in \partial X$ or $C_s \cap X$ is non-empty and D_1-quasi-convex, with closure C_s in $\overline{X}$.

For $(x,t) \in Y$ and $s \leq t$ define now a curve $\eta_s(x,t) : [0,1] \to Y$ connecting $(x,t) = \eta_s(x,t)(0)$ to $\eta_s(x,t)(1) \in X \times \{s\}$ as follows. First, if $t = s$ then let $\eta_s(x,t)(\tau) = (x,t)$ for all $\tau \in [0,1]$. If $y = (x,t)$ for some $t > s$ then choose a minimal geodesic $\gamma_{x,s} : [0,\sigma] \to X$ connecting $\gamma_{x,s}(0) = x$ to the set C_s; if C_s is a single point $\zeta \in \partial X$ then $\sigma = \infty$ and we require that $\gamma_{x,s}$ converges to ζ. We assume that the choice of $\gamma_{x,s}$ only depends on x, s but not on t. Since $\ell_t < \ell_{t'}$ for $t < t'$, there is a smallest number $\nu_{x,t,s} \geq 0$ so that $(\ell_t - \ell_s)(\gamma_{x,s}(\nu_{x,t,s})) \leq A$. Let $\eta_s(x,t)$ be a reparametrization on the interval $[0,1]$ of the horizontal arc $\gamma_{x,s}[0,\nu_{x,t,s}] \times \{t\}$ with a vertical arc of length at most A connecting $(\gamma_{x,s}(\nu_{x,t,s}), t)$ to $(\gamma_{x,s}(\nu_{x,t,s}), s) \in X \times \{s\}$.

Step 2:

In a second step, we show that for every $R > 0$, $s \in J$ and all $y, z \in X \times ([s,\infty) \cap J)$ with $d(y,z) \leq R$ the Hausdorff distance between the curves $\eta_s(y)$ and $\eta_s(z)$ is bounded from above by a number $\tau(R) > 0$ only depending on R (and on κ, n, A) but not on s, y, z. More precisely, we establish that there are reparametrizations $\tilde{\eta}_s(y), \tilde{\eta}_s(z)$ of $\eta_s(y), \eta_s(z)$ on the interval $[0,1]$ such that $d(\tilde{\eta}_s(y)(\sigma), \tilde{\eta}_s(z)(\sigma)) \leq \tau(R)$ for all $\sigma \in [0,1]$.

We first consider the case that the points y, z are contained in $X \times \{t\}$ for some fixed $t \geq s$. Thus let $R > 0$, $s \geq 0$, let $t \geq s$, let $x, u \in X$ with $d(x,u) \leq R$ and let

$y=(x,t), z=(u,t)\in Y$. By hyperbolicity of X, the Hausdorff distance between the two geodesics $\gamma_{x,s},\gamma_{u,s}$ of minimal length connecting the points $x,u\in X$ of distance at most R to the D_1-quasi-convex subset C_s of $\overline{X}$ is bounded from above by a universal constant $\tau_1(R)>0$ only depending on R,δ but not on x,u.

There are smallest numbers $\nu_{x,t,s}\geq 0, \nu_{u,t,s}\geq 0$ such that $(\ell_t-\ell_s)(\gamma_{v,s}(\nu_{v,t,s}))\leq A$ $(v=x,u)$. By the definition of the curves $\eta_s(y)$ it is now enough to show that the distance between $\gamma_{x,s}(\nu_{x,t,s})$ and $\gamma_{u,s}(\nu_{u,t,s})$ is bounded from above by a constant which only depends on R. Note that for $s=t$ we have $\nu_{x,t,s}=0=\nu_{u,t,s}$ and hence there is nothing to show, so assume that $s<t$.

Since by assumption the growth of the vertical distance functions is uniformly exponentially bounded, there is a universal number $\beta>0$ only depending on $\tau_1(R)$ such that for any two points $v,w\in X$ with $d(v,w)\leq\tau_1(R)$ we have $(\ell_t-\ell_s)(v)\leq\beta(\ell_t-\ell_s)(w)$ provided that $(\ell_t-\ell_s)(v)\geq A, (\ell_t-\ell_s)(w)\geq A$. By the definition of $\nu_{x,t,s}$ and the flaring property, this means that there is a number $\xi>0$ only depending on R such that $(\ell_t-\ell_s)(w)>A$ whenever $w\in X$ is such that $d(w,\gamma_{x,s}(\sigma))\leq\tau_1(R)$ for some $\sigma\in[0,\nu_{x,t,s}-\xi]$ (compare the argument in the proof of Lemma 3.2). As a consequence, if $a\geq 0$ is such that $d(\gamma_{x,s}(a),\gamma_{u,s}(\nu_{u,t,s}))\leq\tau_1(R)$ then $a\geq\nu_{x,t,s}-\xi$. Since $\gamma_{x,s},\gamma_{u,s}$ are geodesics of Hausdorff distance at most $\tau_1(R)$ we conclude that $\nu_{u,t,s}=d(\gamma_{u,s}(0),\gamma_{u,s}(\nu_{u,t,s}))\geq d(\gamma_{x,s}(0),\gamma_{x,s}(a))-R-\tau_1(R)=a-R-\tau_1(R)\geq\nu_{x,t,s}-\xi-R-\tau_1(R)$. Exchanging the role of x and u then shows that $|\nu_{x,t,s}-\nu_{u,t,s}|\leq\xi+R+\tau_1(R)$. But $\gamma_{x,s},\gamma_{u,s}$ are geodesics of Hausdorff distance at most $\tau_1(R)$ and therefore the distance between $\gamma_{x,s}(\nu_{x,t,s})$ and $\gamma_{u,s}(\nu_{u,t,s})$ is indeed bounded from above by a universal constant $\tau_2(R)>0$ only depending on R. As a consequence, the Hausdorff distance between $\eta_s(y)$ and $\eta_s(z)$ is bounded from above by a number $\tau_3(R)>0$ only depending on R.

Now consider nearby points y,z contained in the same fibre of our metric fibration. Thus let $x\in X$ and let $t\geq 0, b>0$ be such that the length of the vertical arc $\{x\}\times[t,t+b]$ is at most A. Write $y=(x,t), z=(x,t+b)$ and let $s\leq t$; we claim that the Hausdorff distance between $\eta_s(y)$ and $\eta_s(z)$ is bounded from above by a universal constant.

Namely, let again $\gamma_{x,s}$ be the minimal geodesic connecting x to C_s as in the definition of the arc $\eta_s(x,t)$. There is a minimal number $\sigma_b=\nu_{x,t+b,s}\geq 0$ such that $(\ell_{t+b}-\ell_s)(\gamma_{x,s}(\sigma_b))\leq A$ and hence $\ell_{t+b}(\gamma_{x,s}(\sigma))-\ell_t(\gamma_{x,s}(\sigma))\leq A$ for $\sigma=0,\sigma_b$. By the bounded κ,n- flaring property for vertical distances, this implies that there is a universal number $A'\geq A$ such that $(\ell_{t+b}-\ell_t)(\gamma_{x,s}(\sigma))\leq A'$ for all $\sigma\in[0,\sigma_b]$. Let $\sigma_0=\nu_{x,t,s}\leq\sigma_b$ be the minimal number such that $(\ell_t-\ell_s)(\gamma_{x,s}(\sigma_0))\leq A$. Then $A\leq(\ell_{t+b}-\ell_s)(\gamma_{x,s}(\sigma))\leq 2A'$ for every $\sigma\in[\sigma_0,\sigma_b]$ and consequently an application of the flaring property as in the proof of Lemma 3.2 for the function $\ell_{t+b}-\ell_s$ yields that $\sigma_b\leq\sigma_0+\xi$ for a universal number $\xi>0$. It follows that the Hausdorff distance between $\eta_s(y)$ and $\eta_s(z)$ is bounded from above by a universal constant.

By our assumption that the growth of the vertical distance functions is uniformly exponentially bounded, for every $R>0$ there is a number $\nu(R)>0$ such that for every $y=(x,t)\in Y$ the R-ball about y is contained in the set $\{z=(u,s)\in Y\mid$

$d(x,u) \le \nu(R), |(\ell_t - \ell_s)(y)| \le \nu(R)\}$. Together we conclude that for every $R > 0$ we can find a number $\tau(R) > 0$ with the following property. Let $y = (x,t), y' = (x',t') \in Y$ with $d(y,y') \le R$; then for every $s \le \min\{t,t'\}$ the Hausdorff distance between $\eta_s(y)$ and $\eta_s(z)$ is bounded from above by $\tau(R)$.

Step 3:

Define a system $c(y,z)$ of arcs connecting an arbitrary pair of points $y,z \in Y$ as follows. If $y = (x,t), z = (u,s) \in Y$ with $0 \le s \le t$ then define $c(y,z)$ to be a reparametrization on $[0,1]$ of the composition of the arc $\eta_s(y)$ with a geodesic in $X \times \{s\} \sim X$ connecting $\eta_s(y)(1)$ to z. Define also $c(z,y)$ to be the inverse of $c(y,z)$.

In our third step we show that for every $R > 0$ and all $y,y' \in Y$ with $d(y,y') \le R$, all $z \in Y$ the Hausdorff distance between $c(y,z), c(y',z)$ is bounded from above by a constant $\chi(R) > 0$ only depending on R (and on δ, κ, n, A). For this let $R > 0$ and let $y,y',z \in Y$ with $d(y,y') \le R$. We distinguish 3 cases.

Case 1: $z = (u,s), y = (x,t), y' = (x',t')$ with $0 \le s \le t \le t'$.

By the definition of the curves $c(v,w)$ and Step 2, the curves $c(y,z), c(y',z)$ are composed of the arcs $\eta_s(y), \eta_s(y')$ of Hausdorff distance at most $\tau(R)$ and geodesic arcs in $X \times \{s\} \sim X$ connecting the points $\eta_s(y)(1), \eta_s(y')(1)$ of distance at most $\tau(R)$ to z. By δ-hyperbolicity of $X \times \{s\} \sim X$, the Hausdorff distance between $c(y,z)$ and $c(y',z)$ is bounded from above by a constant $\chi_1(R) > 0$ only depending on R.

Case 2: $z = (u,s), y = (x,t), y' = (x',t')$ with $0 \le t \le s \le t'$.

Since the distance between y and y' is at most R and Y is a geodesic metric space, there is a point $y'' = (x'',s) \in X \times \{s\}$ whose distance to both y,y' is at most R. By Case 1 above, the Hausdorff distance between $c(y',z)$ and $c(y'',z)$ is at most $\chi_1(R)$. Thus we may assume without loss of generality that $t' = s$; then $c(y',z) = c(z,y')$ is the lift to $X \times \{s\}$ of a geodesic in X connecting u to x'. Since $d(y,y') \le R$ and $y' = (x',s), y = (x,t)$, we have $d(x,x') \le R$ and hence by hyperbolicity of X, the Hausdorff distance between a geodesic connecting u to x and a geodesic connecting u to x' is bounded by a uniform constant $\chi_2(R) > 0$. Thus the Hausdorff distance between $c(y',z)$ and $c((x,s),z)$ is at most $\chi_2(R)$ and we may assume without loss of generality that $x = x'$.

Since the distance between $(x,s),(x,t)$ is bounded from above by a constant only depending on R, by the flaring property for vertical distances the point x is contained in a uniformly bounded neighborhood of the set $E = \{\ell_s - \ell_t \le A\}$. Since E is D_1-quasi-convex, by hyperbolicity a geodesic in X connecting u to x is contained in a uniform neighborhood of the composition of a minimal geodesic ζ connecting u to E and a geodesic arc connecting the endpoint of ζ to x. By construction, the curve $c(y,z) = c(z,y)$ is composed of the lift to $X \times \{s\}$ of a minimal geodesic $\zeta : [0,\tau] \to X$ connecting u to E, a vertical arc of length at most A connecting $(\zeta(\tau),s)$ to $(\zeta(\tau),t)$ and the lift to $X \times \{t\}$ of a geodesic ξ in X connecting $\zeta(\tau)$ to x. Since ξ is contained in a uniformly bounded neighborhood of E, it follows that the Hausdorff distance between $c(y,z)$ and the lift of the composition of ζ and ξ to $X \times \{s\}$ is uniformly bounded. Therefore the Hausdorff distance between $c(y,z), c(y',z)$ is bounded from above by a constant $\chi_3(R) > 0$ only depending on R.

Case 3: $z = (u,s), y = (x,t), y' = (x',t')$ with $0 \leq t \leq t' \leq s$.

We claim that $\eta_t(z)$ contains a subarc $\eta_t(z)[0,\sigma] \subset X \times \{s\}$ whose Hausdorff distance to $\eta_{t'}(z)$ is uniformly bounded. Namely, the sets $C_t, C_{t'} \subset X$ are D_1-quasi-convex and $C_t \subset C_{t'}$. By hyperbolicity of X, if $u \in X - C_{t'}$ then a minimal geodesic ξ in X connecting u to C_t is contained in a uniformly bounded neighborhood of the composition of a minimal geodesic $\zeta_1 : [0,a] \to X$ connecting u to $C_{t'}$ and a minimal geodesic ζ_2 connecting $\zeta_1(a) = \zeta_2(0)$ to C_t. From this and the definition of our arcs η_t, the claim is immediate.

Denote by $z'' = \eta_t(z)(\sigma) \in X \times \{s\}$ ($\sigma \in [0,1]$) the endpoint of the above subarc of $\eta_t(z)$ and write $z' = \eta_{t'}(z)(1) \in X \times \{t'\}$. The curve $c(y',z)$ is composed of the arcs $\eta_{t'}(z)$ and $c(y',z')$, and the curve $c(y,z)$ is composed of the arcs $\eta_t(z)[0,\sigma]$ and $c(y,z'')$. Thus up to a constant only depending on R, the Hausdorff distance between $c(y,z)$ and $c(y',z)$ is bounded from above by the Hausdorff distance between $c(y,z'')$ and $c(y',z')$. Now the distance between z' and z'' is uniformly bounded and hence by Case 1 above, the Hausdorff distance between $c(y,z')$ and $c(y,z'')$ is bounded by a constant only depending on R. In other words, for our estimate we may replace z by z', i.e. we may assume without loss of generality that $s = t'$. However, this case is contained in Case 2 above.

Together we established an upper bound $\chi(R) > 0$ for the Hausdorff distance between $c(y,z)$ and $c(y',z)$ whenever $d(y,y') \leq R$.

Step 4:

In a final step, we show that our system of curves satisfies the properties 1)-3) in Lemma 3.3.

Namely, for $y = z$ the curve $c(y,z)$ is constant, and hence if $d(y,z) \leq 1$ then the diameter of $c(y,z)$ is at most $\chi(1)$ where $\chi(1) > 0$ is as in Step 3. This means that property 1 is valid with $D = \chi(1) > 0$.

Similarly, let $y, z \in Y$ and let $0 \leq s \leq t \leq 1$. Then either the restriction of the curve $c(y,z)$ to $[s,t]$ is obtained from our above procedure, i.e. from the same construction used for the curve $c(c(y,z)(s), c(y,z)(t))$, or one of the points $c(y,z)(s), c(y,z)(t)$ is contained in the vertical subarc of $c(y,z)$. In the first case it is immediate from Step 3 above that the Hausdorff distance between $c(y,z)[s,t]$ and $c(c(y,z)(s), c(y,z)(t))$ is uniformly bounded. In the second case, if say the point $c(y,z)(s)$ is contained in the vertical subarc of $c(y,z)$ then there is some $s' \geq s$ such that $c(y,z)[s',t]$ is obtained from the above procedure and that the Hausdorff distance between $c(y,z)[s,t]$ and $c(y,z)[s',t]$ is bounded from above by a universal constant. By Step 3, the Hausdorff distance between $c(c(y,z)(s), c(y,z)(t))$ and $c(y,z)[s',t]$ is bounded from above by a universal constant as well. As a consequence, there is a number $\nu > 0$ such that property 2 is valid with $D = \nu$.

We are left with showing that the δ_0-thin triangle condition for a universal number $\delta_0 > 0$ also holds. For this let y_1, y_2, y_3 be any 3 points in Y. Assume that $y_i = (x_i, s_i)$ with $0 \leq s_1 \leq s_2 \leq s_3$. By construction and Step 3 above, the curves $c(y_1, y_3), c(y_2, y_3)$ both contain a subarc whose Hausdorff distance to $\eta_{s_2}(y_3)$ is uniformly bounded. By our above estimate of Hausdorff distances, this means that for the purpose of establishing the thin triangle condition we may replace y_3 by $\eta_{s_2}(y_3)(1)$, i.e. we may assume that in fact $s_2 = s_3 = s$. Then the arc $c(y_2, y_3)$ is the lift to $X \times \{s\}$ of a geodesic γ in X connecting x_2 to x_3.

Let $E = \{u \in X \mid (\ell_s - \ell_{s_1})(u) \leq A\}$. Recall that E is D_1-quasi-convex. Let $\zeta_i : [0, \sigma_i] \to X$ $(i = 2, 3)$ be a minimal geodesic connecting x_2, x_3 to E. Then γ is contained in a uniformly bounded neighborhood of the union of $\zeta_2[0, \sigma_2] \cup \zeta_3[0, \sigma_3]$ with a geodesic arc connecting $\zeta_2(\sigma_2)$ to $\zeta_3(\sigma_3)$. By construction, for a suitable choice of ζ_i the curve $c(y_2, y_1), c(y_3, y_1)$ contains the arc $\zeta_2 \times \{s\}, \zeta_3 \times \{s\}$. As a consequence, for the purpose of the thin triangle condition we may as well assume that the points x_2, x_3 are contained in E. However, in this case the thin triangle condition is immediate from the definition of the curves $c(x, y)$ and hyperbolicity of X. As a consequence, our system of curves $c(x, y)$ satisfies the properties 1)-3) in Lemma 3.3 for a number $D > 0$ only depending on δ, κ, n, A and hence the space Y is δ'-hyperbolic for a constant $\delta' > 0$ only depending on δ, κ, n, A. $\square$

By Lemma 3.3, there is a number $q > 0$ such that the curves $c(y, z)$ constructed in the proof of Lemma 3.4 are unparametrized q-quasi-geodesics. This fact can be used to determine explicitly the Gromov boundary of a hyperbolic metric fibration $X \times J \to X$ as in Lemma 3.4. We illustrate this in the case that $J = [0, 1]$ is compact (the result is not needed in the sequel). As in the proof of Lemma 3.4, for $t \in J$ let ℓ_t be the function on X which associates to $x \in X$ the vertical length of $\{x\} \times [0, t]$. Let ∂X be the Gromov boundary of X. For $\xi \in \partial X$ define $\rho(\xi) = 0$ if for some (and hence every) geodesic $\gamma : [0, \infty) \to X$ which converges to ξ we have $\liminf_{s \to \infty} \ell_1(\gamma(s)) < \infty$ and define $\rho(\xi) = 1$ otherwise. Then we have.

Corollary 3.5: *Let $X \times [0, 1] \to X$ be a metric fibration over a proper δ-hyperbolic geodesic metric space X with the properties stated in Lemma 3.4. Then the Gromov boundary of $X \times [0, 1]$ is homeomorphic to $\{(\xi, s) \in \partial X \times [0, 1] \mid s \leq \rho(\xi)\}$.*

Proof: Let $X \times [0, 1] \to X$ be as in the corollary and fix a point $x_0 \in X$ such that $\ell_1(x_0) \leq A$. For $y, z \in X \times [0, 1]$ let $c(y, z)$ be the curve connecting y to z constructed in the proof of Lemma 3.4. Then there is a number $p > 0$ such that for every bounded geodesic $\gamma : [0, m] \to X \times [0, 1]$ the Hausdorff distance between $\gamma[0, m]$ and the curve $c(\gamma(0), \gamma(m))$ is at most p.

Let $x_0 \in X$ be a point such that $\ell_1(x_0) \leq A$. It follows from the explicit construction of the curves $c(y, z)$ and quasi-convexity of the set $\{\ell_1 \leq A\}$ that for every geodesic ray $\gamma : [0, \infty) \to X \times [0, 1]$ issuing from $\gamma(0) = (x_0, 0)$ there is a geodesic $\zeta : [0, \infty) \to X$ issuing from $\zeta(0) = x_0$ and a number $s \in [0, 1]$ such that the Hausdorff distance between γ and the curve $t \to (\zeta(t), s)$ is bounded by a constant $R > 0$ not depending on γ. On other hand, for any two numbers $s_1, s_2 \in [0, 1]$ and any two geodesic rays ζ_1, ζ_2 in X issuing from x_0 the Hausdorff distance between the curves $t \to (\zeta_1(t), s_1), t \to (\zeta_2(t), s_2)$ is unbounded if this is the case for the Hausdorff distance between ζ_1, ζ_2. Therefore, if ∂X denotes the Gromov boundary of X then the Gromov boundary of $X \times [0,1]$ is the quotient of $\partial X \times [0,1]$ under the equivalence relation $\sim$ which is defined as follows: $(\xi, s) \sim (\eta, \tilde{s})$ if and only if $\xi = \eta$ and if for one (and hence every) geodesic $\zeta : [0, \infty) \to X$ converging to ξ the function $t \to |(\ell_s - \ell_{\tilde{s}})(\zeta(t))|$ is bounded.

Now if $\rho(\xi) = 0$ then for all $s, t \in [0,1]$ the points $(\xi, s), (\xi, t)$ are equivalent. On the other hand, if $\rho(\xi) = 1$ then for every $s \in [0,1]$ the equivalence class of (ξ, s) is of the form $\{\xi\} \times [a, b]$ for a *proper* closed subinterval $[a, b]$ of $[0,1]$. In particular, the quotient space $\{\xi\} \times [0,1]/\sim$ is connected, infinite and totally ordered and hence it is homeomorphic to $[0,1]$. This shows the corollary. □

Let X be a proper δ-hyperbolic geodesic metric space. Recall that a closed subset E of X is *strictly convex* if every geodesic connecting two points in E is contained in E. The following lemma shows that under suitable assumptions, hyperbolicity is preserved under glueing along stricly convex subsets. For its formulation, for a number $R > 0$ we call two closed strictly convex subsets D, E of a X *R-separated* if D, E are disjoint and if moreover the following holds. Let $\gamma : [0, a] \to X$ be a minimal geodesic connecting D to E; then $\gamma[0, a]$ is contained in the R-neighborhood of *every* geodesic connecting D to E. For example, two non-intersecting geodesics in the hyperbolic plane are R-separated for a constant $R > 0$ which tends to infinity as the distance between the geodesics tends to zero. The two boundary geodesics of a flat strip in $\mathbb{R}^2$ are not R-separated for any $R > 0$. Note also that by the explicit construction of the curves $c(x, y)$ in the proof of Lemma 3.4 the following holds. If $Y = X \times [0,1] \to X$ is a bounded metric fibration over a δ-hyperbolic geodesic metric space such that the vertical distance functions satisfy the κ, n-flaring property with threshold A for some $\kappa > 1, n > 0, A > 0$ and if the infimum of the vertical length of the fibres equals A then Y is hyperbolic and the subsets $X \times \{0\}, X \times \{1\}$ are strictly convex and R-separated for a number $R > 0$ only depending on δ, κ, n, A.

Lemma 3.6: *Let $\delta > 0, R > 0$, let $I \subset \mathbb{Z}$ be any subset and let X be a geodesic metric space with the following properties.*

a) *$X = \cup_{i \in I} X_i$ where for each $i \in I$, X_i is a proper δ-hyperbolic geodesic metric space.*
b) *For each $i \in I$ the intersection $X_i \cap X_{i+1}$ is a strictly convex closed subset of both X_i, X_{i+1}, and $X_i \cap X_j = \emptyset$ for $|i - j| \geq 2$.*
c) *For each $i \in I$, the sets $X_i \cap X_{i-1}$ and $X_i \cap X_{i+1}$ are R-separated in X_i.*

Then X is δ'-hyperbolic for a constant $\delta' > 0$ only depending on δ, R.

Proof: Let X, I, X_i be as in the lemma. We may assume without loss of generality that $I = \mathbb{Z}$. Write $E_i = X_i \cap X_{i+1}$; by our assumption, E_i is a strictly convex subset of the proper δ-hyperbolic spaces X_i, X_{i+1}; moreover, the subsets E_{i-1}, E_i of X_i

are R-separated for a constant $R > 0$ not depending on i. Thus after possibly enlarging R the following properties are satisfied.

i) Every point $x \in X_i$ can be connected to E_i by a geodesic $\zeta_x^+ : [0,1] \to X_i$ of minimal length and to E_{i-1} by a geodesic $\zeta_x^- : [0,1] \to X_i$ of minimal length. If the distance between x, y is at most 1 then the Hausdorff distance between $\zeta_x^\pm$ and $\zeta_y^\pm$ is at most R.
ii) Let $x, y \in X_i$ and let γ be a geodesic connecting x to y. Then γ is contained in the R-neighborhood of the piecewise geodesic which is composed of the arc ζ_x^+, a geodesic in E_i connecting $\zeta_x^+(1)$ to $\zeta_y^+(1)$ and the inverse of ζ_y^+. The geodesic γ is also contained in the R-neighborhood of a piecewise geodesic which is constructed in the same way using the geodesic arcs ζ_x^-, ζ_y^- and a geodesic in E_{i-1}.
iii) Let $\gamma_i : [0,1] \to X_i$ be a minimal geodesic connecting E_{i-1} to E_i; then for every $x \in E_{i-1}$ the Hausdorff distance between a minimal geodesic connecting x to E_i and the composition with γ_i of a geodesic in E_{i-1} connecting x to $\gamma_i(0)$ is not bigger than R.

We use once more the criterion for hyperbolicity from Lemma 3.3. Namely, we define in three steps for any pair of points $x, y \in X$ a curve $c(x, y)$ connecting x to y as follows.

Step 1: If there is some $i \in \mathbb{Z}$ such that $x, y \in X_i$ then define $c(x, y)$ to be a geodesic in X_i connecting x to y.

Step 2: If there is some $i \in \mathbb{Z}$ such that $x \in X_i - E_i, y \in X_{i+1} - E_i$ then define $c(x, y)$ to be the piecewise geodesic which is composed from the geodesic ζ_x^+ connecting x to E_i, a geodesic arc in E_i connecting $\zeta_x^+(1)$ to $\zeta_y^-(1)$ and the inverse of the geodesic ζ_y^-.

Step 3: If $x \in X_i - E_i, y \in X_j - E_{j-1}$ for some $j \geq i+1$ then define inductively $c(x, y)$ to be the piecewise geodesic which consists of the geodesic segment ζ_x^+, a geodesic in E_i connecting $\zeta_x^+(1)$ to $\gamma_{i+1}(0)$ and the arc $c(\gamma_{i+1}(0), y)$.

Assume that the curves $c(x, y)$ are all parametrized on the unit interval $[0,1]$. We claim that there is a number $D > 0$ only depending on δ, κ, n, A such that the curves $c(x, y)$ satisfy the three conditions in Lemma 3.3.

The first property is immediate from the definition of the curves $c(x, y)$. To show that the second condition is valid as well, let $x, y \in X$, let $0 \leq s \leq t \leq 1$ and let $x' = c(x, y)(s), y' = c(x, y)(t)$ be points on the curve $c(x, y)$. We have to show that the Hausdorff distance between $c(x, y)[s, t]$ and $c(x', y')[0, 1]$ is bounded from above by a constant $D_1 > 0$ only depending on δ, κ, n, A. For this we distinguish three cases.

First, if $x', y' \in X_i$ for some $i \in \mathbb{Z}$, then either $c(x, y)[s, t]$ is a geodesic in X_i connecting x' to y' or $x' \in E_{i-1}$ or $y' \in E_i$ and by properties ii) and iii) above for X_i, the Hausdorff distance between the arc $c(x, y)[s, t]$ and the geodesic $c(x', y')$ connecting x' to y' is bounded from above by a universal constant $\chi_1 > 0$.

Next assume that $x' \in X_i - E_i$ for some $i \in \mathbb{Z}$ and that $y' \in X_{i+1} - E_i$. Let $\zeta_{x'}^+$ be a geodesic of minimal length connecting x' to E_i. By the definition of the curve $c(x,y)$ and hyperbolicity of X_i, there is a number $s' > s$ such that $c(x,y)(s') \in E_i$ and that the Hausdorff distance between $c(x,y)[s,s']$ and the geodesic $\zeta_{x'}^+$ is at most R. Similarly, by property iii) above and the definition of the curves $c(v,w)$ there is a number $t' \leq t$ such that $c(x,y)(t') \in E_i$ and that the Hausdorff distance between $c(x,y)[t',t]$ and the geodesic $\zeta_{y'}^-$ of minimal length connecting y' to E_i is bounded from above by R. On the other hand, $c(x',y')$ is composed of the arcs $\zeta_{x'}^+, \zeta_{y'}^-$ and a geodesic arc in E_i connecting $\zeta_{x'}(1)$ to $\zeta_{y'}(1)$; moreover, $c(x,y)[s',t']$ is a geodesic in E_i connecting $c(x,y)(s')$ to $c(x,y)(t')$. Since E_i is δ-hyperbolic for a number $\delta > 0$ not depending on i, the Hausdorff distance between any two compact geodesic arcs in E_i is up to an additive constant bounded from above by the sum of the distances between the endpoints of the arcs. Therefore the Hausdorff distance between $c(x,y)[s,t]$ and $c(x',y')$ is at most χ_2 for a constant $\chi_2 \geq \chi_1$ only depending on δ.

Finally, the case that $x' \in X_i - E_i$ and $y' \in X_j - E_{j-1}$ for some $j \geq i+1$ follows immediately from the above consideration. Namely, in this case there are numbers $s \leq s' < t' \leq t$, $0 \leq \sigma < \tau \leq 1$ such that the arcs $c(x,y)[s',t'], c(x',y')[\sigma,\tau]$ coincide and that moreover the above consideration can be applied to the curves $c(x',y')[0,\sigma], c(x',y')[\tau,1]$ and $c(x,y)[0,s'], c(x,y)[t',1]$. Thus the second condition in the proof of Lemma 3.3 is satisfied for our system of curves with a number $D_1 > 0$ only depending on δ, κ, n, A (note that we can choose $D_1 = 2\chi_2$ where $\chi_2 > 0$ is as above).

We are left with showing the thin triangle condition for our system of curves $c(x,y)$, i.e. we have to find a number $D_2 > 0$ such that for every triple of points $x,y,z \in Y$ the curve $c(x,y)$ is contained in the D_2-neighborhood of $c(y,z) \cup c(z,x)$. Consider first the case that the points x,y,z are all contained in X_i for some $i \in \mathbb{Z}$. Then the curves $c(x,y), c(y,z), c(z,x)$ are geodesics in X_i connecting these three points and hence the curve $c(x,y)$ is contained in a uniformly bounded neighborhood of $c(y,z) \cup c(z,x)$ by hyperbolicity of X_i. Next assume that two of the points, say the points x,y, are contained in X_i but that the third point z is contained in $X_j - E_{j-1}$ for some $j \geq i+1$. Then the intersections with $\cup_{p \geq i+1} X_p - E_i$ of the curves $c(x,z), c(y,z)$ coincide. Let $t_x, t_y \in [0,1]$ be such that $c(x,z)(t_x,1] = c(x,z)[0,1] \cap (\cup_{p \geq i+1} X_p - E_i)$ and similarly for $c(y,z)$; then $v = c(x,z)(t_x) = c(y,z)(t_y)$. Together with property 2 for our curve system established above we conclude that it is enough to show the D_2-thin triangle condition for the curves $c(x,v), c(y,v), c(x,y)$. However, since $x,y,v \in X_i$ this condition holds by our above consideration. The same argument can also be applied in the case that for each i, the set $X \times [t_{i-1}, t_i]$ contains at most one of the points x,y,z. From this we immediately deduce that the third condition for our curve system is valid as well for a universal constant $D_2 > 0$ only depending on δ, R. As a consequence of Lemma 3.3, the space X is δ'-hyperbolic for a constant δ' only depending on δ, R. □

Let T be a simplicial tree of bounded valence. Then for any two points in T, there is a unique simple path connecting these points. For every metric fibration

$Y = X \times T \to X$ and every point $\tau \in T$, the set $X \times \{\tau\} \subset Y$ is strictly convex. We use these facts together with the glueing lemma to extend Lemma 3.4 as follows.

Corollary 3.7: *Let X be a proper δ-hyperbolic geodesic metric space and let $X \times T \to X$ be a metric fibration with fibre a simplicial tree of bounded valence. If vertical distances satisfy the bounded κ, n-flaring property with threshold $A > 0$ for some $\kappa > 1, n > 0, A > 0$ then Y is δ_1-hyperbolic for a number $\delta_1 > 0$ only depending on κ, n, δ, A.*

Proof: We begin with showing the corollary in the particular case that the tree T is just an arbitrary closed connected subset J of the real line $\mathbb{R}$. Thus let X be a δ-hyperbolic geodesic metric space, let $J \subset \mathbb{R}$ be an arbitrary closed connected set and let $Y = X \times J \to X$ be a metric fibration with fibre J. Assume that vertical distances satisfy the bounded κ, n-flaring property with threshold A for some $\kappa > 1, n > 0, A > 0$ and assume without loss of generality that $A \geq 1$.

Let $0 \in J$ and assume that 0 is an endpoint of J if $J \neq \mathbb{R}$. Assume moreover that in this case the set J is contained in $[0, \infty)$. For $t \in J$ let $\ell_t^1 : X \to [0, \infty)$ be the function which associates to a point $x \in X$ the length of the vertical path $\{x\} \times [0, t]$. By assumption, the function ℓ_t^1 is continuous. Write $\mu^1(t) = \inf_{x \in X} \ell_t^1(x)$; the function $t \to \mu^1(t)$ is continuous and monotonously increasing on $[0, \infty)$, monotonously decreasing on $(-\infty, 0]$. Let $t_1 \in (0, \infty]$ be the smallest positive number with $\mu^1(t_1) = A$; here we write $t_1 = \infty$ if $\mu^1(t) < A$ for all $t > 0$. If $t_1 < \infty$ then define for $t \geq t_1$ a new function $\ell_t^2 : X \to [0, \infty)$ by assigning to $x \in X$ the length of the arc $\{x\} \times [t_1, t]$. Let $\mu_2(t) = \inf_{x \in X} \ell_t^2(x)$ and let $t_2 \in (t_1, \infty]$ be the smallest number such that $\mu^2(t_2) = A$. Inductively we construct in this way an increasing sequence $0 < t_1 < t_2 < \dots$ and functions μ^i, ℓ_t^i. The sequence might be trivial, finite or infinite. If $J = \mathbb{R}$ then define in the same way a sequence $0 > t_{-1} > t_{-2} > \dots$ and functions μ^i, ℓ_t^i $(i \leq -1)$.

By Lemma 3.5, there is a constant $\delta_0 > 0$ such that for each $i \in \mathbb{Z}$, the convex subset $X \times [t_{i-1}, t_i]$ of $X \times J$ is δ_0-hyperbolic. The sets $X \times \{t_{i-1}\}, X \times \{t_i\}$ are strictly convex in Y. Moreover, by the remark preceding Lemma 3.6 they are also R-separated for a constant $R > 0$ only depending on δ, κ, n, A. Thus we can apply Lemma 3.6 and conclude that the metric fibration $X \times J$ is δ_1-hyperbolic for a constant $\delta_1 > 0$ only depending on δ, κ, n, A.

Now let T be a simplicial tree of bounded valence. Let $X \times T \to X$ be a metric fibration over a proper δ-hyperbolic geodesic metric space X. Assume that vertical distances satisfy the κ, n-flaring property with threshold $A > 0$ for some $\kappa > 1, n > 0, A > 0$. Our goal is to show that Y is δ_2-hyperbolic for a constant $\delta_2 > 0$ only depending on δ, κ, n, A.

For this let $y_1, y_2, y_3 \in Y$ be a triple of points and let $c(y_i, y_j)$ $(i, j = 1, 2, 3)$ be geodesics in $X \times T$ connecting y_i to y_j. We have to show that $c(y_1, y_2)$ is contained in a uniformly bounded neighborhood of $c(y_2, y_3) \cup c(y_3, y_1)$. Write $y_i = (x_i, \tau_i)$ with $x_i \in X, \tau_i \in T$. For $i = 1, 2, 3$ let J_i be the unique embedded segment in T connecting τ_i to τ_{i+1} (indices are taken mod 3). Then the intersection $\cap_i J_i$ consists of a unique point τ. By our above consideration, the subsets $X \times J_i \subset Y$

of Y are strictly convex and moreover δ_1-hyperbolic for a constant $\delta_1 > 0$ not depending on i; they contain $X \times \{\tau\}$ as a strictly convex subset. Let $\rho(y_i, y_{i+1})$ be a piecewise geodesic which up to orientation and parametrization is composed of a minimal geodesic $\alpha_i : [0,1] \to X \times J_i$ connecting y_i to $X \times \{\tau\}$, a minimal geodesic $\alpha_{i+1} : [0,1] \to X \times J_{i+1}$ connecting y_{i+1} to $X \times \{\tau\}$ and a geodesic in $X \times \{\tau\}$ connecting $\alpha_i(1)$ to $\alpha_{i+1}(1)$. By hyperbolicity, the Hausdorff distance between the geodesic $c(y_i, y_{i+1}) \subset X \times J_i$ and the piecewise geodesic $\rho(y_i, y_{i+1})$ is bounded from above by a constant only depending on δ, κ, n, A. As a consequence, it is enough to show the thin triangle inequality for a geodesic triangle entirely contained in the strictly convex subset $X \times \{\tau\} \subset Y$. Since $X \times \{\tau\} \sim X$ is δ-hyperbolic, this is immediate and shows the corollary. □

We summarize the results of this section as follows.

Corollary 3.8: *Let X be a proper hyperbolic geodesic metric space and let $Y = X \times T \to X$ be a metric fibration with fibre a simplicial tree of bounded valence and such that the growth of the vertical distance functions is uniformly exponentially bounded at large scale. Then Y is hyperbolic if and only if the vertical distance functions satisfy the bounded κ, n-flaring property with threshold A for some $\kappa > 1, n > 0, A > 0$.*

Proof: Let X be a proper hyperbolic geodesic metric space and let $Y = X \times T \to X$ be a metric fibration with fibre a simplicial tree of bounded valence such that the growth of the vertical distance functions is uniformly exponentially bounded at large scale. Lemma 5.2 of [FM02] shows that if Y is hyperbolic, then the vertical distance functions satisfy the bounded κ, n-flaring property with threshold A for some $\kappa > 0, n > 0, A > 0$. By Corollary 3.7, this condition is also sufficient for hyperbolicity of Y. □

Remark: The results in this section are motivated by the work of Bestvina and Feighn [BF92]. Their main result characterizes hyperbolic fibrations whose base is the universal cover of a finite graph and whose fibre is the universal cover of a finite cell complex with word hyperbolic fundamental group by the flaring property for vertical distance functions. Our proof of Corollary 3.8 does not use the arguments in [BF92], and our result does not include the result in [BF92] since we only deal with metric fibrations whose fibres are simplicial trees. However, using the ideas from Section 4 of this paper, Corollary 3.8 can be extended to metric fibrations whose fibre and base are arbitrary proper hyperbolic geodesic spaces. Due to the lack of interesting applications, we only discuss the case when the fibre is the hyperbolic plane; this is sufficient for the proof of the theorem from the introduction.

4. Proof of the theorem

In this final section we consider a *closed* surface of genus $g \geq 2$. Our goal is to show that for a convex cocompact subgoup Γ of the mapping class group $\mathcal{M}_g$ for S, the natural $\pi_1(S)$-extension Γ_S of Γ is word hyperbolic. For this choose a finite symmetric generating set $\mathcal{G}$ for Γ and denote by $\| \, \|$ the induced word norm on Γ

and by $\mathcal{CG}$ the corresponding Cayley graph. Choose a point h in the Teichmüller space $\mathcal{T}_g$ for S which does not admit any nontrivial automorphisms (recall that the set of such points is open and dense in $\mathcal{T}_g$) and define a map $\Theta : \mathcal{CG} \to \mathcal{T}_g$ by mapping a vertex $\varphi \in \Gamma$ to the point $\varphi h \in \mathcal{T}_g$ and by mapping an edge e of $\mathcal{CG}$ to the Teichmüller geodesic arc connecting the image of the endpoints of e. By Lemma 2.8, the map Θ is a quasi-isometric embedding; moreover, the set $\Theta\mathcal{CG}$ is invariant under the action of Γ.

There is a natural smooth marked surface bundle $\mathcal{S} \to \mathcal{T}_g$ whose fibre $\mathcal{S}_z$ at a point $z \in \mathcal{T}_g$ is just the surface S with the marked hyperbolic structure defined by z. This hyperbolic structure is a smooth Riemannian metric on the fibre $\mathcal{S}_z$, and these metrics fit together to a smooth Riemannian metric on the *vertical bundle* of the fibration (i.e. the tangent bundle of the fibres). In other words, the vertical foliation of $\mathcal{S}$ into the fibres of our fibration admits a natural smooth Riemannian metric.

The action of the mapping class group $\mathcal{M}_g$ on $\mathcal{T}_g$ lifts to a unique action on $\mathcal{S}$ which is determined by the requirement that for every $\varphi \in \mathcal{M}_g$ and every $z \in \mathcal{T}_g$, the restriction of the lift of φ to $\mathcal{S}_z$ is the unique isometry of $\mathcal{S}_z$ onto $\mathcal{S}_{\varphi z}$ in the isotopy class determined by φ. In particular, the Riemannian metric on the vertical foliation is invariant under the action of $\mathcal{M}_g$. The restriction $\mathcal{S}_\Gamma$ of the bundle $\mathcal{S}$ to $\Theta\mathcal{CG}$ is invariant under the action of the subgroup Γ of $\mathcal{M}_g$.

We equip now the bundle $\mathcal{S}_\Gamma$ with the following geodesic metric. First, recall that for a given edge b in $\mathcal{CG}$ the arc Θb is a geodesic, and its endpoints are marked hyperbolic metrics on the surface S which are isometric with an isometry in the class determined by the element of $\mathcal{G}$ corresponding to b. If we identify the edge b with the unit interval $[0,1]$ then for all $s,t \in [0,1]$ there is a unique Teichmüller map of minimal quasi-conformal dilatation which maps the fibre $\mathcal{S}_{\Theta(s)}$ to $\mathcal{S}_{\Theta(t)}$, and these maps combine to a smooth fibre preserving horizontal flow on the restriction $\mathcal{S}_{\Theta b}$ of $\mathcal{S}$ to Θb. Defining the tangent of each of these flow-lines to be orthogonal to the fibres and of the same length as its projection to Θb defines a smooth Riemannian metric on $\mathcal{S}_{\Theta b}$ so that the canonical projection $\mathcal{S}_{\Theta b} \to \Theta b$ is a Riemannian submersion. If two edges are incident on the same vertex, then the metrics on the fibres over this vertex coincide. Therefore, the metrics naturally induce a complete length metric d on $\mathcal{S}_\Gamma$ which for every edge b of $\mathcal{CG}$ restricts to the length metric of the above Riemannian structure on $\mathcal{S}_{\Theta b}$.

The universal cover $\mathcal{H}$ of $\mathcal{S}$ is a smooth fibre bundle $\Pi : \mathcal{H} \to \mathcal{T}_g$ whose fibre $\mathcal{H}_z$ at a point $z \in \mathcal{T}_g$ equipped with the lift of the Riemannian metric on $\mathcal{S}_z$ is isometric to the hyperbolic plane $\mathbf{H}^2$. The group Γ_S acts on $\mathcal{H}$ as a group of bundle isomorphisms preserving the metric of the vertical foliation. The pre-image of every Γ-invariant subset of $\mathcal{T}_g$ is invariant under the action of Γ_S. In particular, the set $\mathcal{H}_\Gamma = \Pi^{-1}(\Theta\mathcal{CG})$ is Γ_S-invariant. The metric on $\mathcal{S}_\Gamma$ lifts to a geodesic metric d on $\mathcal{H}_\Gamma$. The group Γ_S acts on the geodesic metric space $(\mathcal{H}_\Gamma, d)$ isometrically, properly and cocompactly and hence we have (see [FM02]).

Lemma 4.1: *$\mathcal{H}_\Gamma$ is quasi-isometric to Γ_S.*

By Lemma 4.1 it is therefore enough to show that the bundle $\mathcal{H}_\Gamma$ with its Γ_S-invariant geodesic metric is hyperbolic.

To show that this is indeed the case, we use the results from Section 3, applied to suitably defined line-subbundles of $\mathcal{H}_\Gamma$. For the construction of these bundles, fix a standard system $a_1, b_1, \dots, a_g, b_g$ of generators for the fundamental group $\pi_1(S)$ of S. For every $z \in \mathcal{T}_g$ there is a unique isomorphism $\rho(z)$ of $\pi_1(S)$ onto a discrete subgroup $\Upsilon(z)$ of $PSL(2, \mathbb{R})$ such that the surface $\mathbf{H}^2/\Upsilon(z)$ is isometric to $\mathcal{S}_z$ and that moreover the following holds (see [IT99]).

a) The conjugacy class of the representation $\rho(z)$ is determined by the marking of $\mathcal{S}_z$.
b) In the upper half-plane model for $\mathbf{H}^2$, the points $0, \infty$ are attracting and repelling fixed points for the action of $\rho(z)(b_g)$, and the point 1 is an attracting fixed point for the action of $\rho(z)(a_g)$.

The representation $\rho(z)$ depends smoothly on z, and the surface bundle $\mathcal{S}$ over $\mathcal{T}_g$ is the quotient of the trivial bundle $\mathcal{T}_g \times \mathbf{H}^2$ under the action of the group $\pi_1(S)$ defined by $\varphi(z,v) = (z, \rho(z)(\varphi)(v))$ $(\varphi \in \pi_1(S), (z,v) \in \mathcal{T}_g \times \mathbf{H}^2)$.

For every $z \in \mathcal{T}_g$ the fibre $\mathcal{H}_z$ of the bundle $\mathcal{H}$ admits a compactification by adding the *ideal boundary* $\partial\mathcal{H}_z$. Every pair of distinct points in $\partial\mathcal{H}_z$ defines uniquely a geodesic line in $\mathcal{H}_z$. Let again $h \in \mathcal{T}_g$ be a point whose Γ-orbit is the vertex set of $\Theta\mathcal{CG}$. For every $z \in \Theta\mathcal{CG}$, the isomorphism $\rho(z) \circ \rho(h)^{-1}$ of $\Upsilon(h)$ onto $\Upsilon(z)$ induces a homeomorphism $\omega(z)$ of $\partial\mathcal{H}_h$ onto $\partial\mathcal{H}_z$. For every pair of distinct points $\xi \neq \eta \in \partial\mathcal{H}_h$ we define a line subbundle $\mathcal{L}^{\xi,\eta}$ of $\mathcal{H}_\Gamma$ by requiring that its fibre $\mathcal{L}^{\xi,\eta}_z$ at z is the geodesic line in $\mathcal{H}_z$ whose endpoints in $\partial\mathcal{H}_z$ are the images of the points ξ, η under the homeomorphism $\omega(z)$. We equip $\mathcal{L}^{\xi,\eta}$ with a complete length metric which is invariant under the natural action of Γ and whose restriction to each fibre coincides with the restriction of the metric on $\mathcal{H}$. More precisely, for $z \in \Theta\mathcal{CG}$ let $\mathcal{B}^{\xi,\eta}_z \subset \mathcal{H}_z$ be the tubular neighborhood of radius one about the fibre $\mathcal{L}^{\xi,\eta}_z$ of $\mathcal{L}^{\xi,\eta}$ at z; then $\mathcal{B}^{\xi,\eta} = \cup_{z\in\Theta\mathcal{CG}}\mathcal{B}^{\xi,\eta}_z$ is a Γ-invariant fibre bundle over $\Theta\mathcal{CG}$ which is an open subset of $\mathcal{H}_\Gamma$. The restriction to $\mathcal{B}^{\xi,\eta}$ of the length structure on $\mathcal{H}_\Gamma$ induces a length metric on $\mathcal{B}^{\xi,\eta}$; we require that the inclusion $\mathcal{L}^{\xi,\eta} \to \mathcal{B}^{\xi,\eta}$ is a p-quasi-isometry for some constant $p \geq 1$ independent of ξ, η. Such a metric can easily be constructed using the fact that the action of Γ on $\Theta\mathcal{CG}$ is cocompact. We have.

Lemma 4.2: *There is a number $q > 0$ so that for every pair of points $\xi \neq \eta \in \partial\mathcal{H}_h$ the following is satisfied.*

(1) *The inclusion $\mathcal{L}^{\xi,\eta} \to \mathcal{H}_\Gamma$ is a q-quasi-isometric embedding.*
(2) *The bundle $\mathcal{L}^{\xi,\eta}$ is q-hyperbolic.*

Proof: Let $\xi \neq \eta \in \partial\mathcal{H}_h$. We begin with showing that there is a number $q_0 > 0$ not depending on ξ, η so that the inclusion $\iota : \mathcal{L}^{\xi,\eta} \to \mathcal{H}_\Gamma$ is a q_0-quasi-isometric embedding. For this we have to show that for any two points $v, w \in \mathcal{L}^{\xi,\eta}$ the distance in $\mathcal{L}^{\xi,\eta}$ between v, w is not bigger than q_0 times the distance between $\iota(v), \iota(w)$. By definition, for the neighborhood $\mathcal{B}^{\xi,\eta}$ of $\iota(\mathcal{L}^{\xi,\eta})$ in $\mathcal{H}_\Gamma$ which intersects each fibre $\mathcal{H}_z$ in the neighborhood of radius one about the geodesic $\iota\mathcal{L}_z^{\xi,\eta}$, we have to find a curve connecting $\iota(v)$ to $\iota(w)$ in $\mathcal{B}^{\xi,\eta}$ whose length is bounded from above by a constant multiple of the distance between $\iota(v)$ and $\iota(w)$ in $\mathcal{H}_\Gamma$.

For this note first that by construction of the metric on $\mathcal{H}_\Gamma$, there is a universal number $a > 0$ with the following property. If $y \in \iota(\mathcal{L}^{\xi,\eta})$ and if $\zeta : [0, a] \to \mathcal{H}_\Gamma$ is a horizontal curve of length at most a issuing from y, then $\zeta[0, a] \subset \mathcal{B}^{\xi,\eta}$.

Let $P : \mathcal{H}_\Gamma \to \mathcal{L}^{\xi,\eta}$ be the unique bundle map whose restriction to a fibre $\mathcal{H}_z$ is the shortest distance projection of $\mathcal{H}_z$ onto $\mathcal{L}_z^{\xi,\eta}$. Let $\zeta : [0, am] \to \mathcal{H}_\Gamma$ be any geodesic of length $am \geq 0$ connecting the points $\iota(v), \iota(w) \in \mathcal{L}^{\xi,\eta}$. Since there is a number $n > 0$ such that the horizontal transport of the fibres of the bundle $\mathcal{H} \to \Theta\mathcal{CG}$ along a geodesic arc in $\Theta\mathcal{CG}$ of length at most ℓ is a bilipschitz map with bilipschitz constant bounded from above by $e^{n\ell}$, there is a curve $\zeta_0 : [0, 2m] \to \mathcal{H}_\Gamma$ connecting $\iota(v)$ to $\iota(w)$ with the property that for every $i < m$ the restriction of ζ_0 to the interval $[2i, 2i + 1]$ is a horizontal arc of length at most a, and the restriction of ζ_0 to $[2i + 1, 2i + 2]$ is vertical and of length at most e^{an}. The length of ζ_0 is bounded from above by $m(e^{an} + a)$. Define a curve $\zeta_1 : [0, 2m] \to \mathcal{H}_\Gamma$ by requiring that for each $i < m$, the restriction of ζ_1 to $[2i, 2i + 1]$ is the horizontal arc issuing from $P\zeta_0(2i)$ whose projection to $\Theta\mathcal{CG}$ coincides with the projection of $\zeta_0[2i, 2i+1]$, and the restriction of ζ_1 to $[2i+1, 2i+2]$ is the vertical geodesic which connects the point $\zeta_1(2i+1)$ to $P\zeta_0(2i+2)$. Note that this curve is entirely contained in $\mathcal{B}^{\xi,\eta}$. To establish our claim it is now enough to show that the distance between the points $\zeta_1(2i+1)$ and $P\zeta_0(2i+2)$ is bounded from above by a universal constant. Namely, if this is the case then the length of ζ_1 is bounded from above by a universal multiple of the length of the geodesic ζ.

For this fix for the moment an arbitrary number $L > 1$. Let $\Psi : \mathbf{H}^2 \to \mathbf{H}^2$ be an L-quasi-isometry which induces the homeomorphism ψ of the ideal boundary $\partial\mathbf{H}^2$ of $\mathbf{H}^2$. Let $\gamma : \mathbb{R} \to \mathbf{H}^2$ be a geodesic line and let $R : \mathbf{H}^2 \to \gamma$ be the shortest distance projection. Let $y \in \mathbf{H}^2$ be such that $R(y) = \gamma(0) = x$ and let $\zeta : [0, \infty) \to \mathbf{H}^2$ be the geodesic ray issuing from $\zeta(0) = x$ and passing through y. Then the concatenation $\sigma_\pm$ of the inverse ζ^{-1} of ζ with the geodesic ray $\gamma[0, \infty)$ and the inverse of the geodesic ray $\gamma(-\infty, 0]$ is a uniform quasi-geodesic in $\mathbf{H}^2$ containing both x and y. Since the isometry group of $\mathbf{H}^2$ acts triply transitive on the ideal boundary, via composing Ψ with an isometry we may assume that ψ fixes the endpoints $\gamma(\pm\infty), \zeta(\infty)$ of γ, ζ. Now Ψ is an L-quasi-isometry and hence the image $L\gamma$ of γ is an L-quasi-geodesic with the same endpoints as γ. Thus the Hausdorff distance between γ and $\Psi\gamma$ is uniformly bounded and hence the distance between Ψx and $R\Psi x$ is bounded from above by a universal constant $p_0 > 0$. Similarly, $\Psi\sigma_\pm$ are uniform quasi-geodesics contained in a uniformly bounded neighborhood of $\sigma_\pm$. Since Ψx is contained in $\Psi\sigma_+ \cap \Psi\sigma_- \cap \Psi\gamma$, we conclude that the distance between x and Ψx is uniformly bounded and that the distance between $R\Psi y$ and Ψx is uniformly bounded as well.

Now for every arc $\nu : [0, a] \to \Theta\mathcal{CG}$ of length at most a, the homeomorphism $\zeta : \mathcal{H}_{\nu(0)} \to \mathcal{H}_{\nu(a)}$ obtained by horizontal transport of the fibres along ν is an L-quasi-isometry for a universal constant $L > 1$ which induces the boundary homeomorphism $\omega(\nu(a)) \circ \omega(\nu(0))^{-1}$. By the definition of the line bundle $\mathcal{L}^{\xi,\eta}$ and the above consideration, we conclude that the distance in $\mathcal{H}_{\zeta_0(2i+1)}$ between the points

$\zeta_1(2i+1)$ and the point $P\zeta_0(2i+1)$ is uniformly bounded. As a consequence, the inclusion $\iota : \mathcal{L}^{\xi,\eta} \to \mathcal{H}_\Gamma$ is a q_0-quasi-isometric embedding for a constant $q_0 > 0$ not depending on ξ, η.

Next we claim that $\mathcal{L}^{\xi,\eta}$ is uniformly quasi-isometric to a metric fibration over $\Theta\mathcal{CG}$ with fibre $\mathbb{R}$. Namely, fix a component A of $\partial\mathcal{H}_h - \{\xi,\eta\}$. For $z \in \Theta\mathcal{CG}$ and $\nu \in A$ define $\Pi(\nu, z) \in \mathcal{L}^{\xi,\eta}_z$ to be the shortest distance projection of $\omega(z)\circ\omega(h)^{-1}(\nu)$ to $\mathcal{L}^{\xi,\eta}_z$. By construction, the map $\Pi : A \times \Theta\mathcal{CG} \to \mathcal{L}^{\xi,\eta}$ is equivariant with respect to the action of Γ on $\Theta\mathcal{CG}$ and $\mathcal{L}^{\xi,\eta}$, and for each fixed $z \in \Theta\mathcal{CG}$ the map $\nu \to \Pi(\nu, z)$ is a homeomorphism of A onto $\mathcal{L}^{\xi,\eta}_z$. It follows from our above consideration that for every $\varphi \in \mathcal{G}$ and every $\nu \in A$ the distance between $\Pi(h,\nu) \in \mathcal{H}_h$ and $\Pi(\varphi h, \nu) \in \mathcal{H}_{\varphi h}$ is uniformly bounded. By equivariance, for every $\psi \in \Gamma$ and $\varphi \in \mathcal{G}$, the distance between $\Pi(\nu, \varphi\psi h)$ and $\Pi(\nu, \psi h)$ is bounded from above by the same constant. As a consequence, for every fixed $\nu \in A$ the map $z \to \Pi(\nu, z)$ is a q_1-quasi-isometric embedding of $\Theta\mathcal{CG}$ into $\mathcal{L}^{\xi,\eta}$ for a number $q_1 > 0$ not depending on ν and on ξ, η, and these quasi-isometric embeddings can be used to define on $\mathcal{L}^{\xi,\eta}$ the structure of a metric fibration whose vertical distance functions coincide with the vertical distance functions induced by the metric on $\mathcal{H}$ and which is quasi-isometric to $\mathcal{L}^{\xi,\eta}$ equipped with the metric induced from the metric on $\mathcal{H}_\Gamma$.

By Corollary 3.7, to show that $\mathcal{L}^{\xi,\eta}$ is q-hyperbolic for a constant $q > 0$ not depending on ξ, η we only have to show that vertical distances for our metric fibration satisfy the bounded κ, n-flaring property with threshold A for numbers $\kappa > 1, n > 0, A > 0$ not depending on ξ, η.

For this let $\zeta : \mathbb{R} \to \Theta\mathcal{CG}$ be any geodesic line. By the discussion in Section 2, ζ is contained in a uniformly bounded neighborhood of a Teichmüller geodesic. By the results of Mosher [Mo03], the restriction $\mathcal{H}_\zeta$ of the bundle $\mathcal{H}_\Gamma$ to ζ is δ_0-hyperbolic for a constant δ_0 not depending on ζ. The above consideration can be applied to the restrictions of the bundles $\mathcal{L}^{\xi,\eta}$ and $\mathcal{H}_\Gamma$ to the geodesic ζ and shows that the inclusion $\mathcal{L}^{\xi,\eta}|\zeta \to \mathcal{H}_\zeta$ is a quasi-isometric embedding. Since $\mathcal{H}_\zeta$ is hyperbolic, the bundle $\mathcal{L}^{\xi,\eta}|\zeta$ is δ_1-hyperbolic for a constant $\delta_1 > 0$ not depending on ζ, ξ, η. Now the geodesic ζ was arbitrary and therefore Lemma 5.2 of [FM02] shows that vertical distances in $\mathcal{L}^{\xi,\eta}$ satisfy the bounded κ, n-flaring property with threshold A for constants $\kappa > 1, n > 0, A > 0$ not depending on ξ, η. As a consequence of Corollary 3.7, the line bundle $\mathcal{L}^{\xi,\eta} \to \Theta\mathcal{CG}$ is q-hyperbolic for a universal constant $q > 0$. □

Now we are ready to show.

Lemma 4.3: *The bundle $\mathcal{H}_\Gamma$ is hyperbolic.*

Proof: Fix a triple of pairwise distinct points $\xi_1, \xi_2, \xi_3 \in \partial\mathcal{H}_h$. Then the line bundles $\mathcal{L}^{\xi_i,\xi_{i+1}}$ bound a subbundle $\mathcal{V}$ of $\mathcal{H}_\Gamma$ whose fibre at a point z is isometric to an ideal triangle in $\mathcal{H}_z$. The arguments in the proof of Lemma 4.2 show that for a suitable choice of a length metric on $\mathcal{V}$ which restricts to the metric on the fibres induced from the metrics on $\mathcal{H}_\Gamma$ the inclusion $\mathcal{V} \to \mathcal{H}_\Gamma$ is a quasi-isometric embedding. We claim that the bundle $\mathcal{V}$ is δ_0-hyperbolic for a number $\delta_0 > 0$

not depending on ξ_i. Namely, an ideal hyperbolic triangle T is uniformly quasi-isometric to the tripod which consists of the unique point in the interior of T of equal distance to each of the three sides and three geodesic rays issuing from this point which make a mutual angle $2\pi/3$. The arguments in the proof of Lemma 4.2 show that the bundle $\mathcal{V}$ is quasi-isometric to a metric fibration over $\Theta\mathcal{C}\mathcal{C}$ whose fibre is precisely this tripod.

Now as before, for every geodesic ζ in $\Theta\mathcal{C}\mathcal{G}$ the restriction of $\mathcal{H}_\Gamma$ to ζ is hyperbolic and hence the same is true for the restriction $\mathcal{V}_\zeta$ of the bundle $\mathcal{V}$ since $\mathcal{V}_\zeta \subset \mathcal{H}_\zeta$ is quasi-isometrically embedded. As a consequence of this and Lemma 5.2 of [FM02], vertical distances in the tripod bundle satisfy the bounded κ, n-flaring property with threshold A for universal numbers $\kappa > 1, n > 0, A > 0$. Corollary 3.7 then shows that the bundle $\mathcal{V}$ is δ_0-hyperbolic for a universal number $\delta_0 > 0$.

We now use once more Lemma 3.3. Namely, for $x, y \in \mathcal{H}_\Gamma$ construct a curve $c(x, y)$ connecting x to y as follows. Fix once and for all a point $\xi \in \partial\mathcal{H}_h$. Then $\cup_{\nu\neq\xi}\mathcal{L}_z^{\xi,\nu} = \mathcal{H}_z$ for every $z \in \Theta\mathcal{C}\mathcal{G}$ and therefore there are unique not necessarily distinct points $\nu_x, \nu_y \in \partial\mathcal{H}_h - \{\xi\}$ such that $x \in \mathcal{L}^{\xi,\nu_x}, y \in \mathcal{L}^{\xi,\nu_y}$. The points ξ, ν_x, ν_y define a bundle over $\Theta\mathcal{C}\mathcal{G}$ whose fibre is a (possibly degenerate) ideal hyperbolic triangle; this bundle is quasi-isometrically embedded in $\mathcal{H}_\Gamma$. We define $c(x, y)$ to be a geodesic in this bundle connecting x to y.

We claim that the system of curves $c(x, y)$ satisfies the properties listed in Lemma 3.3. Namely, property 1) is immediate from the fact that the bundles $\mathcal{V} \subset \mathcal{H}_\Gamma$ as above are uniformly quasi-isometrically embedded. To show property 3), let x, y, z be a triple of points. Then x, y, z are contained in a subbundle of $\mathcal{H}_\Gamma$ whose fibre at the point v is the ideal quadrangle in $\mathcal{H}_v$ with vertices ξ, ν_x, ν_y, ν_z. As before, this bundle is δ_1-hyperbolic for a universal number $\delta_1 > 0$, and the subbundles whose fibres are the ideal triangles with vertices ξ, ν_x, ν_y and ξ, ν_x, ν_z and ξ, ν_y, ν_z are uniformly quasi-isometrically embedded. By definition of our curve system, property 3) now follows from hyperbolicity of our bundle of quadrangles, and property 2) is obtained in the same way. □

As a consequence, we obtain the second part of our theorem.

Corollary 4.4: *For a finitely generated subgroup $\Gamma < \mathcal{M}_g$, the following are equivalent.*

(1) *The natural $\pi_1(S)$-extension of Γ is hyperbolic.*
(2) *Γ is convex cocompact.*

Proof: Farb and Mosher [FM02] show the following. If $\Gamma < \mathcal{M}_g$ is any finitely generated group such that the $\pi_1(S)$-extension of Γ is hyperbolic, then there is a quasi-convex orbit for the action of Γ on $\mathcal{T}_g$. By Theorem 2.9, this is equivalent to saying that Γ is convex cocompact.

On the other hand, by Lemma 4.1 and Lemma 4.3 the $\pi_1(S)$-extension of a convex cocompact subgroup Γ of $\mathcal{M}_g$ is word hyperbolic. □

Remark: As mentioned in the introduction, there is no example known to me of a convex cocompact subgroup of $\mathcal{M}_g$ which is not virtually free. On might ask whether indeed *all* convex cocompact subgroups Γ of $\mathcal{M}_g$ are virtually free. One possible approach to study this question is via the Gromov boundary $\partial\Gamma$ of Γ. Namely, $\partial\Gamma$ embeds into the Gromov boundary of the complex of curves $\mathcal{C}(S)$, and by Theorem 2.9 and the results of Masur [Ma82a], its image is contained in the subset of all minimal geodesic laminations which are *uniquely ergodic*, i.e. which support up to multiple a unique transverse invariant measure.

As a consequence, $\partial\Gamma$ embeds into the set $\mathcal{UE} \subset \mathcal{PML}$ of uniquely ergodic projective measured laminations. Therefore, if $\mathcal{UE}$ is totally disconnected, then the same is true for the Gromov boundary of Γ and consequently Γ is virtually free. However, to my knowledge, the topology of the set $\mathcal{UE}$ is not known. Results of Kerckhoff, Masur and Smillie [KMS86] indicate that the set of foliations which are not uniquely ergodic is small, so it is possible that the set $\mathcal{UE}$ is locally connected.

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After this paper was written, a preprint by Kent and Leininger [KL05] appeared which contains a different proof of the equivalence of properties 1) and 2) in our main theorem.

References

[BF92] M. Bestvina, M. Feighn, *A combination theorem for negatively curved groups*, J. Diff. Geom. 35 (1992), 85–101.

[B02] B. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, preprint 2002.

[BH99] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer Grundlehren 319, Springer, Berlin 1999.

[Bu92] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser, Boston 1992.

[CEG87] R. Canary, D. Epstein, P. Green, *Notes on notes of Thurston*, in "Analytical and geometric aspects of hyperbolic space", edited by D. Epstein, London Math. Soc. Lecture Notes 111, Cambridge University Press, Cambridge 1987.

[FM02] B. Farb, L. Mosher, *Convex cocompact subgroups of mapping class groups*, Geom. Top. 6 (2002), 91-152.

[FLP91] A. Fathi, F. Laudenbach, V. Poénaru, *Travaux de Thurston sur les surfaces,* Astérisque 1991.

[GDH99] G. Gonzalez-Díez, W. Harvey, *Surface subgroups inside mapping class groups*, Topology 38 (1999), 57–69.

[H04] U. Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves*, to appear in "Spaces of Kleinian groups", edited by Y. Minski, M. Sakuma and C. Series, Lond. Math. Soc. Lec. Notes 329, 187–207; arXiv:math.GT/0409611.

[H05] U. Hamenstädt, *Geometry of the complex of curves and of Teichmüller space*, arXiv:math.GT/0502256.

[Ha81] W. J. Harvey, *Boundary structure of the modular group*, in "Riemann Surfaces and Related topics: Proceedings of the 1978 Stony Brook Conference" edited by I. Kra and B. Maskit, Ann. Math. Stud. 97, Princeton, 1981.

[IT99] Y. Imayoshi, M. Taniguchi, *An introduction to Teichmüller spaces*, Springer, Tokyo 1999.

[I02] N. V. Ivanov, *Mapping class groups*, Chapter 12 in "Handbook of Geometric Topology", edited by R.J. Daverman and R.B. Sher, Elsevier Science (2002), 523–633.

[KL05] R. Kent, C. Leininger, *Shadows of mapping class groups: capturing convex cocompactness*, arXiv:math.GT/0505114.

[Ke92] S. Kerckhoff, *Lines of minima in Teichmüller space*, Duke Math. J. 65 (1992), 187-213.

[KMS86] S.Kerckhoff, H. Masur, J. Smillie, *Ergodicity of billiard flows and quadratic differentials*, Ann. Math. 124 (1986), 293-311.

[K99] E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space,* preprint 1999.

[LR05] C. Leininger, A. Reid, *A combination theorem for Veech subgroups of the mapping class group*, arXiv:math.GT/0410041.

[Ma82a] H. Masur, *Interval exchange transformations and measured foliations*, Ann. Math. 115 (1982), 169-200.

[Ma82b] H. Masur, *Two boundaries of Teichmüller space*, Duke Math. J. 49 (1982), 183–190.

[MM99] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103–149.

[MP89] J. McCarthy, A. Papadopoulos, *Dynamics on Thurston's sphere of projective measured foliations*, Comm. Math. Helv. 64 (1989), 133–166.

[Mo96] L. Mosher, *Hyperbolic extensions of groups*, J. Pure Appl. Alg.110 (1996), 305–314.

[Mo03] L. Mosher, *Stable Teichmüller quasigeodesics and ending laminations*, Geom. Top. 7 (2003), 33-90.