# FUNDAMENTAL GROUPS OF NONPOSITIVELY CURVED SQUARE COMPLEXES

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Abstract. We show that a torsion free one-ended hyperbolic group $\Gamma$ which is the fundamental group of a compact special square complex contains quasi-convex surface subgroups.

## 1. Introduction

CAT(0) cube complexes are important spaces in geometric group theory, and they have been studied extensively in recent years. Even *square complexes*, that is, cube complexes of dimension two, are interesting: a celebrated example of Wise [Wi96] shows that there are quotients of CAT(0) square complexes whose fundamental groups are not residually finite. Often one-relator groups are fundamental groups of nonpositively curved square complexes [PW13], that is, square complexes whose universal coverings are CAT(0).

The first goal of this note is to prove the following.

**Theorem 1.** *Let $X$ be a proper* CAT(0) *square complex, and let $\Gamma$ be a torsion free group which acts properly and cocompactly on $X$ by cubical isometries. Then one of the following holds.*

1. $\Gamma$ *is free.*
2. $\Gamma$ *is the fundamental group of a (possibly non-orientable) closed surface.*
3. *The rank of $H^2(\Gamma; \mathbb{Z}\Gamma)$ is infinite.*

Note that a group $\Gamma$ is free if and only if it is of cohomological dimension one [Sw69]. A group $\Gamma$ which acts properly discontinuously on a proper CAT(0) cube complex acts *freely* if and only if it is torsion free. Namely, any finite group of isometries of a proper CAT(0) cube complex fixes the center of the convex hull of an orbit of the finite group and hence does not act freely. Vice versa, point stabilizers of a group acting properly discontinuously are finite. Also, a torsion free group which contains a finite index subgroup which is the fundamental group of a closed surface, called *surface group* in the sequel, is a surface group (Section 5 of [SW79]).

The proof of Theorem 1 is a direct application of the work of Wilton [Wi24]. We do not require that the group $\Gamma$ is hyperbolic. Under this additional assumption,

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Theorem A and Theorem 5.2 of [Wi24] combined show a stronger result: A one-ended torsion free hyperbolic group of cohomological dimension two which is the fundamental group of a non-positively curved cube complex either is a surface group, or $H^2(\Gamma, \mathbb{Z}\Gamma)$ has infinite rank.

The following is a consequence of Theorem 1 and [Be96]. For its formulation, denote by $H^2_c(\Gamma)$ the compactly supported second cohomology group of $\Gamma$ with integral coefficients.

**Corollary.** *If $\Gamma$ is the fundamental group of a compact nonpositively curved square complex and if $\Gamma$ is not free, then $H^2_c(\Gamma) \neq 0$. If in addition $\Gamma$ is not a surface group then* $\dim(H^2_c(\Gamma)) = \infty$.

*Proof.* The geometric boundary of a proper CAT(0)-space $X$ defines an $\mathcal{E}\mathcal{Z}$-structure for any torsion free group $\Gamma$ of isometries of $X$ which acts properly discontinuously [Be96]. If $\Gamma$ acts cocompactly, then the second compactly supported cohomology group $H^2_c(\Gamma)$ of $\Gamma$ can be identified with the second compactly supported cohomology of $X$ and hence with the first Čech cohomology group of the geometric boundary $\partial X$ of $X$. This cohomology in turn coincides with $H^2(\Gamma, \mathbb{Z}\Gamma)$ [Be96]. □

Note that there is no clear relation between the second compactly supported cohomology group of a group $\Gamma$ as in Theorem 1 and its second cohomology. For example, the fundamental group of a square complex obtained from a compact orientable surface $S_0$ of genus at least one with connected boundary by gluing 3 copies of $S_0$ along the boundary is word hyperbolic and has the third property in Theorem 1, but using cellular homology and dualizing, it is easy to see that $\mathrm{rk}\, H^2(\Gamma, \mathbb{Q}) = 2$.

A *special* group is the fundamental group of a nonpositively curved cube complex which is special in the sense of Haglund and Wise [HW08]. Such groups are known to embed into right angled Artin groups, in particular they are residually finite and linear. A well known question asks whether one-ended hyperbolic groups contain surface subgroups. Our second result confirms this question for hyperbolic one-ended special groups of dimension two. The assumption on specialness can be slightly relaxed to a notion we call *weakly special*, but we defer this discussion to Section 3.

**Theorem 2.** *If $\Gamma$ is a one-ended hyperbolic group which is the fundamental group of a compact nonpositively curved special square complex, then $\Gamma$ contains quasi-convex surface subgroups.*

Our approach is elementary, and purely geometric. The proofs of the two theorems are contained in Section 2 and Section 3, respectively. The proof of Theorem 2 heavily depends on (weak) specialness of the square complex and the results of Wilton [Wi18], but it uses the fact that we are working with square complexes only in a mild way. It is conceivable that it can be extended to cover all one-ended fundamental groups of special cube complexes. We leave this question to other authors.

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## 2. The cohomology of compact nonpositively curved square complexes

Normalize an $n$-cube in a CAT(0) cube complex $X$ to be an isometric copy of $[-\frac{1}{2}, \frac{1}{2}]^n$. A *midcube* of an $n$-cube $c$ is an $(n-1)$-cube in $c$ obtained by restricting exactly one coordinate to 0. A *hyperplane* $H$ in $X$ is a connected union of midcubes of cubes in $X$ such that, for any cube $c$ of $X$, either $H \cap c = \emptyset$ or $H \cap c$ is a single midcube of $c$. We define the *carrier* $N(H)$ of the hyperplane $H$ to be the union of all cubes which intersect $H$ in a midcube. Note that $N(H)$ naturally is a cube complex.

The following summarizes several results of Sageev, see Section 4 of [S95].

**Theorem 2.1** ([S95])**.** *Let $H$ be a hyperplane in the* CAT(0) *square complex $X$; then*

1. *$H$ is convex;*
2. *$H$ is two-sided, that is, $N(H) \cong H \times [-\frac{1}{2}, \frac{1}{2}]$;*
3. *$H$ is separating, that is, $X - H$ has exactly two components, called half-spaces associated to $H$;*
4. *any midcube is contained in a unique hyperplane;*
5. *$H$ is a* CAT(0)*-cube complex whose hyperplanes are of the form $V \cap H$ where $V \neq H$ is a hyperplane of $X$ that crosses $H$.*

The goal of this section is to show Theorem 1 from the introduction, which is restated here for convenience.

**Theorem 2.2.** *Let $\Gamma$ be a torsion free group which acts properly and cocompactly on a* CAT(0) *square complex by cubical isometries. Then one of the following mutually exclusive possibilities is satisfied.*

1. *$\Gamma$ is free.*
2. *$\Gamma$ is a surface group.*
3. *The rank of $H^2(\Gamma; \mathbb{Z}\Gamma)$ is infinite.*

**Corollary 2.3.** *If the fundamental group $\Gamma$ of a compact non-positively curved square complex is one-ended, then either $\Gamma$ is a surface group, or $H^2(\Gamma, \mathbb{Z}\Gamma)$ is infinite dimensional.*

Our strategy is to reduce Theorem 2.2 to a result of Wilton [Wi24]. For this and later use we establish the following proposition.

**Proposition 2.4.** *A torsion free group which acts properly and cocompactly on a* CAT(0) *square complex by cubical isometries also acts properly and cocompactly on a geodesically complete* CAT(0) *square complex.*

*Proof.* Let $X$ be a CAT(0) square complex. We claim that this complex is geodesically complete if and only if $X$ does not have a leaf, that is, a vertex on which only one edge is incident, and does not have a square with an *open* side, that is, a side which is contained in precisely one square.

To see this assume first that $X$ has no leaves and open sides of squares, which is equivalent to stating that the link complex of every vertex is a finite union of components, and each of these components are graphs without leaves and no circuit of length less than four. Now that any geodesic segment in $X$ whose endpoint is not a vertex or contained in the closure of an open side can locally be extended by the local CAT(0) property of the complex. A geodesic segment $\alpha$ whose endpoint is a vertex $v$ has an inward pointing direction at the endpoint which is the direction of an edge or of a square. If this direction is the direction of an edge, then it defines a vertex $e$ in the link complex $L(v)$ of $v$. Then there is a vertex $p \in L(v)$ whose combinatorial distance to $e$ is at least two. The concatenation of $\alpha$ with the directed edge defined by $p$ is a geodesic and hence $\alpha$ can be extended past its endpoint. The same argument can also be applied if the direction is the direction of a line segment in the interior of a square and shows that $X$ is geodesically complete.

On the other hand, a segment whose endpoint is the endpoint of a leaf can not be extended. A segment whose endpoint is contained in the interior of an open side and which meets the open side transversely can not be extended as the CAT(0) angle between the oriented direction of the segment and any direction at the endpoint is smaller than $\pi$, so the condition on non-existence of leaves and open sides of squares is also neccesary.

As the torsion free group $\Gamma$ acts freely on a proper CAT(0) square complex by cubical isometries, we can take the quotient $Z = X/\Gamma$, which is a nonpositively curved square complex. If $X$ is not geodesically complete, then either $Z$ has a leaf, or it contains a square $C$ with an open side $e$.

If $Z$ has a leaf, then the removal of the leaf retains the property of nonpositive curvature, as such a leaf yields an isolated vertex in the link complexes at its endpoints, and it does not change the fundamental group of $Z$. Thus we may assume that $Z$ and hence $X$ do not have leaves.

Let $e$ be an open side of a square $C$ in $Z$ and let $Z_0$ be the square complex obtained by removal of the edge $e$ and the interior of $C$. Then $Z_0$ is a deformation retract of $Z$, and $Z_0 = X_0/\Gamma$ where $X_0$ is obtained by removal of the interiors of the squares which project to $C$ and removal of the edges which project to $e$. We claim that $Z_0$ is non-positively curved. To this end we only have to check that the link complex of every vertex of $Z_0$ is a flag complex [BH99].

Let $L$ be the link complex in $Z$ of a vertex $v \in C$. Then $L$ is a graph since $Z$ is a square complex. As $Z$ is nonpositively curved, $L$ is a flag complex and hence this graph does not have cycles of length at most three.

Assume first that the open edge $e$ is incident on $v$. Then $e$ defines a vertex in the link complex $L$ of $v$ in $Z$ which is a leaf, that is, $e$ is contained in the boundary of a unique square. In other words, $e$ is not contained in any cycle of $L$. The link complex of $v$ in $Z_0$ is obtained from $L$ by removal of this leaf and hence it is a flag complex as well. Similarly, if the edge $e$ is not incident on the vertex $v \in C$ then the link complex $L_0$ of $Z_0$ at $v$ is obtained from the link complex $L$ by removal of one edge and once again, as $L$ is a flag complex, the same holds true for $L_0$. The links in $Z_0$ of vertices not contained in $C$ coincide with the links in $Z$.

Proceed inductively and remove in this way successively all edges of $Z$ which are newly formed leaves, and remove all open sides of a square together with the adjacent interiors of these squares. As $Z$ is compact, this is accomplished in finitely many steps. The resulting complex $\hat{Z}$ is a deformation retract of $Z$ which is a nonpositively curved square complex. Its preimage $\hat{X}$ in $X$ is a $\Gamma$-equivariant deformation retract of $X$. Thus $\Gamma$ acts properly and cocompactly on the complex $\hat{X}$, which is a CAT(0) square complex $\hat{X} \subset X$. The complex $\hat{X}$ does not contain any leaves or squares with an open side. But this means that $\hat{X}$ is geodesically complete. □

**Remark 2.5.** Proposition 2.4 is not valid for cube complexes of dimension at least three.

A *cut vertex* in $Z$ is a vertex $v$ such that the link complex $L(v)$ is disconnected. An example of a cut vertex is a vertex $v$ with the property that there exists an edge incident on $v$ that is not contained in the boundary of any square. This edge defines an isolated point in $L(v)$. A cut vertex in the link complex $L(v)$ of some vertex $v$ is defined in the same way. More generally, a *cut simplex* in $L(v)$ is a zero- or one-simplex which disconnects $L(v)$. The following result rests on the work of Wilton [Wi24].

**Proposition 2.6.** *Let $\Gamma$ be the fundamental group of a compact nonpositively curved square complex. Then $\Gamma = \Gamma_0 * \Gamma_1 * \cdots * \Gamma_k$ where $\Gamma_0$ is free, and for $i \geq 1$, $\Gamma_i$ is a one-ended group which is the fundamental group of a compact non-positively curved geodesically complete square complex without cut vertices and so that no link complex has a cut simplex.*

*Proof.* By Proposition 2.4, we may assume that the non-positively curved square complex $Z$ is geodesically complete. Following [Wi24], an *unfolding* of $Z$ is a map of square complexes $\varphi : Z' \to Z$ so that (i) the restriction of $\varphi$ to 1-skeleta is a combinatorial homotopy equivalence of graphs; and (ii) $\varphi$ is a homeomorphism on the complements of the 1-skeleta.

By Proposition 2.31 of [Wi24], there is an unfolding

$$Z' = G \vee Z_1 \vee \cdots \vee Z_k \vee S_1 \vee \cdots \vee S_m$$

where $Z'$ is a compact nonpositively curved square complex and $S_i$ are squares and, furthermore,

(1) $G$ is a graph;

(2) each $Z_i$ has the property that every link is connected without cut vertices or cut edges.

Then $\Gamma = \Gamma_0 * \Gamma_1 * \cdots * \Gamma_k$ where $\Gamma_0$ is free and $\Gamma_i$ is the fundamental group of $Z_i$. The unfolding complex has the same number of squares as $Z$.

If $i \geq 1$ then the group $\Gamma_i$ is nontrivial, and it is torsion free as this holds true for $\Gamma$. Thus $\Gamma_i$ is infinite and hence has at least one end. By Theorem E of [Wi24], it has at most one end, showing that $\Gamma_i$ is one-ended. That we can modify $Z_i$ to a geodesically complete square complex was shown in Proposition 2.4. This modification does not alter the fundamental group. If it is nontrivial, then it decreases the number of squares. We then can unfold the resulting square complex $Z_i'$ further. In finitely many such steps we arrive to a geodesically complete square complex without cut vertices and cut simplices in link complexes with fundamental group the one-ended subgroup $\Gamma_i$ of $\Gamma$. □

We shall use the following simple

**Lemma 2.7.** *Let $G = A * B$ with $A, B$ infinite, of type $FP$ and torsion free. If either $H^2(A; \mathbb{Z}A)$ has infinite rank or $A$ is the fundamental group of a closed surface, then $H^2(\Gamma; \mathbb{Z}\Gamma)$ has infinite rank.*

*Proof.* Regard $\mathbb{Z}G$ as a left $G$-module in the usual way. Then

$$H^2(G; \mathbb{Z}G) = H^2(A; \mathbb{Z}G) \oplus H^2(B; \mathbb{Z}G).$$

Using the fact that $\mathbb{Z}G = \mathbb{Z}G \otimes_{\mathbb{Z}A} \mathbb{Z}A$ and that $\mathbb{Z}G$ is a free right $\mathbb{Z}A$-module, Shapiro's lemma gives

$$H^2(A; \mathbb{Z}G) = \mathbb{Z}G \otimes_{\mathbb{Z}A} H^2(A; \mathbb{Z}A)$$

and similarly for $B$. Therefore

$$H^2(G; \mathbb{Z}G) = (\mathbb{Z}G \otimes_{\mathbb{Z}A} H^2(A; \mathbb{Z}A)) \oplus (\mathbb{Z}G \otimes_{\mathbb{Z}B} H^2(B; \mathbb{Z}B))$$

from which the lemma follows. □

*Proof of Theorem 2.2 .* A free product of free groups is free. If $\Gamma$ is any group with $\mathrm{rk}(H^2(\Gamma; \mathbb{Z}\Gamma)) = \infty$, then by Lemma 2.7, the same holds true for the free product of $\Gamma$ with any other group. Moreover, using again Lemma 2.7, a free product of a surface group with an infinite group also fulfills $\mathrm{rk}(H^2(\Gamma; \mathbb{Z}\Gamma)) = \infty$. Thus by Proposition 2.4 and Lemma 2.6, it suffices to show the theorem for a torsion free group $\Gamma$ which is the fundamental group of a compact geodesically complete nonpositively curved square complex without cut vertices and no cut simplices in link complexes. In particular, we may assume that $\Gamma$ is one-ended and hence its cohomological dimension equals two [Sw69]. Moreover, every edge in $Z$ is contained in the boundary of at least two squares.

Now as $\Gamma$ is the fundamental group of a finite aspherical cell complex and hence of type $FP$ and of cohomological dimension two, we know that $H^0(\Gamma; \mathbb{Z}\Gamma) = 0$ (because $\Gamma$ is infinite), and $H^1(\Gamma; \mathbb{Z}\Gamma) = 0$ (because $\Gamma$ is one-ended). Theorem 2 of [Fa75] then states that either the rank of $H^2(\Gamma; \mathbb{Z}\Gamma)$ is one, or it is infinite. Thus we are left with showing that if this rank equals one then $\Gamma$ is the fundamental group of a closed surface.

If the rank of $H^2(\Gamma;\mathbb{Z}\Gamma)$ is one, then as $\Gamma$ is torsion free, we have $H^2(\Gamma;\mathbb{Z}\Gamma)=\mathbb{Z}$ with the usual orientation character. Thus $\Gamma$ is a two-dimensional Poincaré duality group.

As a two-dimensional Poincaré duality group is the fundamental group of a closed surface [EM80, EL83], this completes the proof of the theorem. □

The following is a consequence of the proof of Theorem 2.2.

**Theorem 2.8.** *The fundamental group $\Gamma$ of compact nonpositively curved square complex is a finite free product $\Gamma=\Gamma_0*\cdots*\Gamma_k$ where $\Gamma_0$ is a finitely generated free group and where for each $i\geq 1$, $\Gamma_i$ is a one ended* CAT(0) *group of cohomological dimension two. Each of the groups $\Gamma_i$ either is a surface group, or $H^2(\Gamma_i;\mathbb{Z}\Gamma_i)$ is infinite dimensional.*

## 3. Fundamental groups of special square complexes

A *special* nonpositively curved square complex is characterized by the following properties [HW08].

(1) Each hyperplane is embedded.
(2) Each hyperplane is 2-sided.
(3) No hyperplane directly or indirectly self-osculates.
(4) No two distinct hyperplanes inter-osculate.

We shall use only a weaker version of specialness as follows. Call a square complex *weakly special* if it satisfies properties (1)-(3). In other words, we allow distinct inter-osculating hyperplanes. Our goal is to analyze fundamental groups $\Gamma$ of a weakly special non-positively curved square complex $Z$ that are in addition one-ended and *hyperbolic*. Note that a hyperbolic group which is the fundamental group of a nonpositively curved square complex has a finite index subgroup which is the fundamental group of a special cube complex [A13]. However, as the construction is somewhat indirect, it is hard to keep track of dimensions.

Following Proposition 2.4, we have.

**Lemma 3.1.** *If $\Gamma$ is the fundamental group of a compact weakly special square complex, then it also is the fundamental group of a geodesically complete weakly special square complex.*

*Proof.* Let $Z$ be a compact nonpositively curved cube complex. Assume that $Z$ is not geodesically complete. We saw in Proposition 2.4 that this is equivalent to stating that either the square complex has a leaf, or there is a square $C$ in $Z$ with an open side. Removal of leaves does not change the fundamental group and specialness, so we may assume that there are no leaves. If there is a square $C$ with an open side $e$, then the proof of Proposition 2.4 shows that removal of the interior of $C$ as well as the open side yields a compact CAT(0) cube complex $Z_0$. By induction, it suffices to show that $Z_0$ is weakly special.

To see this note that a hyperplane in $Z_0$ is a subset of a hyperplane in $Z \supset Z_0$. Hence a hyperplane in $Z_0$ embeds since this holds true for a hyperplane in $Z$ as the embedding property is equivalent to stating that a hyperplane does not have a self-crossing. Two-sidedness of hyperplanes in $Z_0$ is also immediate from two-sidedness of hyperplanes in $Z$.

If $H_0 \subset Z_0$ is a hyperplane that directly or indirectly self-osculates, then the carrier $N(H_0)$ has a self-touching point as in the definition of a self-osculating hyperplane [HW08]. But $H_0$ is a subset of a hyperplane $H$ in $Z$, and the carrier of $H_0$ is a subset of the carrier of $H$. Now an osculation point in $Z_0$ between two hyperplanes $A_0, B_0 \subset Z_0$ which is not an osculation point for the hyperplanes $A, B$ in $Z$ containing $A_0, B_0$ corresponds to an intersection between $A, B$. Thus since $H$ is embedded in $Z$ and does not self-osculate, the same holds true for $H_0$. As a consequence, $Z_0$ is weakly special. □

From now on, we assume that $Z$ is a geodesically complete compact weakly special square complex. We call $Z$ of *pure dimension* 2 if the dimension of $Z$ is 2 and if every cube of dimension at most one is a face of a cube of dimension 2. By the results of Section 2, since $\Gamma$ is one-ended by assumption, we may assume that $Z$ is of pure dimension 2 without cut vertices and such that no link complex of $Z$ has a cut simplex. Namely, in complete analogy to Lemma 3.1, it is immediate from the work of Wilton [Wi24] that an unfolding of a weakly special square complex as specified in Proposition 2.31 of [Wi24] is weakly special. We know that the cohomological dimension of $\Gamma$ equals two [Sw69].

Since $Z$ is a purely two-dimensional square complex, a hyperplane $H \subset Z$ is an embedded finite graph (by weakly specialness) without vertices of valency one (as $Z$ is geodesically complete), that is, $H$ is a finite graph without leaves. In particular, the fundamental group of $H$ is a free group of positive rank, and this group embeds into the fundamental group $\Gamma$ of $Z$ as a quasi-convex subgroup. Recall that $H$ is two-sided as $Z$ is weakly special, and hence the boundary of its carrier $N(H)$ consists of two embedded connected components $N_-(H), N_+(H)$ homotopic to $H$ which do not intersect. Moreover, $N(H)$ admits a natural deformation retraction onto $H$.

Every oriented closed curve which intersects $H$ transversely (in the combinatorial sense) either crosses through $N(H)$ or returns to the same side of $H$ in $N(H)$, Thus we can define the mod 2 intersection $\psi(\gamma)$ of a closed curve $\gamma \subset Z$ with $H$ as the number of crossings of $\gamma$ through $N(H)$ up to homotopy and taken modulo two. Although $H$ is finite graph and may not naturally define a mod 2 cycle in $Z$, a standard inspection of homotopies of loops (or using graph of groups decompositions and the action on the Bass Serre tree) shows that this is well defined, and $\psi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism whose restriction to $\pi_1(H)$ vanishes.

The following is standard and added her for completeness.

**Lemma 3.2.** *If the homomorphism $\psi$ is trivial then the interior of $N(H)$ separates $Z$.*

*Proof.* Let $U$ be the interior of $N(H)$. It admits a deformation retraction onto $H$. As $H$ is connected and two-sided, if $Z \setminus U$ is connected then we can find a loop which crosses through $H$ precisely once. This loop is constructed from an arc in $Z \setminus U$ connecting two points on the different components of the boundary of $U$ and extending this arc by attaching an arc in $U$ with the same endpoints which crosses once through $H$. But this violates the assumption that every loop intersects $H$ an even number of times, counted with multiplicities. □

If the homomorphism $\psi$ is nontrivial then we can pass to the double cover $\hat{Z} \to Z$ whose fundamental group is the kernel of this homomorphism. In other words, up to perhaps passing to a double cover, we may assume that this homomorphism is trivial.

**Lemma 3.3.** *If the homomorphism $\psi$ is nontrivial, then the preimage $\hat{H}$ of $H$ in the covering $\hat{Z} \to Z$ has two connected components.*

*Proof.* If the preimage $\hat{H}$ of $H$ in $\hat{Z}$ is connected, then the restriction of the homomorphism $\psi : Z \to \mathbb{Z}/2\mathbb{Z}$ to $H$ is nontrivial. But the restriction of $\psi$ to $H$ is trivial by its construction (and the discussion preceding this proof). This yields the lemma. □

As a consequence, if the covering $\hat{Z} \to Z$ is nontrivial, then the deck group of the covering exchanges the two components of $\hat{Z} \setminus \operatorname{int} N(\hat{H})$ and hence each of these components has the same number of squares as $Z$.

From now on we denote by $\hat{Z}$ the (possibly trivial) covering of $Z$ whose fundamental group is the kernel of $\psi$. Removal of the interior $\operatorname{int} N(\hat{H})$ of the carrier $N(\hat{H})$ yields a square complex $\hat{Y}$ consisting of two components whose number of squares is strictly smaller than the number of squares of $Z$. Recall that if $I^2$ is a square whose intersection with $N(\hat{H})$ is a side of a square, then this square intersects $\hat{H}$, and $\hat{Y}$ contains $I^2$, but one side of $I^2$ (the side corresponding to the side contained in $N(\hat{H}) \subset \hat{X}$) is attached to precisely one square, and its endpoints are attached to the boundary of $N(\hat{H})$.

**Lemma 3.4.**
1. *$\hat{Y}$ has the structure of a weakly special* CAT(0) *square complex.*
2. *If, in addition,* $\Gamma$ *is hyperbolic, then* $\pi_1(\hat{Y})$ *is a quasi-convex subgroup of* $\Gamma$.

*Proof.* By Lemma 3.3 and the construction, $\hat{Y}$ has two connected components. Furthermore, it has naturally the structure of a square complex. Note that there is a natural cubical embedding $F : \hat{Y} \to \hat{Z}$.

We have to show that $\hat{Y}$ is nonpositively curved. To this end we have to verify the link condition at every vertex. Now a vertex $v$ in $\hat{Y}$ which is mapped by $F$ to a vertex not contained in the boundary of $N(\hat{H})$ in $\hat{Z}$ has the same link as its image under $F$ and hence the link complex is a flag complex as this holds true in $\hat{Z}$. On the other hand, since $Z$ is weakly special and hence carriers of hyperplanes do not self-osculate, the link of a vertex $v \in \hat{Y}$ which is mapped by $F$ to a vertex in the boundary of $N(\hat{H})$ is obtained from the link of $F(v)$ by deleting one vertex (corresponding to the edge of $\hat{Z}$ crossing through $N(\hat{H})$) and all of its adjacent sides. Thus this link complex is a full subcomplex of the link complex of $v$ in $\hat{Z}$ and hence it is a flag complex as this was true for the link complex of $F(v)$. This shows that indeed, $\hat{Y}$ is nonpositively curved.

To show that the fundamental group of $\hat{Y}$ is a quasi-convex subgroup of the fundamental group $\Gamma$ of $\hat{Z}$ (provided that $\Gamma$ is hyperbolic) it suffices to verify that $\hat{Y}$ is a *locally convex* subspace of $\hat{Z}$. This means that the star of each vertex of $\hat{Y}$ is a convex subspace of the star of its image under the inclusion map $\hat{Y} \to \hat{Z}$. But this is an immediate consequence of the fact that the link complex of a vertex $v \in \hat{Y}$ is a full subcomplex of the link complex of $v$ in $\hat{Z}$.

Now a locally convex subcomplex of a weakly special cube complex is weakly special as hyperplanes in the subcomplex map to hyperplanes in the complex by the inclusion map (see the end of the proof of Lemma 3.1). Thus $\hat{Y}$ is weakly special. This completes the proof of the lemma. ☐

*Proof of Theorem 2.* Let $Y_1, Y_2$ be the two components of $\hat{Y}$. Their fundamental groups $\Lambda_1, \Lambda_2$ are the fundamental groups of a weakly special CAT(0) square complex, and by Lemma 3.4, $\Lambda_i$ is a quasiconvex subgroup of $\Gamma$. We claim that $Y_i$ contains fewer squares than $Z$ ($i = 1, 2$).

As every edge of $Z$ is contained in the boundary of at least two squares, this is obvious if $\hat{H}$ is connected. Otherwise note that if we denote by $s(V)$ the number of squares in a square complex, then for the two-sheeted covering $\hat{Z}$ of $Z$ we have $s(\hat{Z}) = 2s(Z)$. But $Y_1$ and $Y_2$ are exchanged by the action of the deck group and hence $s(Y_1) = s(Y_2)$. Since $s(\hat{Z}) > s(Y_1) + s(Y_2)$, this shows the claim.

By Lemma 3.1, we may successively remove from $Y_i$ all leaves and all squares with at least one open side, that is, a side that is only attached to one square, to find a geodesically complete weakly special square complex with the same fundamental group $\Lambda_i$. Denote the resulting complex again by $Y_i$ by abuse of notation. As this modification process does not increase the number of squares, we have $s(Y_i) < s(Z)$. Unfold $Y_i$ as in Proposition 2.31 of [Wi24] (see the proof of Theorem 2.2) to obtain a square complex $Y_i' = G_i \vee V_i^1 \vee \cdots \vee V_i^{k_i} \vee S_i^1 \vee \cdots \vee S_i^{m_i}$ with the same number of squares and the same fundamental group so that the link complex of no vertex has a cut simplex. The fundamental group of $G$ is free, and the cohomological dimension of the fundamental groups $\Psi_i^\ell$ of the complexes $V_i^\ell$ equals two.

There are now three possibilities. In the first case, one of the groups $\Psi_i^\ell$ ($i = 1, 2, \ell \leq m_i$) is a surface group. As $\Psi_i^\ell$ is a quasi-convex subgroup of $\Lambda_i$ and hence $\Gamma$, in this case we are done.

In the second case, none of the groups $\Psi_i^\ell$ is a surface group, but for $i = 1$ or $i = 2$, we have $k_i \geq 1$. Then $\Psi_i^1$ is a one-ended fundamental group of a geodesically complete nonpositively curved square complex $V$ whose number of squares is smaller than the number of squares of $Z$ and so that $V$ neither has cut vertices nor cut simplices in any link complex. Moreover, $\Psi_i^1$ is a quasi-convex subgroup of $\Gamma$, and the rank of $H^2(\Psi_i^1, \mathbb{Z}\Psi_i^1)$ is infinite.

Replace $Z$ by $V$ and $\Gamma$ by $\Psi_i^1$ and proceed by induction. As in each such step, the number of squares of the complexes considered strictly decreases, in finitely many steps we either find a quasi-convex surface subgroup of $\Gamma$, or we find a quasi-convex subgroup $\Lambda$ with a hyperplane as in the remaining third case, when $k_1 = k_2 = 0$.

In this third case, the fundamental group $\Lambda$ is the fundamental group of a weakly special compact geodesically complete square complex $V$ containing a hyperplane $H'$, possibly consisting of two connected components, such that $V \setminus H'$ has two components whose fundamental groups are free. Furthermore, $\Lambda$ is one-ended and $H^2(\Lambda, \mathbb{Z}\Lambda)$ has infinite rank. We are left with finding a surface subgroup of a group $\Lambda$ with these properties.

Construct a graph of groups decomposition for $\Lambda$ as follows. Assume first that $H'$ is connected. Then $H'$ is a connected graph without leaves, and we can collapse a spanning tree in $H'$ so that the resulting graph is a rose $R$ with say $k \geq 1$ petals. Each petal corresponds to a generator $a_i$ $(i \leq k)$ of the fundamental group of $H'$. These generators then define a graph of groups decomposition for $\pi_1(H')$ with $k+1$ vertex groups and cyclic edge groups. There is one vertex group $\pi_1(H')$, and there are $k$ cyclic vertex groups corresponding to the infinite cyclic groups $\langle a_i \rangle$ generated by the elements $a_i$. The edge groups equal the groups $\langle a_i \rangle$ as well. The graph of groups decomposition corresponds to a graph of spaces as follows. Embed the rose $R$ into the plane and let $\hat{R}$ be a closed tubular neighborhood of $R$. This is a compact planar surface with $k+1$ boundary components. There are $k$ boundary components corresponding to the $k$ cyclic groups $\langle a_i \rangle$. An application of the Seifert van Kampen theorem to a small open neighborhood of the union of these components, chosen so that this neighborhood consists of $k$ annuli with disjoint closure, and the complement of the union of smaller compact annuli about these boundary components yields the desired decomposition.

In the same way, we can represent $\pi_1(H')$ as a graph of groups with $2k+1$ vertex groups and $2k$ cyclic edge groups, where for each $i$, two of these vertex groups and two of these edge groups equal the group $\langle a_i \rangle$. Namely, we can replace the planar surface $\hat{R}$ by its product with $[1,2]$ and choose $2k$ open neighborhoods of the boundary circles with fundamental groups the groups $\langle a_i \rangle$. This decomposition is once more a consequence of the Seifert van Kampen theorem.

For $i = 1, 2$ let $G_i$ be the fundamental group of a component $Y_i$ of $Z \setminus H'$. Construct a graph of groups decomposition for $G_i$ with $k+1$ vertex groups and cyclic edge groups corresponding to the groups $\langle a_i \rangle$ in the same way: thicken the boundary of $Y_i$ to obtain a compact planar surface $Q$ with $k+1$ oriented boundary components, where the fundamental groups of $k$ of these oriented boundary circles are the groups $\langle a_i \rangle$ generated by the elements $a_i$ and where the fundamental group of the remaining boundary component $q$ is generated by the product of these generators. Choose a generating set for the free group $G_i$ and a rose $R_i$ with fundamental group $G_i$ whose petals represent these generators. Gluing the boundary component $q$ of $Q$ onto $R_i$

according to the word representing $q$ in the chosen generating set then defines a cell complex $W_i$ with $k$ distinguished boundary components whose fundamental groups are the groups $\langle a_i \rangle$. This cell complex can be glued to $\hat{R} \times \{i\}$ $(i = 1, 2)$ in such a way that the boundary circles whose fundamental groups equal the group $\langle a_i \rangle$ are identified. The resulting space $W$ is a two-dimensional cell complex containing $2k$ distinguished embedded circles. Cutting $W$ open along these circles and using Seifert van Kampen shows that the fundamental group of this complex decomposes as a graph of free groups with cyclic edge groups. There are $3 + 2k$ vertex groups that are the groups $G_i$ and the cyclic groups $\langle a_i \rangle$, and there are $4k$ cyclic edge groups, so that for each $i$, there are two edge groups isomorphic to $\langle a_i \rangle$.

We have to verify that $\Lambda' = \Lambda$. To this end note that by the Seifert van Kampen theorem, applied to $Z \setminus H'$ and the open set $U(H')$, we have $\Lambda = G_1 *_{G_3} G_2$. But then an inspection of the graph of spaces constructed above also shows that this decomposition can be refined to yield the decomposition constructed in the previous paragraph.

If $H'$ is not connected, then we proceed in the same way. In this case we obtain a graph of groups decomposition for $\Lambda$ with two vertices connected by two edges. The vertex groups are the fundamental groups of the components of $V \setminus H'$, and the edge groups are the fundamental groups of the two components of $H'$. In precisely the same way as in the previous paragraph, we construct from the decomposition a graph of groups decomposition for $\Lambda$ with free vertex groups and cyclic edge groups.

To summarize, we found a decomposition of $\Lambda$ as a graph of free groups with cyclic edge groups. As $\Lambda$ is one-ended, Theorem A of [Wi18] shows that $\Lambda$ contains a quasi-convex surface subgroup and hence the same is true for $\Gamma$, as stated in the theorem. □

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