SPOTTED DISK AND SPHERE GRAPHS

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ABSTRACT. The disk graph of a handlebody H of genus $g \geq 2$ with $m \geq 0$ marked points on the boundary is the graph whose vertices are disks disjoint from the marked points and where two vertices are connected by an edge of length one if they can be realized disjointly. We show that for $m = 1$ the disk graph contains quasi-isometrically embedded copies of $\mathbb{R}^2$. For $m = 2$ the disk graph contains for every $n \geq 1$ a quasi-isometrically embedded copy of $\mathbb{R}^n$. The same holds true for sphere graphs of the doubled handlebody with one or two marked points, respectively.

1. INTRODUCTION

The *curve graph* $\mathcal{CG}$ of an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$ is the graph whose vertices are isotopy classes of essential (i.e. non-contractible and not homotopic into a puncture) simple closed curves on S. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a locally infinite hyperbolic geodesic metric space of infinite diameter [MM99].

A handlebody of genus $g \geq 1$ is a compact three-dimensional manifold H which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of g circles. Its boundary ∂H is an oriented surface of genus g. We allow that ∂H is equipped with $m \geq 0$ marked points (punctures) which we call *spots* in the sequel. The group $\mathrm{Map}(H)$ of all isotopy classes of orientation preserving homeomorphisms of H which fix each of the spots is called the *handlebody group* of H. The restriction of an element of $\mathrm{Map}(H)$ to the boundary ∂H defines an embedding of $\mathrm{Map}(H)$ into the mapping class group of ∂H, viewed as a surface with punctures [S77, Wa98].

An *essential disk* in H is a properly embedded disk $(D, \partial D) \subset (H, \partial H)$ whose boundary ∂D is an essential simple closed curve in ∂H disjoint from the marked points. An isotopy of such a disk is supposed to consist of disks disjoint from the marked points.

The *disk graph* $\mathcal{DG}$ of H is the graph whose vertices are isotopy classes of essential disks in H. Two such disks are connected by an edge of length one if and only if they can be realized disjointly.

A metric space X is said to have *asymptotic dimension* $\mathrm{asdim}(X) \leq n$ if for every $R > 0$ there exists a covering of X by uniformly bounded subsets of X so that any ball of radius R intersects at most $n + 1$ sets from the covering. The asymptotic dimension of a curve graph is finite [BF08].

In [MS13, H11, H16] the following is shown.

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Theorem 1. *The disk graph of a handlebody of genus $g \geq 2$ without spots is hyperbolic and has finite asymptotic dimension.*

The main goal of this work is to show that in contrast to the case of curve graphs, Theorem 1 is not true if we allow spots on the boundary.

Theorem 2. *Let H be a handlebody of genus $g \geq 2$ with $m \geq 1$ spots.*

(1) *For $m = 1$ the disk graph of H contains quasi-isometrically embedded copies of $\mathbb{R}^2$. In particular, it is not hyperbolic.*
(2) *For $m = 2$ the disk graph of H contains for every $n \geq 1$ a quasi-isometrically embedded copy of $\mathbb{R}^n$. In particular, it is not hyperbolic, and its asymptotic dimension is infinite.*

The proof of the second part of Theorem 2 uses $m = 2$ in an essential way. I do not know whether or not the asymptotic dimension of the disk graph of a handlebody with a single spot or with $m \geq 3$ spots is finite. In view of the results in [H16], it seems possible that finiteness holds true for all $m \neq 2$.

Theorem 2 implies that disk graphs can not be used effectively to obtain a geometric understanding of the handlebody group of a handlebody of genus $g \geq 3$ paralleling the program developed by Masur and Minsky for the mapping class group [MM00]. Note that the handlebody group of H is an exponentially distorted subgroup of the mapping class group of ∂H [HH12]. The analogue of the strategy of Masur and Minsky would consist in cutting a handlebody open along an embedded disk which yields a (perhaps disconnected) handlebody with two spots on the boundary and studying disk graphs in the cut open handlebody.

Theorem 2 has an analogue for geometric graphs related to the Outer automorphism group $\mathrm{Out}(F_g)$ of the free group on g generators. Namely, doubling the handlebody H yields a connected sum $M = \sharp_g S^2 \times S^1$ of g copies of $S^2 \times S^1$ with m marked points. A doubled disk is an embedded essential sphere in M. The *sphere graph* in M is the graph whose vertices are isotopy classes of essential spheres in M and where two such spheres are connected by an edge of length one if and only if they can be realized disjointly. As before, an isotopy of spheres is required to be disjoint from the marked points. The sphere graph of a doubled handlebody without marked points is hyperbolic [HM13b].

Paralleling the result in Theorem 2 we have

Theorem 3. *Let $g \geq 2$ and let M be a doubled handlebody of genus g with $m \geq 1$ marked points.*

(1) *If $m = 1$ and if g is even then the sphere graph of M contains quasi-isometrically embedded copies of $\mathbb{R}^2$. In particular, it is not hyperbolic.*
(2) *If $m = 2$ then the sphere graph of M contains for every $n \geq 1$ a quasi-isometrically embedded copy of $\mathbb{R}^n$. In particular, it is not hyperbolic, and its asymptotic dimension is infinite.*

As in the case of disk graphs, this indicates that sphere graphs can be not be used to obtain an effective geometric understanding of $\mathrm{Out}(F_g)$ following the program developed in [MM00]. Theorem 3 may be related to the fact that for $g \geq 3$, the Dehn function of $\mathrm{Out}(F_g)$ is exponential [BV12, HM13a].

The first example known to us of a geometric graph of infinite asymptotic dimension is due to Sabalka and Savchuk [SS14]. The vertices of this graph are isotopy classes of essential separating spheres in $\sharp_g S^2 \times S^1$. Two such spheres are connected by an edge of length one if and only if they can be realized disjointly. We use the main idea in [SS14] for the proof of the second part of Theorem 2 and of Theorem 3.

The argument in the proof of the first part of Theorem 3 uses the first part of Theorem 2 and a result in [HH15] which relates the sphere graph in a connected sum $\sharp_g S^2 \times S^1$ for g even to the arc graph of an oriented surface of genus $g/2$ with connected non-empty boundary. A corresponding result for odd g and a non-orientable surface with a single boundary component would yield the first part of Theorem 3 for odd $g \geq 3$, but at the moment, such a result is not available.

2. Once spotted handlebodies and doubled handlebodies

The goal of this section is to construct quasi-isometrically embedded copies of $\mathbb{R}^2$ in the disk graph of a handlebody with a single spot and in the sphere graph of a doubled handlebody of even genus with a single spot.

Thus let H be a handlebody of genus $g \geq 2$ with a single spot. Let H_0 be the handlebody obtained from H by removing the spot and let

$$\Phi : H \to H_0$$

be the spot removal map. The image under Φ of an essential diskbounding simple closed curve in ∂H is an essential diskbounding simple closed curve in ∂H_0.

The handlebody H_0 without spots can be realized as an I-bundle over a surface F with a single boundary component. If the surface F is orientable, then the genus g is even and the I-bundle is trivial. The genus of F equals $g/2$, and the boundary ∂F of F defines an isotopy class of a separating simple closed curve c on ∂H_0 which decomposes ∂H_0 into two surfaces of genus $g/2$, with a single boundary component. If the surface F is non-orientable, then the I-bundle is non-trivial and the boundary ∂F defines a non-separating simple closed curve c in ∂H_0.

Following [H11, H16], define an *I-bundle generator* for H_0 to be a simple closed curve $c \subset \partial H_0$ so that H_0 can be realized as an I-bundle over a compact surface F with connected boundary ∂F and such that c is freely homotopic to $\partial F \subset \partial H_0$. The surface F is called the *base* of the I-bundle. If the I-bundle generator c is separating, then F is oriented of genus $g/2$ where g is the genus of H_0. If c is non-separating, then the surface F is non-orientable, and the complement of an open annulus about c in ∂H_0 is the orientation cover of F. The I-bundle over every simple embedded arc in F with endpoints on ∂F is an essential disk in H_0 which intersects c in precisely two points (up to isotopy).

An I-bundle generator c in ∂H_0 is *diskbusting*, which means that it has an essential intersection with every disk (see [MS13, H11]). Namely, the base F of the I-bundle is a deformation retract of H_0. Thus if γ is any essential simple closed curve on ∂H_0 which does not intersect c then γ projects to an essential closed curve on F. Such a curve is not nullhomotopic in H_0 and hence it can not be diskbounding.

The *arc graph* $\mathcal{A}(X)$ of a compact surface X of genus $n \geq 1$ with connected boundary ∂X is the graph whose vertices are isotopy classes of embedded essential arcs in X with endpoints on the boundary and where two such arcs are connected by an edge of length one if and only if they can be realized disjointly. The arc graph $\mathcal{A}(X)$ of X is hyperbolic, however the inclusion of $\mathcal{A}(X)$ into the arc and curve graph of X is a quasi-isometry only if the genus of X equals one [MS13] (see also [H16]).

A *coarse L-Lipschitz retraction* of a metric space (X,d) onto a subspace Y is a coarse L-Lipschitz map $\Psi : X \to Y$ (this means that $d(\Psi(x),\Psi(y)) \leq Ld(x,y)+L$ for some $L \geq 1$ and all x,y) with the additional property that there exists a number $C>0$ with $d(\Psi(y),y) \leq C$ for all $y \in Y$. If X is a geodesic metric space then the image Y of a coarse Lipschitz retraction is a *coarsely quasi-convex* subspace of X, i.e. any two points in Y can be connected by a uniform quasi-geodesic in X which is entirely contained in Y.

For an I-bundle generator c in H_0 let $\mathcal{RD}(c)$ be the complete subgraph of the disk graph $\mathcal{DG}_0$ of H_0 consisting of disks which intersect c in precisely two points. The boundary of each such disk is an I-bundle over an arc in the base F of the I-bundle corresponding to c. As two such disks are disjoint if and only if the corresponding arcs in F are disjoint, the graph $\mathcal{RD}(c)$ is isometric to the arc graph $\mathcal{A}(F)$ of F.

Lemma 2.1. *There exists a coarse Lipschitz retraction $\Theta_0 : \mathcal{DG}_0 \to \mathcal{RD}(c)$ whose restriction to $\mathcal{RD}(c)$ is the identity.*

Proof. The case that c is a separating I-bundle generator is completely elementary. Namely, in this case the base F of the I-bundle can be identified with a component of $\partial H_0 - c$. As c is diskbusting, the map

$$\Upsilon_0 : \mathcal{DG}_0 \to \mathcal{A}(F)$$

which associates to a disk D a component of $\partial D \cap F$ is coarsely well defined: Although it depends on choices, any other choice Υ_0' maps a disk D to an arc disjoint from $\Upsilon_0(D)$. If we denote by $Q : \mathcal{A}(F) \to \mathcal{RD}(c)$ the map which associates to an arc α in F the I-bundle over α, then the disks $Q(\Upsilon_0(D)), Q(\Upsilon_0'(D))$ are disjoint as well.

Furthermore, if D, D' are disjoint disks then the arcs $\Upsilon_0(D), \Upsilon_0(D')$ are disjoint and hence $d_{\mathcal{DG}_0}(Q\Upsilon_0(D), Q\Upsilon_0(D')) \leq 1$. This shows that $Q \circ \Upsilon_0$ is coarsely one-Lipschitz. As a disk $D \in \mathcal{RD}(c)$ intersects F in a single arc, we have $Q\Upsilon_0(D) = D$. This completes the proof of the lemma in the case that c is separating.

For a nonseparating I-bundle generator c we argue as follows. By Theorem 10.1 of [MS13] (which can only be used indirectly as the "holes" are not precisely specified) and, more specifically, Corollary 4.6 of [H16], the graph $\mathcal{RD}(c)$ is a (uniformly) quasi-convex subgraph of the hyperbolic graph $\mathcal{DG}_0$. In particular, there is a coarsely well defined shortest distance projection $\Theta_0 : \mathcal{DG}_0 \to \mathcal{RD}(c)$, and such a shortest distance projection is a coarse Lipschitz retraction. □

Our goal is to use I-bundle generators in ∂H_0 to construct quasi-isometrically embedded euclidean planes in the disk graph of H. In analogy to [H11], we define an *I-bundle generator* for the spotted handlebody H to be a simple closed curve in ∂H whose image under the map Φ is an I-bundle generator in ∂H_0.

Let $(c_1, c_2) \subset \partial H$ be a pair of non-isotopic disjoint I-bundle generators so that $\partial H - \{c_1 \cup c_2\}$ has a connected component which is an annulus containing the spot in its interior. Then up to isotopy, $\Phi(c_1) = \Phi(c_2) = c$ for an I-bundle generator c in H_0.

The following construction is due to Kra; we refer to [KLS09] for details and for some applications. For its formulation, for a pair (c_1, c_2) of disjoint I-bundle generators on ∂H as in the previous paragraph let $\mathcal{RD}(c_1, c_2)$ be the complete subgraph of the disk graph $\mathcal{DG}$ of H whose vertex set consists of all disks which intersect each of the curves c_1, c_2 in precisely two points. Note that if $D \in \mathcal{RD}(c_1, c_2)$ then the image of D under the spot removing map Φ is contained in $\mathcal{RD}(c)$ where $c = \Phi(c_i)$.

In the next lemma we denote by abuse of notation the map $\mathcal{DG} \to \mathcal{DG}_0$ induced by the spot forgetful map Φ again by Φ. Furthermore, for the remainder of this section we represent a disk by its boundary, i.e. we view the disk graph as the complete subgraph of the curve graph of ∂H whose vertex set is the set of diskbounding curves.

Lemma 2.2. *There exists a simplicial embedding $\iota : \mathcal{DG}_0 \to \mathcal{DG}$ with the following properties.*

(1) *$\Phi \circ \iota$ is the identity.*
(2) *ι maps $\mathcal{RD}(c)$ into $\mathcal{RD}(c_1, c_2)$.*

Proof. Note first that there is a natural orientation reversing involution ρ_0 of ∂H_0 which exchanges the endpoints of the fibres of the interval bundle over the base F. This involution fixes c and preserves up to isotopy each diskbounding simple closed curve which intersects c in precisely two points.

Choose a hyperbolic metric g_0 on ∂H_0 which is invariant under ρ_0 and let $\hat{c}$ be the geodesic representative of c. Choose a point $p \in \hat{c}$ not contained in any diskbounding simple closed geodesic; this is possible since each diskbounding geodesic intersects $\hat{c}$ transversely in finitely many points and hence the set of all points of $\hat{c}$ contained in a diskbounding geodesic is countable. View p as a marked point on ∂H_0; then the geodesic representative of a diskbounding curve α is a diskbounding curve $\iota(\alpha)$ in $\partial H_0 - \{p\}$. Via identification of a disk with its boundary, this construction defines a simplicial embedding

$$\iota : \mathcal{DG}_0 \to \mathcal{DG}$$

with the property that $\Phi \circ \iota$ equals the identity. Furthermore, we clearly have $\iota(\mathcal{RD}(c)) \subset \mathcal{RD}(c_1, c_2)$. □

Let B be the connected component of $\partial H - \{c_1, c_2\}$ containing the spot (this is a once spotted annulus). Let Λ be a diffeomorphism of ∂H which preserves the complement of B (and hence the boundary of B) pointwise and which pushes the spot in B one full turn around a central loop in B. The isotopy class of Λ is contained in the kernel of the homomorphism $\mathrm{Mod}(\partial H) \to \mathrm{Mod}(\partial H_0)$ induced by the spot removal map Φ. The map Λ extends to a diffeomorphism of the handlebody H. This can be seen as in the case of point-pushing in a surface: Identify the image of B under the spot forgetful map Φ with a closed annulus A. Choose a neighborhood N of the punctured annulus B in H which is homeomorphic to $A \times [0,1]$, with one interior point removed from $A \times \{0\}$. Gradually undo the rotation of the marked point as one moves towards $A \times \{1\} \cup \partial A \times [0,1]$. Therefore the diffeomorphism Λ generates an infinite cyclic group of simplicial isometries of $\mathcal{RD}(c_1, c_2)$ which we denote again by Λ.

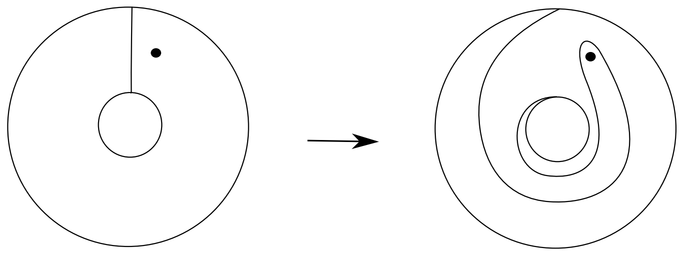
Let $\Theta_0 : \mathcal{DG}_0 \to \mathcal{RD}(c)$ be as in Lemma 2.1. Define

$$\Theta = \Theta_0 \circ \Phi : \mathcal{DG} \to \mathcal{RD}(c).$$

Observe that $\Theta(\iota(D)) = \Theta_0(D)$ for all disks $D \in \mathcal{DG}_0$. This then implies that $\Theta(\iota(D)) = D$ for all $D \in \mathcal{RD}(c)$. Furthermore, Θ is coarsely Lipschitz (compare the proof of Lemma 2.1 for a detailed explanation), and we may assume that

$$\Theta(\Lambda(D)) = \Theta(D)$$

Figure A



for all disks D.

Recall that $\mathcal{RD}(c)$ is isometric to the arc graph $\mathcal{A}(F)$ of F. Define a distance d_0 on $\mathcal{RD}(c)\times\mathbb{Z}$ by

$$d_0((\alpha,a),(\beta,b)) = d_{\mathcal{RD}(c)}(\alpha,\beta)+|a-b|$$

where $d_{\mathcal{RD}(c)}$ denotes the distance in $\mathcal{RD}(c)$. Let moreover

$$\Omega=\cup_k\Lambda^k\iota(\mathcal{RD}(c)).$$

Lemma 2.3. *The map $\Psi:\Omega\to\mathcal{RD}(c)\times\mathbb{Z}$ which maps $D\in\Lambda^k\iota(\mathcal{RD}(c))$ to $\Psi(D)=(\Theta(D),k)$ is a bijective quasi-isometry.*

Proof. Recall that $\Theta(D)=\Theta(\Lambda^k(D))$ for all disks D and all k and that furthermore the restriction of Θ to $\iota(\mathcal{RD}(c))$ is an isometry. In particular, if D_0,E_0 are distinct disks in $\mathcal{RD}(c)$ then $\Theta(\iota(D_0))\neq\Theta(\iota(E_0))$ and hence $\Psi(\iota(D_0))\neq\Psi(\Lambda^k(\iota(E_0)))$ for all k.

We claim that for every disk $D\in\Omega$ the following holds true.

(1) $D\neq\Lambda^k(D)$ for all $k\neq 0$.
(2) The disks D and $\Lambda(D)$ can be realized disjointly.
(3) Two disks $D\in\Lambda^k\iota(\mathcal{RD}(c)),E\in\Lambda^\ell\iota(\mathcal{RD}(c))$ are disjoint only if $|k-\ell|\leq 1$.

To this end let $D\in\Omega$ and for $k\in\mathbb{Z}$ let $D_k=\Lambda^k(D)$. Figure A shows that for $\ell\geq 1$, the disk $D_{k+\ell}$ has precisely $2\ell-2$ essential intersections with D_k, and these intersection points are up to isotopy contained in the annulus B. This yields part (2) of the above claim, and part (3) follows from the same argument. Furthermore, the twist parameter k can be recovered from the geometric intersection numbers between $\Lambda^k(D)$ and $\Lambda^{-1}(D),D,\Lambda(D)$. For example, if $k\geq 2$ then these intersection numbers equal $2k,2k-2,2k-4$, respectively, and if $k\leq -2$ then these intersection numbers are $-2k-4,-2k-2,-2k$. This establishes part (1) of the above claim.

Part (1) of the above claim together with the beginning of this proof yields that the map Ψ is well defined and a bijection. Moreover, as the restriction of the map Θ to $\mathcal{RD}(c_1,c_2)$ is one-Lipschitz, part (3) of the above claim implies that the map Ψ is two-Lipschitz.

As $\Lambda^k\iota(\mathcal{RD}(c))$ is isometric to $\mathcal{A}(F)$ for all k, the inverse of Ψ which associates to a pair $(D,k)\in\mathcal{RD}(c)\times\mathbb{Z}$ the disk $\Lambda^k(\iota(D))$ is coarsely one-Lipschitz. This shows that indeed, the map Ψ is a quasi-isometry. □

The following proposition is the main remaining step towards a proof of first part of Theorem 2.

Proposition 2.4. *There is a coarse Lipschitz retraction of $\mathcal{DG}\to\cup_k\Lambda^k\iota(\mathcal{RD}(c))=\Omega$. In particular, Ω is a coarsely quasi-convex subset of $\mathcal{DG}$.*

Proof. As in the proof of Lemma 2.2, let ρ_0 be an orientation reversing involution of ∂H_0 which fixes the I-bundle generator c pointwise. This involution determines an involution ρ of the complement in ∂H of the interior $\mathrm{int}(B)$ of the annulus B which exchanges the curves c_1 and c_2. Write as before $\Omega = \cup_k \Lambda^k \iota(\mathcal{RD}(c))$.

Choose a complete finite area hyperbolic metric on ∂H (so that the marked point becomes a puncture) with the property that the involution ρ of $\partial H - \mathrm{int}(B)$ is an isometry for this metric which maps the geodesic representative $\hat{c}_1$ of c_1 to the geodesic representative $\hat{c}_2$ of c_2. This metric restricts to a hyperbolic metric on the once punctured annulus B with geodesic boundary.

Choose a geodesic arc α connecting the two boundary components of B which is contained in the geodesic representative of one of the curves from $\iota(\mathcal{RD}(c))$. Cutting B open along α yields a once punctured rectangle with geodesic sides, where two distinguished sides come from the arc α. For any pair of points x_1, x_2 on the remaining two sides, choose a simple arc disjoint from α connecting these two points and let $\alpha(x_1, x_2) \subset B$ be the geodesic representative of this arc. By convexity, $\alpha(x_1, x_2)$ is disjoint from α if its endpoints are disjoint from the endpoints of α.

This construction yields for any pair of points $x_1 \in \hat{c}_1, x_2 \in \hat{c}_2$ an oriented geodesic arc $\alpha(x_1, x_2) \subset B$ with endpoints x_1, x_2 such that any two of these arcs connecting distinct pairs of points on $\hat{c}_1, \hat{c}_2$ intersect in at most two points. Furthermore, each of these arcs intersects a geodesic representative of a curve in $\iota(\mathcal{RD}(c))$ in at most two points.

We use these oriented arcs as follows. Let β be a diskbounding simple closed curve on ∂H. The intersection of β with $\partial H - \mathrm{int}(B)$ consists of a non-empty collection ζ of finitely many pairwise disjoint simple arcs with endpoints on $\hat{c}_1, \hat{c}_2$. Each such arc is freely homotopic relative to $\hat{c}_1, \hat{c}_2$ to a unique geodesic arc which meets $\hat{c}_1, \hat{c}_2$ orthogonally at its endpoints.

We claim that the components of the thus defined collection $\hat{\zeta}$ of geodesic arcs are pairwise disjoint. However, some of these arcs may have nontrivial multiplicities as $\beta \cap (\partial H - \mathrm{int}(B))$ may contain several components which are homotopic relative to the boundary. To verify the claim, double each component X of the hyperbolic surface $\partial H - \mathrm{int}(B)$ along its boundary. The possibly disconnected resulting closed hyperbolic surface S admits an isometric involution σ preserving the components of S whose fixed point set is precisely the image α of the boundary of $\partial H - \mathrm{int}(B)$ in the doubled manifold. The double of the above collection ζ of arcs is a collection of simple closed curves on S which are invariant under σ.

The free homotopy classes of these closed curves are σ-invariant and hence the same holds true for their geodesic representatives: Namely, if γ is the geodesic representative of such a free homotopy class, then γ intersects the geodesic multicurve α in precisely two points. Let γ_1 be the component of $\gamma - \alpha$ of smaller length. Then $\gamma_1 \cup \sigma(\gamma_1)$ is a simple closed curve freely homotopic to γ, and its length is at most the length of γ. But γ is the unique simple closed curve of minimal length in its free homotopy class and hence $\gamma = \gamma_1 \cup \sigma(\gamma_1)$. Thus γ intersects α orthogonally, and $\gamma \cap X$ is a component of the arc system $\hat{\zeta}$. The claim now follows from the well known fact that the geodesic representative of a simple closed multicurve on a hyperbolic surface is a simple closed multicurve.

As a consequence of the above discussion, the order of the endpoints of the components of $\beta - \mathrm{int}(B)$ on $\hat{c}_1 \cup \hat{c}_2$ coincides with the order of the endpoints of the collection of geodesic arcs $\hat{\zeta}$ which meet $\hat{c}_1 \cup \hat{c}_2$ orthogonally at their endpoints and are freely homotopic to the components of $\beta - \mathrm{int}(B)$. This implies that a diskbounding simple closed curve β on ∂H can be homotoped to a curve $\hat\beta$ of the following form. The restriction of $\hat\beta$ to $\partial H - \mathrm{int}(B)$ consists of a finite collection of pairwise disjoint geodesic arcs which meet $\hat c_i$ orthogonally at their endpoints. Some of these arcs may occur more than once. The restriction of $\hat\beta$ to the once punctured annulus B consists of a finite non-empty collection of arcs connecting $\hat c_1$ to $\hat c_2$ and perhaps a finite number of arcs which go around the puncture and return to the same boundary component of B. Distinct such arcs have disjoint interiors. The curve $\hat\beta$ is uniquely determined by β up to a homotopy with fixed endpoints of the components of $\hat\beta \cap B$. By construction of the map ι, if $\beta = \iota(\beta') \in \iota\mathcal{RD}(c)$ then $\hat\beta \cap \partial H - \mathrm{int}(B)$ is just the lift of the geodesic representative of β' to $\partial H - \mathrm{int}(B)$ using the obvious isometry between $\partial H - B$ and $\partial H_0 - c$. In particular, the intersections with B of the representatives $\hat\beta$ of the elements $\beta \in \iota\mathcal{RD}(c)$ are pairwise disjoint.

We use this normal form for diskbounding simple closed curves to define a map

$$\Xi : \mathcal{DG} \to \mathbb{Z}$$

as follows. Let $\hat\beta$ be a closed curve constructed from the simple closed diskbounding curve β as in the previous paragraph. Let b be one of the components of $\hat\beta \cap B$ with endpoints on $\hat c_1$ and $\hat c_2$, oriented in such a way that it connects $\hat c_1$ to $\hat c_2$. Let x_1, x_2 be the endpoints of b on $\hat c_1, \hat c_2$.

Let $a = \alpha(x_1, x_2)$; then b, a are simple arcs in B with the same endpoints which intersect a core curve of the annulus B in precisely one point. Assume that $\hat c_1, \hat c_2$ are oriented and define the boundary orientation of B. Then b is homotopic with fixed endpoints to the arc $\hat c_1^k \cdot a \cdot \hat c_2^\ell$ for unique $k, \ell \in \mathbb{Z}$ (read from left to right). In other words, if we denote by τ_i the positive Dehn twist about $\hat c_i$, viewed as a diffeomorphism of the punctured disk B with fixed boundary, then b is homotopic with fixed endpoints to the arc $\tau_1^k \tau_2^{-\ell} a$. Define $\Xi(\beta) = k$.

Observe that although this definition depends on the choice of the arcs $\alpha(x_1, x_2)$ and on the choice of the component b of $B \cap \hat\beta$, the map Ξ is coarsely well defined. Namely, let b' be a second component of $\hat\beta \cap B$, with endpoints x_1', x_2' on $\hat c_1, \hat c_2$ and distinct from b. Then the interior of b' is disjoint from the interior of b. In particular, if a' is an arc in B with the same endpoints as b' whose interior is disjoint from a, then b' is homotopic with fixed endpoints to $\tau_1^q \tau_2^{-r} a'$ for $|q - k| \leq 1, |r - \ell| \leq 1$. On the other hand, the arc $\alpha(x_1', x_2')$ intersects a at most twice and hence $a' = \tau_1^s \tau_2^{-u} \alpha(x_1', x_2')$ for some $|s| \leq 2, |u| \leq 2$. This shows that the multiplicity k' of the curve $\hat c_1$ in the description of b' relative to $\alpha(x_1', x_2')$ satisfies $|k - k'| \leq 3$. The same reasoning yields that the map Ξ is coarsely three-Lipschitz. Furthermore, we have $\Xi(\iota(\mathcal{RD}(c))) \subset [-2, 2]$.

To summarize, the map

$$(\Theta, \Xi) : \mathcal{DG} \to \mathcal{RD}(c) \times \mathbb{Z}$$

is coarsely Lipschitz, and its composition with the inverse of the map Ψ from Lemma 2.3 is a coarse Lipschitz retraction of $\mathcal{DG}$ onto Ω provided that the map Ξ maps a point in $\Lambda^k \iota(\mathcal{RD}(c))$ into a uniformly bounded neighborhood of k.

However, if $\beta_0 \in \iota\mathcal{RD}(c)$ and if $\beta = \Lambda^k(\beta_0) \in \Lambda^k \iota(\mathcal{RD}(c))$, then the intersections with $H - \mathrm{int}(B)$ of the representatives $\hat\beta, \hat\beta_0$ of β, β_0 constructed above coincide. This implies that up to homotopy with fixed endpoints, $\hat\beta \cap B = \Lambda^k(\hat\beta_0 \cap B)$.

On the other hand, point-pushing along a simple closed curve γ based at p descends to conjugation by γ in $\pi_1(\partial H_0, p)$. Therefore the image under the map Λ of a simple arc b in B with endpoints on the two distinct components of ∂B is homotopic with fixed endpoints to $c_1 b c_2$ (recall that we oriented c_1, c_2 so that they define the boundary orientation of B). As $\Xi(\iota(\mathcal{RD}(c))) \subset [-2, 2]$, it follows that $|\Xi(\beta) - k| \leq 2$. This shows the proposition. □

To summarize, we obtain

Corollary 2.5. *The disk graph of a handlebody H of genus $g \geq 2$ with one spot contains quasi-isometrically embedded copies of* $\mathbb{R}^2$.

Remark 2.6. in [H11] we showed that in contrast to handlebodies without spots, the disk graph of a handlebody H with a single spot on the boundary is *not* a quasi-convex subgraph of the curve graph of ∂H_0.

In the remainder of this section we explain how the above construction can be used to show the first part of Theorem 3.

Namely, consider the double $M_0 = \sharp_g S^2 \times S^1$ of a handlebody H_0 of genus $g \geq 2$ without spots. Let M be the manifold M_0 equipped with a marked point p. As before, we call p a spot in M. There is a natural spot removing map $\Phi : M \to M_0$.

Let $\mathcal{SG}$ be the *sphere graph* of M whose vertices are isotopy classes of embedded spheres in M which are disjoint from the spot. Isotopies are required to be disjoint from the spot as well. Two such spheres are connected by an edge of length one if they can be realized disjointly. Similarly, let $\mathcal{SG}_0$ be the sphere graph of M_0.

Assume from now on that $g = 2n$ for some $n \geq 1$. Choose an embedded oriented surface $F_0 \subset M_0$ of genus n with connected boundary such that the inclusion $F_0 \to M_0$ induces an isomorphism $\pi_1(F_0) \to \pi_1(M_0)$. We may assume that the oriented I-bundle H_0 over F_0 is an embedded handlebody $H_0 \subset M_0$ whose double equals M_0. Thus every embedded essential arc α in F_0 with boundary in ∂F_0 determines a sphere $\Upsilon_0(\alpha)$ in M_0 as follows. The interval bundle over α is an embedded essential disk in H_0, with boundary in ∂H_0, and we let $\Upsilon_0(\alpha)$ be the double of this disk. By construction, the sphere $\Upsilon_0(\alpha)$ intersects the surface F_0 precisely in the arc α. By Lemma 4.17 of [HH15], distinct arcs give rise to non-isotopic spheres, furthermore the map Υ_0 preserves disjointness and hence Υ_0 is a simplicial embedding of the arc graph $\mathcal{A}(F_0)$ of F_0 into the sphere graph $\mathcal{SG}_0$ of M_0.

Now mark a point p on the boundary ∂F_0 of F_0 and view the resulting spotted surface F as a surface in the spotted manifold M. The *arc graph* $\mathcal{A}(F)$ of F is the graph whose vertices are isotopy classes of essential arcs in F with endpoints on the complement of p in the boundary of F. Here we exclude arcs which are homotopic with fixed endpoints to a subarc of ∂F containing the base point p, and we require that an isotopy preserves the marked point p and hence endpoints of arcs can only slide along $\partial F - \{p\}$. Two such arcs are connected by an edge if they can be realized disjointly. Associate to an arc α in F the double $\Upsilon(\alpha)$ of the I-bundle over α.

The spot forgetful map $\Phi : M \to M_0$ induces a simplicial surjection $\mathcal{SG} \to \mathcal{SG}_0$, again denoted by Φ for simplicity. Similarly, if we let $\varphi : F \to F_0$ be the map which forgets the marked point $p \in \partial F$, then φ induces a simplicial surjection

$\mathcal{A}(F) \to \mathcal{A}(F_0)$, denoted as well by φ. We then obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{A}(F) & \xrightarrow{\varphi} & \mathcal{A}(F_0) \\ \big\downarrow \Upsilon & & \big\downarrow \Upsilon_0 \\ \mathcal{SG} & \xrightarrow{\Phi} & \mathcal{SG}_0 \end{array} \tag{1}$$

Similar to the case of the handlebody M_0 without spots and the map Υ_0, we obtain

Lemma 2.7. *The map Υ is a simplicial embedding of the arc graph $\mathcal{A}(F)$ onto a subgraph of the sphere graph*

Proof. We have to show that the map Υ is injective. As Υ_0 is injective and as the diagram (1) commutes, it suffices to show the following. Let $\alpha \neq \beta \in \mathcal{A}(F)$ be such that $\varphi(\alpha) = \varphi(\beta)$; then $\Upsilon(\alpha) \neq \Upsilon(\beta)$.

Now $\varphi(\alpha) = \varphi(\beta)$ means that up to exchanging α and β, there exists a number $k > 0$ such that β can be obtained from α by k half Dehn twists about the boundary ∂F of F. Here the half Dehn twist $T(\alpha)$ of α is defined as follows.

The orientation of F induces a boundary orientation for ∂F which in turn induces an orientation on $\partial F - \{p\}$. With respect to the order defined by this orientation, let x be the bigger of the two endpoints x, y of α. Slide x across p to obtain a new arc $T(\alpha)$, with endpoints x', y. This arc is not homotopic to α. To see this it suffices to show that the double $DT(\alpha)$ of $T(\alpha)$ in the double DF of F (which is a surface with one puncture) is not homotopic to the double $D(\alpha)$ of α. This follows since the homotopy class of $DT(\alpha)$ with respect to the basepoint $y \in F \subset DF$ can be written as uzv where u, z, v are based loops at y, uw is the homotopy class of $D(\alpha)$ and z is a loop encircling the puncture.

The same reasoning also shows that the sphere $\Upsilon(T(\alpha))$ is not homotopic to the sphere $\Upsilon(\alpha)$. Namely, let $\chi \subset \partial F \cup \{p\}$ be the oriented embedded arc connecting the intersection point x of α with ∂F to the point x'. Recall that this arc contains p in its interior. Then the sphere $\Upsilon(T(\alpha))$ is a connected sum of the sphere $\Upsilon(\alpha)$ with the boundary S of a punctured ball which is a thickening of χ. The sphere S can be constructed in such a way that it intersects the sphere $\Upsilon(\alpha)$ in a disk neighborhood of x, and it is not contractible as it encloses the puncture. Consequently the homotopy class of $\Upsilon(T(\alpha))$ with basepoint x equals the sum (at x) of the homotopy classes of $\Upsilon(\alpha)$ and S and $\Upsilon(T(\alpha))$ is not homotopic to $\Upsilon(\alpha)$.

The above construction, applied to the sphere $\Upsilon(T(\alpha))$ instead of the sphere $\Upsilon(\alpha)$ and where the point y takes on the role of the point x in the above discussion, shows that $\Upsilon(T^2(\alpha))$ is obtained from $\Upsilon(\alpha)$ by point-pushing along the oriented loop ∂F with basepoint p. This is a diffeomorphism of M which leaves the complement of a small tubular neighborhood of ∂F pointwise fixed and pushes the basepoint p along ∂F. As in the proof of Lemma 2.3, this argument can be iterated. It shows that the sphere $\Upsilon(T^k(\alpha))$ intersects the sphere $\Upsilon(\alpha)$ in $k-1$ essential intersection circles. Revoking the proof of Lemma 2.3, we conclude that indeed, for $k \neq \ell$, $\Upsilon(T^k(\alpha))$ is not homotopic to $\Upsilon(T^\ell(\alpha))$.

We showed so far that the map Υ is injective. To complete the proof of the lemma, it suffices to observe that disjoint arcs are mapped to disjoint spheres. But this is immediate from the construction. □

Proposition 4.18 of [HH15] shows that there is a one-Lipschitz retraction

$$\Psi_0 : \mathcal{SG}_0 \to \Upsilon_0(\mathcal{A}(F_0))$$

which is of the form $\Psi_0 = \Upsilon_0 \circ \Theta_0$ (read from right to left) where $\Theta_0 : \mathcal{SG}_0 \to \mathcal{A}(F_0)$ is a one-Lipschitz map. In particular, $\Upsilon_0(\mathcal{A}(F_0))$ is a quasi-isometrically embedded subgraph of $\mathcal{SG}_0$ which is quasi-isometric to $\mathcal{A}(F_0)$. Our goal is to show that there also is a coarse Lipschitz retraction of $\mathcal{SG}$ onto $\Upsilon(\mathcal{A}(F))$ of the form $\Psi = \Theta \circ \Upsilon$ where $\Theta : \mathcal{SG} \to \mathcal{A}(F)$ is a coarse Lipschitz map. This then yields the first part of Theorem 3 from the introduction.

To construct the map Θ we use the method from [HH15]. We next explain how this method can be adapted to our needs.

Let as before $F \subset M$ be an embedded oriented surface with connected boundary ∂F so that M is the double of the trivial I-bundle over F. We assume that the marked point p is contained in the boundary ∂F of F. Furthermore, we assume that the boundary ∂F of F is a smoothly embedded circle in $M \cup \{p\}$ (i.e. an embedded compact one-dimensional submanifold). As before, we use the marked point p as the basepoint for the fundamental group of M. Then ∂F equipped with its boundary orientation defines a homotopy class $\beta \in \pi_1(M, p) = F_{2g}$. As β is not contained in any free factor, ∂F intersects every sphere in M. Namely, for any given sphere S in M, the subgroup of $\pi_1(M, p)$ of all homotopy classes of loops which do not intersect S is a proper free factor of $\pi_1(M, p)$.

As in [HH15] and similar to the construction in Lemma 2.1, the strategy is to associate to a sphere S in M a component of the intersection $F \cap S$. However, unlike in the case of curves on surfaces, there is no suitable normal form for intersections of spheres with the surface F, and the main work in [HH15] consists in overcoming this difficulty by introducing a relative normal form which allows to associate to a sphere in M_0 an intersection arc with F_0 so that the resulting map $\mathcal{SG}_0 \to \mathcal{A}(F_0)$ is one-Lipschitz.

For the remainder of this section we outline the main steps in this construction, adapted to the sphere graph $\mathcal{SG}$ of M and the arc graph $\mathcal{A}(F)$ of F. This requires modifying spheres with isotopies not crossing through p, and modifying the surface F with homotopies leaving the boundary ∂F pointwise fixed.

For convenience, we record some definitions from [HH15] (the following combines Definition 4.7 and Definition 4.9 of [HH15]).

Definition 2.8. Let Σ be a sphere or a sphere system.

(1) *∂F intersects Σ minimally* if ∂F intersects Σ transversely and if no component of $\partial F - \Sigma$ not containing the basepoint p is homotopic with fixed endpoints into Σ.
(2) *F is in minimal position with respect to* Σ if ∂F intersects Σ minimally and if moreover each component of $\Sigma \cap F$ is a properly embedded arc which either is essential or homotopic with fixed endpoints to a subarc of ∂F containing the marked point.

A version of the easy Lemma 4.6 of [HH15] states that any closed curve containing the basepoint can be put into minimal position relative to a sphere system Σ as defined in the first part of Definition 2.8. The following is a version of Lemma 4.12 of [HH15]. For its formulation, call a sphere system Σ *simple* if it decomposes M into a collection of balls or balls with one puncture.

Lemma 2.9. *Let Σ be a simple sphere system in M. Suppose that F is in minimal position with respect to Σ. Let σ' be an embedded sphere disjoint from Σ and let Σ' be a simple sphere system obtained from Σ by either adding σ', or removing one sphere $\sigma \in \Sigma$. Then F can be homotoped leaving p fixed to a surface F' which is in minimal position with respect to Σ'.*

Proof. As in the proof of Lemma 4.15 of [HH15], removing a sphere preserves minimal position, so only the case of adding a sphere has to be considered.

Thus let Σ be a simple sphere system and let σ' be a sphere disjoint from Σ. Assume that F is in minimal position with respect to Σ. Let W_Σ be the complement of Σ in M, that is, W_Σ is a compact (possibly disconnected) manifold whose boundary consists of $2k$ boundary spheres $\sigma_1^+, \sigma_1^-, \cdots, \sigma_k^+, \sigma_k^-$. The boundary spheres σ_i^+ and σ_i^- correspond to the two sides of a sphere $\sigma_i \in \Sigma$. The surface F intersects W_Σ in a collection of embedded surfaces with boundaries. Each such surface is a polygonal disk P_i $(i = 1, \dots, m)$. The sides of each such polygon alternate between subarcs of ∂F and arcs contained in Σ. There is at most one bigon, that is, a polygon with two sides, and this polygon then contains the point p in one of its sides. Each rectangle, if any, is homotopic into ∂F.

The proof of Lemma 4.15 of [HH15] now proceeds by studying the intersection of each polygonal component of $F - \Sigma$ with the sphere σ'. This is done by contracting each such polygonal component P to a ribbon tree $T(P)$ in such a way that the boundary components in Σ are contracted to single points in $T(P)$. If P is not a rectangle or bigon, then $T(P)$ has a single vertex which is not univalent. As such ribbon trees are one-dimensional objects, they can be homotoped with fixed endpoints on ∂W_Σ to tres which are in minimal position with respect to σ'. This construction applies without change to rectangles and perhaps the bigon which we can be represented by an interval with one endpoint at p and the second endpoint on a component of Σ. We refer to the proof of Lemma 4.15 of [HH15] for details. No adjustment of the argument is necessary. □

The above construction is only valid for simple sphere systems Σ and not for individual spheres. Furthermore, it is unclear whether the arc system on $F \cap \Sigma$ obtained by putting F into minimal posiiton with respect to Σ is uniquely determined by Σ. To overcome this difficulty, the work [HH15] uses as an auxiliary datum a maximal system A_0 of pairwise disjoint essential arcs on the surface F_0. Here maximal means that any arc which is disjoint from A_0 is contained in A_0. The system A_0 then *binds* F_0, that is, $F - A_0$ is a union of topological disks. Furthermore, ∂F_0 and each arc $\alpha \in A_0$ is equipped with an orientation.

Choose an arc system A for F which binds F. If $F \subset M$ is in minimal position with respect to Σ, then a homotopy assures that no arcs from the arc system A intersects a component of $F - \Sigma$ which is a rectangle or a bigon. Then Lemma 4.15 of [HH15] and its proof applies without modification and shows that with a homotopy, F can be put into normal form with respect to the arc system A, called *A-tight minimal position*. This then yields the statement of Lemma 4.16 of [HH15]: if F is in A-tight minimal position, then for every simple sphere system Σ, the binding arc system $\Sigma \cap F$ is determined by Σ. In particular, two distinct spheres from Σ intersect F in disjoint essential arcs. There may in addtion be inessential arcs, i.e. arcs which are homotopic with fixed endpoints to a subsegment of ∂F containing the basepoint p, but these will be unimportant for our purpose.

Now let σ be an essential sphere in M. Let Σ be a simple sphere system in M containing σ as a component. We put F into A-tight minimal position with respect to Σ. The $\sigma \cap F$ consists of a non-empty collection of essential arcs and perhaps some additional non-essential arcs. Choose one of the essential intersection arcs α and define $\Theta(\sigma) = \alpha$. As in [HH15] and Proposition 2.4 we now obtain

Proposition 2.10. *The map Θ is a coarse Lipschitz map. For each arc $\alpha \in \mathcal{A}(F)$, we have $\Theta(\Upsilon(\alpha)) = \alpha$. As a consequence, if $g = 2n$ is even then the sphere graph $\mathcal{SG}$ of M contains quasi-isometrically embedded copies of $\mathbb{R}^2$.*

Proof. Given the above discussion, the proof that Θ is a Lipschitz map is identical to the proof that the map Θ_0 is a Lipschitz map in [HH15] and will be omitted. Moreover, as for $\alpha \in \mathcal{A}(F)$, the sphere $\Upsilon(\alpha)$ intersects F in the unique arc α, we have $\Theta(\Upsilon(\alpha)) = \alpha$.

As a consequence, $\Theta|\Upsilon(\mathcal{A}(F))$ is a Lipschitz bijection, with inverse Υ. Then the subgraph $\Upsilon(\mathcal{A}(F))$ of $\mathcal{SG}$ is bilipschitz equivalent to $\mathcal{A}(F)$. Furthermore, the map $\Upsilon \circ \Theta$ is a Lipschitz retraction of $\mathcal{SG}$ onto $\Upsilon(\mathcal{A}(F))$. Then $\Upsilon(\mathcal{A}(F))$ is a quasi-isometrically embedded subgraph of $\mathcal{SG}$ which is moreover quasi-isometric to $\mathcal{A}(F)$.

Let as before F_0 be the surface obtained from F by removing the spot. We are left with showing that $\mathcal{A}(F)$ is quasi-isometric to $\mathcal{A}(F_0) \times \mathbb{Z}$. However, this was shown in Lemma 2.3. Namely, in the terminology used before, the boundary ∂F is an I-bundle generator in the trivial interval bundle H over F, and associating to an arce α the I-bundle over α defines an isomophism of $\mathcal{A}(F)$ with the subgraph Ω of the disk graph of H used in Lemma 2.3. The statement now follows from Lemma 2.3. ∎

Remark 2.11. Most likely Proposition 2.10 holds true as well in the case that $g = 2n + 1$ is odd, and furthermore this can be deduced with the above argument using non-oriented surfaces. However, the analogue of Proposition 4.18 of [HH15] for non-orientable surfaces is not available, and we leave the verification of these claims to other authors.

3. Handlebodies and doubled handlebodies with two spots

The goal of this section is to show the second part of Theorem 2 and Theorem 3 from the introduction.

We begin with discussing briefly handlebodies of genus one. A handlebody of genus one with at most one spot on the boundary contains a single disk. This is used to establish

Proposition 3.1. *The disk graph of a solid torus with two spots on the boundary is a tree.*

Proof. Let H be a solid torus with two spots p_1, p_2 on the boundary. The handlebody H_1 obtained from H by removing the spot p_2 is a solid torus with one spot on the boundary. Let $\Phi_1 : \partial H \to \partial H_1$ be the natural spot removal map.

The handlebody H_1 contains a single disk D_1, and this disk is non-separating. If $D \subset H$ is any non-separating disk then $\Phi_1(\partial D) = \partial D_1$. Thus by Theorem 7.1 of [KLS09], the complete subgraph of the disk graph of H whose vertex set is the set of non-separating disks in H is a tree T. This is the Bass-Serre tree for the splitting of $\pi_1(H_1, p_2)$ defined by D_1. Equivalently, it is the tree dual to the curve ∂D_1 with its action of $\pi_1(H_1, p_2)$.

If $D \subset H$ is a separating disk then ∂D decomposes ∂H into a disk with two spots and a torus with the interior of a closed disk removed. In particular, $\Phi_1(\partial D)$ is peripheral. There is a single disk in H which is disjoint from D, and this disk is non-separating. Thus there is a single edge in $\mathcal{DG}$ with one endpoint at D. The second endpoint is a vertex in the tree T.

Now if D is any non-separating disk then cutting H open along D yields a ball with four spots on the boundary. Two of these spots are the two copies of D. Any simple closed curve which separates these two distinguished spots from the remaining two spots is the boundary of a separating disk in H disjoint from D, and any disk disjoint from D arises in this way. There are countably many such disks.

As a consequence, the disk graph of H is an extension of the tree T which attaches to each vertex in T a countable collection of edges whose second endpoints are univalent. Thus this graph is a tree as well. The proposition follows. □

In most of the remainder of this paper we investigate a handlebody H of genus $g \geq 2$ with two spots p_1, p_2 on the boundary. Let H_0 be the handlebody of genus g without spots and let $\Phi : H \to H_0$ be the spot removing map. A disk D in H *encloses the two spots* p_1, p_2 if $\Phi(D) \subset H_0$ is trivial.

We next use the two spots to add a handle to H. The resulting manifold is a handlebody H' of genus $g + 1$ with one spot. To this end slightly enlarge the two spots p_1, p_2 to two small compact disjoint disks B_1, B_2 in ∂H with $p_i \in \partial B_i$. Identifying these two disks with an orientation reversing diffeomorphism $B_1 \to B_2$ which maps p_1 to p_2 yields a handlebody H' of genus $g + 1$. We may view the common image of the points p_1, p_2 as a spot $p \in \partial H'$. The fundamental group of H' is the free group F_{g+1} with $g+1$ generators. We choose the spot p of H' as the basepoint for the fundamental group of H'.

The following simple observation will be used several times later on.

Lemma 3.2. *A disk D in H which encloses the two spots determines a free splitting $\pi_1(H', p) = F_{g+1} = F_g * \mathbb{Z}$.*

Proof. Up to isotopy, we may assume that the disk D is disjoint from the two closed disks B_1 and B_2 used in the construction of H'. Thus D determines a separating disk D' in H' which only depends on D. This disk cuts H' into a handlebody of genus g with fundamental group F_g and a solid torus T with fundamental group $\mathbb{Z}$ which contains the basepoint p.

Van Kampen's theorem now shows that D' defines a free splitting

$$\pi_1(H', p) = F_{g+1} = F_g * \mathbb{Z}.$$

unique up to conjugation with an element of the free factor $\mathbb{Z}$. Namely, the basepoint p is contained in the solid torus T. Thus the splitting of $\pi_1(H', p)$ obtained by van Kampen's theorem is determined by D' up to conjugation with an element of T.

To see that we obtain in fact a uniquely determined splitting, it suffices to observe that the solid torus T is obtained by identifying two disks in the boundary of a ball. As the ball is contractible, any two ways to move the basepoint p_2 to the disk give rise to the same splitting and hence there is no ambiguity in the construction (in other words, the fundamental group of the solid torus T appears only after the gluing). □

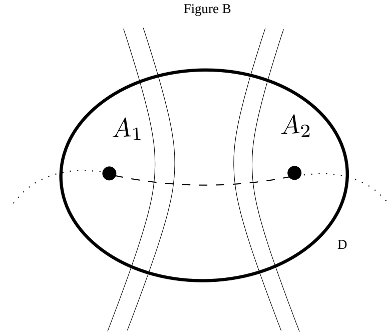
The construction in Lemma 3.2 can be reversed. Namely, observe that the handlebody H' contains a distinguished non-separating disk V which is the image of the two disks in ∂H used in the construction. If $E \subset H'$ is any separating disk disjoint from V which decomposes H' into a solid torus $T \supset V$ and a handlebody of genus g, then E is the image of a disk in H enclosing the two spots under the glueing construction.

We next investigate the dependence of this splitting on the choice of the disk D enclosing the two spots. To this end note first that in the splitting $F_{g+1} = F_g * \mathbb{Z}$, a generator a of the free factor $\mathbb{Z}$ is the image in H' of an embedded arc α_0 in ∂H which is disjoint from D, whose interior is disjoint from $B_1 \cup B_2$ and which connects the spot p_1 to p_2.

Now let $E \subset H$ be another disk which encloses the two spots p_1, p_2. Assume that E is not freely homotopic to D and in minimal position with respect to D. This means in particular that the boundaries $\partial D, \partial E$ intersect in the minimal number of points among all representatives in their isotopy classes.

The simple closed curves $\partial D, \partial E$ are the boundaries of unique disks $\tilde D, \tilde E \subset \partial H$ containing the two spots (thus if we think of the spots as missing points, then $\tilde D, \tilde E$ should be viewed as twice punctured disks). The intersection $\tilde D \cap \tilde E$ consists of two disjoint disks A_1, A_2 with one spot at p_1, p_2, respectively, and a disjoint union of rectangles. In particular, $\partial D \cap \partial E$ consists of at least four points.

Figure B



Use the ordered pair of disks $(\tilde D, \tilde E)$ to construct a loop $\gamma \subset (\partial H \cup \{p_2\}, p_2)$ based at p_2 as follows. First, connect the point p_2 to the point p_1 by an arc α whose interior is embedded in $\tilde D$. The endpoints of α are the two spots of H, and they are precisely the spots of the disk $\tilde E$. Thus there is an embedded arc β in $\tilde E$ connecting p_1 to p_2. The loop γ is homotopic to the concatenation of α with β (which we move off the spot p_1 with a small deformation). Note that the inverse of the loop γ is constructed with exactly the same procedure, but with the roles of the disks D, E exchanged. Furthermore, the homotopy class of γ as a loop in the handlebody H based at p_2 is uniquely determined by the ordered pair (D, E). However, the homotopy class of γ in $\partial H \cup \{p_2\}$ depends on choices and is not well defined.

To summarize, each ordered pair (D, E) of disks in H enclosing the two spots determines uniquely a homotopy class $q(D, E)$ of a loop in H based at p_2. We have $q(E, D) = q(D, E)^{-1}$ for all D, E, in particular, $q(D, D)$ is neutral element. Moreover, the disk E is the image of the disk D under the element of the handlebody group of H induced by pushing the point p_2 along the loop γ. Note that this is true in spite of the fact that the loop γ depends on a choice caused by the need to avoid the marked point p_1. Any of these choices gives rise to the same disk. This reflects the fact that the pure mapping class group of a disk with two punctures is infinite cyclic, and it is generated by the Dehn twist about the boundary circle.

Now recall from Lemma 3.2 that each disk D in H enclosing the two spots defines a free splitting $F_{g+1} = F_g * \mathbb{Z}$. If we fix the splitting defined by D, then the splitting defined by the disk E can be described as follows.

The generator of the free factor $\mathbb{Z}$ of F_{g+1} defined by D is given by the homotopy class a with fixed endpoints of an embedded arc in $\tilde{D} \subset \partial H$ which connects p_1 to p_2. The generator of the free factor $\mathbb{Z}$ in the free splitting defined by the disk E is determined by the homotopy class

$$a \cdot q(D,E)$$

(read from left to right).

There is another way to describe this generator. Namely, we can use the ordered pair $(\tilde{D}, \tilde{E})$ to construct a loop $\gamma' \subset (\partial H \cup \{p_1\}, p_1)$ based at p_1 as follows. Connect the point p_1 to the point p_2 by the arc α^{-1} which is the inverse of the arc α. Its interior is contained in $\tilde{D}$. Compose α^{-1} with the inverse β^{-1} of the arc β contained in the interior of $\tilde{E}$ which connects p_2 to p_1. Define γ' to be the concatenation of α^{-1} with β^{-1}. If we denote by $p(E,D)$ the homotopy class of γ', viewed as a loop in H based at p_1, then the generator of the free factor $\mathbb{Z}$ in the free splitting defined by the disk E is determined by the homotopy class

$$p(E,D) \cdot a$$

(read from left to right). Equivalently, the disk E is the image of D under point-pushing of p_1 along the loop γ'. As before, we have $p(V,E) = p(E,V)^{-1}$.

The homotopy class of γ' can be determined as follows. The disk $\tilde{D}$ determines an isomorphism of $\pi_1(H,p_2)$ with $\pi_1(H,p_1)$ by associating to a loop ζ based at p_2 the loop $\alpha^{-1} \circ \zeta \circ \alpha$ where as before, α is an arc connecting p_2 to p_1 which is embedded in $\tilde{D}$ (read from left to right). The homotopy class of γ' is the image of the homotopy class of γ^{-1} under this isomorphism.

Denoting now by $[a \cdot q(D,E)]$ the disk obtained from D by point-pushing p_2 along the loop $q(D,E)$ based at p_2, and by $[p(E,D) \cdot a]$ the disk obtained from D by point-pushing p_1 along the loop $p(E,D)$ based at p_1, the above discussion can be summarized by the formula

$$[a \cdot \zeta] = [\zeta^{-1} \cdot a]. \tag{2}$$

Remark 3.3. If we fix the disk D then the above discussion shows that every disk E in ∂H which encloses the two spots p_1, p_2 determines an element of the fundamental group $\pi_1(H,p_2)$ of H in such a way that D defines the trivial element.

However, the thus defined map from the set of disks E enclosing the two spots into $\pi_1(H,p_2)$ is not injective. A simple example of a disk E not isotopic to D which however is mapped to the trivial element in the fundamental group of H is a disk E with the following property. $E - D$ is a rectangle which is a thickening of an arc γ with endpoints on ∂D and the property that the union of γ with one of the arcs $\partial D - \gamma$ is essential in ∂H but contractible in H.

From now on we fix a disk D enclosing the two spots in H. This disk defines a free splitting $F_{g+1} = F_g * \mathbb{Z}$ where the free factor $\mathbb{Z}$ is generated by an element

a obtained from an embedded arc in the twice marked disk $\tilde{D}$ in ∂H with the same boundary as D which connects p_2 to p_1. The free factor F_g in the free splitting $F_{g+1} = F_g * \mathbb{Z}$ is naturally isomorphic to $\pi_1(H, p_2)$. Thus a free basis $\mathcal{A} = \{a_1, \dots, a_g\}$ of $F_g = \pi_1(H, p_2)$ extends to a free basis $\hat{\mathcal{A}} = \{a_1, \dots, a_g, a\}$ of F_{g+1}.

We now use a device from [SS14]. Define the *Whitehead graph* $\Gamma_{\mathcal{A}}(x)$ of a word x in a free basis $\mathcal{A} \cup \mathcal{A}^{-1}$ of F_g as follows. The set of vertices of $\Gamma_{\mathcal{A}}(x)$ is identified with the set $\mathcal{A} \cup \mathcal{A}^{-1}$. Each pair of consecutive letters $a_i a_j$ in the unique reduced word representing x contributes one edge from the vertex a_i to the vertex a_j^{-1}. Thus if the reduced length of x equals n then $\Gamma_{\mathcal{A}}(x)$ has $n-1$ edges.

Following [SS14], define the *simple* $g+1$*-length*

$$|w|_{g+1}^{simple}$$

of a word w in the free basis $\mathcal{A} = \{a_1, \dots, a_g\}$ of F_g to be the greatest number t such that w is of the form $w_1 w_2 \cdots w_t$ where the Whitehead graph of w_j with respect to the basis $\mathcal{A}$ has no cut vertex for each $j = 1, \dots, t$. If the Whitehead graph of w has a cut vertex then the simple $g+1$-length of w is defined to be zero. The terminology here is taken from [SS14] although it is not well adapted to the situation at hand.

The next lemma relates simple $g+1$-length to the disk graph $\mathcal{DG}$ of H. To simplify the notation, in the sequel we call a sequence (D_i) of disks in H a *path* in $\mathcal{DG}$ if for all i the disk D_i is disjoint from D_{i+1}. Thus such a sequence is the set of integral points on a simplicial path in $\mathcal{DG}$ connecting its endpoints.

Lemma 3.4. *Let $(D_i)_{0 \le i \le n}$ be a path in $\mathcal{DG}$ which begins and ends with a disk enclosing the two spots p_1, p_2. Let $w = q(D_0, D_n) \in \pi_1(H, p_2)$; then*

$$|w|_{g+1}^{simple} \le 2n.$$

Proof. Assume without loss of generality that the path (D_i) connecting D_0 to D_n is of minimal length in $\mathcal{DG}$. First we modify inductively the sequence (D_i) without increasing its length in such a way that each of the disks D_i $(1 \le i \le n-1)$ either is non-separating or encloses the spots p_1, p_2.

The construction proceeds in two steps. In a first step, we replace each separating disk D_{2i-1} with odd index by a disk which either is non-separating or encloses the two spots. We do not change the disks D_{2i} with even index. In a second step, we then modify the disks with even index and preserve those with odd index.

To carry out the first step, let $\ell \le n/2$ and assume that the disk $D_{2\ell-1}$ is separating and does not enclose the spots; otherwise there is nothing to do. If $D_{2\ell-2}, D_{2\ell}$ are contained in distinct components of $H - D_{2\ell-1}$ then they are disjoint. In this case we can remove $D_{2\ell-1}$ from the path (D_i) and obtain a shorter path with the same endpoints. Since the path (D_i) has minimal length this is impossible.

Thus $D_{2\ell-2}, D_{2\ell}$ are contained in the same component V of $H - D_{2\ell-1}$. Since $D_{2\ell-1}$ does not enclose the spots, none of the two components of $\partial H - D_{2\ell-1}$ is a three-holed sphere. Since H has precisely two spots, this implies that the genus of each of the two components of $\partial H - D_{2\ell-1}$ is positive. Now each component of $H - D_{2\ell-1}$ is a handlebody with spots and therefore the component $H - V$ of $H - D_{2\ell-1}$ contains a non-separating disk $\tilde{D}_{2\ell-1}$. Replace $D_{2\ell-1}$ by $\tilde{D}_{2\ell-1}$.

Replace in this way any disk $D_{2\ell-1}$ with an odd index which is separating but does not enclose p_1, p_2 by a non-separating disk without modifying the disks D_{2i}

with even index. This implements the first step of the construction. The second step is exactly identical after exchanging the roles of even and odd index. To summarize, we may assume from now on that every separating disk in the path (D_i) encloses the two spots.

From the path (D_i) we next construct a path $(E_j)_{0\leq j\leq 2u}$ of disks connecting D_0 to D_n whose length $2u$ is at most four times the length n of the path (D_i) and such that for each j, the disk E_{2j} encloses the spots p_1, p_2 and the disk E_{2j-1} is non-separating.

To this end recall that any two distinct disks which enclose the spots p_1, p_2 intersect. This means that if the disk D_i from the above sequence encloses the spots then the disks D_{i-1}, D_{i+1} are non-separating. Thus for the construction of the path (E_j) it now suffices to replace any consecutive pair D_i, D_{i+1} of disjoint non-separating disks by a path of length at most four with the same endpoints whose vertices alternate between non-separating disks and disks enclosing the spots.

Let $i < n-1$ be such that the disks D_i, D_{i+1} are both non-separating. If $H-(D_i\cup D_{i+1})$ is connected then there is a disk B which encloses the spots p_1, p_2 and which is disjoint from $D_i\cup D_{i+1}$. Such a disk can be obtained by thickening an embedded arc in $\partial H-(D_i\cup D_{i+1})$ which connects p_1 to p_2. Replace the consecutive pair D_i, D_{i+1} by the path D_i, B, D_{i+1} of length two.

If $H-(D_i\cup D_{i+1})$ is disconnected and if the spots p_1, p_2 are both contained in the same component of $\partial H-(D_i\cup D_{i+1})$ then we can proceed as in the previous paragraph. Otherwise there is a component of $\partial(H-(D_i\cup D_{i+1}))$ which is a surface of genus $g\geq 1$ with three holes. One of the holes is a spot, the other two holes are boundary components given by the boundary circles of D_i, D_{i+1}. Hence there is a non-separating disk B which is disjoint from $D_i\cup D_{i+1}$ and such that $H-(D_i\cup B)$ and $H-(D_{i+1}\cup B)$ are both connected. Replace the consecutive pair D_i, D_{i+1} by a path of length 4 of the form $D_i, A_1, B, A_2, D_{i+1}$ so that consecutive disks in this sequence are disjoint and that the disks A_1, A_2 both enclose the spots p_1, p_2. This completes the construction of the sequence (E_j).

For each j the disk E_{2j} defines a free splitting of F_{g+1} of the form $F_g * \mathbb{Z}$. If a is the generator of the free factor $\mathbb{Z}$ for the splitting defined by D_0, then for each j the free factor $\mathbb{Z}$ for the free splitting defined by E_{2j} is generated by $a\cdot q(D_0, E_{2j})$ where $a\cdot q(D_0, E_{2j} \in F_g$.

Let $w_j = q(D_0, E_{2j}^{-1}q(D_0, E_{2j+2}) \in F_g$. By construction, the disk E_{2j+1} in H is non-separating. The set of all loops in $H\cup\{p_2\}$ with basepoint p_2 which do not intersect E_{2j+1} define a free factor Q of F_g of corank one. Since E_{2j+1} is disjoint from E_{2j} and E_{2j+2}, the element w_j is contained in the free factor Q.

Since $w_j\in Q$, by Theorem 2.4 of [S00] the Whitehead graph of w_j has a cut vertex. But this just means that the simple $g+1$-length of w_j vanishes. An inductive application of Lemma 4.7 of [SS14] now shows that the simple $g+1$-length of the word $q(D_0, D_n)\in F_g$ is at most $u\leq 2n$. Namely, this lemma shows that for any two freely reduced words u, v in the letters $\mathcal{A}\cup\mathcal{A}^{-1}$, the word $w = uv$ satisfies

$$|w|_{g+1}^{simple} \leq |u|_{g+1}^{simple} + |v|_{g+1}^{simple} + 1.$$

Although this statement is only formulated in the case that w is freely reduced, due to the fact that $|u|_{g+1}^{simple} \geq |v|_{g+1}^{simple}$ whenever v is a subword of u (this is the easy Lemma 4.6 of [SS14]), it is also valid if the words u, v are freely reduced but this is not the case for $w = uv$. $\square$

Now we are ready to show the second part of Theorem 2 from the introduction.

Proposition 3.5. *For every $n \geq 1$, the disk graph of a handlebody H of genus $g \geq 2$ with two spots contains quasi-isometrically embedded copies of $\mathbb{R}^n$.*

Proof. Let H_2 be the handlebody with the spots p_2 removed (so that H_2 has a single spot p_1). Choose an embedded rose R in $\partial H_2 = \partial H \cup \{p_2\}$ (this notation indicates that we think of the spot p_2 as a missing point) with vertex p_2 and g petals whose fundamental group $\pi_1(R, p_2)$ maps isomorphically onto the fundamental group of H_2 via the inclusion $R \to H_2$. The image of $\pi_1(R, p_2)$ in $\pi_1(\partial H_2, p_2)$ is a free subgroup Q of $\pi_1(\partial H_2, p_2)$. The restriction to Q of the homomorphism $\pi_1(\partial H_2, p_2) \to \pi_1(H_2, p_2)$ induced by inclusion is an isomorphism. Using an abuse of notation, in the sequel we identify $\pi_1(H_2, p_2)$ with the fundamental group of H via the spot removal map, and we write $\pi_1(H, p_2)$ for the fundamental group of H with respect to a fixed choice of a basepoint near p_2.

Orient the petals of R in an arbitrary way. These oriented petals $e_1, \dots, e_g$ then determine a free basis

$$a_1, \dots, a_g$$

for Q and for $\pi_1(H, p_2) = F_g$. For each petal e_i choose a disk E_i in H which intersects the petal e_i in a single point and has no other intersections with R.

Consider the Birman exact sequence

$$0 \to \pi_1(\partial H_2, p_2) \to \mathrm{Map}(H) \to \mathrm{Map}(H_2) \to 0$$

(we refer to [HH12] for a detailed discussion of this sequence). For the homotopy class $[c]$ of a loop c in ∂H_2 based at p_2, the point pushing map $\psi_{[c]} \in \mathrm{Map}(H)$ of $[c]$ is the element of the handlebody group $\mathrm{Map}(H)$ which can be realized by a diffeomorphism fixing the complement in H of a small neighborhood of c pointwise and pushing the point p_2 along the closed curve c. The identification with $\pi_1(\partial H_2, p_2)$ of the fibre of the Birman exact sequence is via point pushing maps. The map $\psi_{[c]}$ acts on $\pi_1(\partial H_2, p_2)$ by conjugation with $[c]$. Here conjugation by $[u]$ is the homomorphism $z \to [u]z[u]^{-1}$, in particular, $\psi_{[uv]}$ is obtained by applying first $\psi_{[v]}$, followed by $\psi_{[u]}$.

As on p.592 in Subsection 5.2 of [SS14], we consider for $t \geq 1$ the element

$$b_t = a_1^{t+1} a_2^{t+1} \cdots a_g^{t+1} a_1^{t+1} a_2^{t+1} a_1^{t+1} \in Q.$$

Let D be a disk in H enclosing the spots p_1, p_2 which intersects the rose R in the boundary of a contractible neighborhood of the vertex p_2. We require that D is disjoint from the disks E_i which intersect a petal of R in a single point. We claim that for every $t \geq 1$ the image of D under the point pushing map by b_t has distance at most 10 to D in the disk graph $\mathcal{DG}$. Recall that we use point-pushing of the point p_2.

Let again $d_{\mathcal{DG}}$ be the distance in the disk graph. We show the claim first in the case that the genus g of H is at least 4. Then $b_t = uv$ where $u = a_1^{t+1} \cdots a_{g-1}^{t+1}$ and $v = a_g^{t+1} a_1^{t+1} a_2^{t+1} a_1^{t+1}$. The word u does not contain the letters a_g, a_g^{-1}, and the word v does not contain the letters a_{g-1}, a_{g-1}^{-1} since $g - 1 \geq 3$.

Thus the word v is represented by a loop on the rose R which does not cross through the petal e_{g-1}. Hence this loop is disjoint from the disk E_{g-1}, and the same holds true for the image $\psi_v(D)$ of the disk D under point pushing along v. In particular, the distance between D and $\psi_v(D)$ in the disk graph $\mathcal{DG}$ is at most two. Similarly, the image $\psi_u(D)$ of D under the point pushing map ψ_u is disjoint from E_g and hence $d_{\mathcal{DG}}(D,\psi_u(D))\leq 2$. But the point pushing map ψ_u acts on the disk graph as a simplicial isometry and consequently $d_{\mathcal{DG}}(\psi_u(D),\psi_u(\psi_v(D)))\leq 2$. Together with the triangle inequality, this yields

$$d_{\mathcal{DG}}(D,\psi_{uv}(D))\leq 4.$$

If $g=3$ then write $b_t=uvw$ where $u=a_1^{t+1}a_2^{t+1}$, where $v=a_3^{t+1}a_1^{t+1}$ and $w=a_2^{t+1}a_1^{t+1}$. Then there is a loop on R representing u,v,w which is disjoint from the petal e_3,e_2,e_1. As in the previous paragraph we conclude that $d_{\mathcal{DG}}(D,\psi_s(D))\leq 2$ for $s=u,v,w$ and by the triangle inequality, the distance between D and $\psi_{q_t}(D)$ is at most 6. In the case $g=2$ we write similarly $b_t=uvwxz$ where $u=a_1^t=w=z$ and $v=a_2^t=x$ and obtain a distance bound of at most 10.

This argument can be used inductively and shows the following. For all $t\geq 1$ and each $k\geq 1$, we have

$$d_{\mathcal{DG}}(D,\psi_{b_t^k}(D))\leq 10k. \tag{3}$$

Recall that the disk D defines a free splitting $\pi_1(H',p)=F_g*\mathbb{Z}$ where as before, $p\in\partial H'$ is the point obtained by identification of p_1 and p_2. Let a be the generator of the infinite cyclic group $\mathbb{Z}$, defined by an arc in ∂H connecting p_2 to p_1 which is disjoint from the boundary of D. As explained in the discussion preceding Remark 3.3, if $u\in Q$ is arbitrary, then the image of D under the point-pushing map ψ_u is a disk enclosing the two spots which defines the free splitting of F_{g+1} where the infinite cyclic free factor in the splitting is generated by $a\cdot q(D,\psi_u(D))$. By the definition of the point pushing map, if we identify Q with $\pi_1(H,p_2)=F_g<\pi_1(H',p)$ as described in the beginning of this proof, the generator of this infinite cyclic free factor is just the element au.

Using the above notations, we follow Section 5.2 of [SS14]. For an arbitrary integer $n\geq 1$, define a map $\Lambda:\mathbb{Z}^n\to\mathcal{DG}$ which associates to $(k_1,\dots,k_n)\in\mathbb{Z}^n$ the image of the disk D under point-pushing of p_2 along the loop $b_1^{k_1}b_2^{k_2}\cdots b_n^{k_n}\in Q$ based at p_2. We claim that

$$d_{\mathcal{DG}}(\Lambda(k_1,\dots,k_n),\Lambda(\ell_1,\dots,\ell_n))\leq 10\sum_{i=1}^n|k_i-\ell_i|. \tag{4}$$

To see this we adapt an argument from p.594 of [SS14]. Our goal is to transform the disk $\Lambda(k_1,\dots,k_n)=\psi_{b_1^{k_1}\cdots b_n^{k_n}}(D)$ to the disk $\Lambda(\ell_1,\dots,\ell_n)=\psi_{b_1^{\ell_1}\cdots b_n^{\ell_n}}(D)$ in a controlled way. These disks are determined by the homotopy classes $b_1^{k_1}b_2^{k_2}\cdots b_n^{k_n}\in Q$ provided that the base disk D is fixed. To take full advantage of this fact we will now consider pairs of disks (E,V) where we view V as a basepoint, and E as a modification of the basepoint. With this viewpoint, our goal will be to transform the pair $(\psi_{b_1^{k_1}\cdots b_n^{k_n}}(D),D)$ to the pair $(\psi_{b_1^{\ell_1}\cdots b_n^{\ell_n}}(D),D)$ in a way which enables us to estimate distances.

To achieve this goal we first claim that

$$d_{\mathcal{CG}}(\psi_{b_1^{k_1}b_2^{k_2}\cdots b_{n-1}^{k_{n-1}}b_n^{k_n}}(D),\psi_{b_1^{\ell_1}b_2^{k_2}\cdots b_n^{k_n}}(D))\leq|\ell_1-k_1|. \tag{5}$$

To show the claim we use the notation from the formula (2). Namely, the estimate (3) and the fact that point-pushing of p_2 induces an isometry on $\mathcal{DG}$ imply that

$$d_{\mathcal{DG}}(\psi_{b_1^{k_1}}(D),\psi_{b_1^{\ell_1}}(D))\leq|\ell_1-k_1|.$$

But for all u, we have $\psi_{b_1^u}(D) = [a \cdot b_1^u] = [b_1^{-u} \cdot a]$ and hence

$$d_{\mathcal{CG}}([b_1^{-k_1} \cdot a], [b_1^{-\ell_1} \cdot a]) \leq |k_1 - \ell_1|.$$

Apply to both disks $[b_1^{-k_1} \cdot a], [b_1^{-\ell_1} \cdot a]$ point-pushing of the point p_1 along a loop based at p_1 representing the homotopy class $b_n^{-k_n} \cdots b_2^{-k_2}$ in the interpretation explained in the text before equality (2). As point-pushing induces an isometry on the disk graph, we obtain

$$d_{\mathcal{CG}}([b_n^{-k_n} \cdots b_2^{-k_2} b_1^{-k_1} \cdot a], [b_n^{-k_n} \cdots b_2^{-k_2} b_1^{-\ell_1} \cdot a]) \leq |k_1 - \ell_1|.$$

Using again equation (2), this is precisely the estimate (5) we wanted to show.

Point-pushing of the point p_2 along the loop $b_1^{-\ell_1}$ transforms the *pair* of disks $(\psi_{b_1^{\ell_1} b_2^{k_2} \cdots b_n^{k_n}}(D), D)$ to the pair $(\psi_{b_2^{k_2} \cdots b_n^{k_n}} D, \psi_{b_1^{-\ell_1}}(D))$. As point-pushing acts as an isometry on the disk graph, we view this operation as a change of basepoints which does not change distances.

In a second step, we use the reasoning which led to the estimate (5) to deduce that

$$d_{\mathcal{CG}}(\psi_{b_2^{k_2} b_3^{k_3} \cdots b_n^{k_n}}(D), \psi_{b_2^{\ell_2} b_3^{k_3} \cdots b_n^{k_n}}(D)) \leq |\ell_2 - k_2|.$$

As a next step, we change the basepoint again. Using point-pushing of the point p_2 along the loop $b_2^{-\ell_2}$, the pair $(\psi_{b_2^{\ell_2} b_3^{k_3} \cdots b_n^{k_n}}(D), \psi_{b_1^{-\ell_1}} D)$ transforms to the pair

$$(\psi_{b_3^{k_3} \cdots b_n^{k_n}}(D), \psi_{b_2^{-\ell_2} b_1^{-\ell_1}} D).$$

Proceeding inductively, in n steps we transform the pair $(\psi_{b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}}(D), D)$ to the pair $(D, \psi_{b_n^{-\ell_n} \cdots b_1^{-\ell_1}}(D))$, changing distances by at most $\sum_i |\ell_i - k_i|$.

Now apply one last time point-pushing by $b_1^{\ell_1} \cdots b_n^{\ell_n}$ to the pair

$$(D, \psi_{b_n^{-\ell_n} \cdots b_1^{-\ell_1}}(D))$$

and obtain the pair $(\psi_{b_1^{\ell_1} \cdots b_n^{\ell_n}}(D), D)$. Using again that point pushing is an isometry, we conclude that the distance between the disk $\Lambda(k_1, \dots, k_n) = \psi_{b_1^{k_1} \cdots b_n^{k_n}}(D)$ and the disk $\Lambda(\ell_1, \dots, \ell_n) = \psi_{b_1^{\ell_1} \cdots b_n^{\ell_n}}(D)$ is at most $\sum_i |\ell_i - k_i|$ as claimed.

Now Lemma 4.15 of [SS14] and the discussion on the bottom of p.592 and on the top of p.594 in [SS14] shows that there is a number $c > 0$ such that

$$\sum_{i=1}^{n} |k_i - \ell_i| \leq c |b_n^{-k_n} b_{n-1}^{-k_{n-1}} \cdots b_1^{-k_1} b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}|_{g+1}^{simple}. \tag{6}$$

We give a short summary of the proof of this fact as found in [SS14]. Namely, following Definition 4.9 of [SS14], we say that a word w in the letters $\mathcal{A} \cup \mathcal{A}^{-1}$ has *conjugate reduced length at most* k if there exist freely reduced words $v_1, \dots, v_\ell, u_1, \dots, u_\ell$ such that.

(a) $w = v_1^{u_1} v_2^{u_2} \cdots v_\ell^{u_\ell}$, where $v_j^{u_j} = u_j^{-1} v_j u_j$, and
(b) $k = (\ell - 1) + |v_1|_{g+1}^{simple} + \cdots + |v_\ell|_{g+1}^{simple}$.

The number k is called the *conjugate reduced* $g+1$*-length associated to the decomposition.* The minimal number k for which such a decomposition exists is called the *conjugate reduced length of* w, and it is denoted by $|w|^{cr}$.

The easy Lemma 4.15 of [SS14] states that $|w|_{g+1}^{simple} \geq |w|^{cr}$, so it suffices to estimate $|w|^{cr}$ from below for $w = b_n^{-k_n} b_{n-1}^{-k_{n-1}} \cdots b_1^{-k_1} b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}$.

Definition 4.10 of [SS14] is geared to this end. A *cancelling pair* in the reduced word w is a pair of subwords of the form u, u^{-1}. A *nested family $\mathcal{F}$ of cancelling pairs* is a finite collection of disjoint cancelling pairs so that if $v, v^{-1} \in \mathcal{F}$ and $u, u^{-1} \in \mathcal{F}$ then v occurs between u, u^{-1} if and only if this is true for v^{-1}. For such a family $\mathcal{F}$ of cancelling pairs let $w - \mathcal{F}$ be the finite collection of subwords of w obtained by erasing the words from $\mathcal{F}$. Define

$$|w-\mathcal{F}|_{g+1}^{simple} = |\mathcal{F}| + \sum_{w' \in w-\mathcal{F}} |w'|_{g+1}^{simple}.$$

The required estimate follows from Lemma 4.11 of [SS14] which states that

$$|w|^{cr} \geq \min_{\mathcal{F}}(\max\{\frac{|\mathcal{F}|}{2} - 1, \frac{1}{5}|w-\mathcal{F}|_{g+1}^{simple} - 3\}). \tag{7}$$

To apply this estimate to the above word w, let $\mathcal{F}$ be a nested family of cancelling pairs for w which minimizes the expression on the right hand side of equation (7) and write $d = \sum |k_i - \ell_i|$. If $|\mathcal{F}| \geq d/10$ then we immediately obtain the required estimate. Otherwise note that by removing a cancelling pair we can at most delete a subword of a string of the form $b_i^{\min\{k_i,\ell_i\}}$. Furthermore it is easy to see that $|b_t^s|_{g+1}^{simple} \geq |s|$ for all s. Thus if $|\mathcal{F}| \leq d/10$ then a rough counting of the simple norm of the subsegments of $w - \mathcal{F}$ as carried out in detail on p.593 of [SS14] yields again the required estimate.

On the other hand, by Lemma 3.4, we have

$$d_{\mathcal{DG}}(\Lambda(k_1,\dots,k_n), \Lambda(\ell_1,\dots,\ell_n)) \geq \frac{1}{2}|b_n^{-k_n} b_{n-1}^{-k_{n-1}} \cdots b_1^{-k_1} b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}|_{g+1}^{simple}. \tag{8}$$

The estimates (4), (6) and (8) together show that the distance in $\mathcal{DG}$ of the images of D under the point pushing of p_2 along $b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}$ and by $b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}$ is bounded from above and below by a fixed positive multiple of $\sum_{i=1}^n |k_i - \ell_i|$. Thus the map $\Lambda : \mathbb{Z}^n \to \mathcal{DG}$ is a quasi-isometric embedding. The proposition follows. □

Remark 3.6. As mentioned earlier, the argument in the proof of Proposition 3.5 indirectly uses the fact that the pure mapping class group of a disk with two punctures is infinite cyclic and consequently point pushing of p_1 leaving p_2 fixed commutes (in an appropriate sense) with point pushing of p_2 leaving p_1 fixed.

The proof of the upper distance bound which appears in the proof of Proposition 3.5 also applies if we view the disks as elements of the curve graph of the boundary surface ∂H of the handlebody, but the resulting estimate is irrelevant in this case. Namely, the distance in the curve graph of all the diskbounding simple closed curves considered equals two as there exists a curve on ∂H (for example, a component of the boundary of a tubular neighborhood of the rose R used in the construction) which is disjoint from all the disks. Such a curve is not diskbounding.

The arguments in the proof of Proposition 3.5 also yield the second part of Theorem 3.

Corollary 3.7. *The sphere graph of a doubled handlebody $\sharp_g S^2 \times S^1$ $(g \geq 2)$ with two spots contains for every $n \geq 2$ a quasi-isometrically embedded copy of $\mathbb{R}^n$.*

Proof. The proof of Proposition 3.5 applies almost verbatim to the doubled handlebody M with two spots, i.e. to the connected sum of g copies of $S^2 \times S^1$ with two spots p_1, p_2. If we identify small disjoint compact embedded balls B_1, B_2 in M

containing the spots p_1, p_2 on the boundary with an orientation reversing diffeomorphism, then we obtain a connected sum N of $g+1$ copies of $S^2 \times S^1$ with one marked point p which is the image of the identified points p_1, p_2.

Any sphere in M enclosing the spots (i.e. a sphere whose image under the spot removal map is contractible) defines a one-edge free splitting of the fundamental group $\pi_1(N,p)$ of N into the free group F_g with g generators, identified with the fundamental group of M, and an infinite cyclic group. Namely, as in the case of a handlebody, such a sphere can be enlarged to an embedded ball in $M \cup \{p_2\}$ which contains p_2 and is disjoint from the ball B_1. The splitting is now a consequence of van Kampen's theorem.

Point pushing of one of the spots along paths in a fixed embedded rose in M (which we may assume to be contained in the boundary of an embedded twice spotted handlebody whose double equals M) acts on these splittings $F_{g+1} = F_g * \mathbb{Z}$ by appending the point-pushing element to the generator of the free factor $\mathbb{Z}$. Note that this is an immediate consequence of the argument for the handlebody whose fundamental group coincides with the fundamental group of M. Now the calculation in the proof of Proposition 3.5 only uses information on splittings of the free group with $g+1$ generators and therefore this calculation is also valid for spheres and yields the corollary. □

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