SPOTTED DISK AND SPHERE GRAPHS

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ABSTRACT. The disk graph of a handlebody H of genus $g \ge 2$ with $m \ge 0$ marked points on the boundary is the graph whose vertices are isotopy classes of disks disjoint from the marked points and where two vertices are connected by an edge of length one if they can be realized disjointly. We show that for m = 1 the disk graph contains quasi-isometrically embedded copies of \mathbb{R}^2 . For m = 2 the disk graph contains for every $n \ge 1$ a quasi-isometrically embedded copy of \mathbb{R}^n . The same holds true for sphere graphs of the doubled handlebody with one or two marked points, respectively.

1. INTRODUCTION

The curve graph $C\mathcal{G}$ of an oriented surface S of genus $g \ge 0$ with $m \ge 0$ punctures and $3g - 3 + m \ge 2$ is the graph whose vertices are isotopy classes of essential (i.e. non-contractible and not homotopic into a puncture) simple closed curves on S. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a locally infinite hyperbolic geodesic metric space of infinite diameter [MM99].

A handlebody of genus $g \ge 1$ is a compact three-dimensional manifold H which can be realized as a closed regular neighborhood in \mathbb{R}^3 of an embedded bouquet of g circles. Its boundary ∂H is an oriented surface of genus g. We allow that ∂H is equipped with $m \ge 0$ marked points (punctures) which we call *spots* in the sequel. The group Map(H) of all isotopy classes of orientation preserving homeomorphisms of H which fix each of the spots is called the *handlebody group* of H. The restriction of an element of Map(H) to the boundary ∂H defines an embedding of Map(H) into the mapping class group of ∂H , viewed as a surface with punctures [S77, Wa98].

An essential disk in H is a properly embedded disk $(D, \partial D) \subset (H, \partial H)$ whose boundary ∂D is an essential simple closed curve in ∂H , viewed as a surface with punctures. An isotopy of such a disk is supposed to consist of such disks.

The disk graph \mathcal{DG} of H is the graph whose vertices are isotopy classes of essential disks in H. Two such disks are connected by an edge of length one if and only if they can be realized disjointly.

A metric space X is said to have asymptotic dimension $\operatorname{asdim}(X) \leq n$ if for every R > 0 there exists a covering of X by uniformly bounded subsets of X so that any ball of radius R intersects at most n + 1 sets from the covering. The asymptotic dimension of a curve graph is finite [BF08].

In [MS13, H19a, H16, H19b] the following is shown.

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Theorem 1. The disk graph of a handlebody of genus $g \ge 2$ without spots is hyperbolic and has finite asymptotic dimension.

The main goal of this work is to show that in contrast to the case of curve graphs, Theorem 1 is not true if we allow spots on the boundary.

Theorem 2. Let H be a handlebody of genus $g \ge 2$ with $m \ge 1$ spots.

- (1) For m = 1 the disk graph of H contains quasi-isometrically embedded copies of \mathbb{R}^2 . In particular, it is not hyperbolic.
- (2) For m = 2 and $g \ge 3$, the disk graph of H contains for every $n \ge 1$ a quasi-isometrically embedded copy of \mathbb{R}^n . In particular, it is not hyperbolic, and its asymptotic dimension is infinite.

The proof of the second part of Theorem 2 uses m = 2 in an essential way. I do not know whether or not the asymptotic dimension of the disk graph of a handlebody with a single spot or with $m \ge 3$ spots is finite. In view of the results in [H16], it seems possible that finiteness holds true for all $m \ne 2$.

Theorem 2 implies that disk graphs can not be used effectively to obtain a geometric understanding of the handlebody group Map(H) of a handlebody H of genus $g \geq 3$ paralleling the program developed by Masur and Minsky for the mapping class group [MM00]. Note that Map(H) is an exponentially distorted subgroup of the mapping class group of ∂H [HH12]. The analogue of the strategy of Masur and Minsky would consist of cutting a handlebody open along an embedded disk which yields a (perhaps disconnected) handlebody with two spots on the boundary and studying disk graphs in the cut open handlebody.

Theorem 2 has an analogue for geometric graphs related to the outer automorphism group $\operatorname{Out}(F_g)$ of the free group on g generators. Namely, doubling the handlebody H yields a connected sum $M = \sharp_g S^2 \times S^1$ of g copies of $S^2 \times S^1$ with m marked points. A doubled disk is an embedded essential sphere in M, which is a sphere which is not homotopically trivial or homotopic into a marked point. The sphere graph of M is the graph whose vertices are isotopy classes of essential spheres in M and where two such spheres are connected by an edge of length one if and only if they can be realized disjointly. As before, an isotopy of spheres is required to be disjoint from the marked points. The sphere graph of a doubled handlebody without marked points is hyperbolic [HM13b].

Paralleling the result in Theorem 2 we have

Theorem 3. Let $g \ge 2$ and let M be a doubled handlebody of genus g with $m \ge 1$ marked points.

- (1) If m = 1 and if g is even then the sphere graph of M contains quasi-isometrically embedded copies of \mathbb{R}^2 . In particular, it is not hyperbolic.
- (2) If m = 2 and $g \ge 3$ then the sphere graph of M contains for every $n \ge 1$ a quasi-isometrically embedded copy of \mathbb{R}^n . In particular, it is not hyperbolic, and its asymptotic dimension is infinite.

As in the case of disk graphs, this indicates that sphere graphs can not be used to obtain an effective geometric understanding of $\operatorname{Out}(F_g)$ following the program developed in [MM00]. Theorem 3 may be related to the fact that in contrast to mapping class groups [M095], for $g \geq 3$ the Dehn functions of $\operatorname{Out}(F_g)$ and of the handlebody group $\operatorname{Map}(H)$ of a handlebody of genus g is exponential [BV12, HM13a, HH19]. The first example known to us of a geometric graph of infinite asymptotic dimension is due to Sabalka and Savchuk [SS14]. The vertices of this graph are isotopy classes of essential separating spheres in $\sharp_g S^2 \times S^1$. Two such spheres are connected by an edge of length one if and only if they can be realized disjointly. We use the main idea in [SS14] for the proof of the second part of Theorem 2 and of Theorem 3.

The argument in the proof of the first part of Theorem 3 uses the first part of Theorem 2 and a result in [HH15] which relates the sphere graph in a connected sum $\sharp_g S^2 \times S^1$ for g even to the arc graph of an oriented surface of genus g/2 with connected non-empty boundary. A corresponding result for odd g and a non-orientable surface with a single boundary component would yield the first part of Theorem 3 for odd $g \geq 3$, but at the moment, such a result is not available.

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2. Once spotted handlebodies and doubled handlebodies

The goal of this section is to construct quasi-isometrically embedded copies of \mathbb{R}^2 in the disk graph of a handlebody with a single spot and in the sphere graph of a doubled handlebody of even genus with a single spot.

Thus let H be a handlebody of genus $g \ge 2$ with a single spot. Let H_0 be the handlebody obtained from H by removing the spot and let

$$\Phi: H \to H_0$$

be the spot removal map. The image under Φ of an essential diskbounding simple closed curve in ∂H is an essential diskbounding simple closed curve in ∂H_0 .

The handlebody H_0 without spots can be realized as an *I*-bundle over a surface F with a single boundary component. If the surface F is orientable, then the genus g is even and the *I*-bundle is trivial. The genus of F equals g/2, and the boundary ∂F of F defines an isotopy class of a separating simple closed curve c on ∂H_0 which decomposes ∂H_0 into two surfaces of genus g/2, with a single boundary component. If the surface F is non-orientable, then the *I*-bundle is non-trivial and the boundary ∂F defines a non-separating simple closed curve c in ∂H_0 .

Following [H19a, H16], define an *I*-bundle generator for H_0 to be a simple closed curve $c \subset \partial H_0$ so that H_0 can be realized as an *I*-bundle over a compact surface F with connected boundary ∂F and such that c is freely homotopic to $\partial F \subset \partial H_0$. The surface F is called the *base* of the *I*-bundle. If the *I*-bundle generator c is separating, then F is orientable of genus g/2 where g is the genus of H_0 . If c is non-separating, then the surface F is non-orientable, and the complement of an open annulus about c in ∂H_0 is the orientation cover of F. The *I*-bundle over every essential simple embedded arc in F with endpoints on ∂F is an essential disk in H_0 which intersects c in precisely two points (up to isotopy).

An *I*-bundle generator c in ∂H_0 is *diskbusting*, which means that it has an essential intersection with every disk (see [MS13, H19a]). Namely, the base F of the *I*-bundle is a deformation retract of H_0 . Thus if γ is any essential closed curve on ∂H_0 which does not intersect c then γ projects to an essential closed curve on F. Such a curve is not nullhomotopic in H_0 and hence it can not be diskbounding.

URSULA HAMENSTÄDT

The arc graph $\mathcal{A}(X)$ of a compact surface X of genus $n \geq 1$ with connected boundary ∂X is the graph whose vertices are isotopy classes of embedded essential arcs in X with endpoints on the boundary, and isotopies are allowed to move the endpoints of an arc along ∂X . Two such arcs are connected by an edge of length one if and only if they can be realized disjointly. The arc graph $\mathcal{A}(X)$ of X is hyperbolic, however the inclusion of $\mathcal{A}(X)$ into the arc and curve graph of X is a quasi-isometry only if the genus of X equals one [MS13] (see also [H16]).

A coarse *L*-Lipschitz retraction of a metric space (X, d) onto a subspace Y is a coarse *L*-Lipschitz map $\Psi : X \to Y$ (this means that $d(\Psi(x), \Psi(y)) \leq Ld(x, y) + L$ for some $L \geq 1$ and all x, y) with the additional property that there exists a number C > 0 with $d(\Psi(y), y) \leq C$ for all $y \in Y$. If X is a geodesic metric space then the image Y of a coarse Lipschitz retraction is a coarsely quasi-convex subspace of X, i.e. any two points in Y can be connected by a uniform quasi-geodesic in X which is entirely contained in Y.

For an *I*-bundle generator c in H_0 let $\mathcal{RD}(c)$ be the complete subgraph of the disk graph \mathcal{DG}_0 of H_0 consisting of disks which intersect c in precisely two points. The boundary of each such disk is an *I*-bundle over an arc in the base F of the *I*-bundle corresponding to c. As two such disks are disjoint if and only if the corresponding arcs in F are disjoint, the graph $\mathcal{RD}(c)$ is isometric to the arc graph $\mathcal{A}(F)$ of F.

Lemma 2.1. There exists a coarsely Lipschitz retraction $\Theta_0 : \mathcal{DG}_0 \to \mathcal{RD}(c)$ whose restriction to $\mathcal{RD}(c)$ is the identity.

Proof. The case that c is a separating *I*-bundle generator is completely elementary. Namely, in this case the base F of the *I*-bundle can be identified with a component of $\partial H_0 - c$. As c is diskbusting, the map

$$\Upsilon_0: \mathcal{DG}_0 \to \mathcal{A}(F)$$

which associates to a disk D a component of $\partial D \cap F$ is coarsely well defined: Although it depends on choices, any other choice Υ'_0 maps a disk D to an arc disjoint from $\Upsilon_0(D)$. If we denote by $Q : \mathcal{A}(F) \to \mathcal{RD}(c)$ the map which associates to an arc α in F the *I*-bundle over α , then the disks $Q(\Upsilon_0(D)), Q(\Upsilon'_0(D))$ are disjoint as well.

Furthermore, if D, D' are disjoint disks then the arcs $\Upsilon_0(D), \Upsilon_0(D')$ are disjoint and hence $d_{\mathcal{D}\mathcal{G}_0}(Q\Upsilon_0(D), Q\Upsilon_0(D')) \leq 1$. This shows that $Q \circ \Upsilon_0$ is coarsely one-Lipschitz. As a disk $D \in \mathcal{RD}(c)$ intersects F in a single arc, we have $Q\Upsilon_0(D) = D$. This completes the proof of the lemma in the case that c is separating.

The above argument does not immediately extend to non-separating *I*-bundle generators. Namely, if c is a non-separating *I*-bundle generator, then the natural orientation reversing involution Φ of the corresponding *I*-bundle which exchanges the two endpoints of a fiber acts as an orientation reversing involution on the boundary ∂H_0 of H_0 . This action preserves an embedded open annulus $A \subset \partial H_0$ about c, and the action of Φ on $\partial H_0 - A$ is free, with quotient a non-orientable surface F with connected boundary ∂F . A disk which intersects c in precisely two points then is the *I*-bundle over an embedded arc in F. Its boundary is a Φ -invariant simple closed curve on ∂H_0 . Thus there is no obvious projection of \mathcal{DG}_0 onto $\mathcal{RD}(c)$ as in the case of a separating *I*-bundle generator.

To show that the lemma holds true in this case as well, it suffices to show that the inclusion $\mathcal{RD}(c) \to \mathcal{DG}_0$ is a quasi-isometric embedding. Namely, if this holds true then as \mathcal{DG}_0 is hyperbolic, there exists a coarsely distance non-increasing coarsely

well defined shortest distance projection $\mathcal{DG}_0 \to \mathcal{RD}(c)$, and such a projection is a coarsely Lipschitz retraction.

That the inclusion $\mathcal{RD}(c) \to \mathcal{DG}_0$ is indeed a quasi-isometric embedding follows from Theorem 10.1 of [MS13] (which can only be used indirectly as the "holes" are not precisely specified) and, more specifically, Corollary 4.6 and Corollary 4.7 of [H16].

To be more precise, in [H16] we constructed from the disk graph \mathcal{DG}_0 of H_0 another graph \mathcal{EDG}_0 with the same vertex set by adding additional edges as follows. If D, E are two disks in H_0 , and if up to homotopy, D, E are disjoint from an essential simple closed curve in ∂H_0 , then we connect D, E by an edge in \mathcal{EDG}_0 . The graph is called the electrified disk graph of H_0 .

Let us denote by $\mathcal{ERD}(c)$ the subgraph of \mathcal{EDG}_0 whose vertex set consists of all disks which intersect the non-separating *I*-bundle generator c in precisely two points. Lemma 4.2 of [H16] shows that the map which associates to an arc in the non-orientable surface F the *I*-bundle over F is a two-quasi-isometry between the arc and curve graph of F and $\mathcal{ERD}(c)$. Furthermore, by Corollary 4.6 of [H16], the inclusion $\mathcal{EDR}(c) \to \mathcal{EDG}_0$ is a uniform quasi-isometric embedding (here uniform means with constants not depending on c).

Let $\zeta : [0, m] \to \mathcal{ERD}(c)$ be a geodesic. Then ζ is a uniform quasi-geodesic in \mathcal{EDG}_0 . Define the *enlargement* ζ_2 of ζ to be the edge path in $\mathcal{ERD}(c)$ obtained from ζ by replacing each edge $\zeta[k, k+1]$ by an edge path $\zeta_2[i_k, i_{k+1}]$ with the same endpoints as follows.

If the disks $\zeta(k)$, $\zeta(k+1)$ are disjoint, then the edge path $\zeta_2[i_k, i_{k+1}]$ just consists of the edge connecting these two points. Otherwise $\zeta(k)$, $\zeta(k+1)$ are disjoint from an essential simple closed curve in ∂H_0 . As each disk $\zeta(j)$ is an *I*-bundles over an arc $\alpha(j)$ in the surface *F*, this means that there is an essential simple closed curve $\beta \subset F$ disjoint from both $\alpha(k), \alpha(k+1)$. We refer to Lemma 4.2 of [H16] for a detailed explanation. Let $X \subset \partial H_0$ be the component of the complement of the preimage of β in ∂H_0 which contains *c*. Then *X* is an essential subsurface in ∂H_0 which contains the boundaries of the disks $\zeta(k), \zeta(k+1)$. No component of its boundary is diskbounding, and it contains *c* as an *I*-bundle generator. Furthermore, no essential simple closed curve in *X* (here essential means non-peripheral) is disjoint from all disks with boundary in *X*. A subsurface *X* of ∂H_0 with these properties is called *thick* in [H16].

By Corollary 4.6 of [H16], the set of all disks with boundary in X defines an electrified disk graph for X. Its subgraph of all disks which intersect c in precisely two points is uniformly quasi-isometrically embedded in the electrified disk graph of X. Furthermore, it is 2-quasi-isometric to the arc and curve graph of $F - \beta$. Define $\zeta_2[i_k, i_{k+1}]$ to be the path in $\mathcal{ERD}(c)$ connecting $\zeta(k)$ to $\zeta(k+1)$ which consists of I-bundles over arcs in $F - \beta$ defined by a geodesic in the arc and curve graph of $F - \beta$. That is, from a geodesic in the arc and curve graph of F - β we construct first an edge path of at most twice the length with the property that among two consecutive vertices, at least one is an arc, and then we view this edge path as an edge path in the subgraph $\mathcal{ERD}(c, X)$ of the electrified disk graph of X consisting of disks which intersect c in precisely two points.

The resulting edge path ζ_2 in $\mathcal{ERD}(c)$ has the property that two consecutive edges are either disjoint, or their boundaries lie in the same proper thick subsurface

X of ∂H_0 containing c as an *I*-bundle generator, and they are connected by an edge in the graph $\mathcal{ERD}(c, X)$.

By the main result of [H16], the path ζ_2 is a uniform quasi-geodesic in the graph $\mathcal{EDG}(2, H_0)$ whose vertex set is the set of disks and where two disks are connected by an edge if either they are disjoint, or if they are disjoint from a multicurve consisting of at least two components. Furthermore, the graph $\mathcal{EDG}(2, H_0)$ is hyperbolic, and it is an electrification of the disk graph of H_0 .

This construction can be iterated. In the next step, we inspect two consecutive vertices $\zeta_2(k), \zeta_2(k+1)$ of ζ_2 . These are disks which intersect c in precisely two points. If they are not disjoint, then they are consecutive vertices of one of the edge paths inserted into ζ to construct ζ_2 . That is, their boundaries are contained in the same thick subsurface X of ∂H_0 containing c as an I-bundle generator, and they are disjoint from the preimage in ∂H_0 of a simple closed curve β in the subsurface F_0 of F which defines X.

The curve β determines a new thick subsurface $\hat{X} \subset X$ of ∂H_0 containing cas an *I*-bundle generator, and this subsurface can be used to connect $\zeta_2(k)$ to $\zeta_2(k+1)$ by an edge path. In finitely many steps we construct in this way a path in the graph $\mathcal{RD}(c)$ connecting the endpoints of ζ . Its length roughly equals the sum of the subsurface projections of its endpoints into all thick subsurfaces of ∂H_0 containing c as an *I*-bundle generator. In particular, by the distance formula in Corollary 4.7 of [H16], its length is uniformly equivalent to the distance in \mathcal{DG}_0 between its endpoints. This also follows as by the main result of [H16], the socalled hierarchy paths, constructed from a geodesic in \mathcal{EDG}_0 in the above fashion, are uniform quasi-geodesics in the disk graph.

As a consequence, taking the *I*-bundle over an arc in *F* defines an isometry between the arc graph of *F* and the graph $\mathcal{RD}(c)$, and this graph is quasi-isometrically embedded in \mathcal{DG}_0 . This is what we wanted to show.

Our goal is to use *I*-bundle generators in ∂H_0 to construct quasi-isometrically embedded euclidean planes in the disk graph of *H*. In analogy to [H19a], we define an *I*-bundle generator for the spotted handlebody *H* to be a simple closed curve in ∂H whose image under the map Φ is an *I*-bundle generator in ∂H_0 .

Let $(c_1, c_2) \subset \partial H$ be a pair of non-isotopic disjoint *I*-bundle generators so that $\partial H - \{c_1 \cup c_2\}$ has a connected component which is an annulus containing the spot in its interior. Then up to isotopy, $\Phi(c_1) = \Phi(c_2) = c$ for an *I*-bundle generator c in H_0 .

The following construction is due to Kra; we refer to [KLS09] for details and for some applications. For its formulation, for a pair (c_1, c_2) of disjoint *I*-bundle generators on ∂H as in the previous paragraph let $\mathcal{RD}(c_1, c_2)$ be the complete subgraph of the disk graph \mathcal{DG} of H whose vertex set consists of all disks which intersect each of the curves c_1, c_2 in precisely two points. Note that if $D \in \mathcal{RD}(c_1, c_2)$ then the image of D under the spot removing map Φ is contained in $\mathcal{RD}(c)$ where $c = \Phi(c_i)$.

In the next lemma we denote by abuse of notation the map $\mathcal{DG} \to \mathcal{DG}_0$ induced by the spot forgetful map Φ again by Φ . Furthermore, for the remainder of this section we represent a disk by its boundary, i.e. we view the disk graph as the complete subgraph of the curve graph of ∂H whose vertex set is the set of diskbounding curves. **Lemma 2.2.** Let (c_1, c_2) be a pair of *I*-bundle generators bounding a punctured annulus and let $c = \Phi(c_1) = \Phi(c_2)$. There exists a simplicial embedding $\iota : \mathcal{DG}_0 \to \mathcal{DG}$ with the following properties.

- (1) $\Phi \circ \iota$ is the identity.
- (2) ι maps $\mathcal{RD}(c)$ into $\mathcal{RD}(c_1, c_2)$.

Proof. Note first that there is a natural orientation reversing involution ρ_0 of ∂H_0 which exchanges the endpoints of the fibres of the interval bundle over the base F. This involution fixes c and preserves up to isotopy each diskbounding simple closed curve which intersects c in precisely two points.

Choose a hyperbolic metric g_0 on ∂H_0 which is invariant under ρ_0 and let \hat{c} be the geodesic representative of c. Choose a point $p \in \hat{c}$ not contained in any diskbounding simple closed geodesic; this is possible since each diskbounding simple closed geodesic intersects \hat{c} transversely in finitely many points and hence the set of all points of \hat{c} contained in a diskbounding closed geodesic is countable. View p as a marked point on ∂H_0 ; then the geodesic representative of a diskbounding curve α in ∂H_0 is a diskbounding curve $\iota(\alpha)$ in $\partial H_0 - \{p\}$. Via identification of a disk with its boundary, this construction defines a simplicial embedding

$$\iota: \mathcal{DG}_0 \to \mathcal{DG}$$

with the property that $\Phi \circ \iota$ equals the identity. Furthermore, we clearly have $\iota(\mathcal{RD}(c)) \subset \mathcal{RD}(c_1, c_2)$.

The situation in the following discussion is illustrated in Figure A. Let B be the connected component of $\partial H - \{c_1, c_2\}$ containing the spot (this is a once spotted annulus). Let Λ be a diffeomorphism of ∂H which preserves the complement of B (and hence the boundary of B) pointwise and which pushes the spot in B one full turn around a central loop in B. The isotopy class of Λ is contained in the kernel of the homomorphism $\operatorname{Mod}(\partial H) \to \operatorname{Mod}(\partial H_0)$ induced by the spot removal map Φ . The map Λ extends to a diffeomorphism of the handlebody H. This can be seen as in the case of point-pushing in a surface: Identify the image of B under the spot removal map Φ with a closed annulus A. Choose a neighborhood N of the punctured annulus B in H which is homeomorphic to $A \times [0, 1]$, with one interior point removed from $A \times \{0\}$. Gradually undo the rotation of the marked point as one moves towards $A \times \{1\} \cup \partial A \times [0, 1]$. Therefore the diffeomorphism Λ generates an infinite cyclic group of simplicial isometries of $\mathcal{RD}(c_1, c_2)$ which we denote again by Λ . With this notation, $\Phi \circ \Lambda = \Phi$.

Let $\Theta_0 : \mathcal{DG}_0 \to \mathcal{RD}(c)$ be as in Lemma 2.1. Define

$$\Theta = \Theta_0 \circ \Phi : \mathcal{DG} \to \mathcal{RD}(c).$$

Observe that $\Theta(\iota(D)) = \Theta_0(D)$ for all disks $D \in \mathcal{DG}_0$. This then implies that $\Theta(\iota(D)) = D$ for all $D \in \mathcal{RD}(c)$. Furthermore, Θ is coarsely Lipschitz (compare the proof of Lemma 2.1 for a detailed explanation), and we have

$$\Theta(\Lambda(D)) = \Theta(D)$$

for all disks D.

Recall that $\mathcal{RD}(c)$ is isometric to the arc graph $\mathcal{A}(F)$ of F. Define a distance d_0 on $\mathcal{RD}(c) \times \mathbb{Z}$ by

$$d_0((\alpha, a), (\beta, b)) = d_{\mathcal{RD}(c)}(\alpha, \beta) + |a - b|$$



where $d_{\mathcal{RD}(c)}$ denotes the distance in $\mathcal{RD}(c)$. Let moreover

$$\Omega = \bigcup_k \Lambda^k \iota(\mathcal{RD}(c)).$$

Lemma 2.3. The map $\Psi : \Omega \to \mathcal{RD}(c) \times \mathbb{Z}$ which maps $D \in \Lambda^k \iota(\mathcal{RD}(c))$ to $\Psi(D) = (\Theta(D), k)$ is a bijective quasi-isometry.

Proof. Recall that $\Theta(D) = \Theta(\Lambda^k(D))$ for all disks D and all k and that furthermore the restriction of Θ to $\iota(\mathcal{RD}(c))$ is an isometry. In particular, if D_0, E_0 are distinct disks in $\mathcal{RD}(c)$ then $\Theta(\iota(D_0)) \neq \Theta(\iota(E_0))$ and hence $\Psi(\iota(D_0)) \neq \Psi(\Lambda^k(\iota(E_0)))$ for all k.

We claim that for every disk $D \in \Omega$ the following hold true.

(1) $D \neq \Lambda^k(D)$ for all $k \neq 0$.

(2) The disks D and $\Lambda(D)$ can be realized disjointly.

(3) Two disks $D \in \Lambda^k \iota(\mathcal{RD}(c)), E \in \Lambda^\ell \iota(\mathcal{RD}(c))$ are disjoint only if $|k-\ell| \leq 1$.

To this end let $D \in \Omega$ and for $k \in \mathbb{Z}$ let $D_k = \Lambda^k(D)$. Figure A shows that for $\ell \geq 1$, the disk $D_{k+\ell}$ has precisely $2\ell - 2$ essential intersections with D_k , and these intersection points are up to isotopy contained in the annulus B. This yields part

(2) of the above claim, and part (3) follows from the same argument. Furthermore, the twist parameter k can be recovered from the geometric intersection numbers between $\Lambda^k(D)$ and $\Lambda^{-1}(D), D, \Lambda(D)$. For example, if $k \ge 2$ then these intersection numbers equal 2k, 2k-2, 2k-4, respectively, and if $k \le -2$ then these intersection numbers are -2k-4, -2k-2, -2k. This establishes part (1) of the above claim.

Part (1) of the above claim together with the beginning of this proof yields that the map Ψ is well defined and a bijection. Now $\Omega \subset \mathcal{RD}(c_1, c_2)$ and the restriction of the map Θ to $\mathcal{RD}(c_1, c_2)$ is just the map induced by the spot forgetful map and hence it is one-Lipschitz. Part (3) of the above claim implies that the map Ψ is two-Lipschitz.

As $\Lambda^k \iota(\mathcal{RD}(c))$ is isometric to $\mathcal{A}(F)$ for all k, the inverse of Ψ which associates to a pair $(D,k) \in \mathcal{RD}(c) \times \mathbb{Z}$ the disk $\Lambda^k(\iota(D))$ is coarsely one-Lipschitz. This shows that indeed, the map Ψ is a quasi-isometry. \Box

The following proposition is the main remaining step towards a proof of the first part of Theorem 2.

Proposition 2.4. There is a coarse Lipschitz retraction $\mathcal{DG} \to \bigcup_k \Lambda^k \iota(\mathcal{RD}(c)) = \Omega$. Moreover, Ω is a coarsely quasi-convex subset of \mathcal{DG} .

Proof. As in the proof of Lemma 2.2, let ρ_0 be an orientation reversing involution of ∂H_0 which fixes the *I*-bundle generator *c* pointwise. This involution determines an involution ρ of the complement in ∂H of the interior $\operatorname{int}(B)$ of the annulus *B* which exchanges the curves c_1 and c_2 . Write as before $\Omega = \bigcup_k \Lambda^k \iota(\mathcal{RD}(c))$.

Choose a complete finite area hyperbolic metric on ∂H (so that the marked point becomes a puncture) with the property that the involution ρ of ∂H – int(B) is an isometry for this metric which maps the geodesic representative \hat{c}_1 of c_1 to the geodesic representative \hat{c}_2 of c_2 . This metric restricts to a hyperbolic metric on the once punctured annulus B with geodesic boundary.

Choose a geodesic arc α connecting the two boundary components of B which is contained in the geodesic representative of one of the curves from $\iota(\mathcal{RD}(c))$. Cutting B open along α yields a once punctured rectangle with geodesic sides, where two distinguished sides come from the arc α . For any pair of points x_1, x_2 on the remaining two sides, choose a simple arc in B connecting these two points which does not cross through α and let $\alpha(x_1, x_2) \subset B$ be the geodesic representative of this arc. By convexity, $\alpha(x_1, x_2)$ is disjoint from α if its endpoints are disjoint from the endpoints of α .

This construction yields for any pair of points $x_1 \in \hat{c}_1, x_2 \in \hat{c}_2$ an oriented geodesic arc $\alpha(x_1, x_2) \subset B$ with endpoints x_1, x_2 such that any two of these arcs connecting distinct pairs of points on \hat{c}_1, \hat{c}_2 intersect in at most two points. Furthermore, each of these arcs intersects a geodesic representative of a curve in $\iota(\mathcal{RD}(c))$ in at most two points.

We use these oriented arcs as follows. Let β be a diskbounding simple closed curve on ∂H . The intersection of β with $\partial H - \operatorname{int}(B)$ consists of a non-empty collection ζ of finitely many pairwise disjoint simple arcs with endpoints on \hat{c}_1, \hat{c}_2 . Each such arc is freely homotopic relative to \hat{c}_1, \hat{c}_2 to a unique geodesic arc which meets \hat{c}_1, \hat{c}_2 orthogonally at its endpoints.

We claim that the components of the thus defined collection $\hat{\zeta}$ of geodesic arcs are pairwise disjoint. However, some of these arcs may have nontrivial multiplicities as $\beta \cap (\partial H - \operatorname{int}(B))$ may contain several components which are homotopic relative

URSULA HAMENSTÄDT

to the boundary. To verify the claim, double each component X of the hyperbolic surface $\partial H - \operatorname{int}(B)$ along its boundary. The possibly disconnected resulting closed hyperbolic surface S admits an isometric involution σ preserving the components of S whose fixed point set is precisely the image C of the boundary of $\partial H - \operatorname{int}(B)$ in the doubled manifold. The double of the above collection ζ of arcs is a collection of simple closed curves on S which are invariant under σ .

The free homotopy classes of these closed curves are σ -invariant and hence the same holds true for their geodesic representatives: Namely, if γ is the geodesic representative of such a free homotopy class, then γ intersects the geodesic multicurve C in precisely two points. Let γ_1 be the component of $\gamma - C$ of smaller length. Then $\gamma_1 \cup \sigma(\gamma_1)$ is a simple closed curve freely homotopic to γ , and its length is at most the length of γ . But γ is the unique simple closed curve of minimal length in its free homotopy class and hence $\gamma = \gamma_1 \cup \sigma(\gamma_1)$. Thus γ intersects C orthogonally, and $\gamma \cap X$ is a component of the arc system $\hat{\zeta}$. The claim now follows from the well known fact that the geodesic representative of a simple closed multicurve on a hyperbolic surface is a simple closed multicurve.

As a consequence of the above discussion, the order of the endpoints of the components of $\beta - int(B)$ on $\hat{c}_1 \cup \hat{c}_2$ coincides with the order of the endpoints of the collection of geodesic arcs $\hat{\zeta}$ which meet $\hat{c}_1 \cup \hat{c}_2$ orthogonally at their endpoints and are freely homotopic to the components of $\beta - int(B)$. This implies that a diskbounding simple closed curve β on ∂H can be homotoped to a curve $\hat{\beta}$ of the following form. The restriction of $\hat{\beta}$ to $\partial H - int(B)$ consists of a finite collection of pairwise disjoint geodesic arcs which meet \hat{c}_i orthogonally at their endpoints. Some of these arcs may occur more than once. The restriction of $\hat{\beta}$ to the once punctured annulus B consists of a finite non-empty collection of arcs connecting \hat{c}_1 to \hat{c}_2 and perhaps a finite number of arcs which go around the puncture and return to the same boundary component of B. Distinct such arcs have disjoint interiors. The curve $\hat{\beta}$ is uniquely determined by β up to a homotopy with fixed endpoints of the components of $\hat{\beta} \cap B$. By construction of the map ι , if $\beta = \iota(\beta') \in \iota \mathcal{RD}(c)$ then $\hat{\beta} \cap \partial H - \operatorname{int}(B)$ is just the lift of the geodesic representative of β' to $\partial H - \operatorname{int}(B)$ for the following hyperbolic metric on $\partial H_0 - c$. Recall that the metric on ∂H was chosen in such a way that the geodesics \hat{c}_1, \hat{c}_2 have the same length. Then $\partial H - B$ can be glued along the two boundary components to a hyperbolic surface which can be viewed as a hyperbolic metric on ∂H_0 . This metric depends on the choice of a twist parameter, but its restriction to the complement of the geodesic representative of the curve c does not. In particular, the intersections with B of the representatives $\hat{\beta}$ of the elements $\beta \in \iota \mathcal{RD}(c)$ are pairwise disjoint.

We use this normal form for diskbounding simple closed curves to define a map

$$\Xi:\mathcal{DG}\to\mathbb{Z}$$

as follows. Let $\hat{\beta}$ be a closed curve constructed from the simple closed diskbounding curve β as in the previous paragraph. Let b be one of the components of $\hat{\beta} \cap B$ with endpoints on \hat{c}_1 and \hat{c}_2 , oriented in such a way that it connects \hat{c}_1 to \hat{c}_2 . Such a component exists since otherwise the image of β under the spot removal map is homotopic to a curve disjoint from the diskbusting curve c on ∂H_0 . Let x_1, x_2 be the endpoints of b on \hat{c}_1, \hat{c}_2 .

Let $a = \alpha(x_1, x_2)$; then b, a are simple arcs in B with the same endpoints which intersect some core curve of the annulus B in precisely one point. Assume that \hat{c}_1, \hat{c}_2 are oriented and define the boundary orientation of B. Then b is homotopic with fixed endpoints to the arc $\hat{c}_1^k \cdot a \cdot \hat{c}_2^\ell$ for unique $k, \ell \in \mathbb{Z}$ (read from left to right). In other words, if we denote by τ_i the positive Dehn twist about \hat{c}_i , viewed as a diffeomorphism of the punctured disk B with fixed boundary, then b is homotopic with fixed endpoints to the arc $\tau_1^k \tau_2^{-\ell} a$. Define $\Xi(\beta) = k$.

Observe that although this definition depends on the choice of the arcs $\alpha(x_1, x_2)$ and on the choice of the component b of $B \cap \hat{\beta}$, the map Ξ is coarsely well defined. Namely, let b' be a second component of $\hat{\beta} \cap B$, with endpoints x'_1, x'_2 on \hat{c}_1, \hat{c}_2 and distinct from b. Then the interior of b' is disjoint from the interior of b. In particular, if a' is an arc in B with the same endpoints as b' whose interior is disjoint from a, then b' is homotopic with fixed endpoints to $\tau_1^q \tau_2^{-r} a'$ for $|q - k| \leq 1, |r - \ell| \leq 1$. On the other hand, both arcs $a, \alpha(x'_1, x'_2)$ do not intersect a fixed arc connecting \hat{c}_1 and hence $a' = \tau_1^s \tau_2^{-u} \alpha(x'_1, x'_2)$ for some $|s| \leq 1, |u| \leq 1$. This shows that the multiplicity k' of the curve \hat{c}_1 in the description of b' relative to $\alpha(x'_1, x'_2)$ satisfies $|k - k'| \leq 2$. The same reasoning yields that the map Ξ is coarsely two-Lipschitz. Furthermore, we have $\Xi(\iota(\mathcal{RD}(c))) \subset [-2, 2]$.

To summarize, the map

$$(\Theta, \Xi) : \mathcal{DG} \to \mathcal{RD}(c) \times \mathbb{Z}$$

is coarsely Lipschitz, and its composition with the inverse of the map Ψ from Lemma 2.3 is a coarse Lipschitz retraction of \mathcal{DG} onto Ω provided that the map Ξ maps a point in $\Lambda^k \iota(\mathcal{RD}(c))$ into a uniformly bounded neighborhood of k.

However, if $\beta_0 \in \iota \mathcal{RD}(c)$ and if $\beta = \Lambda^k(\beta_0) \in \Lambda^k \iota(\mathcal{RD}(c))$, then the intersections with $H - \operatorname{int}(B)$ of the representatives $\hat{\beta}, \hat{\beta}_0$ of β, β_0 constructed above coincide. This implies that up to homotopy with fixed endpoints, $\hat{\beta} \cap B = \Lambda^k(\hat{\beta}_0 \cap B)$.

On the other hand, point-pushing along a simple closed curve γ based at p descends to conjugation by γ in $\pi_1(\partial H_0, p)$. Therefore the image under the map Λ of a simple arc b in B with endpoints on the two distinct components of ∂B is homotopic with fixed endpoints to c_1bc_2 (recall that we oriented c_1, c_2 so that they define the boundary orientation of B). As $\Xi(\iota(\mathcal{RD}(c))) \subset [-2,2]$, it follows that $|\Xi(\beta) - k| \leq 2$. This shows the proposition.

To summarize, we obtain

Corollary 2.5. The disk graph of a handlebody H of genus $g \ge 2$ with one spot contains quasi-isometrically embedded copies of \mathbb{R}^2 .

Remark 2.6. In [H19a] we showed that in contrast to handlebodies without spots, the disk graph of a handlebody H with a single spot on the boundary is *not* a quasi-convex subgraph of the curve graph of ∂H .

In the remainder of this section we explain how the above construction can be used to show the first part of Theorem 3.

Namely, consider the double $M_0 = \sharp_g S^2 \times S^1$ of a handlebody H_0 of genus $g \ge 2$ without spots. Let M be the manifold M_0 equipped with a marked point p. As before, we call p a spot in M. There is a natural spot removing map $\Phi : M \to M_0$.

Let SG be the sphere graph of M whose vertices are isotopy classes of embedded spheres in M which are disjoint from the spot and not isotopic into the spot. Isotopies are required to be disjoint from the spot as well. Two such spheres are connected by an edge of length one if they can be realized disjointly. Similarly, let SG_0 be the sphere graph of M_0 .

URSULA HAMENSTÄDT

Assume from now on that g = 2n for some $n \ge 1$. Choose an embedded oriented surface $F_0 \subset M_0$ of genus n with connected boundary such that the inclusion $F_0 \to M_0$ induces an isomorphism $\pi_1(F_0) \to \pi_1(M_0)$. We may assume that the oriented *I*-bundle H_0 over F_0 is an embedded handlebody $H_0 \subset M_0$ whose double equals M_0 . Thus every embedded essential arc α in F_0 with boundary in ∂F_0 determines a sphere $\Upsilon_0(\alpha)$ in M_0 as follows. The interval bundle over α is an embedded essential disk in H_0 , with boundary in ∂H_0 , and we let $\Upsilon_0(\alpha)$ be the double of this disk. By construction, the sphere $\Upsilon_0(\alpha)$ intersects the surface F_0 precisely in the arc α . By Lemma 4.17 of [HH15], distinct arcs give rise to nonisotopic spheres, furthermore the map Υ_0 preserves disjointness and hence Υ_0 is a simplicial embedding of the arc graph $\mathcal{A}(F_0)$ of F_0 into the sphere graph $S\mathcal{G}_0$ of M_0 .

Now mark a point p on the boundary ∂F_0 of F_0 and view the resulting spotted surface F as a surface in the spotted manifold M. The arc graph $\mathcal{A}(F)$ of F is the graph whose vertices are isotopy classes of essential arcs in F with endpoints on the complement of p in the boundary of F. Here we exclude arcs which are homotopic with fixed endpoints to a subarc of ∂F containing the base point p, and we require that an isotopy preserves the marked point p and hence endpoints of arcs can only slide along $\partial F - \{p\}$. Two such arcs are connected by an edge if they can be realized disjointly. Associate to an arc α in F the double $\Upsilon(\alpha)$ of the I-bundle over α .

The spot removal map $\Phi: M \to M_0$ induces a simplicial surjection $S\mathcal{G} \to S\mathcal{G}_0$, again denoted by Φ for simplicity. Similarly, if we let $\varphi: F \to F_0$ be the map which forgets the marked point $p \in \partial F$, then φ induces a simplicial surjection $\mathcal{A}(F) \to \mathcal{A}(F_0)$, denoted as well by φ . We then obtain a commutative diagram

(1)
$$\begin{array}{ccc} \mathcal{A}(F) & \stackrel{\varphi}{\longrightarrow} \mathcal{A}(F_{0}) \\ & & & \downarrow^{\Upsilon} & & \downarrow^{\Upsilon_{0}} \\ & & \mathcal{S}\mathcal{G} & \stackrel{\Phi}{\longrightarrow} \mathcal{S}\mathcal{G}_{0} \end{array}$$

Similar to the case of the handlebody M_0 without spots and the map Υ_0 , we obtain

Lemma 2.7. The map Υ is a simplicial embedding of the arc graph $\mathcal{A}(F)$ into the sphere graph.

Proof. We have to show that the map Υ is injective. As Υ_0 is injective and as the diagram (1) commutes, it suffices to show the following. Let $\alpha \neq \beta \in \mathcal{A}(F)$ be such that $\varphi(\alpha) = \varphi(\beta)$; then $\Upsilon(\alpha) \neq \Upsilon(\beta)$.

Now $\varphi(\alpha) = \varphi(\beta)$ means that up to exchanging α and β , there exists a number k > 0 such that β can be obtained from α by k half Dehn twists about the boundary ∂F of F. Here the half Dehn twist $T(\alpha)$ of α is defined as follows.

The orientation of F induces a boundary orientation for ∂F which in turn induces an orientation on $\partial F - \{p\}$. With respect to the order defined by this orientation, let x be the bigger of the two endpoints x, y of α . Slide x across p to obtain a new arc $T(\alpha)$, with endpoints x', y. This arc is not homotopic to α . To see this it suffices to show that the double $DT(\alpha)$ of $T(\alpha)$ in the double DF of F (which is a surface with one puncture) is not freely homotopic to the double $D(\alpha)$ of α . This follows since $D(\alpha)$ and $DT(\alpha)$ can be homotoped in such a way that they bound a once punctured annulus in DF.

12

The same reasoning also shows that the sphere $\Upsilon(T(\alpha))$ is not homotopic to the sphere $\Upsilon(\alpha)$. Namely, let $\chi \subset \partial F \cup \{p\}$ be the oriented embedded arc connecting the intersection point x of α with ∂F to the point x'. This arc contains p in its interior. Then the sphere $\Upsilon(T(\alpha))$ is a connected sum of the sphere $\Upsilon(\alpha)$ with the boundary of a punctured ball which is a thickening of χ . Thus $\Upsilon(\alpha)$ and $\Upsilon(T(\alpha))$ can be isotoped in such a way that they bound a subset of M homeomorphic to the complement of an interior point of $S^2 \times [0, 1]$.

The above construction, applied to the sphere $\Upsilon(T(\alpha))$ instead of the sphere $\Upsilon(\alpha)$ and where the point y takes on the role of the point x in the above discussion, shows that $\Upsilon(T^2(\alpha))$ is obtained from $\Upsilon(\alpha)$ by point-pushing along the oriented loop ∂F with basepoint p. This is a diffeomorphism of M which leaves the complement of a small tubular neighborhood of ∂F pointwise fixed and pushes the basepoint p along ∂F . As in the proof of Lemma 2.3, this argument can be iterated. It shows that the sphere $\Upsilon(T^k(\alpha))$ intersects the sphere $\Upsilon(\alpha)$ in k-1 intersection circles. These circles are essential since they cut both $\Upsilon(T^k(\alpha))$ and $\Upsilon(\alpha)$ into two disks and k-2 annuli, where a disk component of $T^k(\alpha) - T(\alpha)$ bounds together with a disk component of $T(\alpha) - T^k(\alpha)$ an embedded sphere enclosing the spot. Invoking the proof of Lemma 2.3, we conclude that indeed, for $k \neq \ell$, $\Upsilon(T^k(\alpha))$ is not homotopic to $\Upsilon(T^{\ell}(\alpha))$.

We showed so far that the map Υ is injective. To complete the proof of the lemma, it suffices to observe that disjoint arcs are mapped to disjoint spheres. But this is immediate from the construction.

Proposition 4.18 of [HH15] shows that there is a one-Lipschitz retraction

$$\Psi_0: \mathcal{SG}_0 \to \Upsilon_0(\mathcal{A}(F_0))$$

which is of the form $\Psi_0 = \Upsilon_0 \circ \Theta_0$ (read from right to left) where $\Theta_0 : S\mathcal{G}_0 \to \mathcal{A}(F_0)$ is a one-Lipschitz map. In particular, $\Upsilon_0(\mathcal{A}(F_0))$ is a quasi-isometrically embedded subgraph of $S\mathcal{G}_0$ which is quasi-isometric to $\mathcal{A}(F_0)$. Our goal is to show that there also is a coarse Lipschitz retraction of $S\mathcal{G}$ onto $\Upsilon(\mathcal{A}(F))$ of the form $\Psi = \Theta \circ \Upsilon$ where $\Theta : S\mathcal{G} \to \mathcal{A}(F)$ is a coarse Lipschitz map. This then yields the first part of Theorem 3 from the introduction.

To construct the map Θ we use the method from [HH15]. We next explain how this method can be adapted to our needs.

Let as before $F \subset M$ be an embedded oriented surface with connected boundary ∂F so that M is the double of the trivial I-bundle over F. We assume that the marked point p is contained in the boundary ∂F of F. Furthermore, we assume that the boundary ∂F of F is a smoothly embedded circle in $M \cup \{p\}$ (i.e. an embedded compact one-dimensional submanifold). As before, we use the marked point p as the basepoint for the fundamental group of M. Then ∂F equipped with its boundary orientation defines a homotopy class $\beta \in \pi_1(M, p) = F_{2g}$. As β is not contained in any free factor, ∂F intersects every sphere in M. Namely, for any given sphere S in M, the subgroup of $\pi_1(M, p)$ of all homotopy classes of loops which do not intersect S is a proper free factor of $\pi_1(M, p)$.

As in [HH15] and similar to the construction in Lemma 2.1, the strategy is to associate to a sphere S in M a component of the intersection $F \cap S$. However, unlike in the case of curves on surfaces, there is no suitable normal form for intersections of spheres with the surface F, and the main work in [HH15] consists of overcoming this difficulty by introducing a relative normal form which allows one to associate to

a sphere in M_0 an intersection arc with F_0 so that the resulting map $\mathcal{SG}_0 \to \mathcal{A}(F_0)$ is one-Lipschitz.

For the remainder of this section we outline the main steps in this construction, adapted to the sphere graph SG of M and the arc graph A(F) of F. This requires modifying spheres with isotopies not crossing through p, and modifying the surface F with homotopies leaving the boundary ∂F pointwise fixed.

For convenience, we record some definitions from [HH15] (the following combines Definition 4.7 and Definition 4.9 of [HH15]).

Definition 2.8. Let Σ be a sphere or a sphere system.

- (1) ∂F intersects Σ minimally if ∂F intersects Σ transversely and if no component of $\partial F \Sigma$ not containing the basepoint p is homotopic with fixed endpoints into Σ .
- (2) F is in minimal position with respect to Σ if ∂F intersects Σ minimally and if moreover each component of $\Sigma \cap F$ is a properly embedded arc which either is essential or homotopic with fixed endpoints to a subarc of ∂F containing the marked point.

A version of the easy Lemma 4.6 of [HH15] states that any closed curve containing the basepoint can be put into minimal position relative to a sphere system Σ as defined in the first part of Definition 2.8. The following is a version of Lemma 4.12 of [HH15]. For its formulation, call a sphere system Σ simple if it decomposes Minto a simply connected components.

Lemma 2.9. Let Σ be a simple sphere system in M. Suppose that F is in minimal position with respect to Σ . Let σ' be an embedded sphere disjoint from Σ and let Σ' be a simple sphere system obtained from Σ by either adding σ' , or removing one sphere $\sigma \in \Sigma$. Then F can be homotoped leaving p fixed to a surface F' which is in minimal position with respect to Σ' .

Proof. As in the proof of Lemma 4.12 of [HH15], removing a sphere preserves minimal position, so only the case of adding a sphere has to be considered.

Thus let Σ be a simple sphere system and let σ' be a sphere disjoint from Σ . Assume that F is in minimal position with respect to Σ . Let W_{Σ} be the complement of Σ in M, that is, W_{Σ} is a compact (possibly disconnected) manifold whose boundary consists of 2k boundary spheres $\sigma_1^+, \sigma_1^-, \cdots, \sigma_k^+, \sigma_k^-$. The boundary spheres σ_i^+ and σ_i^- correspond to the two sides of a sphere $\sigma_i \in \Sigma$. The surface F intersects W_{Σ} in a collection of embedded surfaces with boundaries. Each such surface is a polygonal disk P_i $(i = 1, \ldots, m)$. The sides of each such polygon alternate between subarcs of ∂F and arcs contained in Σ . There is at most one bigon, that is, a polygon with two sides, and this polygon then contains the point p in one of its sides. Each rectangle, if any, is homotopic into ∂F .

The proof of Lemma 4.12 of [HH15] now proceeds by studying the intersection of each polygonal component of $F - \Sigma$ with the sphere σ' . This is done by contracting each such polygonal component P to a ribbon tree T(P) in such a way that the boundary components in Σ are contracted to single points in T(P). If P is not a rectangle or bigon, then T(P) has a single vertex which is not univalent. As such ribbon trees are one-dimensional objects, they can be homotoped with fixed endpoints on ∂W_{Σ} to trees which are in minimal position with respect to σ' . This construction applies without change to rectangles and perhaps the bigon which can be represented by an interval with one endpoint at p and the second endpoint on a component of Σ . We refer to the proof of Lemma 4.12 of [HH15] for details. No adjustment of the argument is necessary.

The above construction is only valid for simple sphere systems Σ and not for individual spheres. Furthermore, it is known that the arc system on $F \cap \Sigma$ obtained by putting F into minimal position with respect to Σ is not uniquely determined by Σ . To overcome this difficulty, the work of [HH15] uses as an auxiliary datum a maximal system A_0 of pairwise disjoint essential arcs on the surface F_0 . Here maximal means that any arc which is disjoint from A_0 is contained in A_0 . The system A_0 then binds F_0 , that is, $F - A_0$ is a union of topological disks. Furthermore, ∂F_0 and each arc $\alpha \in A_0$ is equipped with an orientation.

Choose an arc system A for F which binds F. If $F \subset M$ is in minimal position with respect to Σ , then a homotopy assures that no arc from the arc system A intersects a component of $F - \Sigma$ which is a rectangle or a bigon. Then Lemma 4.12 of [HH15] and its proof applies without modification and shows that with a homotopy, F can be put into normal form with respect to the arc system A, called A-tight minimal position with respect to Σ . This then yields the statement of Lemma 4.16 of [HH15]: if F is in A-tight minimal position with respect to the simple sphere system Σ , then the binding arc system $\Sigma \cap F$ is determined by Σ . In particular, two distinct spheres from Σ intersect F in disjoint essential arcs. There may in addition be inessential arcs, i.e. arcs which are homotopic with fixed endpoints to a subsegment of ∂F containing the basepoint p, but these will be unimportant for our purpose.

Now let σ be an essential sphere in M. Let Σ be a simple sphere system in M containing σ as a component. We put F into A-tight minimal position with respect to Σ . Then $\sigma \cap F$ consists of a non-empty collection of essential arcs and perhaps some additional non-essential arcs. Choose one of the essential intersection arcs α and define $\Theta(\sigma) = \alpha$. As in [HH15] and Proposition 2.4 we now obtain

Proposition 2.10. The map Θ is a coarsely Lipschitz map. For each arc $\alpha \in \mathcal{A}(F)$, we have $\Theta(\Upsilon(\alpha)) = \alpha$. As a consequence, if g = 2n is even then the sphere graph SG of M contains quasi-isometrically embedded copies of \mathbb{R}^2 .

Proof. Given the above discussion, the proof that Θ is a coarsely Lipschitz map is identical to the proof that the map Θ_0 is a coarsely Lipschitz map in Proposition 4.18 of [HH15] and will be omitted. Moreover, as for $\alpha \in \mathcal{A}(F)$, the sphere $\Upsilon(\alpha)$ intersects F in the unique arc α , we have $\Theta(\Upsilon(\alpha)) = \alpha$.

As a consequence, $\Theta|\Upsilon(\mathcal{A}(F))$ is a Lipschitz bijection, with inverse Υ . Then the subgraph $\Upsilon(\mathcal{A}(F))$ of $S\mathcal{G}$ is bilipschitz equivalent to $\mathcal{A}(F)$. Furthermore, the map $\Upsilon \circ \Theta$ is a Lipschitz retraction of $S\mathcal{G}$ onto $\Upsilon(\mathcal{A}(F))$. Then $\Upsilon(\mathcal{A}(F))$ is a quasi-isometrically embedded subgraph of $S\mathcal{G}$ which is moreover quasi-isometric to $\mathcal{A}(F)$.

Let as before F_0 be the surface obtained from F by removing the spot. We are left with showing that $\mathcal{A}(F)$ is quasi-isometric to $\mathcal{A}(F_0) \times \mathbb{Z}$. However, this was shown in Lemma 2.3. Namely, in the terminology used before, the boundary ∂F is an *I*-bundle generator in the trivial interval bundle H over F, and associating to an arc α the *I*-bundle over α defines an isomorphism of $\mathcal{A}(F)$ with the subgraph Ω of the disk graph of H used in Lemma 2.3. The statement now follows from Lemma 2.3. **Remark 2.11.** Most likely Proposition 2.10 holds true as well in the case that g = 2n + 1 is odd, and furthermore this can be deduced with the above argument using non-orientable surfaces. However, the analogue of Proposition 4.18 of [HH15] for non-orientable surfaces is not available, and we leave the verification of these claims to other authors.

3. Handlebodies and doubled handlebodies with two spots

The goal of this section is to show the second part of Theorem 2 and Theorem 3 from the introduction.

We begin with discussing briefly handlebodies of genus one. A handlebody of genus one with at most one spot on the boundary contains a single disk. This is used to establish

Proposition 3.1. The disk graph of a solid torus with two spots on the boundary is a tree.

Proof. Let H be a solid torus with two spots p_1, p_2 on the boundary. The handlebody H_1 obtained from H by removing the spot p_2 is a solid torus with one spot on the boundary. Let $\Phi_1 : \partial H \to \partial H_1$ be the natural spot removal map.

The handlebody H_1 contains a single disk D_1 , and this disk is non-separating. If $D \subset H$ is any non-separating disk then $\Phi_1(\partial D) = \partial D_1$. Thus by Theorem 7.1 of [KLS09], the complete subgraph of the disk graph of H whose vertex set is the set of non-separating disks in H is a tree T. This is the Bass-Serre tree for the splitting of $\pi_1(H_1, p_2)$ defined by D_1 . Equivalently, it is the tree dual to the curve ∂D_1 with its action of $\pi_1(H_1, p_2)$.

If $D \subset H$ is a separating disk then ∂D decomposes ∂H into a disk with two spots and a torus with the interior of a closed disk removed. In particular, $\Phi_1(\partial D)$ is peripheral. There is a single disk in H which is disjoint from D, and this disk is non-separating. Thus there is a single edge in \mathcal{DG} with one endpoint at D. The second endpoint is a vertex in the tree T.

Now if D is any non-separating disk then cutting H open along D yields a ball with four spots on the boundary. Two of these spots are the two copies of D. Any simple closed curve which separates these two distinguished spots from the remaining two spots is the boundary of a separating disk in H disjoint from D, and any disk disjoint from D arises in this way. There are countably many such disks.

As a consequence, the disk graph of H is an extension of the tree T which attaches to each vertex in T a countable collection of edges whose second endpoints are univalent. Thus this graph is a tree as well. The proposition follows.

Section 4 of [HH19] contains some closely related results for handlebodies of genus 2.

In most of the remainder of this paper we investigate a handlebody H of genus $g \geq 2$ with two spots p_1, p_2 on the boundary. Let H_0 be the handlebody of genus g without spots and let $\Phi : H \to H_0$ be the spot removing map. A disk D in H encloses the two spots p_1, p_2 if $\Phi(D) \subset H_0$ is homotopic to a point.

We next use the two spots to add a handle to H. The resulting manifold is a handlebody H' of genus g + 1 with one spot. To this end slightly enlarge the two spots p_1, p_2 to two small compact disjoint disks B_1, B_2 in ∂H with $p_i \in \partial B_i$. Identifying these two disks with an orientation reversing diffeomorphism $B_1 \to B_2$ which maps p_1 to p_2 yields a handlebody H' of genus g + 1. We may view the common image of the points p_1, p_2 as a spot $p \in \partial H'$. The fundamental group of H' is the free group F_{g+1} with g+1 generators. We choose the spot p of H' as the basepoint for the fundamental group of H'.

The following simple observation will be used several times later on.

Lemma 3.2. A disk D in H which encloses the two spots p_1, p_2 and the choice of one of the spots p_i determines a free splitting $\pi_1(H', p) = F_{g+1} = F_g * \mathbb{Z}$. Changing the spot changes the splitting by conjugation with a generator of the \mathbb{Z} -factor.

Proof. Up to isotopy, we may assume that the disk D is disjoint from the two closed disks B_1 and B_2 used in the construction of H'. Thus D determines a separating disk D' in H' which only depends on D. This disk cuts H' into a handlebody of genus g with fundamental group F_g and a solid torus T with fundamental group \mathbb{Z} which contains the basepoint p.

Van Kampen's theorem now shows that D' defines a free splitting

$$\pi_1(H', p) = F_{g+1} = F_g * \mathbb{Z},$$

unique up to conjugation with an element of the free factor \mathbb{Z} . Namely, the basepoint p is contained in the solid torus T. Thus the splitting of $\pi_1(H', p)$ obtained by van Kampen's theorem is determined by D' up to conjugation with an element of $\pi_1(T)$.

To see that if we fix one of the spots p_i then we obtain in fact a uniquely determined splitting, it suffices to observe that the solid torus T is obtained by identifying two disks in the boundary of a ball. This ball is fixed, but the disks are allowed to move within a fixed subdisk D of this boundary. As a disk is contractible, moving the two disks B_1, B_2 freely in D gives rise to the same splitting and hence there is no ambiguity in the construction (in other words, the fundamental group of the solid torus T appears only after the gluing).

The construction in Lemma 3.2 can be reversed. Namely, observe that the handlebody H' contains a distinguished non-separating disk V which is the image of the two disks in ∂H used in the construction. If $E \subset H'$ is any separating disk disjoint from V which decomposes H' into a solid torus $T \supset V$ and a handlebody of genus g, then E is the image of a disk in H enclosing the two spots under the glueing construction.

We next investigate the dependence of this splitting on the choice of the disk D enclosing the two spots. To this end note first that in the splitting $F_{g+1} = F_g * \mathbb{Z}$, a generator a of the free factor \mathbb{Z} is the image in H' of an embedded oriented arc in ∂H which is disjoint from D, whose interior is disjoint from $B_1 \cup B_2$ and which connects the spot p_1 to p_2 . As D decomposes the handlebody H into a handlebody of genus g without spots and a ball with two spots at the boundary, the homotopy class in H of such an arc with fixed endpoints is unique.

Now let $E \subset H$ be another disk which encloses the two spots p_1, p_2 . Assume that E is not freely homotopic to D and in minimal position with respect to D. This means in particular that the boundaries $\partial D, \partial E$ intersect in the minimal number of points among all representatives in their isotopy classes.

The simple closed curves ∂D , ∂E are the boundaries of unique disks $\tilde{D}, \tilde{E} \subset \partial H$ containing the two spots (thus if we think of the spots as missing points, then \tilde{D}, \tilde{E} should be viewed as twice punctured disks). The intersection $\tilde{D} \cap \tilde{E}$ consists of two disjoint disks A_1, A_2 with one spot at p_1, p_2 , respectively, and a disjoint union of rectangles. In particular, $\partial D \cap \partial E$ consists of at least four points.



Use the ordered pair of disks (\tilde{D}, \tilde{E}) to construct a loop $\gamma \subset (\partial H \cup \{p_2\}, p_2)$ based at p_2 as follows. First, connect the point p_1 to the point p_2 by an oriented arc α whose interior is embedded in \tilde{D} . The endpoints of α are the two spots of H, and they are precisely the spots of the disk \tilde{E} . Thus there is an embedded arc β in \tilde{E} connecting p_1 to p_2 . The loop γ is homotopic to the concatenation of α^{-1} with β (which we move off the spot p_1 with a small deformation). Note that the inverse of the loop γ is constructed with exactly the same procedure, but with the roles of the disks D, E exchanged. Furthermore, the homotopy class of γ as a loop in ∂H_0 based at p_2 is uniquely determined by the ordered pair (D, E). Namely, the ambiguity in the above construction consists in the precomposition with the homotopy class of a loop in \tilde{D} based at p_2 which surrounds the marked point p_1 .

To summarize, each ordered pair (D, E) of disks in H enclosing the two spots determines uniquely a homotopy class $\hat{q}(D, E) \in \pi_1(\partial H_0, p_2)$. We have $\hat{q}(E, D) = \hat{q}(D, E)^{-1}$ for all D, E, in particular, $\hat{q}(D, D)$ is the neutral element in $\pi_1(\partial H_0, p_2)$.

Lemma 3.3. The disk E is the image of the disk D under the element of the handlebody group of H induced by pushing the point p_2 along a based loop γ in the homotopy class $\hat{q}(D, E) \in \pi_1(\partial H_0, p_2)$.

Proof. Let $\gamma \subset \partial H \cup \{p_2\}$ be a loop based at p_2 constructed as above from the ordered pair (\tilde{D}, \tilde{E}) of disks and moved off p_1 . Moving this loop off p_1 depends on a choice. We first observe that the disk obtained from D by point pushing p_2 along γ does not depend on the choice made. For this it suffices to observe that point pushing p_2 along a loop ζ based at p_2 which is entirely contained in the disk D and encircles p_1 preserves the disk D.

Now up to homotopy with fixed basepoint, the loop ζ is embedded in a disk $\hat{D} \supset \tilde{D}$ which is isotopic to \tilde{D} . Point pushing of p_2 along ζ can be represented by a diffeomorphism which fixes $\partial H - \hat{D}$ pointwise and hence defines an element of the pure Artin braid group of a disk with two marked points. This pure braid group is just the group of Dehn twists about the boundary of \hat{D} , and it preserves this boundary pointwise. But the boundary of \hat{D} is isotopic to the boundary of D. This implies that indeed, point pushing of p_2 along ζ preserves the disk D.

Let us denote as before by α an embedded arc in \tilde{D} connecting p_1 to p_2 , and let β be an embedded arc in \tilde{E} connecting p_1 to p_2 . Assume first that the loop γ which is the concatenation of α^{-1} with β , moved off p_1 , is simple, that is, it does not have self-intersections. Thus γ is embedded in $\partial H \cup \{p_2\}$ and hence there is an embedded annulus $A \subset \partial H \cup \{p_2\}$ with core curve γ , disjoint from p_1 . Up to isotopy, the arc α intersects A in a single embedded segment α_0 with one endpoint p_2 . Furthermore, we may assume that α_0 intersects the core curve γ of A in the unique point p_2 .

The point pushing diffeomorphism of p_2 along γ equals the identity on $(\partial H \cup \{p_2\} - A) \cup \gamma$. In each of the two components of $A - \gamma$, it is a Dehn twist about the core curve (which is freely homotopic to γ), in one component positive, in the second negative. As α_0 and hence α meets only one component of $A - \gamma$, this diffeomorphism transforms the homotopy class of α with fixed endpoint by concatenation with γ . As a consequence, the image of the disk \tilde{D} by the point pushing map along γ is isotopic to a thickening of the arc with endpoints p_1, p_2 which is homotopic to the concatenation of α with γ (read from left to right). This arc is homotopic to β and hence this disk is the disk \tilde{E} .

This construction extends to the case that the curve γ has self-intersections or, equivalently, that the intersection $\tilde{D} \cap \tilde{E}$ has at least one rectangle component. Namely, parameterize the arc β on the interval [0, 1]. Assuming that β is in minimal position with respect to α , let $0 = t_0 < t_1 < \cdots < t_k = 1$ be such that $\beta(t_i)$ are the intersection points of β with α , with the endpoints included. Let γ_1 be the concatenation of α^{-1} with an arc ζ_1 connecting p_1 to p_2 which is composed of $\beta[0, t_1]$ and the subarc α_1 of α connecting $\beta(t_1)$ back to p_2 . With a small homotopy with fixed endpoint, the arc ζ_1 can be pushed off the interior of α . The resulting loop based at p_2 is simple and can be pushed off p_1 . Denote this loop again by γ_1 .

In a second step, define a based loop γ_2 at p_2 as follows. Let $2 \leq j \leq k$ be such that $\beta(t_j)$ is the first intersection point of $\beta[t_1, 1]$ with the subarc α_1 of α (perhaps this is the endpoint p_2 of β). Let γ_2 be the concatentation of ζ_1^{-1} , the subarc $\beta[0, t_j]$ of β and the subarc α_2 of α connecting $\beta(t_j)$ back to p_2 . This loop is homotopic with fixed endpoints to the concatentation of α_1^{-1} , the arc $\beta[t_1, t_j]$ and the arc α_2 and hence it is simple. Furthermore, it can also be described as the concatentation of ζ_1^{-1} and an arc ζ_2 connecting p_1 to p_2 .

Proceeding inductively, define for $\ell \leq m$ (where $m \leq k$ is a number computed from the order in which the rectangle components of $\tilde{D} \cap \tilde{E}$ are passed through by β) a based loop γ_{ℓ} at p_2 composed of the arc $\zeta_{\ell-1}^{-1}$ and an arc ζ_{ℓ} connecting p_1 to p_2 which is a concatenation of a subarc $\beta[0, t_{j(\ell)}]$ of β and the subarc of α connecting $\beta(t_{j(\ell)})$ back to p_2 . Up to homotopy, the loops γ_{ℓ} are simple, and we have $\gamma = \gamma_1 \circ \cdots \circ \gamma_m$ (read from left to right).

By the first part of this proof, the image of D under point pushing along γ_1 is a disk \tilde{D}_1 which is a thickening of the arc ζ_1 . By the above construction, the arc ζ_1 intersects γ_2 only at the point p_2 . Thus using again the first part of this proof, point pushing of \tilde{D}_1 along the loop γ_2 yields a disk \tilde{D}_2 which is a thickening of the arc ζ_2 . As the point pushing group is a group, we also know that \tilde{D}_2 is the image of \tilde{D} under point pushing along $\gamma_1 \circ \gamma_2$ (read from left to right). In $m \leq k$ such steps we deduce that indeed, the disk \tilde{E} is the image of \tilde{D} under pointpushing of p_2 along γ .

Let us summarize what we obtain from Lemma 3.3 and its proof. Let $D \subset H$ be a disk enclosing the two spots p_1, p_2 . Its boundary ∂D is a simple closed curve in ∂H . It determines the homotopy class of an arc in ∂H_0 with endpoints p_1, p_2 . Let us choose such an arc $\alpha \subset \partial H_0$, oriented in such a way that it connects p_1 to p_2 , and let *a* be its homotopy class with fixed endpoints as an arc in ∂H_0 .

Let $E \subset H$ be another such disk which determines the homotopy class b of an oriented arc $\beta \subset \partial H_0$ connecting p_1, p_2 . The concatenation $\alpha^{-1} \circ \beta$ (read from left to right) is a loop based at p_2 . It defines the homotopy class $a^{-1} \cdot b \in \pi_1(\partial H, p_2)$.

Move as before $\alpha^{-1} \circ \beta$ off p_1 . Point pushing p_2 along $\alpha^{-1} \cdot \beta$ defines the isotopy class of a diffeomorphism of ∂H . It can be represented by a diffeomorphism which equals the identity on the complement of a small neighborhood of $\alpha^{-1} \circ \beta$, and we may assume that the point p_1 is contained in this neighborhood. The image of the disk D under this diffeomorphism is the disk E.

The next lemma reverses this observation by describing the image of a disk D enclosing the two spots under point pushing of the point p_2 along a based loop at p_2 . As before, all homotopies of arcs are homotopies in the boundary surface, however the homotopy classes appearing in the lemma are only homotopy classes with fixed endpoints of arcs where the homotopies are allowed to cross through the endpoints.

Lemma 3.4. Let $D \subset H$ be a disk enclosing the two spots and let $\alpha \subset \partial H_0$ be an arc connecting p_1 to p_2 which does not cross through the boundary of D. Let E be the image of D under point pushing p_2 along a loop γ based at p_2 , constructed as above from a based loop $\gamma \subset \partial H \cup \{p_2\}$ based at p_2 . Then an arc $\beta \subset \partial H_0$ which connects p_1 to p_2 and does not cross through the boundary of E is homotopic in ∂H_0 with fixed endpoints to $\alpha \circ \gamma$ (read from left to right).

Proof. Assume for the moment that the based loop γ is simple, that is, it does not have self-intersections. Recall that $p_1 \notin \gamma$. Let $A \subset \partial H_0$ be an embedded closed annulus with core curve γ . By putting the arc α in minimal position with respect to γ and parameterizing α on [0, 1], we may assume that $A \cap \alpha = \bigcup_{i=1}^{k} \alpha[t_{2i-1}, t_{2i}]$ where $k \geq 1$ and where $0 < t_1 \cdots < t_{2k} = 1$. Furthermore, the arc $\alpha[t_{2k-1}, t_{2k}]$ intersects γ only at p_2 , and for each $i \leq k - 1$ the arc $\alpha[t_{2i-1}, t_{2i}]$ crosses through A and intersects γ in a single point.

Choose a diffeomorphism supported in A which represents the point pushing transformation of p_2 about γ and preserves γ pointwise as in the proof of Lemma 3.3. This diffeomorphism changes the homotopy class of the arc $\alpha[t_{2k-1}, 1]$ with fixed endpoints by concatenation with the homotopy class of γ . It does not change the homotopy class of any of the arcs $\alpha[t_{2i-1}, t_{2i}]$ with fixed endpoints as these arcs are modified by the concatenation of a Dehn twist about γ and its inverse. Thus the homotopy class of the arc connecting p_1 to p_2 which is determined by the image of D under point pushing along γ is the class of the concatenation $\alpha \circ \gamma$.

As any based loop at γ is homotopic with fixed endpoints to a concatentation of simple based loops, applying the group law to both point pushing maps and homotopy classes of arcs in ∂H_0 connecting p_1 to p_2 yields the lemma.

Remark 3.5. Recall from Lemma 3.2 that each disk D in H enclosing the two spots defines a free splitting $F_{g+1} = F_g * \mathbb{Z}$. Lemma 3.4 immediately implies the following. If $a \in F_{g+1}$ is the generator of the \mathbb{Z} -factor in the free splitting of F_{g+1} defined by the disk D and the choice of the basepoint p_1 , viewed as a homotopy class with fixed endpoints of an arc connecting p_1 to p_2 , and if E is the image of Dby point pushing p_2 along a loop in the homotopy class $\hat{q}(D, E) \in \pi_1(\partial H_0, p_2)$, then the \mathbb{Z} -factor defined by p_1 and the disk E is generated by $a \cdot \iota_* \hat{q}(D, E) = q(D, E)$ (read from left to right) where $\iota : \partial H \cup \{p_2\} \to H$ is the inclusion homomorphism.

While Lemma 3.3 and Lemma 3.4 are true for curves in the curve graph of ∂H which enclose the two spots, we now study more specifically disks enclosing the two spots of ∂H as elements of the disk graph of H. Namely, as the proof of Lemma 3.3 shows, the image of a disk D enclosing the two spots under point pushing along a based loop in $\partial H \cup \{p_2\}$ depends in a sensitive way on the loop and not only on its homotopy class in $\pi_1(\partial H \cup \{p_2\}, p_2)$ due to clearing intersections which pushes arcs around the spot p_2 . Furthermore, paths in the curve graph of ∂H whose vertices consist of curves enclosing the two spots may be highly inefficient. In fact, as the curve graph of ∂H is hyperbolic and such paths are also paths in the disk graph of H, Theorem 2 confirms that this is the case.

The main feature of the disk graph of a handlebody is the possibility that two points in the disk graph whose boundaries are close in the curve graph have large distance in the disk graph. An *I*-bundle generator is the source of such a hole as explained earlier, and this is the feature we are going to use now as well.

Assume for the moment that the genus g of H_0 is even. Choose a compact surface $F \subset \partial H_0$ of genus h = g/2 with connected boundary ∂F so that H_0 is the oriented I-bundle over F. Let Φ be the involution of H_0 which exchanges the two endpoints of the intervals in the interval bundle. Choose a point $p_2 \in \partial F$, let $p_1 = \Phi(p_2)$ and define $H = H_0 - \{p_1, p_2\}$. The thickening of the interval with endpoints p_1, p_2 is a disk enclosing the two spots.

The disk D can be modified by point pushing p_2 along any loop in F based at p_2 . It can also be modified by point pushing p_1 along any loop in $\Phi(F)$ based at p_1 . Since a point pushing diffeomorphism along a loop γ is supported in a small neighborhood of γ , these point pushing operations commute.

Denote as before by $d_{\mathcal{DG}}$ the distance in the disk graph. The following observation is geared at relating the effects of these two constructions.

Lemma 3.6. Let $E \subset H$ be a disk enclosing the two spots which is invariant under Φ and intersects the fixed point set of Φ in a single arc; then $d_{\mathcal{CG}}(E, D) \leq 2$.

Proof. Note first that the presence of the two spots in ∂H fills the hole of the *I*bundle generator ∂F . Namely, recall from Lemma 2.1 that the arc graph of F is quasi-isometrically embedded in the disk graph of H_0 by associating to an arc ζ in F the *I*-bundle over ζ . However, as we may assume that none of the two endpoints of the arc equals the point p_2 , up to isotopy this disk is disjoint from the disk Denclosing the two spots (a thickening of an interval of the *I*-bundle) and hence its distance to D equals one.

Now consider a Φ -invariant disk E enclosing the two spots which intersects the fixed point set of Φ in a single arc. We view E as a Φ -invariant thickened arc in ∂H connecting p_1 to p_2 . For the proof of the lemma, it suffices to find an essential arc $\zeta_0 \subset F$ such that the disk E is disjoint from the *I*-bundle over ζ_0 . To this end note first that since E is invariant under Φ and Φ reverses the orientation, the intersection of E with the open annulus $A \subset \partial H$ bounded by $\partial F, \Phi(\partial F)$ is untwisted relative to its foliation by intervals. Namely, by invariance, any twisting of E about the core curve of the annulus A near F is followed by twisting about the core curve of the annulus A near $\Phi(F)$ in the opposite direction and hence twisting can be removed with a homotopy.

Since the disk E intersects the fixed point set of Φ in a single arc, up to homotopy its intersection with the annulus A is the union of a rectangle R which connects ∂F to $\Phi(\partial F)$, the intersection with A of two small disks B_1, B_2 centered at p_1, p_2 and a nested collection of small rectangles surrounding B_1, B_2 , with two opposite sides on the same component of ∂A (see the proof of Lemma 3.4 for an illustration). We may assume that the number of such rectangles is minimal in the isotopy class of E. This is equivalent to stating the E is in minimal position with respect to $\partial F, \Phi(\partial F)$.

Choose a fixed point x for Φ in the boundary of the rectangle R and follow the boundary of E in both directions starting from x until its first intersection point y with the closure of B_i or the closure of one of the rectangles surrounding B_i . One of these points is contained in ∂F , and the second points is its image under Φ . The subsegment of ∂E with endpoints $y, \Phi(y)$ which contains x in its interior is an embedded Φ -invariant arc ζ with one endpoint $y \in \partial F$ and the second endpoint $\Phi(y) \in \Phi(\partial F)$. If E is not isotopic to D, then since E is in minimal position with respect to $\partial F, \Phi(\partial F)$, the arc ζ is not homotopic with fixed endpoints to a fiber of the interval bundle and hence it intersects F in an essential arc ζ_0 with endpoints on ∂F . Then ζ is contained in the boundary of the *I*-bundle H over ζ_0 , and up to isotopy, the disk E is disjoint from the disk H. As $d_{\mathcal{CD}}(H, D) = 1$, this completes the proof of the lemma.

We use Lemma 3.6 to deduce

Corollary 3.7. Let $\gamma \subset F$ be a loop based p_2 and let $E_1, E_2 \subset H$ be the disks enclosing the two spots which are obtained from D by pushing p_2 along γ and pushing p_1 along $\Phi(\gamma)^{-1}$, respectively; then $d_{\mathcal{CG}}(E_1, E_2) \leq 2$.

Proof. Since point pushing disks along loops based at p_1 defines a group of isometries of \mathcal{DG} , it suffices to show that for any loop $\gamma \subset F$ based at p_2 , the distance in \mathcal{DG} between the disk D and the disk E obtained from D by first point pushing p_2 along γ and then point pushing p_1 along $\Phi(\gamma)$ equals at most two.

However, such a disk is Φ -invariant by construction. and it intersects the fixed point set of Φ in a single arc. Hence the corollary follows from Lemma 3.6.

Remark 3.8. Lemma 3.7 extends to handlebodies of odd genus $g = 2h + 1 \ge 3$ as follows. Let F be a non-orientable surface of Euler characteristic -2h with connected boundary ∂F . It can be represented as the connected sum of an orientable surface of genus h with connected boundary and a projective plane. Equivalently, F contains an orientable subsurface $F_0 \subset F$ with two boundary components c_0, c_1 , and F is obtained from F_0 by gluing a Möbius band to the boundary component c_1 . The fundamental group of F_0 is an index two subgroup of the fundamental group of F. As the surface F_0 is oriented, its preimage in the orientation cover Fof F consists of two disjoint copies of F_0 , and \tilde{F} is obtained from these two copies of F_0 by connecting the two components of the preimage of c_1 with an annulus (which is the orientation cover of the Möbius band). The oriented I-bundle over F contains the trivial I-bundle over the bordered subsurface F_0 as a submanifold. We then can apply the construction in Corollary 3.7 to point pushing of a point $p_2 \in \partial F_0 \subset \partial F$ along loops in F_0 and point pushing of $p_1 = \Phi(p_2)$ along loops in $\Phi(F_0)$ where as before, Φ is the orientation reversing involution of H_0 which exchanges the endpoints of the intervals of the *I*-bundle.

22

From now on we fix a disk D enclosing the two spots in H which is a thickening of an interval in an I-bundle over a compact surface F with connected boundary. If the genus of H is even then we assume that F is orientable. This disk defines a free splitting $F_{g+1} = F_g * \mathbb{Z}$ where the free factor \mathbb{Z} is generated by an element aobtained from an embedded oriented arc in the twice spotted disk \tilde{D} in ∂H with the same boundary as D which connects p_1 to p_2 . The free factor F_g in the free splitting $F_{g+1} = F_g * \mathbb{Z}$ is naturally isomorphic to $\pi_1(H \cup p_2, p_2)$. Thus a free basis $\mathcal{A} = \{a_1, \ldots, a_g\}$ of $F_g = \pi_1(H \cup p_2, p_2)$ extends to a free basis $\hat{\mathcal{A}} = \{a_1, \ldots, a_g, a\}$ of F_{g+1} .

We now use a device from [SS14]. Define the Whitehead graph $\Gamma_{\mathcal{A}}(x)$ of a word $x \in F_g$ in a free basis $\mathcal{A} \cup \mathcal{A}^{-1}$ of F_g as follows. The set of vertices of $\Gamma_{\mathcal{A}}(x)$ is identified with the set $\mathcal{A} \cup \mathcal{A}^{-1}$. Each pair of consecutive letters $a_i a_j$ in the word x contributes one edge from the vertex a_i to the vertex a_j^{-1} . Thus if the length of x equals n then $\Gamma_{\mathcal{A}}(x)$ has n-1 edges, and $\Gamma_{\mathcal{A}}(x)$ has a cut vertex if $x \in \mathcal{A}$. Furthermore, if $\Gamma_{\mathcal{A}}(x)$ has a cut vertex, then the same holds true for the unique reduced word which defines the same element of F_g as x.

Following [SS14], define the simple g + 1-length

$$|w|_{q+1}^{simple}$$

of any reduced word w in the free basis $\mathcal{A} = \{a_1, \ldots, a_g\}$ of F_g to be the greatest number t such that w is of the form $w_1w_2\cdots w_t$ where the Whitehead graph of w_j with respect to the basis \mathcal{A} has no cut vertex for each $j = 1, \ldots, t$. If the Whitehead graph of w has a cut vertex then the simple g+1-length of w is defined to be zero. We have that $|w|_{g+1}^{simple}$ is bounded from above by the word length of the reduced word w with respect to the basis \mathcal{A} . Furthermore, $|w^{-1}|_{g+1}^{simple} = |w|_{g+1}^{simple}$. The terminology here is taken from [SS14] although it is not well adapted to the situation at hand.

The following statement combines Lemma 4.6 and Lemma 4.7 of [SS14],

Lemma 3.9. (1) $|u|_{g+1}^{simple} \ge |v|_{g+1}^{simple}$ whenever v is a subword of u. (2) $|w|_{g+1}^{simple} \le |u|_{g+1}^{simple} + |v|_{g+1}^{simple} + 1$

if u, v are freely reduced words in the letters $\mathcal{A} \cup \mathcal{A}^{-1}$ and w = uv.

Proof. The statement of Lemma 4.7 of [SS14] shows the second part of the lemma only in the case that w = uv is freely reduced. To show that it is true as stated, assume that $|v|_{g+1}^{simple} = 0$ and that w is the reduced word representing uv. Then w is obtained from uv by erasing some letters at the end of u and the beginning of v. In particular, by the first part of the lemma, the Whitehead graph of the subword of v which is contained in w has a cut vertex.

As a consequence, if $w = w_1 \cdots w_t$ where the Whitehead graph of w_i does not have a cut vertex, then as u is reduced, $w_1 \cdots w_{t-1}$ is a subword of u. Then $t-1 \leq |u|_{g+1}^{simple}$ by the first part of the lemma and hence $|w|_{g+1}^{simple} \leq |u|_{g+1}^{simple} + 1$ as claimed.

The general case follows from a rather straightforward modification of this argument and will be omitted. Only the case that $|v|_{g+1}^{simple} = 0$ is used in the sequel. \Box

The next lemma relates simple g + 1-length to the disk graph \mathcal{DG} of H. To simplify the notation, in the sequel we call a sequence (D_i) of disks in H a path

in \mathcal{DG} if for all *i* the disk D_i is disjoint from D_{i+1} . Thus such a sequence is the set of integral points on a simplicial path in \mathcal{DG} connecting its endpoints. For its formulation, recall from Remark 3.5 that a pair (D, E) of disks enclosing the two spots p_1, p_2 determines uniquely an element $q(D, E) \in \pi_1(H_0, p_2)$.

Lemma 3.10. Let $(D_i)_{0 \le i \le n}$ be a path in \mathcal{DG} which begins and ends with a disk enclosing the two spots p_1, p_2 . Let $w = \iota q(D_0, D_n) \in \pi_1(H_0, p_1)$; then

$$|w|_{a+1}^{simple} \le 2n$$

Proof. Assume without loss of generality that the path (D_i) connecting D_0 to D_n is of minimal length in \mathcal{DG} . First we modify inductively the sequence (D_i) without increasing its length in such a way that each of the disks D_i $(1 \le i \le n-1)$ either is non-separating or encloses the spots p_1, p_2 .

The construction proceeds in two steps. In a first step, we replace each separating disk D_{2i-1} with odd index by a disk which either is non-separating or encloses the two spots. We do not change the disks D_{2i} with even index. In a second step, we then modify the disks with even index and preserve those with odd index.

To carry out the first step, let $\ell \leq n/2$ and assume that the disk $D_{2\ell-1}$ is separating and does not enclose the spots; otherwise there is nothing to do. If $D_{2\ell-2}, D_{2\ell}$ are contained in distinct components of $H-D_{2\ell-1}$ then they are disjoint. In this case we can remove $D_{2\ell-1}$ from the path (D_i) and obtain a shorter path with the same endpoints. Since the path (D_i) has minimal length this is impossible.

Thus $D_{2\ell-2}, D_{2\ell}$ are contained in the same component V of $H - D_{2\ell-1}$. Since $D_{2\ell-1}$ does not enclose the spots, neither of the two components of $\partial H - D_{2\ell-1}$ is a three-holed sphere. Since H has precisely two spots, this implies that the genus of each of the two components of $\partial H - D_{2\ell-1}$ is positive. Now each component of $H - D_{2\ell-1}$ is a handlebody with spots and therefore the component H - V of $H - D_{2\ell-1}$ contains a non-separating disk $\tilde{D}_{2\ell-1}$. Replace $D_{2\ell-1}$ by $\tilde{D}_{2\ell-1}$.

Replace in this way any disk $D_{2\ell-1}$ with an odd index which is separating but does not enclose p_1, p_2 by a non-separating disk without modifying the disks D_{2i} with even index. This implements the first step of the construction. The second step is exactly identical after exchanging the roles of even and odd index. To summarize, we may assume from now on that every separating disk in the path (D_i) encloses the two spots.

From the path (D_i) we next construct a path $(E_j)_{0 \le j \le 2u}$ of disks connecting D_0 to D_n whose length 2u is at most four times the length n of the path (D_i) and such that for each j, the disk E_{2j} encloses the spots p_1, p_2 and the disk E_{2j-1} is non-separating.

To this end recall that any two distinct disks which enclose the spots p_1, p_2 intersect. This means that if the disk D_i from the above sequence encloses the spots then the disks D_{i-1}, D_{i+1} are non-separating. Thus for the construction of the path (E_j) it now suffices to replace any consecutive pair D_i, D_{i+1} of disjoint non-separating disks by a path of length at most four with the same endpoints whose vertices alternate between non-separating disks and disks enclosing the spots.

Let i < n-1 be such that the disks D_i, D_{i+1} are both non-separating. If $H - (D_i \cup D_{i+1})$ is connected then there is a disk B which encloses the spots p_1, p_2 and which is disjoint from $D_i \cup D_{i+1}$. Such a disk can be obtained by thickening an embedded arc in $\partial H - (D_i \cup D_{i+1})$ which connects p_1 to p_2 . Replace the consecutive pair D_i, D_{i+1} by the path D_i, B, D_{i+1} of length two.

If $H - (D_i \cup D_{i+1})$ is disconnected and if the spots p_1, p_2 are both contained in the same component of $\partial H - (D_i \cup D_{i+1})$ then we can proceed as in the previous paragraph. Otherwise there is a component of $\partial (H - (D_i \cup D_{i+1}))$ which is a surface of genus $g \ge 1$ with three holes. One of the holes is a spot, the other two holes are boundary components given by the boundary circles of D_i, D_{i+1} . Hence there is a non-separating disk B which is disjoint from $D_i \cup D_{i+1}$ and such that $H - (D_i \cup B)$ and $H - (D_{i+1} \cup B)$ are both connected. Replace the consecutive pair D_i, D_{i+1} by a path of length 4 of the form $D_i, A_1, B, A_2, D_{i+1}$ so that consecutive disks in this sequence are disjoint and that the disks A_1, A_2 both enclose the spots p_1, p_2 . This completes the construction of the sequence (E_j) .

For each j the disk E_{2j} defines a free splitting of F_{g+1} of the form $F_g * \mathbb{Z}$. If a is the generator of the free factor \mathbb{Z} for the splitting defined by D_0 (in the sense as before, namely we think of a as a homotopy class of an embedded arc in ∂H which connects p_1 to p_2 which does not intersect the boundary of D_0 , and this homotopy class determines the free factor \mathbb{Z} in the free splitting defined by D_0) then for each j the free factor \mathbb{Z} for the free splitting defined by E_{2j} is generated by $a \cdot q(D_0, E_{2j})$ where $q(D_0, E_{2j}) \in F_q$.

Let $w_j = q(D_0, E_{2j})^{-1}q(D_0, E_{2j+2}) \in F_g$. By construction, the disk E_{2j+1} in H is non-separating and disjoint from E_{2j}, E_{2j+2} . The set of all loops in $H \cup \{p_2\}$ with basepoint p_2 which do not intersect E_{2j+1} define a free factor Q of F_g of corank one. Since E_{2j+1} is disjoint from E_{2j} and E_{2j+2} , the element w_j is contained in the free factor Q.

Since $w_j \in Q$, by Theorem 2.4 of [S00] the Whitehead graph of w_j has a cut vertex. But this just means that the simple g+1-length of w_j vanishes. An inductive application of the second part of Lemma 3.9 now shows that the simple g+1-length of the word $q(D_0, D_n) \in F_g$ is at most $u \leq 2n$. This is what we wanted to show. \Box

Now we are ready to show the second part of Theorem 2 from the introduction.

Proposition 3.11. For every $n \ge 1$, the disk graph of a handlebody H of genus $g \ge 3$ with two spots contains quasi-isometrically embedded copies of \mathbb{R}^n .

Proof. Assume first that the genus g = 2h of H is even. We will explain at the end of this proof how to adjust the argument to handlebodies of odd genus.

Let $F \subset \partial H_0$ be an embedded oriented surface with connected boundary ∂F such that H_0 equals the *I*-bundle over ∂F . Let Φ be the orientation reversing involution of H_0 which exchanges the endpoints of the intervals which make up the interval bundle. Fix a point $p_2 \in \partial F$ and let $p_1 = \Phi(p_2)$.

Arrange the $h = g/2 \ge 2$ handles of the surface F cyclically around ∂F . Choose for each handle of F two oriented disjoint essential arcs in the handle with endpoints on ∂F . We may assume that ∂F is partitioned into h segments I_1, \ldots, I_h with disjoint interior, ordered cyclically along ∂F (that is, if $h \ge 3$ then $I_j \cap I_{j+1}$ consists of a single point for all j) so that each of these segments I_j contains all four endpoints of the arcs $\hat{a}_{2j-1}, \hat{a}_{2j}$ which are embedded in one of the handles of F. A small neighborhood of the union of these 2h arcs and the boundary of F is a ribbon graph, that is, a planar surface $F_0 \subset F$. We require that the inclusion $F_0 \to F$ induces a surjection on fundamental groups. This is equivalent to stating that F can be obtained from F_0 by attaching a disk to the component of the boundary of F_0 distinct from ∂F . If $h \geq 3$ then let p_2 be the intersection $I_h \cap I_1$ and let $x = I_{h-1} \cap I_h$. If h = 2 then we require that $\{p_2, x\} = I_1 \cap I_2$. Slide the endpoints of the arcs \hat{a}_i which define the ribbon graph F_0 along ∂F to p_2 in such a way that this sliding operation does not cross through x. The image of each of the arcs \hat{a}_i under this homotopy is a based oriented loop a_i at p_2 . The union of these loops is an embedded rose R with vertex p_2 (the rose R does not contain the boundary circle of F). As H_0 is an I-bundle over F, the inclusion $R \to H \cup \{p_2\}$ induces an isomorphism of $Q = \pi_1(R, p_2)$ onto the group $\pi_1(H \cup \{p_2\}, p_2)$ which is isomorphic to the fundamental group of H. Thus if we write $H_2 = H \cup \{p_2\}$ then we have $\pi_1(H_2, p_2) = Q$. In the sequel we think of the based loops a_i $(i = 1, \ldots, 2h)$ as generators of the fundamental group Q of R.

As on p.592 in Subsection 5.2 of [SS14], we consider for $t \ge 1$ the element

$$b_t = a_1^{t+1} a_2^{t+1} \cdots a_q^{t+1} a_1^{t+1} a_2^{t+1} a_1^{t+1} \in Q.$$

We claim that for every $t \ge 1$ the image of D under the point pushing map of p_2 along b_t has distance at most 6 to D in the disk graph \mathcal{DG} .

We show the claim first in the case that the genus g of H is at least 6 and hence the genus of F is at least three. Then $b_t = uv$ where $u = a_1^{t+1} \cdots a_{g-2}^{t+1}$ and $v = a_{g-1}^{t+1}a_g^{t+1}a_1^{t+1}a_2^{t+1}a_1^{t+1}$. The word u does not contain the letters $a_{g-1}, a_{g-1}^{-1}, a_g, a_g^{-1}$, and the word v does not contain the letters $a_{g-3}, a_{g-3}^{-1}, a_{g-2}, a_{g-2}^{-1}$ since $g - 3 \ge 3$. As a consequence, the word u is represented by a loop on the rose R whose image

As a consequence, the word u is represented by a loop on the rose R whose image in the ribbon graph F_0 is disjoint from the arcs with endpoints in I_h . Hence up to homotopy, this loop is disjoint from the *I*-bundle over each of these two arcs. Then the same holds true for the image $\psi_u(D)$ of the disk D under point pushing along u. In particular, the distance between D and $\psi_u(D)$ in the disk graph $\mathcal{D}\mathcal{G}$ is at most two (see Lemma 3.6 for an explanation). Similarly, the image $\psi_v(D)$ of D under the point pushing map ψ_v is disjoint from an *I*-bundle over an arc with endpoints in the interval I_{h-1} and hence $d_{\mathcal{D}\mathcal{G}}(D, \psi_v(D)) \leq 2$. But the point pushing map ψ_v acts on the disk graph as a simplicial isometry and consequently $d_{\mathcal{D}\mathcal{G}}(\psi_v(D), \psi_v(\psi_u(D))) \leq 2$. Together with the triangle inequality, this yields

$$d_{\mathcal{DG}}(D,\psi_{uv}(D)) \le 4$$

(here words are read from left to right).

If g = 4 then write $b_t = uvw$ where $u = a_1^{t+1}a_2^{t+1}$, where $v = a_3^{t+1}a_4^{t+1}$ and $w = a_1^{t+1}a_2^{t+1}a_1^{t+1}$. Then there is a loop on R representing u, v, w which is disjoint from an arc with endpoints in I_2, I_1, I_2 . As in the previous paragraph, we conclude that $d_{\mathcal{DG}}(D, \psi_s(D)) \leq 2$ for s = u, v, w and by the triangle inequality, the distance between D and $\psi_{b_t}(D)$ is at most 6.

This argument can be used inductively and shows the following. For all $t \ge 1$ and each $k \ge 1$, we have

(2)
$$d_{\mathcal{DG}}(D, \psi_{b_{k}^{k}}(D)) \leq 6k.$$

Recall that the disk D defines a free splitting $\pi_1(H', p) = F_g * \mathbb{Z}$ where as before, $p \in \partial H'$ is the point obtained by identification of p_1 and p_2 . Let a be the generator of the infinite cyclic group \mathbb{Z} , defined by the homotopy class of the arc α in ∂H connecting p_2 to p_1 which is disjoint from the boundary of D. As explained in the discussion preceding Remark 3.5, if $u \in Q$ is arbitrary, then the image of D under the point-pushing map ψ_u is a disk $\psi_u(D)$ enclosing the two spots which defines the free splitting of F_{g+1} where the infinite cyclic free factor in the splitting is generated

26

by $a \cdot q(D, \psi_u(D))$. By the definition of the point pushing map, if we identify Q with $\pi_1(H, p_2) = F_g < \pi_1(H', p)$ as described in the beginning of this proof, the generator of this infinite cyclic free factor is just the element au. We refer to the discussion before Lemma 3.3 for more details.

Using the above notations, we follow Section 5.2 of [SS14]. For an arbitrary integer $n \geq 1$, define a map $\Lambda : \mathbb{Z}^n \to \mathcal{DG}$ which associates to $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ the image of the disk D under point-pushing of p_2 along the loop $b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n} \in Q$ based at p_2 . We claim that

(3)
$$d_{\mathcal{DG}}(\Lambda(k_1,\ldots,k_n),\Lambda(\ell_1,\ldots,\ell_n)) \le 6\sum_{i=1}^n (|k_i-\ell_i|+8).$$

To see this we adapt an argument from p.594 of [SS14]. Our goal is to transform the disk $\Lambda(k_1, \ldots, k_n) = \psi_{b_1^{k_1} \ldots b_n^{k_n}}(D)$ to the disk $\Lambda(\ell_1, \ldots, \ell_n) = \psi_{b_1^{\ell_1} \ldots b_n^{\ell_n}}(D)$ in a controlled way. These disks are determined by the homotopy classes $b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n} \in Q$ and $b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n} \in Q$, respectively, provided that the base disk D is fixed. To take full advantage of this fact we will now consider pairs of disks (E, V) where we view V as a basepoint, and E as a modification of the basepoint. With this viewpoint, our goal will be to transform the pair $(\psi_{b_1^{k_1} \ldots b_n^{k_n}}(D), D)$ to the pair $(\psi_{b_1^{\ell_1} \ldots b_n^{\ell_n}}(D), D)$ in a way which enables us to estimate distances.

To simplify the discussion, let us introduce the following notation. For an element $u \in Q$, represented up to homotopy by a unique reduced edge path in the rose R, let us denote by $[a \cdot u]_2$ the disk $\psi_u(D)$ obtained from D by point pushing p_2 along u, and denote by $[a^{-1} \cdot u]_1$ the disk obtained from D by point pushing p_1 along $\Phi(u)$. Corollary 3.7 shows that

(4)
$$d_{\mathcal{CG}}([a,u]_2, [a^{-1}, u^{-1}]_1) \le 2.$$

We first claim that

(5)
$$d_{\mathcal{DG}}(\psi_{b_1^{k_1}\cdots b_{n-1}^{k_n-1}b_n^{k_n}}(D),\psi_{b_1^{k_1}\cdots b_{n-1}^{k_n-1}b_n^{\ell_n}}(D)) \le 6|\ell_n-k_n|+8$$

Namely, the estimate (2) and the fact that point-pushing of p_2 induces an isometry on \mathcal{DG} imply that

$$d_{\mathcal{DG}}(\psi_{b_n^{k_n}}(D),\psi_{b_n^{\ell_n}}(D)) \le 6|\ell_n - k_n|.$$

But for all u, we have $\psi_{b_n^u}(D) = [a \cdot b_n^u]_2$ and hence the estimate (4) shows that

$$d_{\mathcal{DG}}([a^{-1} \cdot b_n^{-k_n}]_1, [a^{-1} \cdot b_n^{-\ell_1}]_1) \le 6|k_n - \ell_n| + 4.$$

Apply to both disks $[a^{-1} \cdot b_n^{-k_n}]_1, [a^{-1} \cdot b_n^{-\ell_n}]_1$ point-pushing of the point p_1 along a loop based at p_1 representing the homotopy class $\Phi(b_{n-1}^{-k_n-1} \cdots b_1^{-k_1})$. As pointpushing induces an isometry on the disk graph (and composition is read from left to right), we obtain

$$d_{\mathcal{DG}}([a^{-1} \cdot b_n^{-k_n} b_{n-1}^{k_{n-1}} \cdots b_1^{-k_1}]_1, [a^{-1} \cdot b_n^{-\ell_n} b_{n-1}^{k_{n-1}} \cdots b_1^{-k_1}]_1) \le 6|k_n - \ell_n| + 4.$$

Using again the estimate (4), this is yields the estimate (5) we wanted to show.

Point-pushing of the point p_2 along the loop $b_n^{-\ell_n}$ transforms the *pair* of disks $(\psi_{b_1^{k_1}\cdots b_{n-1}^{k_n-1}b_n^{\ell_n}}(D), D)$ to the pair $(\psi_{b_1^{k_1}\cdots b_{n-1}^{k_n-1}}D, \psi_{b_n^{-\ell_n}}(D))$. As point-pushing acts as an isometry on the disk graph, we view this operation as a change of basepoints which does not change distances.

In a second step, we use the reasoning which led to the estimate (5) to deduce that

$$d_{\mathcal{DG}}(\psi_{b_{1}^{k_{1}}\cdots b_{n-2}^{k_{n-2}}b_{n-1}^{k_{n-1}}}(D),\psi_{b_{1}^{k_{1}}\cdots b_{n-2}^{k_{n-2}}b_{n-1}^{\ell_{n-1}}}(D)) \leq 6|\ell_{n-2}-k_{n-2}|+8.$$

As a next step, we change the basepoint again. Using point-pushing of the point p_2 along the loop $b_2^{-\ell_{n-1}}$, the pair $(\psi_{b_1^{k_1}\cdots b_{n-2}^{k_{n-2}}b_{n-1}^{\ell_{n-1}}(D), \psi_{b_n^{-\ell_n}}D)$ transforms to the pair

$$(\psi_{b_1^{k_1}\cdots b_{n-2}^{k_{n-2}}}(D),\psi_{b_n^{-\ell_n}b_{n-1}^{-\ell_{n-1}}}D).$$

Proceeding inductively, in *n* steps we transform the pair $(\psi_{b_1^{k_1}b_2^{k_2}\cdots b_n^{k_n}}(D), D)$ to the pair $(D, \psi_{b_n^{-\ell_n}\cdots b_1^{-\ell_1}}(D))$, changing distances by at most $\sum_i (6|\ell_i - k_i| + 8)$.

Now apply one last time point-pushing of the point p_2 along the loop $b_1^{\ell_1} \cdots b_n^{\ell_n}$ to the pair

$$(D, \psi_{b_n^{-\ell_n} \cdots b_1^{-\ell_1}}(D))$$

and obtain the pair $(\psi_{b_1^{\ell_1}\dots b_n^{\ell_n}}(D), D)$. Using again that point pushing is an isometry, we conclude that the distance between the disk $\Lambda(k_1, \dots, k_n) = \psi_{b_1^{k_1}\dots b_n^{k_n}}(D)$ and the disk $\Lambda(\ell_1, \dots, \ell_n) = \psi_{b_1^{\ell_1}\dots b_n^{\ell_n}}(D)$ is at most $\sum_i (6|\ell_i - k_i| + 8)$ as claimed.

Now Lemma 4.15 of [SS14] and the discussion on the bottom of p.592 and on the top of p.594 in [SS14] shows that there is a number c > 0 such that

(6)
$$\sum_{i=1}^{n} |k_i - \ell_i| \le c |b_n^{-k_n} b_{n-1}^{-k_{n-1}} \cdots b_1^{-k_1} b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}|_{g+1}^{simple}.$$

We give a short summary of the proof of this fact as found in [SS14]. Namely, following Definition 4.9 of [SS14], we say that a word w in the letters $\mathcal{A} \cup \mathcal{A}^{-1}$ has conjugate reduced length at most k if there exist freely reduced words $v_1, \ldots, v_{\ell}, u_1, \ldots, u_{\ell}$ such that.

(a)
$$w = v_1^{u_1} v_2^{u_2} \cdots v_{\ell}^{u_{\ell}}$$
, where $v_j^{u_j} = u_j^{-1} v_j u_j$, and

(b)
$$k = (\ell - 1) + |v_1|_{q+1}^{simple} + \dots + |v_\ell|_{q+1}^{simple}$$

The number k is called the *conjugate reduced* g + 1-length associated to the decomposition. The minimal number k for which such a decomposition exists is called the *conjugate reduced length of* w, and it is denoted by $|w|^{cr}$.

The easy Lemma 4.15 of [SS14] states that $|w|_{g+1}^{simple} \ge |w|^{cr}$, so it suffices to estimate $|w|^{cr}$ from below for $w = b_n^{-k_n} b_{n-1}^{-k_{n-1}} \cdots b_1^{-k_1} b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}$.

Definition 4.10 of [SS14] is geared to this end. A cancelling pair in the reduced word w is a pair of subwords of the form u, u^{-1} . A nested family \mathcal{F} of cancelling pairs is a finite collection of disjoint cancelling pairs so that if $v, v^{-1} \in \mathcal{F}$ and $u, u^{-1} \in \mathcal{F}$ then v occurs between u, u^{-1} if and only if this is true for v^{-1} . For such a family \mathcal{F} of cancelling pairs let $w - \mathcal{F}$ be the finite collection of subwords of wobtained by erasing the words from \mathcal{F} . Define

$$|w - \mathcal{F}|_{g+1}^{simple} = |\mathcal{F}| + \sum_{w' \in w - \mathcal{F}} |w'|_{g+1}^{simple}.$$

The required estimate follows from Lemma 4.11 of [SS14] which states that

(7)
$$|w|^{cr} \ge \min_{\mathcal{F}} (\max\{\frac{|\mathcal{F}|}{2} - 1, \frac{1}{5}|w - \mathcal{F}|_{g+1}^{simple} - 3\}).$$

28

To apply this estimate to the above word w, let \mathcal{F} be a nested family of cancelling pairs for w which minimizes the expression on the right hand side of equation (7) and write $d = \sum |k_i - \ell_i|$. If $|\mathcal{F}| \ge d/10$ then we immediately obtain the required estimate. Otherwise note that by removing a cancelling pair we can at most delete a subword of a string of the form $b_i^{\min\{k_i,\ell_i\}}$. Furthermore it is easy to see that $|b_t^s|_{g+1}^{simple} \ge |s|$ for all s. Thus if $|\mathcal{F}| \le d/10$ then a rough counting of the simple norm of the subsegments of $w - \mathcal{F}$ as carried out in detail on p.593 of [SS14] yields again the required estimate.

On the other hand, by Lemma 3.10, we have

(8)
$$d_{\mathcal{DG}}(\Lambda(k_1,\ldots,k_n),\Lambda(\ell_1,\ldots,\ell_n)) \ge \frac{1}{2} |b_n^{-k_n} b_{n-1}^{-k_{n-1}} \cdots b_1^{-k_1} b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}|_{g+1}^{simple}.$$

The estimates (3), (6) and (8) together show that the distance in \mathcal{DG} of the images of D under the point pushing of p_2 along $b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}$ and by $b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}$ is bounded from above and below by a fixed positive multiple of $\sum_{i=1}^{n} |k_i - \ell_i|$. Thus the map $\Lambda : \mathbb{Z}^n \to \mathcal{DG}$ is a quasi-isometric embedding. The proposition for handlebodies of even genus is established.

The argument can be adjusted for handlebodies of odd genus as follows. Let H_0 be such a handlebody of genus $g \geq 3$. Choose a non-separating *I*-bundle generator c. Then H_0 is the oriented *I*-bundle over a non-orientable surface F with connected boundary $\partial F = c$. The surface F can be obtained from an orientable surface F_0 of genus (g-1)/2 whose boundary consists of 2 connected components c_0, c_1 by attaching a Möbius band to c_1 . The orientation cover of F equals the complement in ∂H_0 of an open annulus with core curve c, and the preimage of F_0 consists of two copies of F_0 which are glued along an annulus. The fundamental group of F_0 is a free group in g generators, and the inclusion of a component of its preimage in ∂H_0 into H_0 defines an isomorphism on fundamental groups.

The argument in the beginning of this proof now applies verbatim using the surface F_0 instead of F and noting that we may choose disjoint generating arcs for the fundamental group of $F_0 \subset F$ with endpoints on the boundary of F with the property that there is a partition of ∂F into $(g-1)/2+1 \geq 2$ disjoint intervals, each containing the endpoints of one or two arcs. This suffices to control the distance in the disk graph of a disk obtained from the base disk D by point pushing p_2 along a loop defined by the word b_t in the corresponding generators. The rest of the argument is identical to the argument for handlebodies of even genus.

Remark 3.12. The argument in the proof of Proposition 3.11 indirectly uses the fact that the pure mapping class group of a disk with two punctures is infinite cyclic and consequently point pushing of p_1 leaving p_2 fixed commutes (in an appropriate sense) with point pushing of p_2 leaving p_1 fixed.

The proof of the upper distance bound which appears in the proof of Proposition 3.11 also applies if we view the disks as elements of the curve graph of the boundary surface ∂H of the handlebody, but the resulting estimate is irrelevant in this case. Namely, the distance in the curve graph of all the diskbounding simple closed curves considered equals two as there exists a curve on ∂H (for example, a component of the boundary of a tubular neighborhood of the rose R used in the construction) which is disjoint from all the disks. Such a curve is not diskbounding.

The arguments in the proof of Proposition 3.11 also yield the second part of Theorem 3.

Corollary 3.13. The sphere graph of a doubled handlebody $\sharp_g S^2 \times S^1$ $(g \ge 2)$ with two spots contains for every $n \ge 2$ a quasi-isometrically embedded copy of \mathbb{R}^n .

Proof. The proof of Proposition 3.11 applies almost verbatim to the doubled handlebody M with two spots, i.e. to the connected sum of g copies of $S^2 \times S^1$ with two spots p_1, p_2 . If we identify small disjoint compact embedded balls B_1, B_2 in M containing the spots p_1, p_2 on the boundary with an orientation reversing diffeomorphism, then we obtain a connected sum N of g + 1 copies of $S^2 \times S^1$ with one marked point p which is the image of the identified points p_1, p_2 .

Any sphere in M enclosing the spots (i.e. a sphere whose image under the spot removal map is contractible) defines a one-edge free splitting of the fundamental group $\pi_1(N, p)$ of N into the free group F_g with g generators, identified with the fundamental group of M, and an infinite cyclic group. Namely, as in the case of a handlebody, such a sphere can be enlarged to an embedded ball in $M \cup \{p_2\}$ which contains p_2 and is disjoint from the ball B_1 . The splitting is now a consequence of van Kampen's theorem.

Point pushing of one of the spots along paths in a fixed embedded rose in M (which we may assume to be contained in the boundary of an embedded twice spotted handlebody whose double equals M) acts on these splittings $F_{g+1} = F_g * \mathbb{Z}$ by appending the point-pushing element to the generator of the free factor \mathbb{Z} . Note that this is an immediate consequence of the argument for the handlebody whose fundamental group coincides with the fundamental group of M. Now the calculation in the proof of Proposition 3.11 only uses information on splittings of the free group with g + 1 generators and therefore this calculation is also valid for spheres and yields the corollary.

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